Required Project II: The logistic map, Part II

Note: If you are asked to write an applet, you are expected to put a working version of it into the drop box. You are also expected to turn in hard copy solutions which include listings of all the applets.

In this project you will study in detail the properties of the sequence of period-doubling bifurcations of the logistic map

$$x_{i+1} = f_r(x_i) = rx_i(1 - x_i). \quad \text{(RP2.1)}$$

In Required Project 1 you studied a few of these bifurcations by calculating the orbits of period 1, 2, and 4. Now you will examine the properties of the period-doubling sequence in detail. Chapter 4 of the class notes presents a strategy for computing orbits with very long periods. We expect you to use this strategy when you write the applets for this project probing the properties of these long-period orbits (unless, of course, you come up with a computational scheme that works even better!).

The reason why it is interesting to study these long-period orbits is that they have universal scaling properties. First you will calculate the high-period orbits in the period-doubling sequence and find properties of these orbits that obey scaling laws. "Universality" means that these scaling laws are characterized by exponents that are exactly the same for a huge class of different mapping functions. You will explore this universality and understand why it arises by using renormalization-group equations.

First you will demonstrate scaling empirically, by explicit computation of orbits in the period-doubling sequence.

Scaling in parameter space.

RP2.1. Write an applet that computes a sequence of r values, r_0, r_1, . . ., r_n, for the superstable 2^n-cycles of the logistic map using Newton's method. You should be able to get as far as n = 10. You should see that the r values are converging to something close to 3.57.

This sequence of r-values converges to an accumulation value, r_∞, at a geometric rate. That is,

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$$r_n \approx r_\infty - \delta^{-n} \qquad \text{(RP2.2)}$$

where δ is a number that we want to calculate. We can rewrite equation (RP2.2) as[1]

$$\delta \approx \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} \qquad \text{(RP2.3)}$$

Since you have already computed the r_n's, it is a simple matter to plug them into equation (RP2.3) to get an estimate of δ. Please have your applet do so.

Scaling in state space. In RP2.1 you calculated how the parameter value r changes as one goes from a 2^n to a 2^{n+1} superstable cycle¾it gets scaled by a factor δ. This is a scaling in "parameter space," since r is a parameter of the map. Now you will characterize the scaling in x (which is called "state space," since x describes the state of the system).

Again consider the superstable 2^n cycles. Every superstable cycle must contain x = 1/2, the point where the function reaches its maximum value (and thus its derivative is zero). Now we ask, for a cycle of length 2^n that starts at $x = 1/2$, what is $f_{r_n}^{2^{n-1}}(x = 1/2)$, the value of x exactly halfway through the cycle?

RP2.2. We claim that $f_{r_n}^{2^{n-1}}(x = 1/2)$ is closer to 1/2 than any other point in the orbit. Please write an applet that checks this claim numerically for a few values of n.

RP2.3. Have your applet compute the $y_n = x_{2^{n-1}} = f_{r_n}^{2^{n-1}}(x = 1/2)$ for n = 2 through 10. You should find that the y_n's also converge geometrically with n:

$$\frac{1}{2} - y_n \approx (-\alpha)^{-n}, \qquad \text{(RP2.4)}$$

where

[1] Equation (RP2.3) is obtained by taking equation (RP2.2) for the values n and n-1, and subtracting them, giving $r_n - r_{n-1} = \delta^{-(n-1)} - \delta^{-n} = \delta^{-n}(\delta - 1)$. Do the same thing for n-1 and n-2: $r_{n-1} - r_{n-2} = \delta^{-(n-1)}(\delta - 1)$. The ratio of these two equations is $\frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} = \frac{\delta^{-(n-1)}(\delta - 1)}{\delta^{-n}(\delta - 1)} = \delta$, which is equation (RP2.3).

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$$-\alpha \approx \frac{y_{n-1} - y_{n-2}}{y_n - y_{n-1}}. \qquad \text{(RP2.5)}$$

Please find α.

Notice that -α is negative. This means that these points are falling on alternate sides of $x = 1/2$.

RP2.4 . At the parameter value r = r_∞ ≈ 3.569945669, the period-doubling sequence of the logistic map accumulates and there is a "cycle" of length 2^∞. At this value of r, the long term iterates of the map fall on a fractal. First generate some graphical output to show that the fractal is self-similar (keep expanding ever smaller regions of the iterates, and show that they replicate the shape of the whole set). Then estimate numerically the box-counting fractal dimension of the set. Try to relate this result for the fractal dimension to the values of α and/or δ that you calculated above.

Universality. Now we ask you to look at the question of universality empirically by computing the exponents for the period-doubling sequences of other functions.

Consider the following functions:

$$F_r(x) = r\sin(\pi x) \quad \text{for } 0 < x < 1,\ 0 < r < 1; \qquad \text{(RP2.6)}$$

$$G_r(x) = re^{-(x-1)^2} \quad \text{for } 0 < x < 4,\ 0 < r < 4. \qquad \text{(RP2.7)}$$

Both these functions have single humps; (RP2.6) has its maximum at x = 0.5 (like the logistic map), while (RP2.7) has its maximum at x = 1.[2]

RP2.5 Choose one of the functions above, (RP2.6) or (RP2.7), and, using modified versions of the programs you used for the logistic map: (a) Draw the bifurcation diagram. (b) Compute α and δ. If you are using equation (RP2.7), be sure to take into account that the maximum is at x = 1, not x = 0.5.

RP2.6. Write an applet that generates the time series for the logistic map with r = r_∞, starting with $x_0 = \frac{1}{2}$ (it should be easy to

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[2]Both of these functions have quadratic maxima. You can verify this for yourself by taking two derivatives of each function, and plugging in the x value at the maximum (i.e., the top of the hump).

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modify your applet from RP2.4 to do this). Let $k=2^n j$ (in other words, consider only every 2^nth iterate) with, say, n=3. Demonstrate that the x's near the maximum $x = \frac{1}{2}$ appear to obey the equation

$$-\alpha\left(x_{2k} - \frac{1}{2}\right) = x_k - \frac{1}{2}. \qquad (5.1)$$

The renormalization group. The class notes show that a function g(z) (with z=x-1/2) which generates a time series with the self-similarity property (5.1) must obey the renormalization group equation

$$g(z) = -\alpha g\left(g\left(-\frac{z}{\alpha}\right)\right). \qquad (5.6)$$

RP2.7. In the class notes the exponent α characterizing the period-doubling route to chaos is calculated for functions which have a quadratic maximum (the generic case, which includes the logistic map). Here you will calculate the exponents that characterize the period-doubling route to chaos for a function with a particular type of cusp, specifically

$$f(x) = r\left[\left(\frac{1}{2}\right)^{3/2} - \left|x - \frac{1}{2}\right|^{3/2}\right].$$

(a) Estimate the values of α and δ for this map using the method you used for the logistic map in RP2.1 and RP2.3 (i.e., by calculating the sequence of superstable 2^n cycles, and finding δ from the r_n values and α from distance that the $(2^{n-1})^{th}$ iterate of x=1/2 is from x = 1/2).

(b) The renormalization group equation for a function g(z) (where z=x-1/2) that embodies the self-similarity of the time series at $r = r_\infty$ follows solely from self-similarity and does not care at all about whether the function has a cusp, so it is identical to that for the logistic map:

$$-\alpha g\left(g\left(-\frac{z}{\alpha}\right)\right) = g(z).$$

For functions with a $\frac{3}{2}$-power cusp, we can expand g(z) near z=0 as:

$$g(z) = A - B|z|^{3/2} + Cz^2 + \dots ,$$

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and solve the RG equation approximately by ignoring terms of order higher than $|z|^{3/2}$. Find an approximate value for α using this method. (Hint: When you equate the coefficients of the powers of z, you will find an algebraic equation that α must satisfy. You may solve this equation numerically using your Newton-Raphson routine, or a package such as Maple, Matlab, or Mathematica.) How does it compare with the result you obtained by direct computation of the orbits? How would you improve the accuracy of your approximate solution to the renormalization-group equation?

Symmetry of the renormalization group equation. In the class notes, the renormalization-group equation is solved by expanding g(z) in Taylor series and truncating the series at order z^2. There were two equations and two Taylor-series coefficients, but the coefficients entered into both equations in only one combination, and α needed to be fixed in order to satisfy both equations. You should have found a similar phenomenon when you solved the renormalization-group equation for the map with a 3/2-cusp (2 equations, 2 coefficients in the expansion of g(z), and yet a solution exists only for a special value of α). Now we ask you to investigate a symmetry of the renormalization-group equation that underlies this behavior.

RP2.8. Verify that if g(z) satisfies the renormalization group equation (5.6), then βg(z/β) does also. Explain why this result justifies setting g(0) = 1, and why this in turn means that the renormalization group equation cannot be solved unless α has a particular value.

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