Chapter 8:

Higher-Dimensional Dynamical Systems

Goals:

- To explore the behavior of nonlinear Hamiltonian systems, and, in particular, the standard map.
- To classify possible orbits of two-dimensional dynamical systems.
- To understand Lyapunov exponents and how to calculate them.

Chapters 6 and 7 focused on dynamical systems described by Newton's laws, all of which have at least two degrees of freedom. In this chapter we will compare these higher-dimensional systems to the one-dimensional systems that we classified in Chapter 3. First, we will examine the special properties of Hamiltonian systems (all of which have at least two degrees of freedom). Then we will classify the possible behaviors of smooth two-dimensional dynamical systems, and compare them to those that we found for one-dimensional maps in Chapter 3.

A. Hamiltonian Systems. At the end of the last chapter we saw that Hamiltonian systems are qualitatively different than damped systems because their dynamics conserve volumes in phase space. In this chapter we explore the consequences of this difference. We start by examining the pendulum, described by the second order ordinary differential equation:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta . \qquad (8.1)$$

To make things a bit simpler, we fix $g/L = 1$ (we can do this by scaling time: we define $\tau = t(g/L)^{1/2}$ and write the equation in terms of τ). The motion of the pendulum is periodic (in other words, the orbits close); you calculated the period in Problem 6.4.

A.1. The importance of conserving phase space areas: two different approximations. As we have seen, the way to solve a

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second order equation like (8.1) is to break it into two first order equations:

$$\frac{d\theta}{dt} = p ,$$
$$\frac{dp}{dt} = -\sin\theta \ . \qquad (8.2)$$

In the last two chapters, we solved equations of this type using the fourth-order Runge-Kutta method, but here we will use a simpler technique to highlight what happens when our approximations violate conservation of phase space areas. We will do the most obvious thing: write θ_{n+1} and p_{n+1} in terms of θ_n and p_n by approximating the time derivatives as finite differences:

$$\theta_{n+1} = \theta_n + \Delta p_n$$
$$p_{n+1} = p_n - \Delta \sin\theta_n , \qquad (8.3)$$

where Δ is the time step (make sure you understand why this is the obvious thing to do).

Problem 8.1. Write the program that implements (8.3) and make phase space plots of the pairs (θ,p) (when calculating p_{n+1}, be sure you use θ_n and not θ_{n+1}!). Start with initial conditions $p_0 = 0$, $\theta_0 = \pi/3$. Try Δ = 0.01. Does the orbit close? How about for Δ = 0.1? How does the "error" (in this case, the distance by which the orbit fails to close) depend on the step size Δ? This divergence occurs even though equation (8.1) should conserve energy. Why is this?

You see that with this scheme, the orbits don't close. No matter how small the step size, if you wait long enough, the orbit will diverge.

In the last chapter we saw that for a map defined by:

$$\mathbf{z}_{j+1} \equiv \begin{pmatrix} x_{j+1} \\ y_{j+1} \end{pmatrix} = \mathbf{f}(\mathbf{z}_j) \equiv \begin{pmatrix} g(x_j, y_j) \\ h(x_j, y_j) \end{pmatrix} , \qquad (8.4)$$

the change in phase space area A when j is incremented by 1 obeys:

$$\frac{\delta A}{A} = \left(\frac{\partial g}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial h}{\partial x} - 1 \right) . \qquad (8.5)$$

As you may recall, the Jacobian matrix J of the map (8.4) is defined as:

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$$\mathbf{J} = \begin{pmatrix} \partial g/\partial x & \partial g/\partial y \\ \partial h/\partial x & \partial h/\partial y \end{pmatrix}. \tag{8.6}$$

Therefore we can write (8.5) as:

$$\frac{\delta A}{A} = \det \mathbf{J} - 1 \ . \tag{8.7}$$

Exercise 8.1. Compute J for (8.3), and find its determinant. You should find that it does not always equal 1, rather it has a term proportional to Δ^2 .

Thus the approximation (8.3) does not agree with an important qualitative property of all Hamiltonian systems: the conservation of phase space volume. Hence we should not be surprised if the approximation fails to capture all kinds of qualitative properties of the true solution. In particular, as we have seen, orbits which we know are closed refuse to close within this approximation.

For this particular equation there is a simple way of fixing it up so that it is area preserving. (This is a trick that works in only a few cases. More generally, if the first order solution analogous to (8.3) doesn't work well enough for your purposes, you have to go to a more complicated, higher order method of solving the equation. We won't go into these methods now, but see Numerical Recipes, chapter 15, if you are interested). The trick is to write

$$\begin{aligned} \theta_{n+1} &= \theta_n + \Delta p_{n+1} \\ p_{n+1} &= p_n - \Delta \sin \theta_n \end{aligned} \tag{8.8}$$

Exercise 8.2: Compute J for (8.8), and verify that its determinant is unity. Fix up your program from Problem 8.1. The difference should be dramatic, especially for large Δ.

A.2. The Standard Map. We introduced equations (8.8) as an approximation to the equations of motion of the pendulum. But these equations are worth investigating in their own right.

If we define $r_j = \Delta p_j$ and $k = \Delta^2$, we can rewrite (8.8) as:

$$\begin{aligned} \theta_{j+1} &= \theta_j + r_{j+1}, \\ r_{j+1} &= r_j - k\sin(\theta_j) \,. \end{aligned} \tag{8.9}$$

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In interpreting equations (8.9) it is sometimes (though not always) useful to think of θ as a phase, so that θ_j can be shifted by an amount equal to 2πn, where n is an integer, without changing the physics. With this interpretation, we choose to write equations (8.9) as:

$$\theta_{j+1} = \theta_j + r_{j+1} \mod 2\pi \qquad (-\pi \le \theta_j < \pi \text{ for all j})$$

$$r_{j+1} = r_j - k\sin(\theta_j). \qquad (8.10)$$

Equations (8.10) are known as the "standard map." One physical situation in which they appear is a circular accelerator in which a beam goes around so that the magnitude of its momentum is constant except in a small acceleration section. Let p_j be the momentum of the beam during the j^{th} circuit. In the acceleration section, the beam sees an oscillating (AC) field which gives it an impulsive kick that depends on the phase, θ_j, at which it enters the acceleration section. If the kick is proportional to $\sin(\theta_j)$, we have

$$p_{j+1} = p_j - K\sin(\theta_j). \qquad (8.11)$$

But the phase at which the beam enters the next time, θ_{j+1}, is advanced by an amount proportional to its momentum, so that

$$\theta_{j+1} = \theta_j + Cp_{j+1}. \qquad (8.12)$$

The pair (8.11) and (8.12) is identical to the set (8.9) if we define $r_j = Cp_j$ and $k = CK$.

There are three kinds of phase space orbits for the Standard Map:

i) Periodic--a finite number of points.

ii) Chaotic--the orbit apparently fills an area.

iii) KAM curve--the orbit falls on a curve (not necessarily continuous).

For a given k, chaotic, periodic, and KAM orbits can all be observed, depending on the initial conditions.

Problem 8.2 . Exploring the behavior of the Standard Map. By plotting a phase space portrait, you should be able to locate all three kinds of behavior. Create an applet to do this. Use k = 0.75. The easiest way to observe all three kinds of behavior is to start at, say, r = 0.1, θ = 0, and then keep increasing the initial r. What does the portrait look like for small r? At what value of r do things change? What does it change to? As k is decreased, do the chaotic regions become easier or harder to find?

> Menu Project. Mapping Phase Space Regions for the Standard Map. In this project, you will characterize the evolution of areas in phase space for the standard map (8.10). Take a whole bunch of points (like maybe a thousand) bunched into a fairly small area, and iterate them all by (8.10), and see what the resulting region looks like. Typically, it will be a distortion of the original region, but it should have the same area. Try to determine if the area seems to be preserved after several iterations. Then see what you can say about how the shape of the area evolves as the map is iterated. Do this also with mapping (8.3), and compare the two cases.

> Menu Project. Dissipative Standard Map. In this project, you will study the map defined by:
>
> $$\theta_{j+1} = \theta_j + p_{j+1},$$
> $$p_{j+1} = bp_j + (1-b)\Omega - k\sin(\theta_j),$$
>
> where $0 \le b \le 1$. This map, called the dissipative standard map, is identical to the standard map when the parameter b is equal to 1. By calculating how phase space areas evolve, you should be able to understand the physical significance of the parameter b. What is the physical significance of the parameter Ω? Calculate the phase space portraits for this map, and characterize the different types of behavior.

Although, as we saw, the Standard Map can be viewed as an approximation to the equations of motion of a pendulum, the Standard Map has chaotic orbits (for some initial conditions) whereas an undamped undriven pendulum does not (recall from chapter 6 that conservation of energy implies that the motion of an undamped undriven pendulum is always periodic). The results of problem 8.2 should help you to see why there is no inconsistency. For a pendulum to exhibit chaos, it must be subject to a driving force. This is the subject of the next project.

> Menu Project. The Undamped Driven Pendulum. This project concerns the dynamics of the undamped driven pendulum, described by the equation of motion:
>
> $$\frac{d^2\theta}{dt^2} = -\sin\theta + F_0 \sin(\omega t).$$
>
> Make some phase space plots for this system, being careful to solve the equations of motion accurately enough so that the orbits appear to close when $F_0 = 0$. How does increasing F_0 change the behavior? If you plot the (p,θ) pairs at discrete times separated by the drive period, how do these phase space plots compare to those for the standard map?

B. Classifying orbits for 2d maps. In Chapter 3 we classified all the possible behaviors for a smooth one-dimensional map near a fixed point. Orbits were classified as stable, superstable, marginally stable, or unstable. We now examine stability for two dimensional maps and systems of differential equations.

B.1. Linearization of 2d maps. We ask the same question for 2d maps that we asked in the 1d case: How do orbits behave near a fixed point? We will answer the question in the same way, by linearizing the map in the vicinity of the fixed point. So, assume z* is a fixed point of the map f(z*), i.e.

$$\mathbf{z}^* \equiv \begin{pmatrix} x^* \\ y^* \end{pmatrix} = \mathbf{f}(\mathbf{z}^*) \equiv \begin{pmatrix} g\begin{pmatrix} x^* \\ y^* \end{pmatrix} \\ h\begin{pmatrix} x^* \\ y^* \end{pmatrix} \end{pmatrix}$$

(we will use vector notation and column vector/matrix notation interchangeably). To linearize the map f, we use the 2 dimensional Taylor expansion, so that if z_0 is close to z*, we have

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$$\mathbf{z}_1 \equiv \mathbf{f}(\mathbf{z}_0) = \mathbf{f}(\mathbf{z}^* + (\mathbf{z}_0 - \mathbf{z}^*))$$
$$\approx \mathbf{z}^* + \mathbf{J}(\mathbf{z}^*)(\mathbf{z}_0 - \mathbf{z}^*) \qquad (8.11)$$

where J(z*) is the Jacobian matrix evaluated at z = z* :

$$\mathbf{J} = \begin{pmatrix} \partial g/\partial x & \partial g/\partial y \\ \partial h/\partial x & \partial h/\partial y \end{pmatrix}. \qquad (8.12)$$

(Equations (8.11) and (8.12) are equivalent to equation (7.4) from chapter 7.) If you are unclear on the derivation of these equations, you can work it out yourself without the vector notation. Use the fact that $x_1 \approx x^* + \frac{\partial g}{\partial x}(x_0 - x^*) + \frac{\partial g}{\partial y}(y_0 - y^*)$ for a function of two variables (likewise for y). Putting x and y back into vector notation should lead you to equations (8.11) and (8.12).

Equation (8.11) asserts that the initial difference vector z_0 - z* is transformed by the matrix J(z*) to the iterated difference vector, z_1 - z*. So to understand what happens to points close to the fixed point, we need to study the structure of the matrix J(z*). Recall that any vector z can be written in terms of the eigenvectors of J(z*) (for reasonable J's). Call the eigenvectors e_i. Then the vector z_0 - z* can be written as $\mathbf{z}_0 - \mathbf{z^*} = \sum_i A_i \mathbf{e_i}$, with unknown amplitudes A_i for each eigenvector. If we apply J(z*) to e_i, we get $\lambda_i \mathbf{e_i}$, where λ_i is the eigenvalue. Hence, applying J(z*) to z_0 - z* gives us the new iterate

$$\mathbf{z}_1 - \mathbf{z^*} = \mathbf{J}(\mathbf{z}_0 - \mathbf{z^*}) = \sum_i A_i \mathbf{J}\mathbf{e_i} = \sum_i A_i \lambda_i \mathbf{e_i}.$$

Now, if we apply N iterations, we have

$$\mathbf{z}_N - \mathbf{z^*} = \mathbf{J}^N(\mathbf{z}_0 - \mathbf{z^*}) = \sum_i A_i \mathbf{J}^N \mathbf{e_i} = \sum_i A_i \lambda_i^N \mathbf{e_i}$$

which tells us that the eigenvalues of J(z*) determine the behavior of the iterates. Roughly speaking, if J(z*) has eigenvalues that exceed 1 in magnitude, the iterates will diverge from the fixed point because λ^N grows with N (analogous to the 1-dimensional case). However, the classification of fixed points is more complex in two dimensions.

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Let λ_1 and λ_2 be the eigenvalues (which may be complex), with associated eigenvectors e_1 and e_2. Then the following are possible (all are illustrated below):

1) λ_1 and λ_2 both real. Then

i) if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then nearby points are attracted to the fixed point, and approach it along hyperbolas whose axes are e_1 and e_2. The fixed point is attracting.

ii) if $|\lambda_1| < 1$ and $|\lambda_2| > 1$, then the fixed point is called a saddle node. Points that start close to e_1 initially move closer to the fixed point, but ultimately diverge away from it.

iii) if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then nearby points diverge away from the fixed point. The fixed point is repelling.

2) if λ_1 and λ_2 are complex, then they must be complex conjugates of each other. (This is because det J $= \lambda_1\lambda_2$ is real.) Then

iv) if λ_1 and λ_2 lie inside the unit circle, points close to the fixed point spiral into it. The fixed point is called an attracting focus.

v) if the eigenvalues lie on the unit circle, nearby points circle about the fixed point in ellipses. The fixed point is said to be a center.

vi) if they lie outside the unit circle, points near the fixed point spiral away from it; it is called a repelling focus.

3) if $|\lambda_i| = 1$ for some i, and no other eigenvalue has absolute value greater than 1, then whether the orbit eventually diverges from the fixed point depends on higher derivatives, in a fashion similar to the one-dimensional case that we discussed in Chapter 3.

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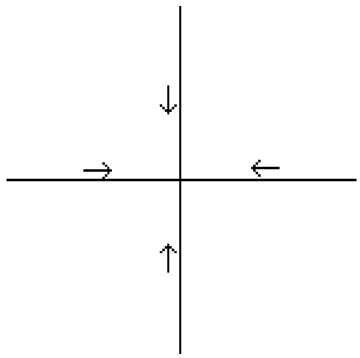
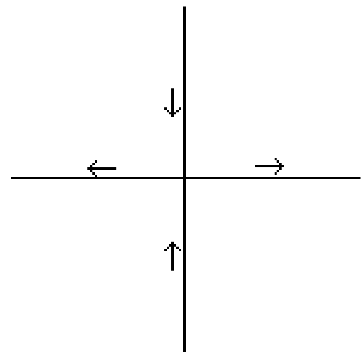
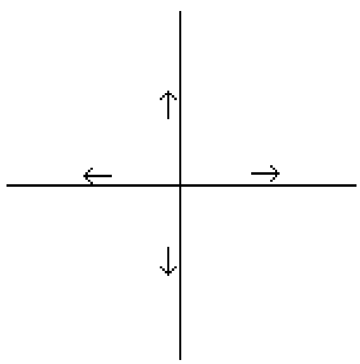


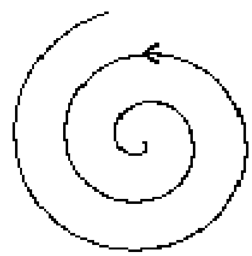
i) Attracting fixed point



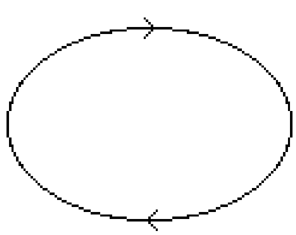
ii) Saddle Node



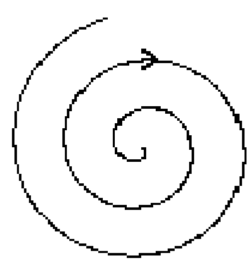
iii) Repelling fixed point



iv) Attracting Focus



v) Center



vi) Repelling Focus

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(Recall that the eigenvalues λ of a matrix J can be found be solving the equation $\det(\mathbf{J}-\lambda\mathbf{I})=0$, where I is the identity matrix.)

Exercise 8.3. Linear maps. One way to observe the cases listed above is to use linear maps. Thus take a general 2-d linear map:

$$\mathbf{x}_{n+1}=\mathbf{f}(\mathbf{x}_n)=\begin{pmatrix}1+\alpha & \beta \\ \gamma & 1+\delta\end{pmatrix}\mathbf{x}_n$$

Choose values of α, β, γ, and δ so as to obtain all six cases listed above, and verify that the behavior is as you expect. For best results, choose small values for α, β, γ, and δ so that the matrix is close to the identity matrix. Give an example (a combination of α, β, γ, and δ) of each type of point. (Hint: avoid the combination $\beta\gamma=\alpha\delta$. Can you figure out why?)

Problem 8.3. Henon Map. The Henon map is

$$\mathbf{z}_{n+1}=\begin{pmatrix}x_{n+1}\\ y_{n+1}\end{pmatrix}=\mathbf{f}(\mathbf{z}_n)=\begin{pmatrix}ab+by_n-x_n^2/b\\ x_n\end{pmatrix}$$

where a and b are parameters, often taken to be a = 1.4, b = 0.3. Find the fixed points of this map, determine their eigenvalues, and verify that nearby orbits behave as expected (you will need to look very close to the fixed points to see the expected behavior).

B.2. Area preserving maps. Recall that Hamiltonian systems have Jacobians with determinant equal to unity. Since det $\underline{J}=\lambda_1\lambda_2=1$, this rules out some of the cases listed above. If one of the eigenvalues is λ, the other must be 1/λ. This eliminates cases i, iii, iv and vi, leaving only the possibility of a saddle node (often called a hyperbolic fixed point in the context of area preserving maps) or a center (often called an elliptic fixed point).

Exercise 8.4 . Using the (area preserving) standard map (8.9), try to find the two possible types of fixed points. Use k = 0.0025. You should have no trouble locating a center; can you also find a saddle node? Plot trajectories starting near the various fixed points in

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phase space. Can you see why the two types of fixed points are called hyperbolic and elliptic?

B.3. Limit Cycles. Up to now we have been talking about stability properties of differential equations

$$\frac{d}{dt}\mathbf{z} = \mathbf{f}(\mathbf{z})$$

or maps (8.13)

$$\mathbf{z}_{j+1} = \mathbf{f}(\mathbf{z}_j)$$

near fixed points (Case 1) or fixed points and cycles (Case 2). There is one more very important possible behavior of a differential equation, which we can illustrate by considering the case of two variables, x and y which obey the ordinary differential equations

$$\frac{d}{dt}x = -y[1+\alpha(x^2+y^2-1)] - kx(x^2+y^2-1)$$

$$\frac{d}{dt}y = x[1+\beta(x^2+y^2-1)] - ky(x^2+y^2-1) \qquad (8.14)$$

These equations have two very simple solutions:

a. A fixed point in which x = y = 0

b. A cycle in which x and y simple rotate around the unit circle

$$x = \cos(t)$$

$$y = \sin(t) \qquad (8.15)$$

Case b can be expressed in polar coordinates with the statement that $r \equiv \sqrt{x^2+y^2}$ is always equal to 1.

Exercise 8.5. For the case in which α = β, show that these are the only possible behaviors as t → ∞ by converting equations (8.14) to polar coordinates. For which initial values of x and y is the fixed point the limiting behavior? For which initial values is the cycle (called a limit cycle) the limiting behavior?

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To see this solution in more general form we suggest you carry out a numerical solution of (8.14) using the fourth order Runge-Kutta method and thereby generate phase plane portraits.

Problem 8.4. Construct a phase plane portrait of the solution to equation (8.14) with $\alpha = 1$, $\beta = 0$, $k = 0.2$. Notice how the solution approaches solution (8.15) for almost any starting value of x and y. By how much does the period you have numerically determined differ from the exact period of 2π? How does this error vary with Δ for small Δ? What is the effect of changing the constant k?

B.4. Flow Diagrams. Flow diagrams are a pictorial way of representing the stability (or instability) of fixed points. When constructing them, we mark the fixed points and then use arrows to indicate the movement of nearby points when iterated. We give some examples below.

In exercise 8.5 you should have recognized that the equation

$$\frac{d}{dt} r = -kr(r^2 - 1)$$

for $r \equiv \sqrt{x^2 + y^2}$ determines the nature of the limit cycle. To see how this works first notice the fixed points of this flow (where dr/dt=0).

r = -1 r = 0 r = 1

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Put arrows on this flow pointing in the direction of dr/dt, i.e. to the right when dr/dt is positive and to the left when dr/dt is negative. Notice that dr/dt can change sign only at the fixed points (where dr/dt=0). Assume k>0.

r = -1 r = 0 r = 1

→ →O← ← ← ← ← ← ← ← ← ←O→ → → → → → → → →O← ← ← ← ←

This picture tells you that the fixed point at r = 0 is unstable and the other two are stable.

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For k<0, the situation is reversed (changing the sign of k changes the sign of dr/dt everywhere).

r = -1 r = 0 r = 1

← ← ←o→ → → → → → → → → →o← ← ← ← ← ← ← ← ← ←o→ → → → → →

Now r=0 is the stable fixed point; the others are unstable. Initial conditions of r>1 or r<-1 will result in the magnitude of r growing without bound.

C. Lyapunov Exponents. One of the hallmarks of a chaotic system is that orbits are extremely sensitive to the initial conditions. Two points initially close together will rapidly diverge from each other as their orbits progress. We have already seen such behavior in the logistic map for r = 4. A quantitative measure of this divergence is the Lyapunov exponent.

One can analyze the stability of any kind of orbit (chaotic or not). To do this, imagine one has constructed a sequence of orbit points

$$x_o, x_1, x_2, \ldots, x_n, \ldots$$

starting from an initial x_o. Then one constructs another orbit starting from an initial value $x_o' = x_o + \delta x_o$, where δx_o is very small. These two orbits initially lie very close together (separated only by δx_o). Then one has x_o', $x_1' = f(x_o')$, . . . , and if δx_o is very small, then:

$$x_n' = x_n + A_n \delta x_o$$

where (using the chain rule):

$$A_1 = \left.\frac{df}{dx}\right|_{x_0}$$

$$A_2 = A_1 \left.\frac{df}{dx}\right|_{x_1}$$

...

$$A_n = \prod_{j=0}^{n-1} \left.\frac{df}{dx}\right|_{x_j} .$$

There is a theorem which states that for large n the limit

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln \left| \left(\frac{df}{dx}\bigg|_{x_j} \right) \right| \tag{8.16}$$

exists for almost all starting values, x_0. Here λ is called the Lyapunov exponent for the motion. If the motion is a stable cycle of length N, then λ will be negative, and in fact

$$\lambda = (\ln|\Lambda|)/N$$

where Λ is the Floquet multiplier for the cycle.

If the motion is chaotic, λ > 0 and e^{λ} represents the average amount (per step) by which the orbits separate from one another. Another say of stating this is that, after N steps, the difference between two orbits will grow as

$$\delta x_N = \delta x_0 e^{\lambda N} \tag{8.17}$$

where δx_N is the separation between the orbits after N iterations. Equation (8.17) applies only when the separation between the orbits is small. If λ is positive, then eventually the orbits will be separated by a distance approaching the size of the attractor, and equation (8.17) will no longer be valid.

Exercise 8.6. For the logistic map $x_{j+1} = rx_j(1-x_j)$ find the value of the Lyapunov index for r = 0.5, 1.5, and 4 by numerical methods. Check your result by using several values of n and x_0. Check them against exact answers you derive. (For r = 4 use the solution $x_j = (1-\cos\theta_j)/2$, $\theta_j = 2^j\theta_o$ to evaluate λ.)

Problem 8.5. Lyapunov Index for Logistic Map. Plot λ against r for the logistic map. Interpret the result. Estimate the error.

For systems of differential equations, if we start from two initial conditions $\mathbf{z}_1(t=0)$ and $\mathbf{z}_2(t=0) = \mathbf{z}_1(t=0) + \delta\mathbf{z}$, the Lyapunov index is defined as:

$$\lambda = \lim_{t\to\infty} \lim_{\delta\mathbf{z}\to 0} \frac{1}{t} \ln|\mathbf{z}_1(t) - \mathbf{z}_2(t)|.$$

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Menu Project. Lyapunov Index for the Lorenz Equations. Recall from Chapter 7 the Lorenz equations:

$$\frac{d}{dt}x = p(y-x)$$

$$\frac{d}{dt}y = -xz + rx - y$$

$$\frac{d}{dt}z = xy - bz \ .$$

Estimate λ for the Lorenz equations for the parameter values p = 10, b = 8/3, and different values of r. What do you think is the error in your estimate?

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