10231: Nonlinear dynamical systems, fall 1998
Homework assignment 4

I.

We want to use Newton's method to get the superstable cycle points of a map of the unit interval and thereby verify Feigenbaum's scaling.

1. To solve the equation $f(x) = 0$, Newton's method starts with a guess, say x_0. Assuming that x_0 is close to an actual root $x^\star$, the function $f(x)$ is expanded to linear order in the deviation $x^\star - x_0$ to find the next guess x_1 (See Strogatz Exercise 10.1.12-13).

$$x_{j+1} = x_j + \Delta x_j = x_j - \frac{f(x_j)}{f'(x_j)} \tag{1}$$

and show that the process is "superconvergent", i.e. $\Delta x_{j+1} \leq \mathrm{const}|\Delta \mathrm{x_j}|^2$, -if it converges at all!

2. Use Newton's method to locate the values R_p where a family of maps $f_R(x) = Rg(x)$ has superstable 2^p-periodic orbits (2^p-cycles). This is easier than finding the bifurcation point, since, as you will see, only one equation needs to be satisfied. Superstability means that

$$\frac{d}{dx} f^{2^p}(x^\star) = 0 \tag{2}$$

for the cycle-point $x^\star$. Show that this implies that the extremum ($x = \frac{1}{2}$) is a cycle-point. Thus the problem can be reduced to finding solutions of

$$F_M(R) - \frac{1}{2} = 0 \tag{3}$$

where $F_M(R) = f_R^M(\frac{1}{2})$. This can be done by Newton's method, which requires the derivative of F_M *with respect to* R. Show that this can be found iteratively as

$$\frac{dF_M(R)}{dR} = \frac{F_M(R)}{R} + f_R'(F_{M-1}(R)) \frac{dF_{M-1}(R)}{dR} \tag{4}$$

3. Implement the process numerically for the logistic map $f_R(x) = Rx(1-x)$ (where $f_R'(x) = R(1-2x)$). Note that a periodic orbit of period, say 2^p is also a solution of (3) for any $M = 2^q$ when $q > p$. Thus you have to make sure you find a *new* solution every time you increase p. This can be done because your knowledge of the value of δ from the text book allows you to make a very good guess for the next

value of R. It is probably a good idea to calculate the first two values, R_1 and R_2 separately and then continue using the scaling. Use the values R_p where the map has a superstable 2^p-cycle to estimate Feigenbaum's δ as

$$\delta = lim_{p\to\infty}\frac{R_p - R_{p-1}}{R_{p+1} - R_p} \tag{5}$$

With the knowledge of R_p up to $p = 5$ you should get two correct decimal places in δ, up to $p = 9$ you should get 5 decimal places.

4. Do the same for the map $f_R(x) = R\sin(\pi x)$ and thereby verify the *universality* of the results.

5. (Extra question, that doesn't count in the evaluation). Can you find α as well?

II.

Consider a "two-scale" Cantor set between 0 and 1. At each level in the construction a piece is cut out of every interval such that an interval of fraction ℓ_1 remains at the left end and one of fraction ℓ_2 remains at the right end. For $\ell_1 = \ell_2 = \frac{1}{3}$ we recover the standard Cantor set.

1. Calculate the similarity dimension, d, of the "single-scale" Cantor set in which $\ell_1 = \ell_2 = \frac{1}{4}$. Verify that it equals the box-dimension, D.

2. Let now $\ell_1 = \frac{1}{4}$ and $\ell_2 = \frac{1}{2}$. Argue that the similarity dimension, $d = \log 3/\log 4$, with a scale factor determined by the **smallest** segment in the Cantor set at a given level in the construction.

3. The similarity dimension does not always give the correct value of the box dimension. The correct value is the so-called Hausdorff dimension determined by the equation

$$\ell_1^D + \ell_2^D = 1 \tag{6}$$

Solve for D (again with $\ell_1 = \frac{1}{4}$ and $\ell_2 = \frac{1}{2}$.)

4. Why do you think the similarity dimension *overestimates* the box dimension? Hint: The similarity argument assumes that the set found on the nth level of construction, is "resolved" down to scale $1/4^n$.

Due at the lecture *Tuesday, December 8.*

Have fun!