10231: Nonlinear dynamical systems, fall 1998
Homework assignment 3

I.

Consider the system

$$\dot{x} = -2xy^2 - x^3, \;\; \dot{y} = -y + x^2 y.$$

1. Show that the system has one fixed point.
2. Can you determine the stability of the fixed point on the basis of the Jacobian matrix?
3. Use the Liapunov function $V = kx^2 + y^2$ for a suitable value of k to determine the stability of the fixed point.

II.

Consider the system

$$\dot{x} = x(1 - 3x^2 - 2y^2) - y, \;\; \dot{y} = y(1 - 3x^2 - 2y^2) + x.$$

1. Show that the system has only one fixed point. Determine its type.
2. Show that the set $C_r = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leqslant r^2\}$ is a trapping region for suitable choices of r.
3. Show that the system has a periodic solution. Confirm by drawing the phase portrait numerically.

III.

Consider a closed container with three chemicals P, A, B. They react according the following scheme:

$$\begin{aligned} P &\to A, \\ A + 2B &\to 3B, \\ B &\to C. \end{aligned}$$

Here C is an end-product which does not react with the other chemicals. Using the law of mass action, and with a suitable scaling, the following differential equations are obtained for the concentrations p, a, b:

$$\dot{p} = -kp, \;\; \dot{a} = kp - ab^2, \;\; \dot{b} = ab^2 - b. \tag{1}$$

In case P is present from the start in a large amount, it is customary to invoke the *pooled chemical approximation* and regard its concentration as constant. Then the equations reduce to

$$\dot{a} = \mu - ab^2, \;\; \dot{b} = ab^2 - b, \tag{2}$$

where μ now is the constant concentration of P.

TURN OVER!

Consider first the pooled equations (2).

1. Show that the system has one fixed point. Determine its stability and show that a Hopf bifurcation occurs at $\mu = 1$.

2. Determine if the bifurcation is super- or subcritical. You can use numerical simulations and/or compute the stability number a (Strogatz, p. 289). The MAPLE programme `selkov.mws` on the course homepage can easily be modified.

3. Make some numerical simulations in the range $\mu \in [0.8, 1.2]$ and locate yet another bifurcation phenomenon of a type we have not encountered before. Describe what you see. Does it make sense physically? A detailed analysis of this bifurcation is somewhat complicated. Can you indicate the source for this unexpected behavior?

Return to the full system (1).

4. Determine the fixed points.

5. Show that all solutions are bounded. You may proceed as follows: First show that $p(t) \to 0$ for $t \to \infty$ (easy!). Then show that if a is sufficiently large, $\dot{a} < 0$. Derive a differential equation for $a + b$ and show that that $\dot{a} + \dot{b} < 0$ for b sufficiently large. Finally, draw the desired conclusion and comment on the physical validity of the two models.

Due at the lecture *Friday, November 20.*