The Hopf bifurcation theorem

$$\dot{\mathbf{x}} = f(\mathbf{x},\mu), \ \ \mathbf{x} \in \mathbb{R}^2, \, \mu \in \mathbb{R}. \tag{1}$$

Assume the system (1) has a fixed point $(\mathbf{x}_0,\mu_0)$ where the Jacobian

$$\mathbf{A} = D_{\mathbf{x}} f(\mathbf{x}_0,\mu_0)$$

has a pair of purely imaginary eigenvalues, $\lambda = \pm i\beta_0, \beta_0 > 0$.

Since $\mathbf{A}$ is regular there is a branch of fixed points $\mathbf{x}(\mu)$ with $\mathbf{x}(\mu_0) = \mathbf{x}_0$ for μ close to μ_0.

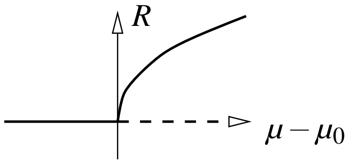
Denote the eigenvalues of the Jacobian at critical point at μ by $\lambda(\mu) = \alpha(\mu) \pm i\beta(\mu)$. We have

$$\alpha(\mu_0) = 0, \beta(\mu_0) = \beta_0.$$

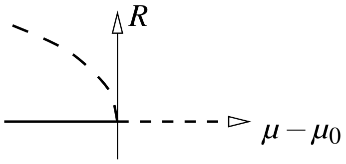
Assume further that the eigenvalues make a strict crossing of the imaginary axis at μ_0, i.e.

$$\frac{d\alpha}{d\mu}(\mu_0) \neq 0.$$

Then a branch of periodic solutions with period approximately $2\pi/\beta_0$ is born at μ_0. Depending on the nonlinearities, there are two generic possibilities, a super- or subcritical bifurcation. In both cases, the amplitude grows as $\sqrt{|\mu - \mu_0|}$.



supercritical



subcritical

Determining the type of a Hopf bifurcation

1. Write down the equations at the bifurcation point μ_0.

2. Make a linear transformation to move the fixed point to the origin and to put the Jacobian in normal form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\beta_0 \\ \beta_0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

3. The type of bifurcation is determined by a given by

$$\begin{aligned} 16a &= f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ &\quad + \frac{1}{\beta_0} \left(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy} \right). \end{aligned}$$

   where $f_{xx} = \frac{\partial^2 f}{\partial x^2}(0,0)$ etc. The bifurcation is supercritical if $a < 0$ and subcritical if $a > 0$. If $a = 0$ higher-order terms determine the type of bifurcation.

If the linearised system *before* the linear transformation is on the form

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \mathbf{A} \begin{pmatrix} u \\ v \end{pmatrix}$$

and $\mathbf{w} = \mathbf{v_1} + i\mathbf{v_2}$ is an eigenvector to $\lambda = i\beta_0$, the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{T} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\mathbf{v_1} & \mathbf{v_2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

will bring the system in the desired form.