mathematical methods - week 9

Fourier transform

Georgia Tech PHYS-6124

Homework HW #9 due Thursday, October 22, 2020

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 9.2 *d-dimensional Gaussian integrals* 5 points
Exercise 9.3 *Convolution of Gaussians* 5 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

edited October 19, 2020

Week 9 syllabus

Tuesday, October 13, 2020

> There is only one thing which interests me vitally now, and that is the recording of all that which is omitted in books. Nobody, as far as I can see, is making use of those elements in the air which give direction and motivation to our lives.
>
> — Henry Miller, *Tropic of Cancer*

This week's lectures are related to AWH Chapter 19 *Fourier Series* (click here), but I prefer Stone and Goldbart [3] (click here) Appendix B exposition, which I follow closely in the online recorded lectures. The fastest way to watch any week's lecture videos is by letting YouTube run the course playlist (click here).

- ChaosBook Sect. A24.4 *Continuum field theory*: Fourier transform as the limit of a discrete Fourier transform.
  - *Propagator in continuum limit*
- Stone and Goldbart (click here) Appendix B.1 *Fourier Series*
  - *Propagator in continuum limit*
  - *Fourier representation, circular Kronecker delta, take #2*
  - *Fourier series*
  - *Circular Dirac delta function*
- Stone and Goldbart (click here) Appendix B.2 *Fourier integral transforms*
  - *Fourier integral transform*
  - *Persival identity*
  - *Fourier transform of a Gaussian*
  - – Exercise 9.3 *Convolution of Gaussians*
  - *Convolution of Gaussians*
  - *Covariance evolution*
    - Cigar is sometimes just a cigar
      - ∗ sect. 9.2 *A bit of noise.*
    - Noise : seminars and papers
  - *The fearful power of symmetry - translational invariance*
  - – example 9.1 *Circulant matrices.*
  - – example 9.2 *Convolution theorem for matrices.*

Optional reading

Discussion: Verbotten! We will not prove Reimann Hypothesis, nor will we explain Wiles proof of Fermat Conjecture in this course. No other course offers intuition. You do not know how lucky you are, boy. You could be back in US back in US back in USSR. As a rule, I do not approve of abuse of children, but Prof. Z is for your own good. Learning from your mistakes is the only way to learn. Countable infinity of professorial opinions. Getting a beating from a class in uprising.

Farazmand notes on Fourier transforms.

Grigoriev notes
4. Integral transforms, 4.3-4.4 square wave, Gibbs phenomenon;
5. Fourier transform: 5.1-5.6 inverse, Parseval's identity, ..., examples

Roger Penrose [2] (click here) chapter on Fourier transforms is sophisticated, but too pretty to pass up.

Alex Kontorovich, on the history of Fourier series: As often happens in mathematics, Fourier was trying to do something completely unrelated when he stumbled on Fourier series. What was it? He was studying the propagation of heat in a uniform medium.

Bernard Maurey, *Fourier, One Man, Several Lives* (2019).

Question 9.1. Henriette Roux asks
Q You usually explain operations by finite-matrix examples, but in exercise 9.3 you asked us to show that the Fourier transform of the convolution corresponds to the product of the Fourier transforms only for continuum integrals. The exercise gives me no intuition for what a convolution *is*.
A "Convolution" is a matrix multiplication for translationally invariant matrix operators. For what that is for discrete Fourier transforms, and what is a "convolution theorem" for matrices, see example 9.2 and *The fearful power of symmetry - translational invariance.*

9.1 Examples

Example 9.1. ***Circulant matrices.*** *An $[L \times L]$ circulant matrix*

$$C = \begin{bmatrix} c_0 & c_{L-1} & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{L-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{L-2} & & \ddots & \ddots & c_{L-1} \\ c_{L-1} & c_{L-2} & \dots & c_1 & c_0 \end{bmatrix}, \tag{9.1}$$

has eigenvectors (discrete Fourier modes) and eigenvalues $Cv_k = \lambda_k v_k$

$$\begin{aligned} v_k &= \frac{1}{\sqrt{L}}(1, \omega^k, \omega^{2k}, \dots, \omega^{k(L-1)})^{\mathrm{T}}, \qquad k = 0, 1, \dots, L-1 \\ \lambda_k &= c_0 + c_{L-1}\omega^k + c_{L-2}\omega^{2k} + \dots + c_1\omega^{k(L-1)}, \end{aligned} \tag{9.2}$$

where

$$\omega = e^{2\pi \mathrm{i}/L} \tag{9.3}$$

is a root of unity. The familiar examples are the one-lattice site shift matrix ($c_1 = 1$, all other $c_k = 0$), and the lattice Laplacian $\Box$.

Example 9.2. ***Convolution theorem for matrices.*** *Translation-invariant matrices can only depend on differences of lattice positions,*

$$C_{ij} = C_{i-j,0} \tag{9.4}$$

All content of a translation-invariant matrix is thus in its first row C_{n0}, all other rows are its cyclic translations, so translation-invariant matrices are always of the circulant form (9.1). A product of two translation-invariant matrices can be written as

$$A_{im} = \sum_j B_{ij} C_{jm} \quad \Rightarrow \quad A_{i-m,0} = \sum_j B_{i-j,0}\, C_{j-m,0}\,,$$

i.e., in the "convolution" form

$$A_{n0} = (BC)_{n0} = \sum_\ell B_{n-\ell,0}\, C_{\ell 0} \tag{9.5}$$

which only uses a single row of each matrix; N operations, rather than the matrix multiplication N^2 operations for each of the N components A_{n0}.

A circulant matrix is constructed from powers of the shift matrix, so it is diagonalized by the discrete Fourier transform, i.e., unitary matrix U. In the Fourier representation, the convolution is thus simply a product of kth Fourier components (no sum over k):

$$UAU^\dagger = UBU^\dagger UCU^\dagger \quad \rightarrow \quad \tilde{A}_{kk} = \tilde{B}_{kk} \tilde{C}_{kk}\,. \tag{9.6}$$

That requires only 1 multiplication for each of the N components A_{n0}.

9.2 A bit of noise

Fourier invented Fourier transforms to describe the diffusion of heat. How does that come about?

Consider a noisy *discrete time* trajectory

$$x_{n+1} = x_n + \xi_n\,, \qquad x_0 = 0\,, \tag{9.7}$$

where x_n is a d-dimensional state vector at time n, $x_{n,j}$ is its jth component, and ξ_n is a noisy kick at time n, with the prescribed probability distribution of zero mean and the covariance matrix (diffusion tensor) Δ,

$$\langle \xi_{n,j} \rangle = 0\,, \qquad \langle \xi_{n,i}\, \xi^T_{m,j} \rangle = \Delta_{ij}\, \delta_{nm}\,, \tag{9.8}$$

where $\langle \cdots \rangle$ stands for average over many realizations of the noise. Each 'Langevin' trajectory $(x_0, x_1, x_2, \cdots)$ is an example of a Brownian motion, or diffusion.

In the Fokker-Planck description individual noisy trajectories (9.7) are replaced by the evolution of a density of noisy trajectories, with the action of discrete one-time step *Fokker-Planck operator* on the density distribution ρ at time n,

$$\rho_{n+1}(y) = [\mathcal{L}\rho_n](y) \; = \int dx \, \mathcal{L}(y,x) \, \rho_n(x) \,, \tag{9.9}$$

given by a normalized Gaussian (work through exercise 9.2)

$$\mathcal{L}(y,x) = \frac{1}{N} \, e^{-\frac{1}{2}(y-x)^T \frac{1}{\Delta}(y-x)} \,, \quad N = (2\pi)^{d/2}\sqrt{\det(\Delta)} \,, \tag{9.10}$$

which smears out the initial density ρ_n diffusively by noise of covariance (9.8). The covariance Δ is a symmetric $[d \times d]$ matrix which can be diagonalized by an orthogonal transformation, and rotated into an ellipsoid with d orthogonal axes, of different widths (covariances) along each axis. You can visualise the Fokker-Planck operator (9.9) as taking a δ-function concentrated initial distribution centered on $x = 0$, and smearing it into a cigar shaped noise cloud.

As $\mathcal{L}(y,x) = \mathcal{L}(y-x)$, the Fokker-Planck operator acts on the initial distribution as a *convolution*,

$$[\mathcal{L}\rho_n](y) \; = \; [\mathcal{L} * \rho_n](y) \; = \int dx \, \mathcal{L}(y-x) \, \rho_n(x)$$

Consider the action of the Fokker-Planck operator on a normalized, cigar-shaped Gaussian density distribution

$$\rho_n(x) = \frac{1}{N_n} e^{-\frac{1}{2}x^T \frac{1}{\Delta_n} \, x} \,, \qquad N_n = (2\pi)^{d/2}\sqrt{\det(\Delta_n)} \,. \tag{9.11}$$

That is also a cigar, but in general of a different shape and orientation than the Fokker-Planck operator (9.10). As you can check by working out exercise 9.3, a convolution of a Gaussian with a Gaussian is again a Gaussian, so the Fokker-Planck operator maps the Gaussian $\rho_n(x_n)$ into the Gaussian

$$\rho_{n+1}(x) = \frac{1}{N_{n+1}} e^{-\frac{1}{2}x^T \frac{1}{\Delta_n + \Delta} \, x} \,, \qquad N_{n+1} = (2\pi)^{d/2}\sqrt{\det(\Delta_n + \Delta)} \tag{9.12}$$

one time step later.

In other words, covariances Δ_n add up. This is the d-dimensional statement of the familiar fact that cumulative error squared is the sum of squares of individual errors. When individual errors are small, and you are adding up a sequence of them in time, you get Brownian motion. If the individual errors are small and added independently to a solution of deterministic equations (so-called 'drift'), you get the Langevin and the Fokker-Planck equations.

References

[1] N. Bleistein and R. A. Handelsman, *Asymptotic Expansions of Integrals* (Dover, New York, 1986).

[2] R. Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe* (A. A. Knopf, New York, 2005).

[3] M. Stone and P. Goldbart, *Mathematics for Physics: A Guided Tour for Graduate Students* (Cambridge Univ. Press, Cambridge UK, 2009).

Exercises

9.1. **Who ordered $\sqrt{\pi}$?** Derive the Gaussian integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2a}} = \sqrt{a}\,, \qquad a > 0\,.$$

assuming only that you know to integrate the exponential function e^{-x}. Hint, hint: x^2 is a radius-squared of something. π is related to the area or circumference of something.

9.2. **d-dimensional Gaussian integrals.** Show that the Gaussian integral in d-dimensions is given by

$$\begin{aligned} Z[J] &= \int d^d x\, e^{-\frac{1}{2} x^\top \cdot M^{-1} \cdot x + x^\top \cdot J} \\ &= (2\pi)^{d/2} |\det M|^{\frac{1}{2}}\, e^{\frac{1}{2} J^\top \cdot M \cdot J}\,, \end{aligned} \tag{9.13}$$

where M is a real positive definite $[d \times d]$ matrix, i.e., a matrix with strictly positive eigenvalues, x and J are d-dimensional vectors, and $(\cdots)^\top$ denotes the transpose.

This integral you will see over and over in statistical mechanics and quantum field theory: it's called 'free field theory', 'Gaussian model', 'Wick expansion', etc.. This is the starting, 'propagator' term in any perturbation expansion.

Here we require that the real symmetric matrix M in the exponent is strictly positive definite, otherwise the integral is infinite. Negative eigenvalues can be accommodated by taking a contour in the complex plane [1], see exercise 6.3 *Fresnel integral*. Zero eigenvalues require stationary phase approximations that go beyond the Gaussian saddle point approximation, typically to the Airy-function type stationary points, see exercise 7.2 *Airy function for large arguments*.

9.3. **Convolution of Gaussians.**

(a) Show that the Fourier transform of the convolution

$$[f * g](x) = \int d^d y\, f(x-y) g(y)$$

corresponds to the product of the Fourier transforms

$$[f * g](x) = \frac{1}{(2\pi)^d} \int d^d k\, F(k) G(k) e^{+ik\cdot x}\,, \tag{9.14}$$

where

$$F(k) = \int \frac{d^d x}{(2\pi)^{d/2}} f(x)\, e^{-ik\cdot x}\,, \quad G(k) = \int \frac{d^d x}{(2\pi)^{d/2}} g(x)\, e^{-ik\cdot x}\,.$$

(b) Consider two normalized Gaussians

$$
\begin{aligned}
f(x) &= \frac{1}{N_1}\, e^{-\frac{1}{2} x^\top \cdot \frac{1}{\Delta_1} \cdot x}\,, \quad N_1 = \sqrt{\det(2\pi\Delta_1)} \\
g(x) &= \frac{1}{N_2}\, e^{-\frac{1}{2} x^\top \cdot \frac{1}{\Delta_2} \cdot x}\,, \quad N_2 = \sqrt{\det(2\pi\Delta_2)} \\
1 &= \int d^d k\, f(x) = \int d^d k\, g(x)\,.
\end{aligned}
$$

Evaluate their Fourier transforms

$$
F(k) = \frac{1}{(2\pi)^{d/2}} e^{\frac{1}{2} k^\top \cdot \Delta_1 \cdot k}\,, \qquad G(k) = \frac{1}{(2\pi)^{d/2}} e^{\frac{1}{2} k^\top \cdot \Delta_2 \cdot k}\,.
$$

Show that the convolution of two normalized Gaussians is a normalized Gaussian

$$
[f * g](x) \quad = \quad \frac{(2\pi)^{-d/2}}{\sqrt{\det(\Delta_1 + \Delta_2)}}\, e^{-\frac{1}{2} x^\top \cdot \frac{1}{\Delta_1 + \Delta_2} \cdot x}\,.
$$

In other words, covariances Δ_j add up. This is the d-dimenional statement of the familiar fact that cumulative error squared is the sum of squares of individual errors. When individual errors are small, and you are adding up a sequence of them in time, you get Brownian motion. If the individual errors are small and added independently to a solution of a deterministic equation, you get Langevin and Fokker-Planck equations.