mathematical methods - week 6

Cauchy theorem at work

Georgia Tech PHYS-6124

Homework HW #6 due Thursday, October 1, 2020

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 6.1 *Complex integration* (a) 4; (b) 2; (c) 2; and (d) 3 points
Exercise 6.2 *Fresnel integral* 7 points

Bonus points
Exercise 6.4 *Cauchy's theorem via Green's theorem in the plane* 6 points

Total of 16 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

edited September 26, 2020

Week 6 syllabus

September 22, 2020

> Mephistopheles knocks at Faust's door and says, "Du mußt es dreimal sagen!"
>
> *"You have to say it three times"*
>
> — Johann Wolfgang von Goethe
>
> *Faust I - Studierzimmer 2. Teil*

Arfken, Weber & Harris [1] (click here) Chapter 11 *Complex variable theory*

- *The essence of complex; Taylor, Laurent series; residue calculus part 1*
  - AWH 11.5 *Laurent expansion*
  - AWH 11.6 *Singularities*
  - AWH 11.7 *Calculus of residues*
- *Calculus of residues. A few integrals, evaluated by Cauchy contours*
  - AWH 11.8 *Evaluation of definite integrals*
- Grigoriev examples worked out in the lecture:
  - *Meromorphic in upper half-plane*
  - *Singularity on the contour*
  - *Pole in upper half-plane*
  - *Singularity on the contour*

Optional reading

- Stone and Goldbart [2] (click here) Chapter 17
  - SG 17.4 *Applications of Cauchy's theorem*
  - SG 17.4.2 *Taylor and Laurent series*
  - SG 17.4.3 *Zeros and singularities*
  - SG 17.4.4 *Analytic continuation*
- *Spatiotemporal cat and the end of time*
  - Turbulence in spacetime : website, talks
- *Wolfram rant: the wunderkid vs. Gradshteyn and Ryzhik; opinions of blackest reactionary professor on graduate educations (the kids are OK). Click on this at your own risk - 30 minutes! Absolutely no science.*
- *The meaning of the things complex rant: The power of visual thinking; Data and dimension reduction; AI, hype and morality. Click on this at your own risk.*

References

[1] G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide*, 7th ed. (Academic, New York, 2013).

[2] M. Stone and P. Goldbart, *Mathematics for Physics: A Guided Tour for Graduate Students* (Cambridge Univ. Press, Cambridge UK, 2009).

Exercises

6.1. **Complex integration.**

(a) Write down the values of $\oint_C (1/z)\,dz$ for each of the following choices of C:
(i) $|z| = 1$, (ii) $|z-2| = 1$, (iii) $|z-1| = 2$.
Then confirm the answers the hard way, using parametric evaluation.

(b) Evaluate parametrically the integral of $1/z$ around the square with vertices $\pm 1 \pm i$.

(c) Confirm by parametric evaluation that the integral of z^m around an origin centered circle vanishes, except when the integer $m = -1$.

(d) Evaluate $\int_{1+i}^{3-2i} dz\,\sin z$ in two ways: (i) via the fundamental theorem of (complex) calculus, and (ii) (bonus) by choosing any path between the end-points and using real integrals.

6.2. **Fresnel integral.**
We wish to evaluate the $I = \int_0^\infty \exp\left(ix^2\right) dx$. To do this, consider the contour integral $I_R = \int_{C(R)} \exp\left(iz^2\right) dz$, where $C(R)$ is the closed circular sector in the upper half-plane with boundary points 0, R and $R\exp(i\pi/4)$. Show that $I_R = 0$ and that $\lim_{R\to\infty} \int_{C_1(R)} \exp\left(iz^2\right) dz = 0$, where $C_1(R)$ is the contour integral along the circular sector from R to $R\exp(i\pi/4)$. [Hint: use $\sin x \geq (2x/\pi)$ on $0 \leq x \leq \pi/2$.] Then, by breaking up the contour $C(R)$ into three components, deduce that

$$\lim_{R\to\infty} \left(\int_0^R \exp\left(ix^2\right) dx - \mathrm{e}^{i\pi/4} \int_0^R \exp\left(-r^2\right) dr \right) = 0$$

and, from the well-known result of real integration $\int_0^\infty \exp\left(-x^2\right) dx = \sqrt{\pi}/2$, deduce that $I = \mathrm{e}^{i\pi/4}\sqrt{\pi}/2$.

6.3. **Fresnel integral.**

(a) Derive the Fresnel integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\; e^{-\frac{x^2}{2ia}} = \sqrt{ia} = |a|^{1/2} e^{i\frac{\pi}{4}\frac{a}{|a|}}\,.$$

Consider the contour integral $I_R = \int_{C(R)} \exp\left(iz^2\right) dz$, where $C(R)$ is the closed circular sector in the upper half-plane with boundary points 0, R and $R\,\exp(i\pi/4)$. Show that $I_R = 0$ and that $\lim_{R\to\infty} \int_{C_1(R)} \exp\left(iz^2\right) dz = 0$, where $C_1(R)$ is the contour integral along the circular sector from R to $R\exp(i\pi/4)$. [Hint: use

$\sin x \geq (2x/\pi)$ on $0 \leq x \leq \pi/2$.] Then, by breaking up the contour $C(R)$ into three components, deduce that

$$\lim_{R\to\infty}\left(\int_0^R \exp\left(ix^2\right)dx - \mathrm{e}^{i\pi/4}\int_0^R \exp\left(-r^2\right)dr\right)$$

vanishes, and, from the real integration $\int_0^\infty \exp\left(-x^2\right)dx = \sqrt{\pi}/2$, deduce that

$$\int_0^\infty \exp\left(ix^2\right)dx = \mathrm{e}^{i\pi/4}\sqrt{\pi}/2\,.$$

Now rescale x by real number $a \neq 0$, and complete the derivation of the Fresnel integral.

(b) In exercise 9.2 the exponent in the d-dimensional Gaussian integrals is real, so the real symmetric matrix M in the exponent has to be strictly positive definite. However, in quantum physics one often has to evaluate the d-dimenional Fresnel integral

$$\frac{1}{(2\pi)^{d/2}}\int d^d\phi e^{-\frac{1}{2i}\phi^\top\cdot M^{-1}\cdot\phi+i\,\phi\cdot J}\,,$$

with a Hermitian matrix M. Evaluate it. What are conditions on its spectrum in order that the integral be well defined?

6.4. **Cauchy's theorem via Green's theorem in the plane.** Express the integral $\oint_C dz\, f(z)$ of the analytic function $f = u+iv$ around the simple contour C in parametric form, apply the two-dimensional version of Gauss' theorem (a.k.a. Green's theorem in the plane), and invoke the Cauchy-Riemann conditions. Hence establish Cauchy's theorem $\oint_C dz\, f(z) = 0$.