mathematical methods - week 5

Complex integration

Georgia Tech PHYS-6124

Homework HW #5 due Thursday, September 24, 2020

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 5.1 *More holomorphic mappings* 10 (+6 bonus) points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

edited September 12, 2020

Week 5 syllabus

September 15, 2020

Arfken, Weber & Harris [1] (click here) Chapter 11 *Complex variable theory*

Complex integration : full ***Tue*** *lecture*

- AWH Sect. 11.3 *Cauchy's integral theorem*

Cauchy contour integral : full ***Thu*** *lecture*

- AWH Sect. 11.4 *Cauchy's integral formula*
- AWH 11.5 *Laurent expansion*
- AWH 11.6 *Singularities*
- AWH 11.7 *Calculus of residues*
- *Everything is allowed in XXX*

Optional reading

- Grigoriev pages 3.1 - 3.3 (Cauchy's contour integral)
- Stone and Goldbart [2] (click here)
  - SG 17.2, 17.3 *Complex integration: Cauchy and Stokes*
  - SG 17.2.2 *Cauchy's theorem*
  - SG 17.2.3 *The residue theorem*
  - SG 17.4 *Applications of Cauchy's theorem*
  - SG 17.4.2 *Taylor and Laurent series*
  - SG 17.4.3 *Zeros and singularities*
  - SG 17.4.4 *Analytic continuation*

References

[1] G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide*, 7th ed. (Academic, New York, 2013).

[2] M. Stone and P. Goldbart, *Mathematics for Physics: A Guided Tour for Graduate Students* (Cambridge Univ. Press, Cambridge UK, 2009).

Exercises

5.1. **More holomorphic mappings.** **Needham**, pp. 211-213

(a) **(bonus)** Use the Cauchy-Riemann conditions to verify that the mapping $z \mapsto \bar{z}$ is not holomorphic.

(b) The mapping $z \mapsto z^3$ acts on an infinitesimal shape and the image is examined. It is found that the shape has been rotated by π, and its linear dimensions expanded by 12. Determine the possibilities for the original location of the shape, i.e., find all values of the complex number z for which an infinitesimal shape at z is rotated by π, and its linear dimensions expanded by 12. Hint: write z in polar form, first find the appropriate $r = |z|$, then find all values of the phase of z such that arg$(z^3) = \pi$.

(c) Consider the map $z \mapsto \bar{z}^2/z$. Determine the geometric effect of this mapping. By considering the effect of the mapping on two small arrows emanating from a typical point z, one arrow parallel and one perpendicular to z, show that the map fails to produce an *amplitwist*.

(d) The interior of a simple closed curve $\mathcal{C}$ is mapped by a holomorphic mapping into the exterior of the image of $\mathcal{C}$. If z travels around the curve counterclockwise, which way does the image of z travel around the image of $\mathcal{C}$?

(e) Consider the mapping produced by the function $f(x + iy) = \big(x^2 + y^2\big) + i\big(y/x\big)$.

  (i) Find and sketch the curves that are mapped by f into horizontal and vertical lines. Notice that f appears to be conformal.

  (ii) Now show that f is *not* in fact a conformal mapping by considering the images of a pair of lines (e.g. , one vertical and one horizontal).

  (iii) By using the Cauchy-Riemann conditions confirm that f is not conformal.

  (iv) Show that no choice of $v(x, y)$ makes $f(x + iy) = \big(x^2 + y^2\big) + iv(x, y)$ holomorphic.

(f) **(bonus)** Show that if f is holomorphic on some connected region then each of the following conditions forces f to reduce to a constant:
(i) $\mathrm{Re}\, f(z) = 0$; (ii) $|f(z)| = \mathrm{const.}$; (iii) $\bar{f}(z)$ is holomorphic too.

(g) **(bonus)** Suppose that the holomorphic mapping $z \mapsto f(z)$ is expressed in terms of the modulus R and argument Φ of f, i.e.,
$f(z) = R(x, y) \exp i\Phi(x, y)$.
Determine the form of the Cauchy-Riemann conditions in terms of R and Φ.

(h) (i) By sketching the image of an infinitesimal rectangle under a holomorphic mapping, determine the the local magnification factor for the area and compare it with that for a infinitesimal line. Re-derive this result by examining the Jacobian determinant for the transformation.

  (ii) Verify that the mapping $z \mapsto \exp z$ satisfies the Cauchy-Riemann conditions, and compute $\big(\exp z\big)'$.

  (iii) **(bonus)** Let S be the square region given by $A - B \le \mathrm{Re}\, z \le A + B$ and $-B \le \mathrm{Im}\, z \le B$ with A and B positive. Sketch a typical S for which $B < A$ and sketch the image $\tilde{S}$ of S under the mapping $z \mapsto \exp z$.

  (iv) **(bonus)** Deduce the ratio $(\text{area of } \tilde{S})/(\text{area of } S)$, and compute its limit as $B \to 0^+$.

  (v) **(bonus)** Compare this limit with the one you would expect from part (i).