mathematical methods - week 13

Probability

Georgia Tech PHYS-6124

Homework HW #13 due Thursday, November 19, 2020

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Bonus points
Exercise 13.1 *Lyapunov equation* 12 points

This week there are no required exercises. Whatever you do, you get bonus points.

edited November 12, 2020

Week 13 syllabus

November 10, 2020

This week's lectures are related to AWH Chapter 23 *Probability and Statistics* (click here). The fastest way to watch any week's lecture videos is by letting YouTube run the course playlist (click here).

- A summary of key concepts
  - ChaosBook appendix A20.1 Moments, cumulants
  - *Clip 1 - Averages, moments (45 min)*
  - *Clip 2 - Why a Gaussian? It's the maximum entropy distribution (8 min)*
- Why Gaussians again?
  - ChaosBook 33.2 Brownian diffusion
  - *Clip 3 - diffusion; Fokker-Planck density evolution (43 min)*
  - ChaosBook 33.3 Noisy trajectories
  - *Clip 4 - I don't like Langevin equation (3 min)*
- A glimpse of Orstein-Uhlenbeck, the "harmonic oscillator" of the theory of stochastic processes. And the one "Lyapunov" thing Lyapunov actually did:)
  - Noise is your friend
  - ChaosBook 33.4 Noisy maps
  - ChaosBook 33.5 All nonlinear noise is local
  - *Clip 5 - noise is your friend (23 min)*

Optional reading

Discussion 1 - Density is evaluated in configuration space, but the Laplacian is diagonalized in the Fourier space; a random walk from stochastic to quantum mechanics; Wiener path integrals; Schrödinger harmonic oscillator is imaginary time relative of the Ornstein-Uhlenbeck; Liouville theorem; Predrag's lecturing is a Gaussian process - on average you learn zero. (20 min)

Clip 6 - negative dimensions (6 min)

13.1 Other sources

- MIT 16-90 Computational methods is a typical mathematical methods in engineering course. Probabilistic methods and optimization are discussed here.

Really going into the Ornstein-Uhlenbeck equation might take too much of your time, so this week we skip doing exercises, and if you are curious, and want to try your hand at solving exercise 13.1 *Lyapunov equation*, you probably should first skim through our lectures on the Ornstein-Uhlenbeck spectrum, Sect. 4.1 and Appen. B.1

here. Finally! we get something one expects from a math methods course, an example of why orthogonal polynomials are useful, in this case the Hermite polynomials :) .

The reason why I like this example is that again the standard 'physics' intuition misleads us. Brownian noise spreads with time as $\sqrt{t}$, but the diffusive dynamics of nonlinear flows is fundamentally different - instead of spreading, in the Ornstein-Uhlenbeck example the noise contained and balanced by the nonlinear dynamics.

- D. Lippolis and P. Cvitanović [3], *How well can one resolve the state space of a chaotic map?*; arXiv:0902.4269

- P. Cvitanović and D. Lippolis [1], *Knowing when to stop: How noise frees us from determinism*; arXiv:1206.5506

- J. M. Heninger, D. Lippolis and P. Cvitanović [2], *Neighborhoods of periodic orbits and the stationary distribution of a noisy chaotic system*; arXiv:1507.00462

Question 13.1. Henriette Roux asks

Q What percentage score on problem sets is a passing grade?

A That might still change, but currently it looks like 60% is good enough to pass the course. 70% for C, 80% for B, 90% for A. Very roughly - will alert you if this changes. Here is the percentage score as of week 10 in the 2019 course.

Question 13.2. Henriette Roux asks

Q How do I subscribe to the nonlinear and math physics and other seminars mailing lists?

A click here

References

[1] P. Cvitanović and D. Lippolis, Knowing when to stop: How noise frees us from determinism, in Let's Face Chaos through Nonlinear Dynamics, edited by M. Robnik and V. G. Romanovski (2012), pp. 82–126.

[2] J. M. Heninger, P. Cvitanović, and D. Lippolis, "Neighborhoods of periodic orbits and the stationary distribution of a noisy chaotic system", Phys. Rev. E **92**, 062922 (2015).

[3] D. Lippolis and P. Cvitanović, "How well can one resolve the state space of a chaotic map?", Phys. Rev. Lett. **104**, 014101 (2010).

Exercises

13.1. **Lyapunov equation.** Consider the following system of ordinary differential equations,

$$\dot{Q} = A\,Q + Q\,A^\top + \Delta\,, \tag{13.1}$$

in which $\{Q, A, \Delta\} = \{Q(t), A(t), \Delta(t)\}$ are $[d\times d]$ matrix functions of time t through their dependence on a deterministic trajectory, $A(t) = A(x(t))$, etc., with stability matrix A and noise covariance matrix Δ given, and density covariance matrix Q sought. The superscript $(\ )^\top$ indicates the transpose of the matrix. Find the solution $Q(t)$, by taking the following steps:

(a) Write the solution in the form $Q(t) = J(t)[Q(0) + W(t)]J^\top(t)$, with Jacobian matrix $J(t)$ satisfying

$$\dot{J}(t) = A(t)\,J(t)\,, \qquad J(0) = \mathbf{1}\,, \tag{13.2}$$

with $\mathbf{1}$ the $[d\times d]$ identity matrix. The Jacobian matrix at time t

$$J(t) = \hat{T} e^{\int_0^t d\tau\, A(\tau)}\,, \tag{13.3}$$

where $\hat{T}$ denotes the 'time-ordering' operation, can be evaluated by integrating (13.2).

(b) Show that $W(t)$ satisfies

$$\dot{W} = \frac{1}{J}\,\Delta\,\frac{1}{J^\top}\,, \qquad W(0) = 0\,. \tag{13.4}$$

(c) Integrate (13.1) to obtain

$$Q(t) = J(t)\left[Q(0) + \int_0^t d\tau\,\frac{1}{J(\tau)}\,\Delta(\tau)\,\frac{1}{J^\top(\tau)}\right]J^\top(t)\,. \tag{13.5}$$

(d) Show that if $A(t)$ commutes with itself throughout the interval $0 \le \tau \le t$ then the time-ordering operation is redundant, and we have the explicit solution $J(t) = \exp\left\{\int_0^t d\tau\, A(\tau)\right\}$. Show that in this case the solution reduces to

$$Q(t) = J(t)\;Q(0)\,J(t)^\top + \int_0^t d\tau'\, e^{\int_{\tau'}^t d\tau\, A(t)}\,\Delta(\tau')\,e^{\int_{\tau'}^t d\tau\, A^\top(t)}\,. \tag{13.6}$$

(e) It is hard to imagine a time dependent $A(t) = A(x(t))$ that would be commuting. However, in the neighborhood of an equilibrium point x^* one can approximate the stability matrix with its time-independent linearization, $A = A(x^*)$. Show that in that case (13.3) reduces to

$$J(t) = e^{t\,A}\,,$$

and (13.6) to what?