mathematical methods - week 12

SO(3) and SU(2)

Georgia Tech PHYS-6124

Homework HW #12 due Thursday, November 12, 2020

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 12.1 *Irreps of SO*(2) 3 points
Exercise 12.2 *Conjugacy classes of SO*(3) 4 points (+ 2 bonus points, if complete)
Exercise 12.3 *The character of SO*(3) *3-dimensional representation* 3 points

Bonus points
Exercise 12.4 *The orthonormality of SO*(3) *characters* 2 points

Total of 10 points = 100 % score.

edited November 5, 2020

Week 12 syllabus

Tuesday, November 3, 2020

This week's lectures are related to AWH Chapter 3 *Vector Analysis* (click here) and Chapter 16 *Angular Momentum* (click here). The fastest way to watch any week's lecture videos is by letting YouTube run the course playlist (click here).

There is way too much material in this week's notes. Watch the main sequence of video clips, that and recommended reading should suffice. The rest is optional. You can glance through sect. 12.1 *Linear algebra*, and sect. 12.2 *SO(3) character orthogonality*, but I do not expect you to master this material.

Clip 1 - Rotations in 3 dimensions (30 min)

- OK, I see that formally $\mathrm{SU}(2) \simeq \mathrm{SO}(3)$, but who ordered "spin?"

  Clip 4 - Rotations in 2 complex dimensions (42 min)

  – Read sect. 12.3 *SU(2) Pauli matrices*
  – Read sect. 12.4 $\mathrm{SU}(2) \simeq \mathrm{SO}(3)$

Optional reading

Clip 2 - Lie algebra (21 min)

Clip 3 - Birdtracks (6 min)

For overall clarity and pleasure of reading, I like Schwichtenberg [8] (click here) discussion best. If you read anything for this week's lectures, read Schwichtenberg.

Reading: Chen, Ping and Wang [2] *Group Representation Theory for Physicists*, Sect 5.2 *Definition of a Lie group, with examples* (click here).

- What's the payback? While for you the geometrically intuitive representation is the set of rotation $[2{\times}2]$ matrices, group theory says no! They split into pairs of 1-dimensional irreps, and the basic building blocks of *our* 2-dimensional rotations on our kitchen table (forget quantum mechanics!) are the $\mathrm{U}(1)$ $[1\times1]$ complex unit vector phase rotations.

  Reading: C. K. Wong *Group Theory* notes, Chap 6 *1D continuous groups*, Sects. 6.1-6.3 Irreps of $\mathrm{SO}(2)$.

  Reading: C. K. Wong *Group Theory* notes, Chap 6 *1D continuous groups*, Sect. 6.6 completes discussion of Fourier analysis as continuum limit of cyclic groups C_n, compares $\mathrm{SO}(2)$, $\mathrm{O}(2)$, discrete translations group, and continuous translations group.

- Dirac belt trick applet

If still anxious, maybe this helps: Mark Staley, *Understanding quaternions and the Dirac belt trick* arXiv:1001.1778.

I have enjoyed reading Mathews and Walker [7] Chap. 16 *Introduction to groups* (click here). Goldbart writes that the book is “based on lectures by Richard Feynman at Cornell University.” Very clever. In particular, work through the example of fig. 16.2: it is very cute, you get explicit eigenmodes from group theory alone. The main message is that if you think things through first, you never have to go through using explicit form of representation matrices - thinking in terms of invariants, like characters, will get you there much faster.

Any book, of 100s available, like Cornwell [3] *Group Theory in Physics: An introduction* that covers group theory might be more to your taste.

Hamilton’s quaternions

Stone and Goldbart [9] (click here) Chapter 17 Sect 17.6 *Analytic functions and topology* (wherein stereographic projection is revealed to be the geometric origin of the spinor representations of the rotation group)

Question 12.1. Predrag asks

Q You are graduate students now. Are you ready for The Talk?

A Henriette Roux: I’m ready!

12.1 Linear algebra

In this section we collect a few basic definitions. A sophisticated reader might prefer skipping straight to the definition of the Lie product (12.8), the big difference between the group elements product used so far in discussions of finite groups, and what is needed to describe continuous groups.

Vector space. A set V of elements $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ is called a *vector* (or *linear*) *space* over a field $\mathbb{F}$ if

(a) *vector addition* “+” is defined in V such that V is an abelian group under addition, with identity element $\mathbf{0}$;

(b) the set is *closed* with respect to *scalar multiplication* and vector addition

$$\begin{aligned} a(\mathbf{x}+\mathbf{y}) &= a\mathbf{x}+a\mathbf{y}\,, \qquad a,b\in\mathbb{F}\,, \quad \mathbf{x},\mathbf{y}\in V \\ (a+b)\mathbf{x} &= a\mathbf{x}+b\mathbf{x} \\ a(b\mathbf{x}) &= (ab)\mathbf{x} \\ 1\,\mathbf{x} &= \mathbf{x}\,, \qquad 0\,\mathbf{x}=\mathbf{0}\,. \end{aligned} \tag{12.1}$$

Here the field $\mathbb{F}$ is either $\mathbb{R}$, the field of reals numbers, or $\mathbb{C}$, the field of complex numbers. Given a subset $V_0 \subset V$, the set of all linear combinations of elements of V_0, or the *span* of V_0, is also a vector space.

A basis. $\{\mathbf{e}^{(1)}, \cdots, \mathbf{e}^{(d)}\}$ is any linearly independent subset of V whose span is V. The number of basis elements d is the *dimension* of the vector space V.

Dual space, dual basis. Under a general linear transformation $g \in GL(n, \mathbb{F})$, the row of basis vectors transforms by right multiplication as $\mathbf{e}^{(j)} = \sum_k (\mathbf{g}^{-1})^j{}_k \, \mathbf{e}^{(k)}$, and the column of x_a's transforms by left multiplication as $x' = \mathbf{g}x$. Under left multiplication the column (row transposed) of basis vectors $\mathbf{e}_{(k)}$ transforms as $\mathbf{e}_{(j)} = (\mathbf{g}^\dagger)_j{}^k \mathbf{e}_{(k)}$, where the *dual rep* $\mathbf{g}^\dagger = (\mathbf{g}^{-1})^\top$ is the transpose of the inverse of $\mathbf{g}$. This observation motivates introduction of a *dual* representation space $\bar{V}$, the space on which $GL(n, \mathbb{F})$ acts via the dual rep $\mathbf{g}^\dagger$.

Definition. If V is a vector representation space, then the *dual space* $\bar{V}$ is the set of all linear forms on V over the field $\mathbb{F}$.

If $\{\mathbf{e}^{(1)}, \cdots, \mathbf{e}^{(d)}\}$ is a basis of V, then $\bar{V}$ is spanned by the *dual basis* $\{\mathbf{e}_{(1)}, \cdots, \mathbf{e}_{(d)}\}$, the set of d linear forms $\mathbf{e}_{(k)}$ such that

$$\mathbf{e}_{(j)} \cdot \mathbf{e}^{(k)} = \delta_j^k \,,$$

where δ_j^k is the Kronecker symbol, $\delta_j^k = 1$ if $j = k$, and zero otherwise. The components of dual representation space vectors $\bar{y} \in \bar{V}$ will here be distinguished by upper indices

$$(y^1, y^2, \ldots, y^n) \,. \tag{12.2}$$

They transform under $GL(n, \mathbb{F})$ as

$$y'^a = (\mathbf{g}^\dagger)^a{}_b y^b \,. \tag{12.3}$$

For $GL(n, \mathbb{F})$ no complex conjugation is implied by the † notation; that interpretation applies only to unitary subgroups $U(n) \subset GL(n, \mathbb{C})$. In the index notation, $\mathbf{g}$ can be distinguished from $\mathbf{g}^\dagger$ by keeping track of the relative ordering of the indices,

$$(\mathbf{g})_a^b \to g_a{}^b \,, \qquad (\mathbf{g}^\dagger)_a^b \to g^b{}_a \,. \tag{12.4}$$

Algebra. A set of r elements $\mathbf{t}_\alpha$ of a vector space $\mathcal{T}$ forms an algebra if, in addition to the vector addition and scalar multiplication,

(a) the set is *closed* with respect to multiplication $\mathcal{T} \cdot \mathcal{T} \to \mathcal{T}$, so that for any two elements $\mathbf{t}_\alpha, \mathbf{t}_\beta \in \mathcal{T}$, the product $\mathbf{t}_\alpha \cdot \mathbf{t}_\beta$ also belongs to $\mathcal{T}$:

$$\mathbf{t}_\alpha \cdot \mathbf{t}_\beta = \sum_{\gamma=0}^{r-1} \tau_{\alpha\beta}{}^\gamma \mathbf{t}_\gamma \,, \qquad \tau_{\alpha\beta}{}^\gamma \in \mathbb{C} \,; \tag{12.5}$$

(b) the multiplication operation is *distributive*:

$$\begin{aligned} (\mathbf{t}_\alpha + \mathbf{t}_\beta) \cdot \mathbf{t}_\gamma &= \mathbf{t}_\alpha \cdot \mathbf{t}_\gamma + \mathbf{t}_\beta \cdot \mathbf{t}_\gamma \\ \mathbf{t}_\alpha \cdot (\mathbf{t}_\beta + \mathbf{t}_\gamma) &= \mathbf{t}_\alpha \cdot \mathbf{t}_\beta + \mathbf{t}_\alpha \cdot \mathbf{t}_\gamma \,. \end{aligned}$$

The set of numbers ${\tau_{\alpha\beta}}^{\gamma}$ are called the *structure constants*. They form a matrix rep of the algebra,

$$(\mathbf{t}_\alpha)_\beta{}^\gamma \equiv \tau_{\alpha\beta}{}^\gamma \,, \tag{12.6}$$

whose dimension is the dimension r of the algebra itself.

Depending on what further assumptions one makes on the multiplication, one obtains different types of algebras. For example, if the multiplication is associative

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta) \cdot \mathbf{t}_\gamma = \mathbf{t}_\alpha \cdot (\mathbf{t}_\beta \cdot \mathbf{t}_\gamma) \,,$$

the algebra is *associative*. Typical examples of products are the *matrix product*

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta)_a^c = (t_\alpha)_a^b (t_\beta)_b^c \,, \qquad \mathbf{t}_\alpha \in V \otimes \bar{V} \,, \tag{12.7}$$

and the *Lie product*

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta)_a^c = (t_\alpha)_a^b (t_\beta)_b^c - (t_\alpha)_c^b (t_\beta)_b^a \,, \qquad \mathbf{t}_\alpha \in V \otimes \bar{V} \tag{12.8}$$

which defines a *Lie algebra*.

12.2 SO(3) character orthogonality

In 3 Euclidean dimensions, a rotation around z axis is given by the $\mathrm{SO}(2)$ matrix

$$R_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp\varphi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,. \tag{12.9}$$

An arbitrary rotation in $\mathbb{R}^3$ can be represented by

$$R_{\boldsymbol{n}}(\varphi) = e^{-i\varphi\, \boldsymbol{n}\cdot\boldsymbol{L}} \qquad \boldsymbol{L} = (L_1, L_2, L_3) \,, \tag{12.10}$$

where the unit vector $\boldsymbol{n}$ determines the plane and the direction of the rotation by angle φ. Here L_1, L_2, L_3 are the generators of rotations along x, y, z axes respectively,

$$L_1 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} , \quad L_2 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \quad L_3 = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{12.11}$$

with Lie algebra relations

$$[L_i, L_j] = i\varepsilon_{ijk} L_k \,. \tag{12.12}$$

All $\mathrm{SO}(3)$ rotations by the same angle θ around different rotation axis $\boldsymbol{n}$ are conjugate to each other,

$$e^{i\phi \boldsymbol{n}_2\cdot\boldsymbol{L}} e^{i\theta \boldsymbol{n}_1\cdot\boldsymbol{L}} e^{-i\phi \boldsymbol{n}_2\cdot\boldsymbol{L}} = e^{i\theta \boldsymbol{n}_3\cdot\boldsymbol{L}} \,, \tag{12.13}$$

with $e^{i\phi \boldsymbol{n}_2\cdot\boldsymbol{L}}$ and $e^{-i\theta \boldsymbol{n}_2\cdot\boldsymbol{L}}$ mapping the vector $\boldsymbol{n}_1$ to $\boldsymbol{n}_3$ and back, so that the rotation around axis $\boldsymbol{n}_1$ by angle θ is mapped to a rotation around axis $\boldsymbol{n}_3$ by the same θ. The conjugacy classes of $\mathrm{SO}(3)$ thus consist of rotations by the same angle about all distinct rotation axes, and are thus labelled the angle θ. As the conjugacy class depends only on θ, the characters can only be a function of θ. For the 3-dimensional special orthogonal representation, the character is exercise 12.3

$$\chi = 2\cos(\theta) + 1\,. \tag{12.14}$$

For an irrep labeled by j, the character of a conjugacy class labeled by θ is

$$\chi^{(j)}(\theta) = \frac{\sin(j+1/2)\theta}{\sin(\theta/2)} \tag{12.15}$$

To check that these characters are orthogonal to each other, one needs to define the group integration over a parametrization of the SO(3) group manifold. A group element is parametrized by the rotation axis $\boldsymbol{n}$ and the rotation angle $\theta \in (-\pi, \pi]$, with $\boldsymbol{n}$ a unit vector which ranges over all points on the surface of a unit ball. Note however, that a π rotation is the same as a $-\pi$ rotation ($\boldsymbol{n}$ and $-\boldsymbol{n}$ point along the same direction), and the $\boldsymbol{n}$ parametrization of SO(3) is thus a 2-dimensional surface of a unit-radius ball with the opposite points identified.

The Haar measure for SO(3) requires a bit of work, here we just note that after the integration over the solid angle (characters do not depend on it), the Haar measure is

$$dg = d\mu(\theta) = \frac{d\theta}{2\pi}(1-\cos(\theta)) = \frac{d\theta}{\pi}\sin^2(\theta/2)\,. \tag{12.16}$$

exercise 12.4 With this measure the characters are orthogonal, and the character orthogonality theorems follow, of the same form as for the finite groups, but with the group averages replaced by the continuous, parameter dependant group integrals

$$\frac{1}{|G|}\sum_{g\in G} \to \int_G dg\,.$$

The good news is that, as explained in ChaosBook.org Chap. *Relativity for cyclists* (and in *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [5]), one never needs to actually explicitly construct a group manifold parametrizations and the corresponding Haar measure.

12.3 SU(2) Pauli matrices

A lightning, bullet points review.

- U(n): unitary transformation $U = e^{iH}$
- Unitarity: $U^\dagger U = \mathbf{1} \Rightarrow H^\dagger = H$, the generator is hermitian.
- SU(n): special unitary transformation det $U = 1$
- Must know: $\ln\det\ = $ tr $\ln$ for any matrix, so the generator is traceless
  $\ln\det U = $ tr $\ln U\ = $ tr $H = 0$
- SU(2) : $H = \begin{pmatrix} a & c \\ e & b \end{pmatrix}\,,\quad a,b,c,e \in \mathbb{C}$, eight real numbers in all.
- H is hermitian: $H = \begin{pmatrix} a & c+id \\ c-id & b \end{pmatrix}\,,\quad a,b,c,d \in \mathbb{R}\,,$
- H is traceless: $0 = $ tr $H \Rightarrow a + b = 0$, three real rotation parameters in all, so

$$\begin{aligned} H &= c\sigma_x + d\sigma_y + a\sigma_z \\ &= c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{12.17}$$

  where σ_j are known as Pauli matrices.

12.4 SU(2) and SO(3)

K. Y. Short

An element of SU(2) can be written as

$$U_{\mathbf{n}}(\phi) = e^{i\phi\,\sigma\cdot\hat{\mathbf{n}}/2} \tag{12.18}$$

where σ_j is a Pauli matrix and ϕ is a real number. What is the significance of the $1/2$ factor in the argument of the exponential?

Consider a generic position vector $\boldsymbol{x} = (x, y, z)$ and construct a Hermitian matrix of the form

$$\begin{aligned} \sigma\cdot\boldsymbol{x} &= \sigma_x x + \sigma_y y + \sigma_z z \\ &= \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iy \\ iy & 0 \end{pmatrix} + \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \\ &= \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \end{aligned} \tag{12.19}$$

Its determinant

$$\det\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} = -(x^2+y^2+z^2) = -\boldsymbol{x}^2 \tag{12.20}$$

gives the length of a vector. Consider a SU(2) transformation (12.18) of this matrix, $U^\dagger(\sigma \cdot \boldsymbol{x})U$. Taking the determinant, we find the same expression as before:

$$\det U(\sigma \cdot \boldsymbol{x})U^\dagger = \det U \det(\sigma \cdot \boldsymbol{x}) \det U^\dagger = \det(\sigma \cdot \boldsymbol{x}) \,. \tag{12.21}$$

Just as SO(3), SU(2) preserves the lengths of vectors.

To make the correspondence between SO(3) and SU(2) more explicit, consider a SU(2) transformation on a complex two-component *spinor*

$$\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{12.22}$$

related to $\boldsymbol{x}$ by

$$x = \frac{1}{2}(\beta^2 - \alpha^2), \quad y = -\frac{i}{2}(\alpha^2 + \beta^2), \quad z = \alpha\beta \tag{12.23}$$

Check that a SU(2) transformation of ψ is equivalent to a SO(3) transformation on $\boldsymbol{x}$. From this equivalence, one sees that a SU(2) transformation has three real parameters that correspond to the three rotation angles of SO(3). If we label the "angles" for the SU(2) transformation by α, β, and γ, we observe, for a "rotation" about $\hat{x}$

$$U_x(\alpha) = \begin{pmatrix} \cos\alpha/2 & i\sin\alpha/2 \\ i\sin\alpha/2 & \cos\alpha/2 \end{pmatrix} , \tag{12.24}$$

for a "rotation" about $\hat{y}$,

$$U_y(\beta) = \begin{pmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{pmatrix} , \tag{12.25}$$

and for "rotation" about $\hat{z}$,

$$U_z(\gamma) = \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix} . \tag{12.26}$$

Compare these three matrices to the corresponding SO(3) rotation matrices:

$$R_x(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\zeta & \sin\zeta \\ 0 & -\sin\zeta & \cos\zeta \end{pmatrix} , \quad R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12.27}$$

They're equivalent! Result: *Half the rotation angle generated by SU(2) corresponds to a rotation generated by SO(3).*

What does this mean? At this point, probably best to switch to Schwichtenberg [8] (click here) who explains clearly that SU(2) is a simply-connected group, and thus the "mother" or covering group, or the double cover of SO(3). This means there is a two-to-one map from SU(2) to SO(3); an SU(2) turn by 4π corresponds to an SO(3) turn by 2π. So, the building blocks of your 3-dimensional world are not 3-dimensional real vectors, but the 2-dimensional complex spinors! Quantum mechanics chose electrons to be spin 1/2, and there is nothing Fox Channel can do about it.

Question 12.2. Henriette Roux asks

Q Why is this complex 2-dimensional vector called a 'spinor'?

A Historical, as Arfken, Weber & Harris [1] explain: "It turns out that half-integral angular momentum states are needed to describe the intrinsic angular momentum of the electron and many other particles. Since these particles also have magnetic moments, an intuitive interpretation is that their charge distributions are spinning about some axis; hence the term spin. It is now understood that the spin phenomena cannot be explained consistently by describing these particles as ordinary charge distributions undergoing rotational motion, [...] "

Schwichtenberg [8]: "[...] spinors have properties that usual vectors do not have. For instance, the factor $1/2$ in the exponent. This factor shows us that a spinor1 is after a rotation by 2π not the same, but gets a minus sign. This is a pretty crazy property, because all objects we deal with in everyday life are exactly the same after a rotation by $360^o = 2\pi$.

Question 12.3. Henriette Roux asks

Q What' relation of Pauli exclusion principle to the spinor 2π rotation amounting to overall minus sign?

A I think of fermion/Grassmann statistics as Archimedes principle + linearity, see my Field Theory [4] *chap. 4 Fermions*. Basically, usually a constraint is imposed by eliminating a variable, for example, given the constraint is $x^2 + y^2 + z^2 = 1$, one gets rid of z by replacing it everywhere with $z \to \sqrt{1 - x^2 - y^2}$. This makes a fully symmetric theory asymmetric and ugly. In linear setting, another option is to keep all the variables and the symmetry, but add a new variable which by construction subtracts a degree of freedom, what I call [6] a "negative dimension". In quantum field theory such variable is called a 'ghost'; it needs to be anti-commuting or Grassmann.

12.5 What *really* happened

They do not make Norwegians as they used to. In his brief biographical sketch of Sophus Lie, Burkman writes: "I feel that I would be remiss in my duties if I failed to mention avery interesting event that took place in Lie's life. Klein (a German) and Lie had moved to Paris in the spring of 1870 (they had earlier been working in Berlin). However, in July 1870, the Franco-Prussian war broke out. Being a German alien in France, Klein decided that it would be safer to return to Germany; Lie also decided to go home to Norway. However (in a move that I think questions his geometric abilities), Lie decided that to go from Paris to Norway, he would walk to Italy (and then presumably take a ship to Norway). The trip did not go as Lie had planned. On the way, Lie ran into some trouble–first some rain, and he had a habit of taking off his clothes and putting them in his backpack when he walked in the rain (so he was walking to Italy in the nude). Second, he ran into the French military (quite possibly while walking in the nude) and they discovered in his sack (in addition to his hopefully dry clothing) letters written to Klein in German containing the words 'lines' and 'spheres' (which the French interpreted as meaning 'infantry' and 'artillery'). Lie was arrested as a (insane) German spy. However, due to intervention by Gaston Darboux, he was released four weeks later and returned to Norway to finish his doctoral dissertation."

Question 12.4. Henriette Roux asks

Q

A This is a math methods course. Why are you not teaching us Bessel functions?

Blame Feynman: On May 2, 1985 my stay at Cornell was to end, and Vinnie of college town *Italian Kitchen* made a special dinner for three of us regulars. Das Wunderkind noticed Feynman ambling down Eddy Avenue, kidnapped him, and here we were, two wunderkinds, two humans.

Feynman was a very smart, forever driven wunderkind. He naturally bonded with our very smart, forever driven wunderkind, who suddenly lurched out of control, and got very competive about at what age who summed which kind of Bessel function series. Something like age twelve, do not remember which one did the Bessels first. At that age I read " Palle Alone in the World," while my nonwunderkind friend, being from California, watched television 12 hours a day.

When Das Wunderkind taught graduate E&M, he spent hours crafting lectures about symmetry groups and their representations as various eigenfunctions. Students were not pleased.

So, fuggedaboutit! if you have not done your Bessels yet, they are eigenfunctions, just like your Fourier modes, but for a spherical symmetry rather than for a translation symmetry; wiggle like a cosine, but decay radially.

When you need them you'll figure them out. Or sue me.

References

[1] G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide*, 7th ed. (Academic, New York, 2013).

[2] J.-Q. Chen, J. Ping, and F. Wang, *Group Representation Theory for Physicists* (World Scientific, Singapore, 1989).

[3] J. F. Cornwell, *Group Theory in Physics: An Introduction* (Academic, New York, 1997).

[4] P. Cvitanović, *Field Theory*, Notes prepared by E. Gyldenkerne (Nordita, Copenhagen, 1983).

[5] P. Cvitanović, *Group Theory: Birdtracks, Lie's and Exceptional Groups* (Princeton Univ. Press, Princeton NJ, 2004).

[6] P. Cvitanović and A. D. Kennedy, "Spinors in negative dimensions", Phys. Scr. **26**, 5–14 (1982).

[7] J. Mathews and R. L. Walker, *Mathematical Methods of Physics* (W. A. Benjamin, Reading, MA, 1970).

[8] J. Schwichtenberg, *Physics from Symmetry* (Springer, Berlin, 2015).

[9] M. Stone and P. Goldbart, *Mathematics for Physics: A Guided Tour for Graduate Students* (Cambridge Univ. Press, Cambridge UK, 2009).

Exercises

12.1. **Irreps of SO(2).** Matrix

$$T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad (12.28)$$

is the generator of rotations in a plane.

(a) Use the method of projection operators to show that for rotations in the kth Fourier mode plane, the irreducible $1D$ subspaces orthonormal basis vectors are

$$\mathbf{e}^{(\pm k)} = \frac{1}{\sqrt{2}} \left(\pm \mathbf{e}_1^{(k)} - i\, \mathbf{e}_2^{(k)} \right) .$$

How does T act on $\mathbf{e}^{(\pm k)}$?

(b) What is the action of the $[2\times 2]$ rotation matrix

$$D^{(k)}(\theta) = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} , \qquad k = 1, 2, \cdots$$

on the $(\pm k)$th subspace $\mathbf{e}^{(\pm k)}$?

(c) What are the irreducible representations characters of SO(2)?

12.2. **Conjugacy classes of SO(3):** Show that all SO(3) rotations (12.10) by the same angle θ around any rotation axis $\boldsymbol{n}$ are conjugate to each other:

$$e^{i\phi \boldsymbol{n}_2 \cdot \boldsymbol{L}} e^{i\theta \boldsymbol{n}_1 \cdot \boldsymbol{L}} e^{-i\phi \boldsymbol{n}_2 \cdot \boldsymbol{L}} = e^{i\theta \boldsymbol{n}_3 \cdot \boldsymbol{L}} \qquad (12.29)$$

Check this for infinitesimal ϕ, and argue that from that it follows that it is also true for finite ϕ. Hint: use the Lie algebra commutators (12.12).

12.3. **The character of SO(3) 3-dimensional representation:** Show that for the 3-dimensional special orthogonal representation (12.10), the character is

$$\chi = 2\cos(\theta) + 1 . \qquad (12.30)$$

Hint: evaluate the character explicitly for $R_x(\theta)$, $R_y(\theta)$ and $R_z(\theta)$, then explain what is the intuitive meaning of 'class' for rotations.

12.4. **The orthonormality of SO(3) characters:** Verify that given the Haar measure (12.16), the characters (12.15) are orthogonal:

$$\langle \chi(j) | \chi(j') \rangle = \int_G dg\, \chi^{(j)}(g^{-1})\, \chi^{(j')}(g) = \delta_{jj'} . \qquad (12.31)$$