mathematical methods - week 10

Discrete symmetries

Georgia Tech PHYS-6124

Homework HW #10 due Thursday, October 29, 2020

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 10.1 *1-dimensional representation of anything* 1 point
Exercise 10.2 *2-dimensional representation of* S_3 4 points
Exercise 10.3 D_3*: symmetries of an equilateral triangle* 5 points

Bonus points
Exercise 10.4 (a), (b) and (c) *Permutation of three objects* 2 points
Exercise 10.5 *3-dimensional representations of* D_3 3 points

Total of 10 points = 100 % score.

edited October 28, 2020

Week 10 syllabus

Tuesday, October 20, 2020

Tyger Tyger burning bright,
In the forests of the night:
What immortal hand or eye,
Dare frame thy fearful symmetry?
—William Blake, *The Tyger*

This week's lectures are related to AWH Chapter 17 *Group Theory* (click here). The fastest way to watch any week's lecture videos is by letting YouTube run the course playlist (click here).

There is way too much material in this week's notes. Watch the main sequence of video clips, that and recommended reading should suffice to do the problems. The rest is optional. You can glance through sect. 10.1 *Group presentations*, and sect. 10.3 *Literature*, but I do not expect you to understand this material.

Group theory and why I love 808,017,· · · ,000 is a great video on group theory from 3Blue1Brown, writes Andrew Wu. I agree: Well worth of your time, more motivational than my lectures. What it actually focuses on - the monster group - is totally useless to us. My focus this week is narrow and technical:

1. theory of finite groups are a natural generalization of discrete Fourier representations
2. it is all about class and character. "Character", in particular, I find very surprising - one complex number suffices to characterize a matrix!

Hang in there! And relax. None of this will be on the test. As a matter of fact, there will be no test.

- It's all about class: Groups, permutations, $D_3 \cong C_{3v} \cong S_3$ symmetries of equilateral triangle, rearrangement theorem, subgroups, cosets, classes.

  - Dresselhaus *et al.* [3] Chapter 1 *Basic Mathematical Background: Introduction* (click here). The MIT course 6.734 online version contains much of the same material.
  - ChaosBook Chapter 10. Flips, slides and turns
  - *Clip 1 - discrete symmetry, an example: 3-disk pinball*
  - *Clip 2 - what is a group?*
    - *Clip 2a - discussion : permutations, symmetric group, simple groups, Italian renaissance, French revolution, Galois*
  - *by Socratica:*
    a delightful introduction to group multiplication (or Cayley) tables.
  - *Clip 3 - active, passive coordinate transformations*
  - *Clip 4 - following Mefisto: symmetry defined three (3) times*

Clip 5 - subgroups, classes, group orbits, reduced state space

- Hard work builds character: Irreps, unitary reps, Schur's Lemma.

  Chapter 2 *Representation Theory and Basic Theorems*
  Dresselhaus *et al.* [3], up to and including
  Sect. 2.4 *The Unitarity of Representations* (click here)

  Clip 6 - this requires character

  Clip 7 - hard work builds character

  Clip 8 - the symmetry group of a propeller

  Clip 9 - irreps of C_3

  Clip 10 - rotation in the plane: irreps of D_3

    Clip 10a - Discussion : more symmetries, fewer invariant subspaces

    Clip 10b - Discussion : abelian vs. nonabelian

- "Wonderful Orthogonality Theorem."

  In this course, we learn about full reducibility of finite and compact continuous groups in two parallel ways. On one hand, I personally find the multiplicative *projection operators* (1.19), coupled with the notion of class algebras (Harter [4] (click here) appendix C) most intuitive - a block-diagonalized subspace for each distinct eigenvalue of a given all-commuting matrix. On the other hand, the character weighted sums (here related to the multiplicative projection operators as in ChaosBook Example A24.2 *Projection operators for discrete Fourier transform*) offer a deceptively 'simple' and elegant formulation of full-reducibility theorems, preferred by all standard textbook expositions:

  Dresselhaus *et al.* [3] Sects. 2.5 and 2.6 *Schur's Lemma*.
  a first go at sect. 2.7

  Clip 11 - irreps

  Clip 12 - Frobenius character formula

  Clip 13 - character orthogonality relations

  Clip 14 - the summary: it is all about class and character

    Clip 14a - discussion : class and character

Optional reading

For a deep dive into this material, here is your rabbit hole.

For deeper insights, read Roger Penrose [8] (click here).

For a typical (but for this course advanced) application see, for example, Stone and Goldbart [11], *Mathematics for Physics: A Guided Tour for Graduate Students*, Section 14.3.2 *Vibrational spectrum of H_2O* (click here).

- Harter's Sect. *3.2 First stage of non-Abelian symmetry analysis*
  group multiplication table (3.1.1); class operators; class multiplication table (3.2.1b); all-commuting or central operators;

- Harter's Sect. *3.3 Second stage of non-Abelian symmetry analysis*
  projection operators (3.2.15); 1-dimensional irreps (3.3.6); 2-dimensional irrep (3.3.7); Lagrange irreps dimensionality relation (3.3.17)

- *An example: a 1-dimensional system with a symmetry*
- *Fundamental domain*
- *Tiling of state space by a finite group*
- *Make the "fundamental tile" your hood*
- *Symmetry-reduced dynamics*
- *Regular representation of permuting tiles*
- *Group theory voodoo*
- *Tell no Lie to plumbers*
- There is no need to learn all these "Greek" words.
- Bedside crocheting.

Question 10.1. Henriette Roux asks
Q What are cosets good for?
A Apologies for glossing over their meaning in the lecture. I try to minimize group-theory jargon, but cosets cannot be ignored.

Dresselhaus *et al.* [3] (click here) Chapter 1 *Basic Mathematical Background: Introduction* needs them to show that the dimension of a subgroup is a divisor of the dimension of the group. For example, C_3 of dimension 3 is a subgroup of D_3 of dimension 6.

In ChaosBook *Chapter 10. Flips, slides and turns* cosets are absolutely essential. The significance of the coset is that if a solution has a symmetry, then the elements in a coset act on the solution the same way, and generate all equivalent copies of this solution. Example 10.7. *Subgroups, cosets of* D_3 should help you understand that.

10.1 Group presentations

> Group theory? It is all about class & character.
> — Predrag Cvitanović, *One minute elevator pitch*

Group multiplication (or Cayley) tables, such as Table 10.1, *define* each distinct discrete group, but they can be hard to digest. A Cayley graph, with links labeled by generators, and the vertices corresponding to the group elements, has the same information as the group multiplication table, but is often a more insightful presentation of the group.

D_3	e	C	C^2	$\sigma^{(1)}$	$\sigma^{(2)}$	$\sigma^{(3)}$
e	e	C	C^2	$\sigma^{(1)}$	$\sigma^{(2)}$	$\sigma^{(3)}$
C	C	C^2	e	$\sigma^{(3)}$	$\sigma^{(1)}$	$\sigma^{(2)}$
C^2	C^2	e	C	$\sigma^{(2)}$	$\sigma^{(3)}$	$\sigma^{(1)}$
$\sigma^{(1)}$	$\sigma^{(1)}$	$\sigma^{(2)}$	$\sigma^{(3)}$	e	C	C^2
$\sigma^{(2)}$	$\sigma^{(2)}$	$\sigma^{(3)}$	$\sigma^{(1)}$	C^2	e	C
$\sigma^{(3)}$	$\sigma^{(3)}$	$\sigma^{(1)}$	$\sigma^{(2)}$	C	C^2	e

Table 10.1: The D_3 group multiplication table.

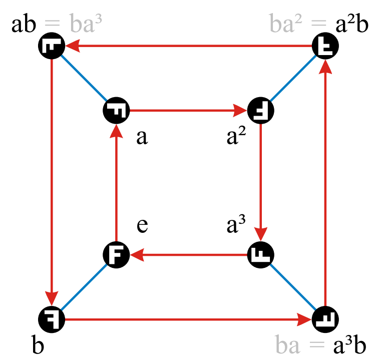


Figure 10.1: A Cayley graph presentation of the dihedral group D_4. The 'root vertex' of the graph, marked e, is here indicated by the letter $\mathbb{F}$, the links are multiplications by two generators: a cyclic rotation by left-multiplication by element a (directed red link), and the flip by b (undirected blue link). The vertices are the 8 possible orientations of the transformed letter $\mathbb{F}$.

For example, the Cayley graph figure 10.1 is a clear presentation of the dihedral group D_4 of order 8,

$$D_4 = (e, a, a^2, a^3, b, ba, ba^2, ba^3)\,, \quad \text{generators } a^4 = e\,,\ b^2 = e\,. \tag{10.1}$$

Quaternion group is also of order 8, but with a distinct multiplication table / Cayley graph, see figure 10.2. For more of such, see, for example, mathoverflow Cayley graph discussion.

Example 10.1. ***Projection operators for cyclic group C_N.***

Consider a cyclic group $C_N = \{e, g, g^2, \cdots g^{N-1}\}$, and let $M = D(g)$ be a $[2N\times 2N]$ representation of the one-step shift g. In the projection operator formulation (1.19), the N distinct eigenvalues of M, the Nth roots of unity $\lambda_n = \lambda^n$, $\lambda = \exp(i\,2\pi/N)$, $n = 0, \ldots N-1$, split the $2N$-dimensional space into N 2-dimensional subspaces by means of projection operators

$$P_n = \prod_{m\neq n} \frac{M - \lambda_m I}{\lambda_n - \lambda_m} = \prod_{m=1}^{N-1} \frac{\lambda^{-n} M - \lambda^m I}{1 - \lambda^m}\,, \tag{10.2}$$

where we have multiplied all denominators and numerators by λ^{-n}. The numerator is now a matrix polynomial of form $(x-\lambda)(x-\lambda^2)\cdots(x-\lambda^{N-1})$, with the zeroth root $(x-\lambda^0) = (x-1)$ quotiented out from the defining matrix equation $M^N - 1 = 0$. Using

$$\frac{1-x^N}{1-x} = 1 + x + \cdots + x^{N-1} = (x-\lambda)(x-\lambda^2)\cdots(x-\lambda^{N-1})$$

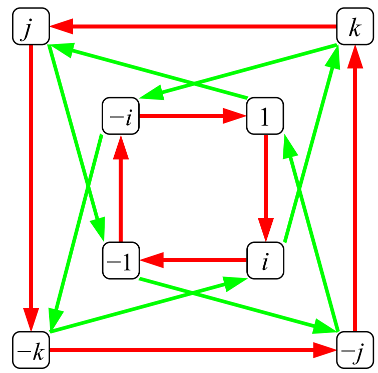


Figure 10.2: A Cayley graph presentation of the quaternion group Q_8. It is also of order 8, but distinct from D_4.

we obtain the projection operator in form of a discrete Fourier sum (rather than the product (1.19)),

$$P_n = \frac{1}{N} \sum_{m=0}^{N-1} e^{i\,\frac{2\pi}{N} nm} \, M^m \,.$$

This form of the projection operator is the simplest example of the key group theory tool, projection operator expressed as a sum over characters,

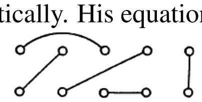
$$P_n = \frac{1}{|G|} \sum_{g \in G} \bar{\chi}(g) D(g) \,.$$

(B. Gutkin and P. Cvitanović)

10.1.1 Permutations in birdtracks

The text that follows is a very condensed extract of chapter *6 Permutations* from *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [2]. I am usually reluctant to use birdtrack notations in front of graduate students indoctrinated by their professors in the 1890's tensor notation, but I'm emboldened by the very enjoyable article on *The new language of mathematics* by Dan Silver [10]. Your professor's notation is as convenient for actual calculations as -let's say- long division using roman numerals. So leave them wallowing in their early progressive rock of 1968, King Crimsons of their youth. You chill to beats younger than Windows 98, to grime, to trap, to hardvapour, to birdtracks.

In 1937 R. Brauer [1] introduced diagrammatic notation for the Kronecker δ_{ij} operation, in order to represent "Brauer algebra" permutations, index contractions, and matrix multiplication diagrammatically. His equation (39)

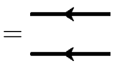


(send index 1 to 2, 2 to 4, contract ingoing (3·4), outgoing (1·3)) is the earliest published diagrammatic notation I know about. While in kindergarten (disclosure: we were too poor to afford kindergarten) I sat out to revolutionize modern group theory [2]. But I suffered a terrible setback; in early 1970's Roger Penrose pre-invented my "birdtracks," or diagrammatic notation, for symmetrization operators [7], Levi-Civita tensors [9], and "strand networks" [6]. Here is a little flavor of how one birdtracks:

We can represent the operation of permuting indices (d "billiard ball labels," tensors with d indices) by a matrix with indices bunched together:

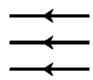
$$\sigma_\alpha^\beta = \sigma_{b_1 \ldots b_p}^{a_1 a_2 \ldots a_q}{}_{,c_q \ldots c_2 c_1}^{d_p \ldots d_1} \, . \tag{10.3}$$

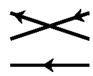
To draw this, Brauer style, it is convenient to turn his drawing on a side. For 2-index tensors, there are two permutations:

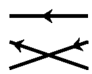
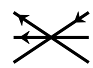
identity: $\mathbf{1}_{ab,}{}^{cd} = \delta_a^d \delta_b^c =$ 

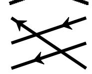
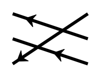
flip: $\sigma_{(12)ab,}{}^{cd} = \delta_a^c \delta_b^d =$  . (10.4)

For 3-index tensors, there are six permutations:

$\mathbf{1}_{a_1 a_2 a_3,}{}^{b_3 b_2 b_1} = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} =$ 

$\sigma_{(12)a_1 a_2 a_3,}{}^{b_3 b_2 b_1} = \delta_{a_1}^{b_2} \delta_{a_2}^{b_1} \delta_{a_3}^{b_3} =$ 

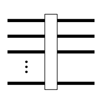
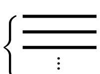
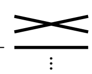
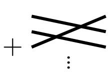
$\sigma_{(23)} =$  , $\sigma_{(13)} =$ 

$\sigma_{(123)} =$  , $\sigma_{(132)} =$  . (10.5)

Here group element labels refer to the standard permutation cycles notation. There is really no need to indicate the "time direction" by arrows, so we omit them from now on.

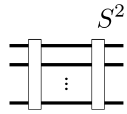
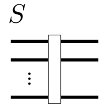
The symmetric sum of all permutations,

$$S_{a_1 a_2 \ldots a_p,}{}^{b_p \ldots b_2 b_1} = \frac{1}{p!} \left\{ \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \ldots \delta_{a_p}^{b_p} + \delta_{a_2}^{b_1} \delta_{a_1}^{b_2} \ldots \delta_{a_p}^{b_p} + \ldots \right\}$$

$S =$  $= \frac{1}{p!} \Big\{$  $+$  $+$  $+ \ldots \Big\}$, (10.6)

yields the symmetrization operator S. In birdtrack notation, a white bar drawn across p lines [7] will always denote symmetrization of the lines crossed. A factor of $1/p!$ has been introduced in order for S to satisfy the projection operator normalization

$S^2 = S$

 $=$  . (10.7)

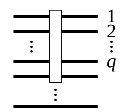
You have already seen such "fully-symmetric representation," in the discussion of discrete Fourier transforms, ChaosBook Example A24.3 *'Configuration-momentum'*

Fourier space duality, but you are not likely to recognize it. There the average was not over all permutations, but the zero-th Fourier mode $\tilde{\phi}_0$ was the average over only cyclic permutations. Every finite discrete group has such fully-symmetric representation, and in statistical mechanics and quantum mechanics this is often the most important state (the 'ground' state).

A subset of indices $a_1, a_2, \ldots a_q$, $q < p$ can be symmetrized by symmetrization matrix $S_{12\ldots q}$

$$(S_{12\ldots q})_{a_1a_2\ldots a_q\ldots a_p},^{b_p\ldots b_q\ldots b_2b_1} = \frac{1}{q!}\left\{\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_q}^{b_q} + \delta_{a_2}^{b_1}\delta_{a_1}^{b_2}\ldots\delta_{a_q}^{b_q} + \ldots\right\}\delta_{a_q+1}^{b_q+1}\ldots\delta_{a_p}^{b_p}$$

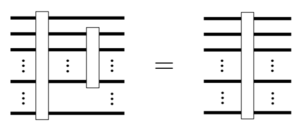
$$S_{12\ldots q} = $$



(10.8)

Overall symmetrization also symmetrizes any subset of indices:

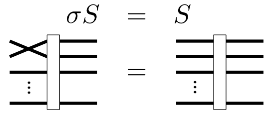
$$SS_{12\ldots q} = S$$



(10.9)

Any permutation has eigenvalue 1 on the symmetric tensor space:

$$\sigma S = S$$



(10.10)

Diagrammatically this means that legs can be crossed and uncrossed at will.

One can construct a projection operator onto the fully antisymmetric space in a similar manner [2]. Other representations are trickier - that's precisely what the theory of finite groups is about.

10.2 It's all about class

You might want to have a look at Harter [4] *Double group theory on the half-shell* (click here). Read appendices B and C on spectral decomposition and class algebras. Article works out some interesting examples.

See also remark 1.2 *Projection operators* and perhaps watch Harter's online lecture from Harter's online course.

There is more detail than what we have time to cover here, but I find Harter's Sect. *3.3 Second stage of non-Abelian symmetry analysis* particularly illuminating. It shows how physically different (but mathematically isomorphic) higher-dimensional irreps are constructed corresponding to different subgroup embeddings. One chooses the irrep that corresponds to a particular sequence of physical symmetry breakings.

10.3 Literature

It's a matter of no small pride for a card-carrying dirt physics theorist to claim full and total ignorance of group theory (read sect. A.6 *Gruppenpest* of ref. [5]). The exposition (or the corresponding chapter in Tinkham [12]) that we follow here largely comes from Wigner's classic *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* [13], which is a harder going, but the more group theory you learn the more you'll appreciate it. Eugene Wigner got the 1963 Nobel Prize in Physics, so by mid 60's gruppenpest was accepted in finer social circles.

The structure of finite groups was understood by late 19th century. A full list of finite groups was another matter. The complete proof of the classification of all finite groups takes about 3 000 pages, a collective 40-years undertaking by over 100 mathematicians, read the wiki. Not all finite groups are as simple or easy to figure out as D_3. For example, the order of the Ree group ${}^2F_4(2)'$ is $212(26+1)(24-1)(23+1)(2-1)/2 = 17\,971\,200$.

From Emory Math Department: A pariah is real! The simple finite groups fit into 18 families, except for the 26 sporadic groups. 20 sporadic groups AKA the Happy Family are parts of the Monster group. The remaining six loners are known as the pariahs.

Question 10.2. Henriette Roux asks

Q What did you do this weekend?

A The same as every other weekend - prepared week's lecture, with my helpers Avi the Little, Edvard the Nordman, and Malbec el Argentino, under Master Roger's watchful eye, see here.

References

[1] R. Brauer, "On algebras which are connected with the semisimple continuous groups", Ann. Math. **38**, 857 (1937).

[2] P. Cvitanović, *Group Theory: Birdtracks, Lie's and Exceptional Groups* (Princeton Univ. Press, Princeton NJ, 2004).

[3] M. S. Dresselhaus, G. Dresselhaus, and A. Jorio, *Group Theory: Application to the Physics of Condensed Matter* (Springer, New York, 2007).

[4] W. G. Harter and N. dos Santos, "Double-group theory on the half-shell and the two-level system. I. Rotation and half-integral spin states", Amer. J. Phys. **46**, 251–263 (1978).

[5] R. Mainieri and P. Cvitanović, "A brief history of chaos", in *Chaos: Classical and Quantum*, edited by P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay (Niels Bohr Inst., Copenhagen, 2020).

[6] R. Penrose, "Angular momentum: An approach to combinatorical space-time", in *Quantum Theory and Beyond*, edited by T. Bastin (Cambridge Univ. Press, Cambridge, 1971).

[7] R. Penrose, "Applications of negative dimensional tensors", in *Combinatorial mathematics and its applications*, edited by D. J.A. Welsh (Academic, New York, 1971), pp. 221–244.

[8] R. Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe* (A. A. Knopf, New York, 2005).

[9] R. Penrose and M. A. H. MacCallum, "Twistor theory: An approach to the quantisation of fields and space-time", Phys. Rep. **6**, 241–315 (1973).

[10] D. S. Silver, "The new language of mathematics", Amer. Sci. **105**, 364 (2017).

[11] M. Stone and P. Goldbart, *Mathematics for Physics: A Guided Tour for Graduate Students* (Cambridge Univ. Press, Cambridge UK, 2009).

[12] M. Tinkham, *Group Theory and Quantum Mechanics* (Dover, New York, 2003).

[13] E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1931).

Exercises

10.1. **1-dimensional representation of anything.** Let $D(g)$ be a representation of a group G. Show that $d(g) = \det D(g)$ is one-dimensional representation of G as well.

(B. Gutkin)

10.2. **2–dimensional representation of S_3.**

(i) Show that the group S_3 of permutations of 3 objects can be generated by two permutations, a transposition and a cyclic permutation:

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

(ii) Show that matrices:

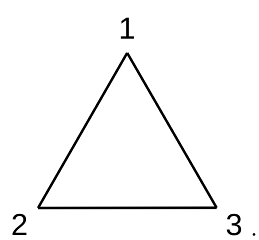
$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; D(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; D(d) = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix},$$

with $z = e^{i2\pi/3}$, provide proper (faithful) representation for these elements and find representation for the remaining elements of the group.

(iii) Is this representation irreducible?
One of those tricky questions so simple that one does not necessarily get them. If it were reducible, all group element matrices could be simultaneously diagonalized. A motivational (counter)example: as multiplication tables for D_3 and S_3 are the same, consider D_3. Is the above representation of its C_3 subgroup irreducible?

(B. Gutkin)

10.3. **D_3: symmetries of an equilateral triangle.** Consider group $D_3 \cong C_{3v} \cong S_3$, the symmetry group of an equilateral triangle:



(a) List the group elements and the corresponding geometric operations

(b) Find the subgroups of the group D_3.

(c) Find the classes of D_3 and the number of elements in them, guided by the geometric interpretation of group elements. Verify your answer using the definition of a class.

(d) List the conjugacy classes of subgroups of D_3. (continued as exercise 11.2 and exercise 11.3)

10.4. **Permutation of three objects.** Consider S_3, the group of permutations of 3 objects.

(a) Show that S_3 is a group.

(b) List the equivalence classes of S_3?

(c) Give an interpretation of these classes if the group elements are substitution operations on a set of three objects.

(c) Give a geometrical interpretation in case of group elements being symmetry operations on equilateral triangle.

10.5. **3-dimensional representations of D_3.** The group D_3 is the symmetry group of the equilateral triangle. It has 6 elements

$$D_3 = \{E, C, C^2, \sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}\},$$

where C is rotation by $2\pi/3$ and $\sigma^{(i)}$ is reflection along one of the 3 symmetry axes.

(i) Prove that this group is isomorphic to S_3

(ii) Show that matrices

$$D(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D(C) = \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^2 \end{pmatrix}, D(\sigma^{(1)}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{10.11}$$

generate a 3-dimensional representation $D(g)$ of D_3. Hint: Calculate products for representations of group elements and compare with the group table (see lecture).

(iii) Show that this is a reducible representation which can be split into one dimensional A and two-dimensional representation Γ. In other words find a matrix R such that

$$\mathbf{R} D(g) \mathbf{R}^{-1} = \begin{pmatrix} A(g) & 0 \\ 0 & \Gamma(g) \end{pmatrix}$$

for all elements g of D_3. (Might help: D_3 has only one (non-equivalent) 2-dim irreducible representation).

(B. Gutkin)