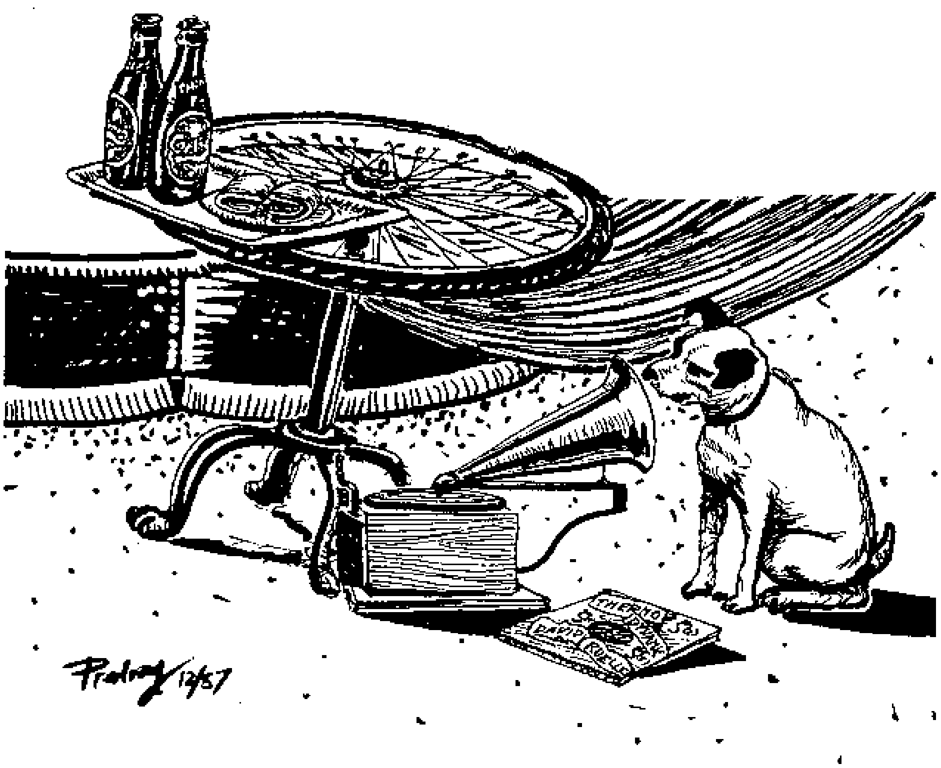
Chaos: Classical and Quantum



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1.1 Why ChaosBook?

All you need to know about chaos is contained in the introduction of [ChaosBook]. However, in order to understand the introduction you will first have to read the rest of the book.
—Gary Morriss

The problem has been with us since Newton's first frustrating (and unsuccessful) crack at the 3-body problem, lunar dynamics. Nature is rich in systems governed by simple deterministic laws whose asymptotic dynamics are complex beyond belief, systems which are locally unstable (almost) everywhere but globally recurrent. How do we describe their long term dynamics?

The answer turns out to be that we have to evaluate a determinant, take a logarithm. It would hardly merit a learned treatise, were it not for the fact that this determinant that we are to compute is fashioned out of infinitely many infinitely small pieces. The feel is of statistical mechanics, and that is how the problem was solved; in the 1960's the pieces were counted, and in the 1970's they were weighted and assembled in a fashion that in beauty and in depth ranks along with thermodynamics, partition functions and path integrals amongst the crown jewels of theoretical physics.

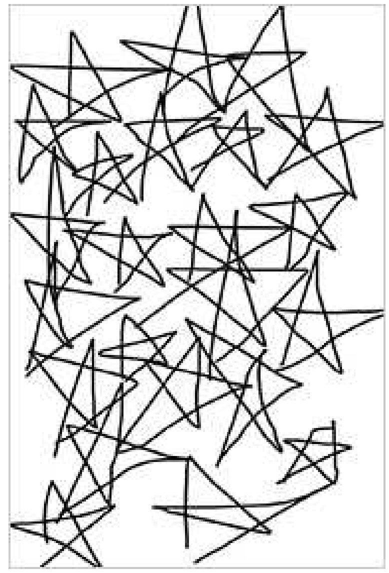

Figure 1.5: Katherine Jones-Smith, 'Untitled 5', the drawing used by K. Jones-Smith and R.P. Taylor to test the fractal analysis of Pollock's drip paintings [22].

Benoit B. Mandelbrot: "I would be perfectly happy being Kepler" [to a coming fractals' Newton]. Referring to the broad array of things now described by fractals, he added, "I have been Kepler many times over."

—J. Gleick, New York Times, January 22, 1985

In 1980's something happened that might be without parallel; this is an area of science where the advent of cheap computation had actually subtracted from our collective understanding. The computer pictures and numerical plots of fractal science of the 1980's have overshadowed the deep insights of the 1970's, and these pictures have since migrated into textbooks. By a regrettable oversight, ChaosBook has none, so 'Untitled 5' of figure 1.5 will have to do as the illustration of the power of fractal analysis. Fractal science posits that certain quantities (Lyapunov exponents, generalized dimensions, ...) can be estimated on a computer. While some of the numbers so obtained are indeed mathematically sensible characterizations of fractals, they are in no sense observable and measurable on the length-scales and time-scales dominated by chaotic dynamics.

Remark 1.8. Sorry, no schmactals!

> After all, it's impossible to read a single tweet, or hear him speak a sentence or two, without staring deep into the abyss. He turns being artless into an art form; he is a Picasso of pettiness; a Shakespeare of s**t. His faults are fractal: even his flaws have flaws, and so on ad infinitum.
>
> —Nate White

On a hype-free planet, the totality of what Hale & Koçak [17] have to say about this baby-boomer phenomenon would suffice: "No exposition of planar maps would be complete without mentioning fractals; so we mention them. Some of the popular resources are Barnsley [7] and Peitgen & Richter [32]."

Question 1.3. Henriette Roux asks
Q Before any serious study of the topic, fractals would have been the first word to come to my mind at the mention of chaos theory. So, if I may, why are fractals on the outs?
A We try to explain why in sect. 1.3.3: it's a regrettable historical accident – fractal pictures are cute, but not how the theory of chaotic dynamics actually works, which is a subject much deeper and intellectually more beautiful – hence ChaosBook. Basically, in the 1980's physicist were trying to learn the new subject, and spent much time on 1-, 2-, 3-dimensional systems that they could visualize playing with computers. Some insights were fruitful in understanding high-dimensional, physical problems. Fractals were not one of them.

But, as people ask, we must say something about them. The word 'fractal' was coined by Mandelbrot [27]. Addison's introduction to fractal dimensions [1] offers a well-motivated entry into this field. ChaosBook skirts mathematics and empirical practice of fractal analysis, such as Hausdorff and fractal dimensions. For reasons that remain mysterious to the authors - perhaps so that Mandelbrot could refer to himself both as the mother of fractals and the grandmother of multifractals - some physics literature refers to any fractal generated by more than one scale as a 'multi'-fractal. This usage divides fractals into 2 classes; one consisting of the canonical 1/3's Cantor set and the Serapinski gasket, and the second consisting of anything else, including all cases of physical interest. A bit like naming all one-legged creatures 'monopeds', and then claiming the credit for the sole discovery of all two- or more long-leggedy beasties, and claiming the honor of naming them 'multipeds'. Even though the experimental evidence for the fractal geometry of nature is circumstantial [4], in studies of probabilistically assembled fractal aggregates such as diffusion limited aggregates (DLA) better measures of 'complexity' are lacking. For deterministic systems, however, we can do *much* better, by studying physically motivated and experimentally measurable quantities (escape rates, diffusion coefficients, energy dissipation rates of turbulent flows, semiclassical atomic spectra, ...). That's what the ChaosBook is about.

remark A1.5

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Remark A1.5. Is the geometry of nature fractal? By 1983 some physicists were starting to learn that there is a thing called "chaos" [125], a thing stressful, nasty and hard to understand (see this book), so they tried to bypass this whole bit of unpleasantness by getting instead an easy, diagnostic number out of it [108]. They were told that Hausdorff dimension is the way to go. Dimension of the canonical 1/3's Cantor set can be explained to a school child, so they tried it out for size on many low-dimensional chaotic attractors, and some crazy high-dimenional ones as well. Our all-time favorite (beyond the 'Untitled 5' of figure 1.5) was the claim that the dimension of climate is 3.1 (remember, the policy of ChaosBook is not to pump up citation numbers for silly or plainly wrong papers): In *Deterministic chaos: the science and the fiction*, David Ruelle [160] comments: "[...] one should not believe dimension estimates that are not well below $2\log_{10} N$. [Authors of ...] claim to find a dimension 3.1 for a 'climatic attractor' with $N = 500$ data points. [...] The 'dimensions' of the order 6 that are obtained are very close to the upper bound $2\log_{10} N$ permitted by the Grassberger-Procaccia algorithm [85] (N is the length of the time series used, of the order of 103). The 'proof' that one has low dimensional dynamics is therefore inconclusive, and the suspicion is that the time evolutions under discussion do not correspond to low-dimensional dynamics. [...] Readers of *The Ultimate Hitchhiker's Guide to the Galaxy*, that masterpiece of British literature by D. Adams [1], know that a huge supercomputer has answered 'the great problem of life, the universe, and everything'. The answer obtained after many years of computation is 42. Unfortunately, one does not know to what precise question this is the answer, and what to make of it. It think that what happened is this. The supercomputer took a very long time series describing all it knew about 'life, the universe, and everything' and proceeded to compute the correlation dimension of the corresponding dynamics, using the Grassberger-Procaccia algorithm. This time series had a length N somewhat larger than 10^{21}. And you can imagine what happened. After many years of computation the answer came: dimension is approximately $2\log_{10} \approx 42$." In 1998 Avnir, Biham, Lidar, and Malcai [12] explored how much support for the fractal self-similarity hypothesis was there, actually. They found that "the majority of the data that was interpreted in terms of fractality in the surveyed Physical Review journals does not seem to be linked (at least in an obvious way) to existing models and, in fact, does not have theoretical backing. Most of the data represent results from nonequilibrium processes. The common situation is this: An experimentalist performs a resolution analysis and finds a limited-range power law with a value of D smaller than the embedding dimension. Without necessarily resorting to special underlying mechanistic arguments, the experimentalist then often chooses to label the object for which she or he finds this power law a 'fractal'. This is the fractal geometry of nature." Their plot says it all: the number of decades (factors of 10) spanned by experimentally derived scaling exponents peaks at 10 (i.e., one decade).

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