4.7 Neighborhood volume

Consider an infinitesimal state space volume $\Delta V = dx_1 dx_2 \cdots dx_d$ centered around the point x_0 at time $t = 0$. The volume $\Delta V'$ around the point $x' = x(t)$ time t later is (see figure 4.1)

$$\Delta V' = \frac{\Delta V'}{\Delta V} \Delta V = \left| \det \frac{\partial x'}{\partial x} \right| \Delta V = \left| \det\ J^t(x_0) \right| \Delta V \,, \qquad (4.27)$$

so the $|\det J|$ is the ratio of the initial and the final volumes. The determinant $\det J^t(x_0) = \prod_{i=1}^{d} \Lambda_i(x_0, t)$ is the product of the Jacobian matrix (4.5) stability multipliers (4.8). We shall refer to this determinant as the *Jacobian* of the flow. To evaluate it, use the matrix identity $\ln\det J = \mathrm{tr}\,\ln J$, take the time derivative and substitute the J evolution equation (4.10):

$$\frac{d}{dt} \ln \Delta V(t) = \frac{d}{dt} \ln\det J = \mathrm{tr}\, \frac{d}{dt} \ln J = \mathrm{tr} \left(\frac{1}{J} \frac{dJ}{dt} \right) = \mathrm{tr}\, A = \partial_i v_i \,.$$

(Here, as elsewhere in this book, a repeated index implies summation.) Integrate both sides to obtain the time evolution of an infinitesimal volume (see Liouville's formula wiki)

$$\det J^t(x_0) = \exp \left[\int_0^t d\tau\, \mathrm{tr}\, \mathbf{A}(x(\tau)) \right] = \exp \left[\int_0^t d\tau\, \partial_i v_i(x(\tau)) \right] . \qquad (4.28)$$

The divergence $\partial_i v_i$ characterizes the behavior of a state space volume in the infinitesimal neighborhood of the trajectory. As this is a scalar quantity, the integral in the exponent (4.19) needs *no time ordering*. So all we need to do is evaluate the time average

$$\begin{aligned} \overline{\partial_i v_i} &= \lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau \sum_{i=1}^{d} A_{ii}(x(\tau)) \\ &= \lim_{t\to\infty} \frac{1}{t} \ln \left| \prod_{i=1}^{d} \Lambda_i(x_0, t) \right| = \sum_{i=1}^{d} \mu^{(i)}(x_0, \infty) \end{aligned} \qquad (4.29)$$

along the trajectory. If the flow is not singular, the stability matrix elements are everywhere bounded from above, $A_{ij} < M$, and so is the trace $\sum_i A_{ii}$. The time integral in (4.28) thus grows at most linearly with t, $\partial_i v_i$ is bounded for all times, and numerical estimates of the $t \to \infty$ limit in (4.29) are not marred by any blowups. In numerical evaluations of stability exponents, the sum rule (4.29) can serve as a helpful check on the accuracy of the computation.

example 4.8
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If $\partial_i v_i < 0$ at a given state space point x, the flow is *locally* contracting, and the trajectory might be falling into an attractor. If $\partial_i v_i(x) < 0$ for all $x \in \mathcal{M}$, the flow is *globally* contracting, with the dimension of the attractor necessarily smaller than the dimension of state space $\mathcal{M}$. For ∞-dimensional dissipative flows, such as Navier-Stokes, the ∞ of stability exponents $\mu^{(i)}$ in (4.29) can be arbitrarily negative; as such exponents represent damping of arbitrarily kinky modes of a viscous fluid, they are of no interest for study of steady turbulence. So the sum (4.29) should be truncated to a finite number d_{phys} of leading stability exponents. We shall refer to this integer as a *physical* dimension of a *strange attractor*, in fluid dynamics often referred to as the *inertial manifold*. Every expanding or marginal direction contributes 1 to d_{phys}, and then to get a lower bound on d_{phys}, one has to keep at least as many negative $\mu^{(i)}$ as needed to ensure that the sum (4.29) is globally contracting. As nonlinear terms can mix various terms in such a way that expansion in some directions overwhelms the strongly contracting ones, d_{phys} is larger than this bound, but still a finite number.

This is an amazing result: a fluid's state space is ∞-dimensional, but its long term dynamics is confined to a finite-dimensional(!) subspace, the reason why we can apply the few degrees of freedom technology developed here to ∞-dimensional field theories.

If $\partial_i v_i = 0$, the flow preserves state space volume, $\det J^t = \mathbf{1}$, and the flow is *incompressible*. An important class of such flows are the Hamiltonian flows considered in sect. 8.3.

question 4.3

But before we can get to that, Henriette Roux, the perfect student and always alert, pipes up, asks question 4.3.

4.8 Examples

10. Try to leave out the part that readers tend to skip.
— Elmore Leonard's Ten Rules of Writing.

Example 4.1. Rössler and Lorenz flows, linearized. (Continued from exam-ple 3.4) For the Rössler (2.28) and Lorenz (2.23) flows, the stability matrices are re-spectively

$$A_{Ross} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-c \end{pmatrix}, \qquad A_{Lor} = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -b \end{pmatrix}. \tag{4.31}$$

Example 4.8. Lorenz flow state space contraction. (Continued from example 4.6) It follows from (4.31) and (4.29) that Lorenz flow is volume contracting,

$$\partial_i v_i = \sum_{i=1}^{3} \lambda^{(i)}(x,t) = -\sigma - b - 1\,, \tag{4.42}$$

at a constant, coordinate- and ρ-independent rate, set by Lorenz to $\partial_i v_i = -13.66$. For periodic orbits and long time averages, there is no contraction/expansion along the flow, $\lambda^{(\|)} = 0$, and the sum of $\lambda^{(i)}$ is constant by (4.42). Thus, we compute only one independent exponent $\lambda^{(i)}$. (continued in example 11.8)

Commentary

Remark 4.1. Linear flows. The subject of linear algebra generates innumerable tomes of its own; in sect. 4.3 we only sketch, and in appendix A4 recapitulate a few facts that our narrative relies on. A useful reference book is Meyer [16]. The basic facts are presented at length in many textbooks. Frequently cited linear algebra references are Golub and Van Loan [7], Coleman and Van Loan [4], and Watkins [24, 25]. The standard references that exhaustively enumerate and explain all possible cases are Hirsch and Smale [9] and Arnol'd [2]. A quick overview is given by Izhikevich [11]; for different notions of orbit stability see Holmes and Shea-Brown [10]. For ChaosBook purposes, we enjoyed the discussion in chapter 2 Meiss [15], chapter 1 of Perko [17] and chapters 3 and 5 of Glendinning [5]; we also liked the discussion of norms, least square problems, and differences between singular value and eigenvalue decompositions in Trefethen and Bau [22]. *Appendix A* of Stone and Goldbart [20] is an advanced summary of almost everything a graduate student needs to know about linear algebra. More pedestrian and perhaps easier to read is *Chapter 3* of Arfken and Weber [1]. Truesdell [23] and Gurtin [8] are excellent references for the continuum mechanics perspective on state space dynamics. For a gentle introduction into parallels between dynamical systems and continuum mechanics, see Christov *et al.* [3] .

The nomenclature tends to be a bit confusing. A Jacobian matrix (4.5) is sometimes referred to as the *fundamental solution matrix* or simply *fundamental matrix*, a name inherited from the theory of linear ODEs, or the *Fréchet derivative* of the nonlinear mapping $f^t(x)$, or *the 'tangent linear propagator'*, or even as the 'error matrix' (Lorenz [13]). The formula (4.22) for the linearization of nth iterate of a d-dimensional map is called a *linear cocyle*, a *multiplicative cocyle*, a *derivative cocyle* or simply a *cocyle* by some. Since matrix J describes the deformation of an infinitesimal neighborhood at a finite time t in the co-moving frame of $x(t)$, it is called a *deformation gradient* or a *transplacement gradient* in continuum mechanics. It is often denoted Df, but for our needs (we shall have to sort through a plethora of related Jacobian matrices) matrix notation J is more economical. Single discrete time-step Jacobian $J_{jl} = \partial f_j/\partial x_l$ in (4.22) is referred to as the 'tangent map' by Skokos [18, 19]. For a discussion of 'fundamental matrix' see appendix A4.2.

We follow Tabor [21] in referring to A in (4.3) as the 'stability matrix'; it is also referred to as the 'velocity gradients matrix' or 'velocity gradient tensor'. Matrix A is used to describe stability of equilibria, time-invariant points in state space; whereas, the stability of trajectories is described by Jacobian matrices. Goldhirsch, Sulem, and Orszag [6] call it the 'Hessenberg matrix', and refer to the equations of variations (4.1) as 'stability equations.' Manos *et al.* [14] refer to (4.1) as the 'variational equations'.

Sometimes A, which describes the instantaneous shear of the neighborhood of $x(x_0, t)$, is referred to as the 'Jacobian matrix', a particularly unfortunate usage when one considers linearized stability of an equilibrium point (5.1). A is not a Jacobian matrix, just as a generator of SO(2) rotation is not a rotation; A is a generator of an infinitesimal time step deformation, $J^{\delta t} \simeq \mathbf{1} + A\delta t$. What Jacobi had in mind in his 1841 fundamental paper [12] on determinants (today known as 'Jacobians') were transformations between different coordinate frames. These are dimensionless quantities, while dimensionally A_{ij} is 1/[time].

More unfortunate still is referring to the Jacobian matrix $J^t = \exp(tA)$ as an 'evolution operator', which here (see sect. 20.2) refers to something altogether different. In this book Jacobian matrix J^t always refers to (4.5), the linearized deformation after a finite time t, either for a continuous time flow, or a discrete time mapping.

Question 4.1. Henriette Roux is confused

Q What's the difference between the *stability matrix* A and the *Jacobian matrix* J^t?

A The velocity gradients matrix A is the *instantaneous* shear rate of a neighborhood of a point x. Dimensionally it is (1/time). The Jacobian matrix J^t is a dimensionless matrix of ratios of distances across the neighborhood after a *finite time t*, divided by initial distances. Stability matrix A is a matrix of spatial derivatives. J^t is obtained by a finite time integration over A.

Question 4.3. Henriette Roux does not like our Jacobian matrix

Q I do not like our definition of the Jacobian matrix in terms of the time-ordered exponential (4.19). Depending on the signs of multipliers, the left hand side of (4.28) can be either positive or negative. But the right hand side is an exponential of a real number, and that can only be positive. What gives?

A As we shall see much later on in this text, in discussion of topological indices arising in semiclassical quantization, this is not at all a dumb question.

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