Chapter 2

Go with the flow

> Dynamical systems theory includes an extensive body of knowledge about qualitative properties of generic smooth families of vector fields and discrete maps. The theory characterizes structurally stable invariant sets [...] The logic of dynamical systems theory is subtle. The theory abandons the goal of describing the qualitative dynamics of all systems as hopeless and instead restricts its attention to phenomena that are found in selected systems. The subtlety comes in specifying the systems of interest and which dynamical phenomena are to be analyzed.
>
> — John Guckenheimer

Geometry of chaos

The fate has handed you a law of nature. What are you to do with it?

1. Define your *dynamical system* $(\mathcal{M}, f)$: the space $\mathcal{M}$ of its possible states, and the law f^t of their evolution in time.

2. Pin it down locally–is there anything about it that is stationary? Try to determine its *equilibria / fixed points*.

3. Cut across it, represent as a return map from a section to a section.

4. Explore the neighborhood by *linearizing* the flow; check the *linear stability* of its equilibria / fixed points, their stability eigen-directions.

5. Does your system have a *symmetry*? If so, you must use it.

6. Go global: Label the regions by *symbolic dynamics*.

7. Now venture global distances across the system by continuing local tangent space into *stable / unstable manifolds*. Their intersections *partition the state space* in a dynamically invariant way.

8. Next: ChaosBook.org.

what does Lai-Sang Young mean by "chaotic behavior"?

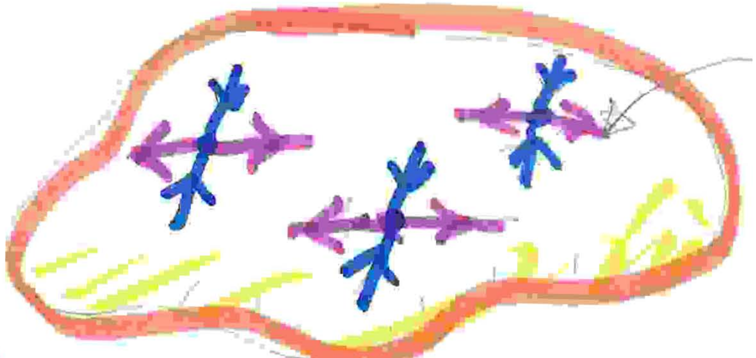
Nov 24, 2021

geometric viewpoint

(1) hyperbolicity = exponential divergence of neighbors

Hadamard (1900)
Hopf, Hedlund (1930s)
Anosov (1968)
Smale axiomatization (1960's)

Axiom A:



statespace $\mathcal{M}$

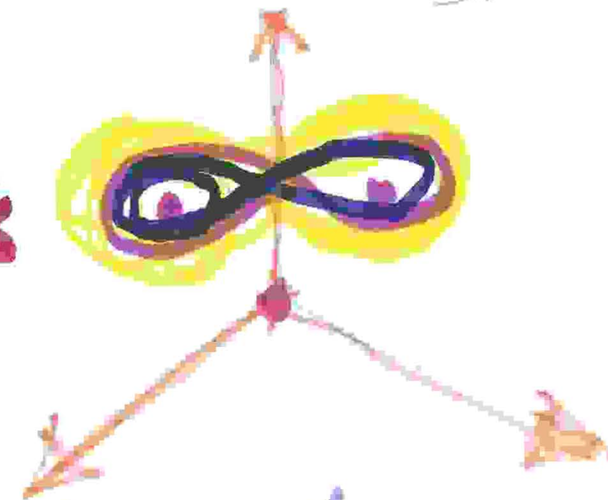
tangent space $T\mathcal{M}$ E E^s

exp contraction

geometric viewpoint

(2) positive topological entropy

exponential growth of distinguishable n-orbits



Lorenz attractor

Ergodic viewpoint

(1) positive Lyapunov exponentials

$$\lambda_i(x, v_{(i)}) = \lim_{n\to\infty} \frac{1}{n} \ln |J^n v_{(i)}(x)|$$

↑ eigenvect of Jacobian matrix

$$\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{d-1}\}$$

(2) positive entropy

$$M = M_1^{(1)} \cup M_2^{(2)} \cup \ldots \cup M_s^{(1)};$$

$$M_2 = M_{21}^{(1)} \cup M_{22}^{(2)} \cup M_{23}^{(3)}$$

$$h_\mu = \sup_{\{part\}} \lim_{|n|\to+\infty} \left[\sum_i^{s\,parts} |M_i| \ln |M_i| \right]$$

↑ uncertainty in locating $x \in M_i$

(Lai-Sang Young slide, Nov 24, 2021)

Setting

M = Riemannian manifold or region of $\mathbb{R}^n$, $U \subset M$ cpct closure

$f : U \to M$ C^2 embedding with $\overline{f(U)} \subset U$

$\Lambda = \cap_{n \geq 0} f^u(U)$ the attractor and U the basin of attraction

"Ideal dynamical picture" :

$m =$ Leb meas on U, $f_*^n m \to \mu$ = natural invariant meas

Let λ_i = Lyap exp wrt μ

Then if $\lambda_i < 0 \; \forall i$, μ supported on a periodic sink ,

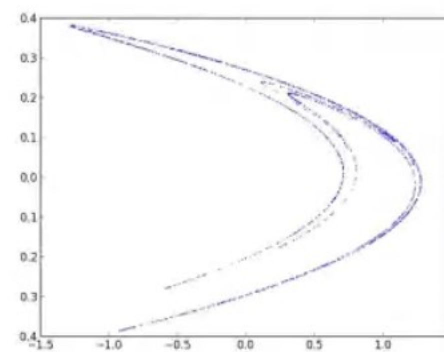
and if $\lambda_i > 0$ some i , then μ has smooth conditional densities on unstable manifolds called an *SRB measure***

SRB measures for "dissipative" chaotic dyn sys

~ Liouville meas for Hamiltonian systems

Distinctive geometry: supp on lower dim'l manifolds

No inv density possib



Chapter 6

Lyapunov exponents

> [...] people should be taught linear algebra a lot earlier than they are now, because it short-circuits a lot of really stupid and painful and idiotic material.
>
> — Stephen Boyd

Let us apply our newly acquired tools to the fundamental diagnostics in dynamics: Is a given system 'chaotic'? And if so, how chaotic? If all points in a neighborhood of a trajectory converge toward the same orbit, the attractor is a fixed point or a limit cycle. However, if the attractor is strange, any two trajectories $x(t) = f^t(x_0)$ and $x(t) + \delta x(t) = f^t(x_0 + \delta x_0)$ that start out very close to each other separate exponentially with time, and in a finite time their separation attains the size of the accessible state space.

This *sensitivity to initial conditions* can be quantified as

$$\|\,\delta x(t)\,\| \approx e^{\lambda t}\,\|\,\delta x_0\,\| \tag{6.1}$$

where λ, the mean rate of separation of trajectories of the system, is called the leading *Lyapunov exponent*.

Part II

Chaos rules

1. Partition the state space and describe all allowed ways of getting from ‘here’ to ‘there’
2. Learn to count
3. Learn how to measure what’s important
4. Learn how to evolve the measure, compute averages
5. and how the short-time / long-time duality is encoded by spectral determinant expression for its spectrum
6. Learn how to use short period cycles to describe chaotic world at times much beyond the Lyapunov time

Chapter 14

Charting the state space

> The classification of the constituents of a chaos, nothing less is here essayed.
>
> —Herman Melville, *Moby Dick*, chapter 32

IN THIS CHAPTER we learn to *partition* state space in a topologically invariant way, and *identify* topologically distinct orbits.

It does not say in the Bible that all laws of nature are expressible linearly.

— Enrico Fermi

So far we have concentrated on describing the trajectory of a single initial point. Our next task is to define and determine the size of a *neighborhood* of $x(t)$. We shall do this by assuming that the flow is locally smooth and by describing the local geometry of the neighborhood by studying the flow linearized around $x(t)$. Nearby points aligned along the stable (contracting) directions remain in the neighborhood of the trajectory $x(t) = f^t(x_0)$; the ones to keep an eye on are the points which leave the neighborhood along the unstable directions.

The repercussions are far-reaching. As long as the number of unstable directions is finite, the same theory applies to finite-dimensional ODEs, state space volume preserving Hamiltonian flows, and dissipative, volume contracting infinite-dimensional PDEs.

The best of all possible theories of deterministic chaos, and the strategy is:

1) count, 2) weigh, 3) add up.

In a chaotic system any open ball of initial conditions, no matter how small, will spread over the entire accessible state space. Hence the theory focuses on describing the geometry of the space of possible outcomes, and evaluating averages over this space, rather than attempting the impossible: precise prediction of individual trajectories. The dynamics of densities of trajectories is described in terms of evolution operators. In the evolution operator formalism the dynamical averages are given by exact formulas, extracted from the spectra of evolution operators.

Chapter 20

Averaging

Why think when you can compute?
—Maciej Zworski

WE DISCUSS the necessity of studying the averages of observables in chaotic dynamics. A time average of an observable is computed by integrating its value along a trajectory. The integral along trajectory can be split into a sum of over integrals evaluated on trajectory segments; if the observable is exponentiated, this yields a *multiplicative* weight for successive trajectory segments.

This elementary observation will enable us to recast the formulas for averages in a multiplicative form that motivates the introduction of evolution operators and further formal developments to come. The main result is that any *dynamical* average measurable in a chaotic system can be extracted from the spec-trum of an appropriately constructed evolution operator.

In physically realistic settings the initial state of a system can be specified only to a finite precision. If the dynamics is chaotic, it is not possible to calculate the long time trajectory of a given initial point.

The study of long-time dynamics thus requires trading in the evolution of a single state space point for the evolution of a *measure*, or the *density* of representative points in state space, acted upon by an *evolution operator*. Essentially this means trading in *nonlinear* dynamical equations on a finite dimensional space $x = (x_1, x_2 \cdots x_d)$ for a *linear* equation on an infinite dimensional vector space of density functions $\rho(x)$. For finite times and for maps such densities are evolved by the *Perron-Frobenius operator*,

$$\rho(x,t) = \mathcal{L}^t \circ \rho\,(x)\,,$$

and in a differential formulation they satisfy the *continuity equation*:

$$\partial_t \rho + \partial \cdot (\rho v) = 0\,.$$

The most physical of stationary measures is the natural measure, a measure robust under perturbations by weak noise.

Reformulated this way, classical dynamics takes on a distinctly quantum-mechanical flavor. If the Lyapunov time (1.1), the time after which the notion of an individual deterministic trajectory loses meaning, is much shorter than the observation time, the “sharp” observables are those dual to time, the eigenvalues of evolution operators. This is very much the same situation as in quantum mechanics; as atomic time scales are so short, what is measured is the energy, the quantum-mechanical observable dual to the time.

The expectation value $\langle a \rangle$ of an observable $a(x)$ integrated, $A^t(x) = \int_0^t d\tau\, a(x(\tau))$, and time averaged, A^t/t, over the trajectory $x \to x(t)$ is given by the derivative

$$\langle a \rangle = \left.\frac{\partial s}{\partial \beta}\right|_{\beta=0}$$

of the leading eigenvalue $e^{ts(\beta)}$ of the evolution operator $\mathcal{L}^t$.

By computing the leading eigenfunction of the Perron-Frobenius operator, one obtains the expectation value of any observable $a(x)$.

The good news is that the scaffolding will be removed, both $\mathcal{L}$'s and their eigenfunctions will be gone, and only the explicit and exact formulas for expectation values of observables will remain.

The next question is: How do we evaluate the eigenvalues of $\mathcal{L}$?

19.3 Why not just leave it to a computer?

Another subtlety in the [dynamical systems] theory is that topological and measure-theoretic concepts of genericity lead to different results.

— John Guckenheimer

(R. Artuso and P. Cvitanović)

To a student with a practical bent: the choice of function space for ρ is crucial, and the physically motivated choice is a space of smooth functions, rather than the space of piecewise constant functions.

1.8 Chaos: what is it good for?

> Happy families are all alike; every unhappy family is unhappy in its own way.
>
> — *Anna Karenina*, by Leo Tolstoy

With initial data accuracy $\delta x = |\delta \mathbf{x}(0)|$ and system size L, a trajectory is predictable only up to the *finite* Lyapunov time (1.1), $T_{\mathrm{Lyap}} \approx \lambda^{-1} \ln |L/\delta x|$. Beyond that, chaos rules. And so the most successful applications of ‘chaos theory’ have so far been to problems where observation time is much longer than a typical ‘turnover’ time, such as statistical mechanics, quantum mechanics, and questions of long term stability in celestial mechanics, where the notion of tracking accurately a given state of the system is nonsensical.

27.10

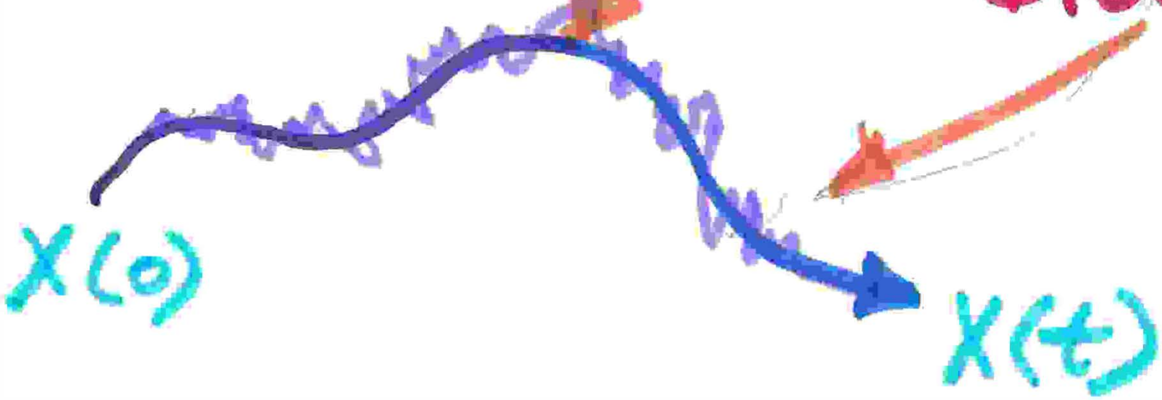
random dynamical system

stochastic flow

$$dX_t = a(x_t)\,dt + b(x_t)\cdot dW_t$$

deteministic "drift"

Brownian motion



Summary

(Lai-Sang Young slide, Nov 24, 2021)

Observations from deterministic dynamical systems that are sufficiently chaotic (hyperbolic) resemble (genuinely random) stochastic processes

Random dynamical systems have a nicer ergodic theory than deterministic systems

(Predrag: "Noise is your friend")

e.g. ideal dynamical picture for deterministic systems still true

Applicability of theory of chaotic dynamical systems hinges on verification of basic assumptions (requiring detailed info), and a little bit of randomness can go a long way.