Appendix A25

Discrete symmetry factorization

A25.1 C_{4v} factorization

If an N-disk arrangement has C_N symmetry, and the disk visitation sequence is given by disk labels $\{\epsilon_1\epsilon_2\epsilon_3\ldots\}$, only the relative increments $\rho_i = \epsilon_{i+1} - \epsilon_i \mod N$ matter. Symmetries under reflections across axes increase the group to C_{Nv} and add relations between symbols: $\{\epsilon_i\}$ and $\{N - \epsilon_i\}$ differ only by a reflection. As a consequence of this reflection increments become decrements until the next reflection and vice versa. Consider four equal disks placed on the vertices of a square (figure A25.1). The symmetry group consists of the identity **e**, the two reflections σ_x, σ_y across x, y axes, the two diagonal reflections σ_{13}, σ_{24}, and the three rotations C_4, C_2 and C_4^3 by angles $\pi/2$, π and $3\pi/2$. We start by exploiting the C_4 subgroup symmetry in order to replace the absolute labels $\epsilon_i \in \{1, 2, 3, 4\}$ by relative increments $\rho_i \in \{1, 2, 3\}$. By reflection across diagonals, an increment by 3 is equivalent to an increment by 1 and a reflection; this new symbol will be called $\underline{1}$. Our convention will be to first perform the increment and then to change the orientation due to the reflection. As an example, consider the fundamental domain cycle 112. Taking the disk 1 → disk 2 segment as the starting segment, this symbol string is mapped into the disk visitation sequence $1_{+1}2_{+1}3_{+2}1\cdots = \overline{123}$, where the subscript indicates the increments (or decrements) between neighboring symbols; the period of the cycle $\overline{112}$ is thus 3 in both the fundamental domain and the full space. Similarly, the cycle $\overline{\underline{1}12}$ will be mapped into $1_{+1}2_{-1}1_{-2}3_{-1}2_{+1}3_{+2}1 = \overline{121323}$ (note that the fundamental domain symbol $\underline{1}$ corresponds to a flip in orientation after the second and fifth symbols); this time the period in the full space is twice that of the fundamental domain. In particular, the fundamental domain fixed points correspond to the following 4-disk cycles:

4-disk		reduced
12	↔	$\underline{1}$
1234	↔	1
13	↔	2

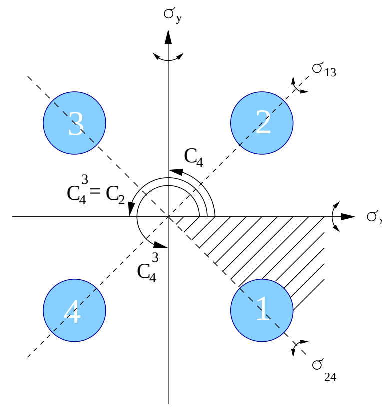

Figure A25.1: Symmetries of four disks on a square. A fundamental domain indicated by the shaded wedge.

Conversions for all periodic orbits of reduced symbol period less than 5 are listed in table A25.1.

This symbolic dynamics is closely related to the group-theoretic structure of the dynamics: the global 4-disk trajectory can be generated by mapping the fundamental domain trajectories onto the full 4-disk space by the accumulated product of the C_{4v} group elements $g_1 = C$, $g_2 = C^2$, $g_{\underline{1}} = \sigma_{diag} C = \sigma_{axis}$, where C is a rotation by $\pi/2$. In the $\overline{\underline{1}12}$ example worked out above, this yields $g_{\underline{1}12} = g_2 g_1 g_{\underline{1}} = C^2 C \sigma_{axis} = \sigma_{diag}$, listed in the last column of table A25.1. Our convention is to multiply group elements in the reverse order with respect to the symbol sequence. We need these group elements for our next step, the dynamical zeta function factorizations.

The C_{4v} group has four 1-dimensional representations, either symmetric (A_1) or antisymmetric (A_2) under both types of reflections, or symmetric under one and antisymmetric under the other (B_1, B_2), and a degenerate pair of 2-dimensional representations E. Substituting the C_{4v} characters

C_{4v}	A_1	A_2	B_1	B_2	E
e	1	1	1	1	2
C_2	1	1	1	1	-2
C_4, C_4^3	1	1	-1	-1	0
σ_{axes}	1	-1	1	-1	0
σ_{diag}	1	-1	-1	1	0

into (25.20) we obtain:

$$
\begin{array}{rlcccccc}
h_{\tilde{p}} & & & A_1 & A_2 & B_1 & B_2 & E \\
e: & (1-t_{\tilde{p}})^8 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}})^4 \\
C_2: & (1-t_{\tilde{p}}^2)^4 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}})^4 \\
C_4, C_4^3: & (1-t_{\tilde{p}}^4)^2 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}^2)^2 \\
\sigma_{axes}: & (1-t_{\tilde{p}}^2)^4 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}^2)^2 \\
\sigma_{diag}: & (1-t_{\tilde{p}}^2)^4 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}^2)^2
\end{array}
$$

Table A25.1: C_{4v} correspondence between the ternary fundamental domain prime cycles $\tilde{p}$ and the full 4-disk {1,2,3,4} labeled cycles p, together with the C_{4v} transformation that maps the end point of the $\tilde{p}$ cycle into an irreducible segment of the p cycle. For typographical convenience, the symbol $\underline{1}$ of sect. A25.1 has been replaced by 0, so that the ternary alphabet is $\{0, 1, 2\}$. The degeneracy of the p cycle is $m_p = 8n_{\tilde{p}}/n_p$. Orbit $\overline{2}$ is the sole boundary orbit, invariant both under a rotation by π and a reflection across a diagonal. The two pairs of cycles marked by (a) and (b) are related by time reversal, but cannot be mapped into each other by C_{4v} transformations.

$\tilde{p}$	p	$\mathbf{h}_{\tilde{p}}$
0	1 2	σ_x
1	1 2 3 4	C_4
2	1 3	C_2, σ_{13}
01	12 14	σ_{24}
02	12 43	σ_y
12	12 41 34 23	C_4^3
001	121 232 343 414	C_4
002	121 343	C_2
011	121 434	σ_y
012	121 323	σ_{13}
021	124 324	σ_{13}
022	124 213	σ_x
112	123	e
122	124 231 342 413	C_4

$\tilde{p}$	p	$\mathbf{h}_{\tilde{p}}$
0001	1212 1414	σ_{24}
0002	1212 4343	σ_y
0011	1212 3434	C_2
0012	1212 4141 3434 2323	C_4^3
0021 (a)	1213 4142 3431 2324	C_4^3
0022	1213	e
0102 (a)	1214 2321 3432 4143	C_4
0111	1214 3234	σ_{13}
0112 (b)	1214 2123	σ_x
0121 (b)	1213 2124	σ_x
0122	1213 1413	σ_{24}
0211	1243 2134	σ_x
0212	1243 1423	σ_{24}
0221	1242 1424	σ_{24}
0222	1242 4313	σ_y
1112	1234 2341 3412 4123	C_4
1122	1231 3413	C_2
1222	1242 4131 3424 2313	C_4^3

The possible irreducible segment group elements $\mathbf{h}_{\tilde{p}}$ are listed in the first column; σ_{axes} denotes a reflection across either the x-axis or the y-axis, and σ_{diag} denotes a reflection across a diagonal (see figure A25.1). In addition, degenerate pairs of boundary orbits can run along the symmetry lines in the full space, with the fundamental domain group theory weights $\mathbf{h}_p = (C_2 + \sigma_x)/2$ (axes) and $\mathbf{h}_p = (C_2 + \sigma_{13})/2$ (diagonals) respectively:

$$\begin{array}{rcl} & & \quad A_1 \qquad A_2 \qquad B_1 \qquad B_2 \qquad E \\ \text{axes:} \quad (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}})(1-0t_{\tilde{p}})(1-t_{\tilde{p}})(1-0t_{\tilde{p}})(1+t_{\tilde{p}})^2 \\ \text{diagonals:} \quad (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}})(1-0t_{\tilde{p}})(1-0t_{\tilde{p}})(1-t_{\tilde{p}})(1+t_{\tilde{p}})^2 \end{array} \tag{A25.1}$$

(we have assumed that $t_{\tilde{p}}$ does not change sign under reflections across symmetry axes). For the 4-disk arrangement considered here only the diagonal orbits $\overline{13}$, $\overline{24}$ occur; they correspond to the $\overline{2}$ fixed point in the fundamental domain.

The A_1 subspace in C_{4v} cycle expansion is given by

$$\begin{aligned}
1/\zeta_{A_1} &= (1-t_0)(1-t_1)(1-t_2)(1-t_{01})(1-t_{02})(1-t_{12}) \\
&\quad (1-t_{001})(1-t_{002})(1-t_{011})(1-t_{012})(1-t_{021})(1-t_{022})(1-t_{112}) \\
&\quad (1-t_{122})(1-t_{0001})(1-t_{0002})(1-t_{0011})(1-t_{0012})(1-t_{0021})\ldots \\
&= 1-t_0-t_1-t_2-(t_{01}-t_0t_1)-(t_{02}-t_0t_2)-(t_{12}-t_1t_2) \\
&\quad -(t_{001}-t_0t_{01})-(t_{002}-t_0t_{02})-(t_{011}-t_1t_{01}) \\
&\quad -(t_{022}-t_2t_{02})-(t_{112}-t_1t_{12})-(t_{122}-t_2t_{12}) \\
&\quad -(t_{012}+t_{021}+t_0t_1t_2-t_0t_{12}-t_1t_{02}-t_2t_{01})\ldots
\end{aligned} \tag{A25.2}$$

(for typographical convenience, $\underline{1}$ is replaced by 0 in the remainder of this section). For 1-dimensional representations, the characters can be read off the symbol strings: $\chi_{A_2}(\mathbf{h}_{\tilde{p}}) = (-1)^{n_0}$, $\chi_{B_1}(\mathbf{h}_{\tilde{p}}) = (-1)^{n_1}$, $\chi_{B_2}(\mathbf{h}_{\tilde{p}}) = (-1)^{n_0+n_1}$, where n_0 and n_1 are the number of times symbols 0, 1 appear in the $\tilde{p}$ symbol string. For B_2 all t_p with an odd total number of 0's and 1's change sign:

$$\begin{aligned}
1/\zeta_{B_2} &= (1+t_0)(1+t_1)(1-t_2)(1-t_{01})(1+t_{02})(1+t_{12}) \\
&\quad (1+t_{001})(1-t_{002})(1+t_{011})(1-t_{012})(1-t_{021})(1+t_{022})(1-t_{112}) \\
&\quad (1+t_{122})(1-t_{0001})(1+t_{0002})(1-t_{0011})(1+t_{0012})(1+t_{0021})\ldots \\
&= 1+t_0+t_1-t_2-(t_{01}-t_0t_1)+(t_{02}-t_0t_2)+(t_{12}-t_1t_2) \\
&\quad +(t_{001}-t_0t_{01})-(t_{002}-t_0t_{02})+(t_{011}-t_1t_{01}) \\
&\quad +(t_{022}-t_2t_{02})-(t_{112}-t_1t_{12})+(t_{122}-t_2t_{12}) \\
&\quad -(t_{012}+t_{021}+t_0t_1t_2-t_0t_{12}-t_1t_{02}-t_2t_{01})\ldots
\end{aligned} \tag{A25.3}$$

The form of the remaining cycle expansions depends crucially on the special role played by the boundary orbits: by (A25.1) the orbit t_2 does not contribute to A_2 and B_1,

$$\begin{aligned}
1/\zeta_{A_2} &= (1+t_0)(1-t_1)(1+t_{01})(1+t_{02})(1-t_{12}) \\
&\quad (1-t_{001})(1-t_{002})(1+t_{011})(1+t_{012})(1+t_{021})(1+t_{022})(1-t_{112}) \\
&\quad (1-t_{122})(1+t_{0001})(1+t_{0002})(1-t_{0011})(1-t_{0012})(1-t_{0021})\ldots \\
&= 1+t_0-t_1+(t_{01}-t_0t_1)+t_{02}-t_{12} \\
&\quad -(t_{001}-t_0t_{01})-(t_{002}-t_0t_{02})+(t_{011}-t_1t_{01}) \\
&\quad +t_{022}-t_{122}-(t_{112}-t_1t_{12})+(t_{012}+t_{021}-t_0t_{12}-t_1t_{02})\ldots
\end{aligned} \tag{A25.4}$$

and

$$\begin{aligned}
1/\zeta_{B_1} &= (1-t_0)(1+t_1)(1+t_{01})(1-t_{02})(1+t_{12}) \\
&\quad (1+t_{001})(1-t_{002})(1-t_{011})(1+t_{012})(1+t_{021})(1-t_{022})(1-t_{112}) \\
&\quad (1+t_{122})(1+t_{0001})(1-t_{0002})(1-t_{0011})(1+t_{0012})(1+t_{0021})\ldots \\
&= 1-t_0+t_1+(t_{01}-t_0t_1)-t_{02}+t_{12} \\
&\quad +(t_{001}-t_0t_{01})-(t_{002}-t_0t_{02})-(t_{011}-t_1t_{01}) \\
&\quad -t_{022}+t_{122}-(t_{112}-t_1t_{12})+(t_{012}+t_{021}-t_0t_{12}-t_1t_{02})\ldots
\end{aligned} \tag{A25.5}$$

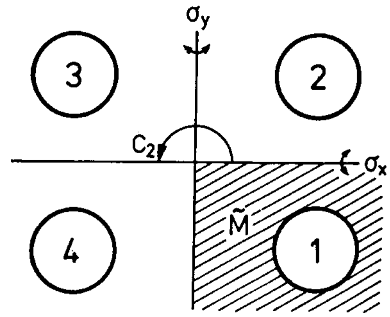

Figure A25.2: Symmetries of four disks on a rectangle. A fundamental domain indicated by the shaded wedge.

In the above we have assumed that t_2 does not change sign under C_{4v} reflections. For the mixed-symmetry subspace E the curvature expansion is given by

$$\begin{aligned}1/\zeta_E \;=\;& 1 + t_2 + (-{t_0}^2 + {t_1}^2) + (2t_{002} - t_2{t_0}^2 - 2t_{112} + t_2{t_1}^2)\\ &+(2t_{0011} - 2t_{0022} + 2t_2t_{002} - {t_{01}}^2 - {t_{02}}^2 + 2t_{1122} - 2t_2t_{112}\\ &+{t_{12}}^2 - {t_0}^2{t_1}^2) + (2t_{00002} - 2t_{00112} + 2t_2t_{0011} - 2t_{00121} - 2t_{00211}\\ &+2t_{00222} - 2t_2t_{0022} + 2t_{01012} + 2t_{01021} - 2t_{01102} - t_2{t_{01}}^2 + 2t_{02022}\\ &-t_2{t_{02}}^2 + 2t_{11112} - 2t_{11222} + 2t_2t_{1122} - 2t_{12122} + t_2{t_{12}}^2 - t_2{t_0}^2{t_1}^2\\ &+2t_{002}(-{t_0}^2 + {t_1}^2) - 2t_{112}(-{t_0}^2 + {t_1}^2))\end{aligned} \qquad \text{(A25.6)}$$

A quick test of the $\zeta = \zeta_{A_1}\zeta_{A_2}\zeta_{B_1}\zeta_{B_2}\zeta_E^2$ factorization is afforded by the topological polynomial; substituting $t_p = z^{n_p}$ into the expansion yields

$$1/\zeta_{A_1} = 1 - 3z\,, \quad 1/\zeta_{A_2} = 1/\zeta_{B_1} = 1\,, \quad 1/\zeta_{B_2} = 1/\zeta_E = 1 + z\,,$$

in agreement with (18.46).

exercise 23.8

A25.2 C_{2v} factorization

An arrangement of four identical disks on the vertices of a rectangle has C_{2v} symmetry, see figure A25.2. C_{2v} consists of $\{e, \sigma_x, \sigma_y, C_2\}$, i.e., the reflections across the symmetry axes and a rotation by π.

This system affords a rather easy visualization of the conversion of a 4-disk dynamics into a fundamental domain symbolic dynamics. An orbit leaving the fundamental domain through one of the axis may be folded back by a reflection on that axis; with these symmetry operations $g_0 = \sigma_x$ and $g_1 = \sigma_y$ we associate labels 1 and 0, respectively. Orbits going to the diagonally opposed disk cross the boundaries of the fundamental domain twice; the product of these two reflections is just $C_2 = \sigma_x\sigma_y$, to which we assign the label 2. For example, a ternary string 0 0 1 0 2 0 1 . . . is converted into 12143123. . . , and the associated group-theory weight is given by $\ldots g_1g_0g_2g_0g_1g_0g_0$.

Short ternary cycles and the corresponding 4-disk cycles are listed in table A25.2. Note that already at length three there is a pair of cycles (012 = 143 and 021 = 142) related by time reversal, but *not* by any C_{2v} symmetries.

Table A25.2: C_{2v} correspondence between the ternary $\{0, 1, 2\}$ fundamental domain prime cycles $\tilde{p}$ and the full 4-disk {1,2,3,4} cycles p, together with the C_{2v} transformation that maps the end point of the $\tilde{p}$ cycle into an irreducible segment of the p cycle. The degeneracy of the p cycle is $m_p = 4n_{\tilde{p}}/n_p$. Note that the 012 and 021 cycles are related by time reversal, but cannot be mapped into each other by C_{2v} transformations. The full space orbit listed here is generated from the symmetry reduced code by the rules given in sect. A25.2, starting from disk 1.

$\tilde{p}$	p	**g**
0	1 4	σ_y
1	1 2	σ_x
2	1 3	C_2
01	14 32	C_2
02	14 23	σ_x
12	12 43	σ_y
001	141 232	σ_x
002	141 323	C_2
011	143 412	σ_y
012	143	e
021	142	e
022	142 413	σ_y
112	121 343	C_2
122	124 213	σ_x

$\tilde{p}$	p	**g**
0001	1414 3232	C_2
0002	1414 2323	σ_x
0011	1412	e
0012	1412 4143	σ_y
0021	1413 4142	σ_y
0022	1413	e
0102	1432 4123	σ_y
0111	1434 3212	C_2
0112	1434 2343	σ_x
0121	1431 2342	σ_x
0122	1431 3213	C_2
0211	1421 2312	σ_x
0212	1421 3243	C_2
0221	1424 3242	C_2
0222	1424 2313	σ_x
1112	1212 4343	σ_y
1122	1213	e
1222	1242 4313	σ_y

The above is the complete description of the symbolic dynamics for 4 sufficiently separated equal disks placed at corners of a rectangle. However, if the fundamental domain requires further partitioning, the ternary description is insufficient. For example, in the stadium billiard fundamental domain one has to distinguish between bounces off the straight and the curved sections of the billiard wall; in that case five symbols suffice for constructing the covering symbolic dynamics.

The group C_{2v} has four 1-dimensional representations, distinguished by their behavior under axis reflections. The A_1 representation is symmetric with respect to both reflections; the A_2 representation is antisymmetric with respect to both. The B_1 and B_2 representations are symmetric under one and antisymmetric under the other reflection. The character table is

C_{2v}	A_1	A_2	B_1	B_2
e	1	1	1	1
C_2	1	1	-1	-1
σ_x	1	-1	1	-1
σ_y	1	-1	-1	1

Substituted into the factorized determinant (25.19), the contributions of periodic orbits split as follows

$$
\begin{array}{rlclccc}
g_{\tilde{p}} & & & A_1 & A_2 & B_1 & B_2 \\
e: & (1-t_{\tilde{p}})^4 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) \\
C_2: & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) \\
\sigma_x: & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) \\
\sigma_y: & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}})
\end{array}
$$

Cycle expansions follow by substituting cycles and their group theory factors from table A25.2. For A_1 all characters are +1, and the corresponding cycle expansion is given in (A25.2). Similarly, the totally antisymmetric subspace factorization A_2 is given by (A25.3), the B_2 factorization of C_{4v}. For B_1 all t_p with an odd total number of 0's and 2's change sign:

$$
\begin{aligned}
1/\zeta_{B_1} &= (1+t_0)(1-t_1)(1+t_2)(1+t_{01})(1-t_{02})(1+t_{12}) \\
&\quad (1-t_{001})(1+t_{002})(1+t_{011})(1-t_{012})(1-t_{021})(1+t_{022})(1+t_{112}) \\
&\quad (1-t_{122})(1+t_{0001})(1-t_{0002})(1-t_{0011})(1+t_{0012})(1+t_{0021})\ldots \\
&= 1+t_0-t_1+t_2+(t_{01}-t_0t_1)-(t_{02}-t_0t_2)+(t_{12}-t_1t_2) \\
&\quad -(t_{001}-t_0t_{01})+(t_{002}-t_0t_{02})+(t_{011}-t_1t_{01}) \\
&\quad +(t_{022}-t_2t_{02})+(t_{112}-t_1t_{12})-(t_{122}-t_2t_{12}) \\
&\quad -(t_{012}+t_{021}+t_0t_1t_2-t_0t_{12}-t_1t_{02}-t_2t_{01})\ldots
\end{aligned}
\qquad \text{(A25.7)}
$$

For B_2 all t_p with an odd total number of 1's and 2's change sign:

$$
\begin{aligned}
1/\zeta_{B_2} &= (1-t_0)(1+t_1)(1+t_2)(1+t_{01})(1+t_{02})(1-t_{12}) \\
&\quad (1+t_{001})(1+t_{002})(1-t_{011})(1-t_{012})(1-t_{021})(1-t_{022})(1+t_{112}) \\
&\quad (1+t_{122})(1+t_{0001})(1+t_{0002})(1-t_{0011})(1-t_{0012})(1-t_{0021})\ldots \\
&= 1-t_0+t_1+t_2+(t_{01}-t_0t_1)+(t_{02}-t_0t_2)-(t_{12}-t_1t_2) \\
&\quad +(t_{001}-t_0t_{01})+(t_{002}-t_0t_{02})-(t_{011}-t_1t_{01}) \\
&\quad -(t_{022}-t_2t_{02})+(t_{112}-t_1t_{12})+(t_{122}-t_2t_{12}) \\
&\quad -(t_{012}+t_{021}+t_0t_1t_2-t_0t_{12}-t_1t_{02}-t_2t_{01})\ldots
\end{aligned}
\qquad \text{(A25.8)}
$$

Note that all of the above cycle expansions group long orbits together with their pseudo-orbit shadows, so that the shadowing arguments for convergence still apply.

The topological polynomial factorizes as

$$
\frac{1}{\zeta_{A_1}} = 1-3z \quad , \quad \frac{1}{\zeta_{A_2}} = \frac{1}{\zeta_{B_1}} = \frac{1}{\zeta_{B_2}} = 1+z,
$$

consistent with the 4-disk factorization (18.46).

Commentary

Remark A25.1. C_{4v} labeling conventions While there is a variety of labeling conventions [2, 3] for the reduced C_{4v} dynamics, we prefer the one introduced here because of its close relation to the group-theoretic structure of the dynamics: the global 4-disk trajectory can be generated by mapping the fundamental domain trajectories onto the full 4-disk space by the accumulated product of the C_{4v} group elements.

Remark A25.2. C_{2v} symmetry C_{2v} is the symmetry of several systems studied in the literature, such as the stadium billiard [1], and the 2-dimensional anisotropic Kepler potential [4].

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[2] F. Christiansen, Analysis of Chaotic Dynamical Systems in Terms of Cycles, MA thesis (Univ. of Copenhagen, Copenhagen, 1989).

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