Appendix A14

Charting the state space

PERIODIC ORBITS of unimodal mappings are studied in sect. A14.1. Pruning theory for Bernoulli shifts (an exercise mostly of formal interest) is discussed in sect. **??**.

A14.1 Periodic orbits of unimodal maps

A *periodic point* (*cycle* point) x_k belonging to a cycle of period n is a real solution of

$$f^n(x_k) = f(f(\ldots f(x_k)\ldots)) = x_k\,, \quad k = 0, 1, 2, \ldots, n-1\,. \tag{A14.1}$$

The nth iterate of a unimodal map has at most 2^n monotone segments, and therefore there will be 2^n or fewer periodic points of length n. Similarly, the backward and the forward Smale horseshoes intersect at most 2^n times, and therefore there will be 2^n or fewer periodic points of length n. A periodic orbit of length n corresponds to an infinite repetition of a length $n = n_p$ symbol string, customarily indicated by a line over the string:

$$S_p = (s_1 s_2 s_3 \ldots s_n)^\infty = \overline{s_1 s_2 s_3 \ldots s_n}\,.$$

As all itineraries are infinite, we shall adopt convention that a finite string itinerary $S_p = s_1 s_2 s_3 \ldots s_n$ stands for infinite repetition of a finite block, and routinely omit the overline. x_0, its cyclic permutation
$\overline{s_k s_{k+1} \ldots s_n s_1 \ldots s_{k-1}}$ corresponds to the point x_{k-1} in the same cycle. A cycle p is called *prime* if its itinerary S cannot be written as a repetition of a shorter block S'.

Each cycle p is a set of n_p rational-valued full tent map periodic points γ. It follows from (14.4) that if the repeating string $s_1 s_2 \ldots s_n$ contains an odd number "1"s, the string of well ordered symbols $w_1 w_2 \ldots w_{2n}$ has to be of the double length before it repeats itself. The cycle-point γ is a geometrical sum which we can rewrite as the fraction

$$\gamma(\overline{s_1 s_2 \ldots s_n}) = \frac{2^{2n}}{2^{2n}-1} \sum_{t=1}^{2n} w_t/2^t \tag{A14.2}$$

Using this we can calculate the $\hat{\gamma}(S)$ for all short cycles. For orbits up to length 5 this is done in table 14.1.

Here we give explicit formulas for the topological coordinate of a periodic point, given its itinerary. For the purpose of what follows it is convenient to compactify the itineraries by replacing the binary alphabet $s_i = \{0, 1\}$ by the infinite alphabet

$$\{a_1, a_2, a_3, a_4, \cdots ; \overline{0}\} = \{1, 10, 100, 1000, \ldots ; \overline{0}\}\,. \tag{A14.3}$$

In this notation the itinerary $S = a_i a_j a_k a_l \cdots$ and the corresponding topological coordinate (14.4) are related by $\gamma(S) = .1^i 0^j 1^k 0^l \cdots$. For example:

$$\begin{array}{lclcl} S & = & 111011101001000\ldots & = & a_1 a_1 a_2 a_1 a_1 a_2 a_3 a_4 \ldots \\ \gamma(S) & = & .101101001110000\ldots & = & .1^1 0^1 1^2 0^1 1^1 0^2 1^3 0^4 \ldots \end{array}$$

Cycle points whose itineraries start with $w_1 = w_2 = \cdots = w_i = 0$, $w_{i+1} = 1$ remain on the left branch of the tent map for i iterations, and satisfy $\gamma(0 \ldots 0S) = \gamma(S)/2^i$.

Periodic points correspond to rational values of γ, but we have to distinguish *even* and *odd* cycles. The even (odd) cycles contain even (odd) number of a_i in the repeating block, with periodic points given by

$$\gamma(a_i a_j \cdots a_k a_\ell) = \begin{cases} \frac{2^n}{2^n-1} .1^i 0^j \cdots 1^k & \text{even} \\ \frac{1}{2^n+1}\left(1 + 2^n \times .1^i 0^j \cdots 1^\ell\right) & \text{odd} \end{cases}, \tag{A14.4}$$

where $n = i + j + \cdots + k + \ell$ is the cycle period. The maximal value periodic point is given by the cyclic permutation of S with the largest a_i as the first symbol, followed by the smallest available a_j as the next symbol, and so on. For example:

$$\begin{array}{lclclclcl} \hat{\gamma}(1) & = & \gamma(a_1) & = & .10101\ldots & = & .\overline{10} & = & 2/3 \\ \hat{\gamma}(10) & = & \gamma(a_2) & = & .1^2 0^2 \ldots & = & .\overline{1100} & = & 4/5 \\ \hat{\gamma}(100) & = & \gamma(a_3) & = & .1^3 0^3 \ldots & = & .\overline{111000} & = & 8/9 \\ \hat{\gamma}(101) & = & \gamma(a_2 a_1) & = & .1^2 0^1 \ldots & = & .\overline{110} & = & 6/7 \end{array}$$

An example of a cycle where only the third symbol determines the maximal value periodic point is

$$\hat{\gamma}(1101110) = \gamma(a_2 a_1 a_2 a_1 a_1) = .\overline{11011010010010} = 100/129\,.$$

Maximal values of all cycles up to length 5 are given in table!?

S	$\hat{\gamma}(S)$			S	$\hat{\gamma}(S)$		
$\overline{0}$	$.\overline{0}$	=	0	$\overline{10111}$	$.\overline{11010}$	=	26/31
$\overline{1}$	$.\overline{10}$	=	2/3	$\overline{10110}$	$.\overline{1101100100}$	=	28/33
$\overline{10}$	$.\overline{1100}$	=	4/5	$\overline{10010}$	$.\overline{11100}$	=	28/31
$\overline{101}$	$.\overline{110}$	=	6/7	$\overline{10011}$	$.\overline{1110100010}$	=	10/11
$\overline{100}$	$.\overline{111000}$	=	8/9	$\overline{10001}$	$.\overline{11110}$	=	30/31
$\overline{1011}$	$.\overline{11010010}$	=	14/17	$\overline{10000}$	$.\overline{1111100000}$	=	32/33
$\overline{1001}$	$.\overline{1110}$	=	14/15				
$\overline{1000}$	$.\overline{11110000}$	=	16/17				

Table A14.1: The maximal values of unimodal map cycles up to length 5. (K.T. Hansen)

A14.2 Unimodal map bifurcation sequences

(K.T. Hansen and P. Cvitanović)

Periodic orbits in smooth unimodal maps are generically created either as a pair with one stable and one unstable length n orbit in a saddle node bifurcation point, or as a period $2n$ orbit in a bifurcation where a period n orbit becomes unstable. Immediately after a saddle node bifurcation the two created orbits both have the same itinerary $\overline{s_1 s_2 \dots s_n}$ with an even number of symbols 1 and with the topological parameter value $\kappa(\overline{s_1 s_2 \dots s_n}) = \hat{\gamma}(\overline{s_1 s_2 \dots s_n})$. Orbits with this itinerary exist for all unimodal maps with $\kappa \geq \hat{\gamma}(\overline{s_1 s_2 \dots s_n})$. As the parameter in the smooth unimodal map increases the stable orbit passes a superstable point and changes its symbolic dynamics. If we now assume that the symbol string $\overline{s_1 s_2 \dots s_n}$ is the cyclic permutation giving the maximum γ value, then the itinerary of the stable orbit after the superstable point is $\overline{s_1 s_2 \dots s_{n-1}(1 - s_n)}$, since the point closest to the critical point passes through the critical point. The topological parameter value of the map is then $\kappa(\overline{s_1 s_2 \dots s_{n-1}(1 - s_n)})$. The inadmissible topological parameter interval $(\kappa(\overline{s_1 s_2 \dots s_n}), \kappa(\overline{s_1 s_2 \dots s_{n-1}(1 - s_n)}))$ is then uniquely related to the parameter interval in a between the saddle node bifurcation and the superstable point, or more loosely speaking; to the a interval where the orbit $\overline{s_1 s_2 \dots s_{n-1}(1 - s_n)}$ is stable.

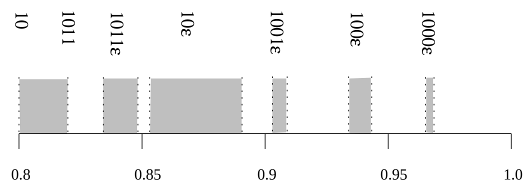
In the same way there will be an interval

$$(\kappa(\overline{s_1 s_2 \dots s_{n-1}(1 - s_n)}), \kappa(\overline{s_1 s_2 \dots s_{n-1}(1 - s_n) s_1 s_2 \dots s_n}))$$

corresponding to the interval in a from where the orbit $\overline{s_1 s_2 \dots s_{n-1}(1 - s_n)}$ is superstable to the point where the orbit $\overline{s_1 s_2 \dots s_{n-1}(1 - s_n) s_1 s_2 \dots s_n}$ is superstable. This interval includes the period doubling bifurcation where the $2n$ orbit $\overline{s_1 s_2 \dots s_{n-1}(1 - s_n) s_1 s_2 \dots s_n}$ is created.

From table A14.1 we can find some of the largest intervals in κ corresponding to the stability windows in a smooth unimodal map. The stable period 3 orbit window on the parameter a-axis corresponds to the interval $(6/7, 8/9)$ on the κ line and so on, see figure A14.1.

Figure A14.1: Bifurcation points from table A14.1 plotted as a function of the topological parameter κ. Gray areas are inadmissible intervals of κ corresponding to stable windows in a smooth unimodal map. As a shorthand notation for pairs of orbits we use the letter ϵ to denote either a 0 or a 1. (K.T. Hansen)



A14.3 Pruned Bernoulli shift

In this section we illustrate extraction of a symbolic dynamics on a piecewise linear repeller for which the itinerary of a repeller point x is given by its binary expansion. The main result is the Algorithm 1 which converts recursively a given value of the "pruning point" x_p into the symbolic dynamics of the map. We shall apply this symbolic dynamics in the next section to construction of explicit examples of zeta functions.

The simplest example of a map with complete binary dynamics is the Bernoulli shift

$$x_{n+1} = 2x_n \quad mod\ 1\ . \tag{A14.5}$$

The map acts by shifting the binary point to the right, so the itinerary of $x = x_0$ (if $x_n < 1/2$, then $s_n = 0$; if $x_n > 1/2$, then s_n=1) is simply its binary expansion $x = .s_1 s_2 s_3 \cdots$. The periodic points $\overline{s_1 s_2 \cdots s_n}$ correspond to rational x:

$$x_{s_1 s_2 \cdots s_n} = \sum_{k=1}^{n} \frac{s_k}{2^k} \sum_{m=0}^{\infty} \frac{1}{2^{nm}} = \frac{2^n}{2^n - 1} .s_1 s_2 \cdots s_n = \frac{\sum_{k=1}^{n} s_k 2^{n-k}}{2^n - 1}\ . \tag{A14.6}$$

This is just the binary version of the familiar fact that the decimal expansion of a rational number is (eventually) periodic.

The *clipped Bernoulli shift* obtained by slicing off all $x \geq x_p$ is a two-branch linear map of form

$$\begin{array}{rclrcl} f_0(x) &=& 2x & 0 \leq & x & \leq 1/2 \\ f_1(x) &=& 2x - 1 & 1/2 \leq & x & < x_p\ . \end{array} \tag{A14.7}$$

We shall refer to x_p as the *pruning point*. All trajectories that land in the $x_p < x \leq 1$ interval escape; only those x whose binary expansion contains no subsequence $\_s_m s_{m+1} s_{m+2} \cdots s_{m+n}\_$ such that $.s_m s_{m+1} s_{m+2} \cdots s_{m+n} > x_p$ survive the pruning. The surviving trajectories are unstable (with the Floquet multiplier $\Lambda_c = 2^{n_c}$, where n_c is the length of the trajectory c), and form a repelling strange set. The symbolic dynamics of the clipped Bernoulli shift is specified by the binary expansion of x_p, in the same sense that the symbolic dynamics of a unimodal map is specified by the trajectory (the kneading invariant [6]) of its critical value $x_p = f(x_c)$.

We find this map pedagogically convenient, as the technique for extracting the symbolic dynamics is essentially the same as for the unimodal maps of sect. **??**, but the conversion of the parameter x_p into symbolic dynamics is somewhat simpler (as the itineraries are ordered monotonically by the ordinary binary tree rather than the alternating binary tree).

Our strategy for converting x_p into symbolic dynamics is to check recursively whether the x_p falls into a window at successive levels of resolution. If it does, we obtain the exact alphabet; if it does not, the last letter in the approximate alphabet has to be refined, and the procedure repeated. We first phrase this recursive procedure as a general algorithm and then illustrate it by a few examples (the reader might prefer to glance at those first).

Algorithm 1: Pruning the symbolic dynamics from above

Given the pruning point x_p, the symbolic dynamics is determined recursively as follows:

1. expand $x_p = .s_1 s_2 s_3 \cdots$ in binary $s_i = \{0, 1\}$.

2. compactify the binary symbol sequences by rewriting them in the alphabet $\{a_1, a_2, a_3, a_4, \cdots\} = \{0, \underline{10}, \underline{110}, \underline{1110}, \cdots\}$ where a_n stands for a block of n-1 binary 1's followed by 0. At this level of resolution x_p is bracketed by

$$.a_n < x_p \leq .a_{n+1}$$

and the approximate alphabet is $\{a_1, a_2, a_3, a_4, \cdots, a_n\}$.

2. prohibition of a_{n+1} implies that the rightmost surviving point is $.\overline{a_n}$. If the pruning point x_p is in the window $\Delta_{a_n} = (.\overline{a_n}, .a_{n+1}]$, the exact alphabet is $\{a_1, a_2, a_3, \cdots, a_n\}$, and the algorithm stops.

3. If $x_p < .\overline{a_n}$, the pruning point falls somewhere within the $l_{a_n} = (.a_n, .\overline{a_n}]$ interval, and not all sequences starting with $.a_n$ are allowed. Subdivide the $.a_n$ interval into finer subintervals by appending to $.a_n$ all allowed basic blocks: $\{a_n\} = \{a_n a_1, a_n a_2, a_n a_3, \cdots, a_n a_m\}$. Here $m \leq n$ is determined by the value of x_p, $.a_n a_m < x_p \leq .a_n a_{m+1}$. The refined approximate alphabet is given by $\{a_1, a_2, a_3, \cdots, a_{n-1}, b_1, b_2, , \cdots, b_m\} = \{a_1, a_2, a_3, \cdots, a_{n-1}, \underline{a_n a_1}, \underline{a_n a_2}, \cdots, \underline{a_n a_m}\}$.

4. repeat step 2: if $x_p \in \Delta_{b_m} = (.\overline{b_m}, .b_{m+1}]$, the above alphabet is exact, otherwise continue refining the $l_{b_m} = (.b_m, .\overline{b_m}]$ interval by replacing b_m by new letters $\{c_1, c_2, , \cdots, c_k\}$, as in step 3.

Clearly any x_p whose binary expansion is finite yields a finite alphabet, and so does any x_p that falls into a Δ_{c_n} window. Otherwise the algorithm generates a monotone sequence of $l_{c_n} = (.c_n, .\overline{c_n}]$ covering intervals, $x_p \in l_{c_n}$, together with the associated approximate alphabets.

Example 1: the "golden mean" pruning $x_p = .11$

The simplest example of pruning for the clipped Bernoulli shift is given by the $x_p = .11$ pruning point value. As the substring _11_ is forbidden, 1 must always be followed by 0, so the allowed sequences can be built from any number of consecutive 0's and 10 blocks, and the alphabet is simply $\{0, 10\}$. Note that if x_p is set to .11, the rightmost surviving point of the repeller is not .11, but $.10101010\cdots$, *ie.* the periodic point $.\overline{10}$. Hence any x_p value in the window $\Delta_{10} = (.\overline{10}, .11] = (2/3, 3/4]$ leads to the same symbolic dynamics.

Example 2: The right ascending staircase $\Delta_0, \Delta_{10}, \Delta_{110}, \ldots,$

By the same argument as the above, $x_p = .11\ldots1$ (n binary "1"s, followed by a "0") pruning leads to the n letter alphabet

$\{a_1, a_2, a_3, \cdots, a_n\} = \{0, 10, 110, 1110, \ldots, 11\cdots10\}$,

and the symbolic dynamics unchanged over windows

$$\Delta_{11\cdots10} = (.\overline{11\cdots10}, .11\cdots1] = (\frac{2^{n-1}-2}{2^n-1}, \frac{2^n-1}{2^n}] , \tag{A14.8}$$

whose width is shrinking as $|\Delta_{11\cdots10}| = \frac{1}{2^n(2^n-1)}$.

Example 3: a "typical" pruning front value: $x_p = .11101101110$

1. At the first level of resolution x_p is bracketed by

$.111 < x_p \leq .1111$

so 1 can appear only within blocks 10, 110 and 1110. Rewrite x_p in the new alphabet $\{a_1, a_2, a_3, a_4\} = \{0, \underline{10}, \underline{110}, \underline{1110}\}$, where a_n stands for a block of $n-1$ binary 1's followed by 0: $x_p = .a_4a_3a_4a_2$

2. As in the above example, prohibition of $a_5 = 11110$ implies that the rightmost surviving point is $.\overline{a_4} = .\overline{1110}$. Were $x_p \in \Delta_{a_4} = (.\overline{a_4}, .a_5]$, the alphabet $\{a_1, a_2, a_3, a_4\}$ would be exact, and we would be finished. However, as $x_p < .\overline{a_4}$, the pruning point falls somewhere within $.a_4 < x_p \leq .\overline{a_4}$, and not all sequences starting with a_4 are allowed.

3. Therefore we subdivide the .1110 interval into finer subintervals by appending to $.a_4$ all allowed basic blocks: $\{a_4\} = \{a_4a_1, a_4a_2, a_4a_3, a_4a_4\}$. As $x_p = .a_4a_3a_4a_2 < .a_4a_4$, the $.a_4a_4$ interval is pruned. x_p can be rewritten in the new alphabet $\{a_1, a_2, a_3, b_1, b_2, b_3\} = \{a_1, a_2, a_3, \underline{a_4a_1}, \underline{a_4a_2}, \underline{a_4a_3}\}$ as $x_p = .b_3b_2$.

4. repeat step 2: is $x_p > .\overline{b_3}$? It is not, so

5. repeat step 3: subdivide $\{b_3\} = \{b_3a_1, b_3a_2, b_3a_3, b_3b_1, b_3b_2\}$. The b_3b_2 block is forbidden by the pruning point value, so we are done; the alphabet consists of 9 letters

$$\{a_1, a_2, a_3, b_1, b_2, c_1, c_2, c_3, c_4\} = \{a_1, a_2, a_3, b_1, b_2, \underline{b_3 a_1}, \underline{b_3 a_2}, \underline{b_3 a_3}, \underline{b_3 b_1}\}$$

We could have kept the binary notation throughout, but a two-letter alphabet makes for rather rather tedious reading; in the binary notation the fundamental blocks are

$$\{0, 10, 110, 11100, 111010, 11101100, 111011010, 1110110110, 111011011100\}$$

.

This finishes the list of examples.

For the clipped Bernoulli map the fraction of the x_p parameter values for which the alphabet is finite can be estimated analytically. If c is a sequence corresponding to one of the windows unfolded recursively in the above, the symbolic dynamics is unchanged over the window

$$\Delta_c = (.\overline{c}, .c1] = (\frac{.c}{1 - 2^{-n_c}}, .c1] \tag{A14.9}$$

whose width shrinks with n_c, the length of the binary string c, as

$$|\Delta_c| = .c1 - .\overline{c} = \frac{1 - .c_{n+1}}{2^{n_c} - 1} = \frac{.c^+ 1}{2^{n_c} - 1} \,, \tag{A14.10}$$

where c^+ is the binary complement of c. This follows from summing the successive images of the pruned interval $1 - .c1$ within the $(.c, .c1)$ interval.

The widths of the fatest $c = .1000 \cdots 0$ and the thinest $c = .11 \cdots 1$ steps in the devil staircase corresponding to strings of length n are, respectively,

$$\frac{2^{n-1} - 1}{2^n (2^n - 1)} \geq \Delta_c \geq \frac{1}{2^n (2^n - 1)} \,.$$

The lower bound follows from (A14.8) and the upper bound from $1 - .c1 = 1 - .10 \ldots 01 = 1/2 - 1/2^n$.

We leave the evaluation of the total measure $\sum \Delta_c$ taken up by finite alphabets as an exercise for the student. What is the measure taken up by the infinite alphabets? Is there a set of non-integer Hausdorff dimension, and what is its significance?

A14.3.1 Topological entropy

The symbolic dynamics considered in the preceeding section gives a class of rather simple topological polynomials. If the symbolic dynamics can be written as a complete (unrestricted, unpruned) alphabet in N symbols, then $t_f = 1$ if $f \in$ alphabet, $t_f = 0$ otherwise. According to the results of the preceeding section, for the clipped Bernoulli shift the symbolic dynamics is given by a finite (or infinite alphabet) built up of blocks of increasing binary length:

$$\{a_1, a_2, a_3, \cdots, a_n, b_1, b_2, , \cdots, b_m, c_1, c_2, \cdots\}$$

For a finite unrestricted alphabet, the topological entropy is given by the smallest root of the corresponding topological polynomial:

$$\begin{aligned} 0 &= 1 - z^{n_{a_1}} - z^{n_{a_2}} - z^{n_{a_3}} - \cdots - z^{n_{a_n}} \\ & -z^{n_{b_1}} - z^{n_{b_2}} - \cdots - z^{n_{b_m}} \\ & -z^{n_{c_1}} - z^{n_{c_2}} - \cdots - z^{n_{c_k}} \; . \end{aligned} \tag{A14.11}$$

The simplest example of a not entirely trivial topological polynomial follows from the 3-cycle pruning example 1. The fundamental cycles 0, $\underline{01}$ are of length 1 and 2, so the topological polynomial is simply

$$\prod_p (1 - z^{n_p}) = 1 - z - z^2 \; , \tag{A14.12}$$

and the topological entropy is $h = \log \frac{1+\sqrt{5}}{2}$.

The topological polynomial for the example 2. is given by

$$\prod_p (1 - z^{n_p}) = 1 - z - z^2 - \cdots - z^n = \frac{1 - 2z + z^{n+1}}{1 - z} \; . \tag{A14.13}$$

The topological entropy is $h = \log \lambda_0$, where λ_0 is the leading eigenvalue $1 < \lambda_0 \leq 2$. The remaining roots of (A14.13) lie (for large n) close to the unit circle in the complex plane and are of no physical interest.

The alphabet above was generated by resolving the longest fundamental string at a given level by a set of longer strings; so even if the grammar is not finite and the cycle expansion is not a polynomial, the convergence of the cycle expansion should be good, as the errors are bounded from below and above by truncating the expansions with terms z^n and z^{n+1}, where n is the length of the longest binary string in the alphabet. With increasing resolution n typically grows in leaps and bounds. The entropy is given by the isolated real zero $1/2 \leq z < 1$; the remaining zeros of the polynomial approximations to the entropy function for infinite grammar bunch on the unit circle. The rate of convergence depends on the separation of the leading, entropy eigenvalue from the non-leading eigenvalues; as long as there is a gap, the convergence will be exponential, though situations without gap also arise and are interesting (*cf.* period doubling $1/\zeta$ in ref. [6]).

The above considerations, in spite of the restriction to mere cycle counting, reveal a great deal about the spectra of more general transfer operators. For linear systems with a single scale Λ, $1/\zeta_0(z)$ is given by (A14.11), simply by rescaling $z \to z/|\Lambda|$. For nonlinear mappings, polynomial approximations to $1/\zeta_k$ have a rather similar structure; there is a physically significant $\lambda_0^{(k)}$, together with a family of unphysical poles in the complex plane, placed roughly on a circle of radius $|1/c|$, where c controls the asymptotic behavior of curvatures, $c_n \approx c^n$. The extraneous zeros delineate the boundary of the convergence of the cycle expansion of $1/\zeta_k$; and for longer and longer truncated Selberg products $1/\zeta_0\zeta_1 \cdots \zeta_k$ this boundary is pushed further and further out, allowing determination of a finite number of leading eigenvalues of $\mathcal{L}$.

Commentary

Remark A14.1. Proving the kneading sequence – topological zeta function relations. The explicit relation between the kneading sequence and the coefficients of the topological zeta function is not commonly seen in the literature. The result can proven by combining some theorems of Milnor and Thurston [6]. That approach is hardly instructive in the context of sect. A18.1. Our derivation was inspired by Metropolis, Stein and Stein classical paper [5]. For further details, consult ref. [2]. (P. Dahlqvist)

Remark A14.2. The XXX inversion formula. One gray day in 1990, in the Bristol University library a graduate student whose name was Jon Keating found an interesting *Physical Review Letter* by Dr. XXX (the policy of ChaosBook.org is not to up citation counts for plagiarized or wrong papers), entitled *"Modified Möbius inverse formula and its applications in physics."* The article starts with the Theorem 268 of Hardy and Wright [3], then derives the generalized inversion formula. By stroke of luck Keating owned the same edition of Hardy and Wright; the generalized inversion formula turned out to be precisely the Theorem 269, the page overleaf. By the evening Keating penned and faxed off a comment to *Phys. Rev. Letters*. The editors response was that *Phys. Rev. Letters* "does not publish comments that are mere factual corrections," and got instead Dr. XXX to publish an erratum saying that "Equation (7) in the text is equivalent to Theorem 270 in Hardy and Wright [...]. If one starts from this theorem, instead of from the original Möbius theorem, the paper becomes more concise."

Two weeks later Sir John Maddox, Nature editor, wrote an entire page editorial on the XXX inversion formula, and how marvelous it was that a physicist discovered all this new mathematics. Six months later, *Physical Review A* published a Rapid Communication entitled *"On XXX's inversion formula"* by a group from New Zealand. Keating requested *Physical Review* to equip all its referees with a copy of Hardy and Wright, but the proposal was turned down, and ever since there has been a stream of papers on the subject; as of 2017, the paper had over 170 citations.

Remark A14.3. What are Manning's multiples? According to Viviane Baladi the *Red Book [7], Proposition 2.4* explains Manning's argument to count periodic points. The idea is that you have to be careful with the boundary of the Markov partition and all of its iterates, as the preimages of the boundary are everywhere dense. Manning's paper [4] is explained in a very un-Bourbaki way in Bowen [1], middle of page 14. This is pure combinatorics and $x_{\vec{i}_0,\ldots,\vec{i}_{m-1}}$ is simply a point in $\mathcal{M}$. If you agree that $x_{\vec{i}_0,\ldots,\vec{i}_{m-1}}$ is simply a way to name a point x in $\mathcal{M}$, then you should not be surprised that $T^*_x f^{-m}$ of *Red Book, Proposition 7.1* denotes the linear bundle map over f^m on the cotangent bundle $T^*\mathcal{M}$. (This map already appeared in *Proposition 6.2* with its ℓ-forms brothers and sisters).

In other words, if $\mathcal{M}_i \cap \mathcal{M}_j \neq \emptyset$, you are double-counting the border points. Therefore you first count all periodic points in $\{\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_{m-1}\}$, then subtract all double counts in all pairs of border overlaps $\mathcal{M}_i \cap \mathcal{M}_j$, then add all triple counts in $\mathcal{M}_i \cap \mathcal{M}_j \cap \mathcal{M}_k$ 3-tuples, and so on.

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Exercises

A14.1. **Lefschetz zeta function.** Elucidate the relation betveen the topological zeta function and the Lefschetz zeta function.

A14.2. **Counting the 3-disk pinball counterterms.** Verify that the number of terms in the 3-disk pinball curvature expansion (25.53) is given by

$$\prod_p \left(1+t_p\right) = \frac{1-3z^4-2z^6}{1-3z^2-2z^3} = 1+3z^2+2z^3+\frac{z^4(6+12z+2z^2)}{1-3z^2-2z^3}$$
$$= 1+3z^2+2z^3+6z^4+12z^5+20z^6+48z^7+84z^8+184z^9+\ldots \quad (A14.14)$$

This means that, for example, c_6 has a total of 20 terms, in agreement with the explicit 3-disk cycle expansion (25.54).

A14.3. **Cycle expansion denominators.** Prove that the denominator of c_k is indeed D_k, as asserted (A18.10).

A14.4. **Counting subsets of cycles.** The techniques developed above can be generalized to counting subsets of cycles. Consider the simplest example of a dynamical system with a complete binary tree, a repeller map (14.20) with two straight branches, which we label 0 and 1. Every cycle weight for such map factorizes, with a factor t_0 for each 0, and factor t_1 for each 1 in its symbol string. The transition matrix traces (18.28) collapse to $tr(T^k) = (t_0+t_1)^k$, and $1/\zeta$ is simply

$$\prod_p \left(1-t_p\right) = 1-t_0-t_1 \quad (A14.15)$$

Substituting into the identity

$$\prod_p \left(1+t_p\right) = \prod_p \frac{1-{t_p}^2}{1-t_p}$$

we obtain

$$\prod_p \left(1+t_p\right) = \frac{1-t_0^2-t_1^2}{1-t_0-t_1} = 1+t_0+t_1+\frac{2t_0t_1}{1-t_0-t_1}$$
$$= 1+t_0+t_1+\sum_{n=2}^{\infty}\sum_{k=1}^{n-1} 2\binom{n-2}{k-1} t_0^k t_1^{n-k} \quad (A14.16)$$

Hence for $n \geq 2$ the number of terms in the expansion ?! with k 0's and $n-k$ 1's in their symbol sequences is $2\binom{n-2}{k-1}$. This is the degeneracy of distinct cycle eigenvalues in fig.?!; for systems with non-uniform hyperbolicity this degeneracy is lifted (see fig. ?!).

In order to count the number of prime cycles in each such subset we denote with $M_{n,k}$ $(n = 1,2,\ldots;\ k = \{0,1\}$ for $n=1$; $k = 1,\ldots,n-1$ for $n \geq 2)$ the number of prime n-cycles whose labels contain k zeros, use binomial string counting and Möbius inversion and obtain

$$M_{1,0} = M_{1,1} = 1$$
$$nM_{n,k} = \sum_{m\left|\frac{n}{k}\right.} \mu(m)\binom{n/m}{k/m}, \quad n \geq 2\,, k = 1,\ldots,n-1$$

where the sum is over all m which divide both n and k.