Commentary

Remark A1.2. A brief history of period doubling universality. Mitchell J. Feigenbaum discovered universality in one-dimensional iterative maps in August 1975. Following Feigenbaum's functional formulation of the problem, in March 1976 Cvitanović derived, in collaboration with Feigenbaum, the equation $g(x) = \alpha g(g(x/\alpha))$ for the period doubling fixed point function (not a big step, it is the limit of Feigenbaum functional recursion sequence), which has since played a key role in the theory of transitions to turbulence. The first published report [62] on Feigenbaum's discovery is dated August 1976 (Los Alamos Theoretical Division Annual Report 1975-1976, pp. 98-102, read it here). By that time the work had became widely known through many seminars Feigenbaum gave in US and Europe. His first paper, submitted to Advances in Mathematics in Nov 1976 was rejected. The second paper was submitted to *SIAM Journal of Applied Mathematics* in April 1977 and rejected in October 1977. Finally, J. Lebowitz published both papers [63, 64] without further referee pain (M. J. Feigenbaum, *J. Stat. Phys.* 19, 25 (1978) and 21, 6 (1979)).

A very informative 1976 review by May [118] describes what was known before Feigenbaum's contribution. The geometric parameter convergence was first noted in 1958 by Myrberg [13, 145], and independently of Feigenbaum, by Grossmann and Thomae [79] in 1977 (without noting the universality of δ). The theory of period-doubling universal equations and scaling functions is developed in Kenway's notes of Feigenbaum 1984 Edinburgh lectures [66] (trifle hard to track down). The elegant unstable manifold formulation of universality given in ChaosBook.org is due to Vul, Khanin, Sinai and Gol'dberg [75, 156, 157] in 1982. The most thorough exposition available is the Collet and Eckmann [30] monograph. For a more recent introduction into renormalization theory that starts out with period doubling before moving on to Quantum Field Theory, see Gurau, Rivasseau and Sfondrini [80].

In 1978 Coullet and Tresser [32, 33] have formulated similar equations, in 1979 Derrida, Gervois and Pomeau [53] have extracted a great many metric universalities from the asymptotic regime, and in 1981 Daido [51] has introduced a different set of universal equations. Grassberger [76] has computed the Hausdorff dimension of the asymptotic attractor. Following up on Grossmann and Thomae [79], Lorenz [113] and Daido [52] have found a universal ratio relating bifurcations and reverse bifurcations. If $f(x)$ is not quadratic around the maximum, the universal numbers will be different - see Vilela Mendés [155] and Hu and Mao [92] for their values. According to Kuramoto and Koga [103] such mappings can arise in chemical turbulence. Nonlinear oscillator; quadratic potential with damping and harmonic driving force exhibit cascades of period-doubling bifurcations [105, 122]. Refs. [22–24] compute solutions of the period-doubling fixed point equation using methods of Schöder and Abel, yielding what are so far the most accurate δ and α. See also Weisstein [159].

Since then the universal equations have been generalized to period n-tuplings [46, 47]; universal scaling functions for all winding numbers in circle maps constructed [48], and universality of the Hausdorff dimension of the critical staircase established [44]. A nice discussion of circle maps and their physical applications is given in refs. [10, 94, 95]. The universality theory for golden mean scalings is developed in refs. [65, 126, 135, 149].

The theory would have remained a curiosity, were it not for the beautiful experiment by Libchaber and Maurer [117], and many others that followed. Crucial insights came from Collet and Eckmann [30] and Collet, Eckmann and Koch [31] who explained how the dynamics of dissipative system (such as a viscous fluid) can become 1-dimensional. The experimental and theoretical developments up to 1990's are summarized in reprint collections by Cvitanović [37] and Hao [87]. We also recommend Hu [91], Crutchfield, Farmer and Huberman [35], Eckmann [58] and Ott [127]. The period-doubling route to turbulence that is by no means the only way to get there; see Eckmann [58] discussion of other routes to chaos.

Remark A1.3. Should one attach names to equations? .

Q : Name the 2nd person who invented General Relativity?
A : Who remembers?
—Professore Dottore Gatto Nero

By 1979 mathematicians understood that the numerical methods used by Feigenbaum and Cvitanović to solve the universal equations were in fact convergent. They did what they do; they attached various names to the equations, they changed letters around. The re-lettering did not stick, but the renamings did.

Feigenbaum [62] discovered and formulated period-doubling universality in 1975: you can read about it and find his 1976 report by clicking here and here. In 1981 Lanford [108] satisfied himself that the iterative method Feigenbaum and Cvitanović used and knew was contracting was indeed contracting. Lanford refers only to the Feigenbaum paper [63]. Coullet and Tresser [32, 33] refer to the Feigenbaum paper [63].

In 1995 Lyubich [115, 116] rechristened the equations to "Feigenbaum-Coullet-Tresser," omitting Cvitanović (the first to formulate the period-doubling fixed point equation), and adding Coullet and Tresser (who rediscovered it a couple of years later). These are all very fine physicists / mathematicians, creative and crazy as bats. But why rename an equation that was widely known and publicized well before 1978? Is there something essential that is missing in the 1976 formulation?

We asked Lyubich why? He wrote back: "In 1990s, I talked to both Feigenbaum and Tresser, and my conclusion was that Coullet-Tresser discovered the phenomenon independently, though slightly later. Also, they seemed to recognize better importance of the dynamical universality (while Feigenbaum focused more on the parameter phenomenon). I felt that Coullet-Tresser did not receive a proper credit for their insights, so I attached all three names to the phenomenon." That's sweet. Turns out Feigenbaum and Cvitanović invented but did not recognize "importance of the dynamical universality", whatever that might mean. While we are at it, why not credit the person who actually wrote the fixed point equation first? Or he's just dog meat?

People reinvent stuff all the time. For example, Myrheim and Cvitanović [46, 47] generalized period doubling to infinity of renomalizations in the complex plane, but once they were told that Golberg, Sinai and Khanin [75] did it first (for period tripling), they gave credit to them, even though both groups discovered the phenomenon independently in 1983.

Why attach names to equations anyway? Pretty soon the attribution problems will sort themselves out by themselves - heart attacks and homicidal Atlanta drivers running down cyclists will take care of that.

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