

Appendix A16

Finding cycles

(C. Chandre)

A16.1 Newton-Raphson method

A16.1.1 Contraction rate

CONSIDER A d -DIMENSIONAL MAP $x' = f(x)$ with an unstable fixed point x_* . The Newton-Raphson algorithm is obtained by iterating the following map

$$x' = g(x) = x - (J(x) - \mathbf{1})^{-1} (f(x) - x).$$

The linearization of g near x_* leads to

$$x_* + \epsilon' = x_* + \epsilon - (J(x_*) - \mathbf{1})^{-1} (f(x_*) + J(x_*)\epsilon - x_* - \epsilon) + O(\|\epsilon\|^2),$$

where $\epsilon = x - x_*$. Therefore,

$$x' - x_* = O\left((x - x_*)^2\right).$$

After n steps and if the initial guess x_0 is close to x_* , the error decreases super-exponentially

$$g^n(x_0) - x_* = O\left((x_0 - x_*)^{2^n}\right).$$

A16.1.2 Computation of the inverse

The Newton-Raphson method for finding n -cycles of d -dimensional mappings using the multi-shooting method reduces to the following equation

$$\begin{pmatrix} \mathbf{1} & & & -Df(x_n) \\ -Df(x_1) & \mathbf{1} & & \\ & \dots & & \\ & & \mathbf{1} & \\ & & -Df(x_{n-1}) & \mathbf{1} \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \dots \\ \delta_n \end{pmatrix} = - \begin{pmatrix} F_1 \\ F_2 \\ \dots \\ F_n \end{pmatrix}, \quad (\text{A16.1})$$

where $Df(x)$ is the $[d \times d]$ Jacobian matrix of the map evaluated at the point x , and $\delta_m = x'_m - x_m$ and $F_m = x_m - f(x_{m-1})$ are d -dimensional vectors. By some straightforward algebra, the vectors δ_m are expressed as functions of the vectors F_m :

$$\delta_m = - \sum_{k=1}^m \beta_{k,m-1} F_k - \beta_{1,m-1} (\mathbf{1} - \beta_{1,n})^{-1} \left(\sum_{k=1}^n \beta_{k,n} F_k \right), \quad (\text{A16.2})$$

for $m = 1, \dots, n$, where $\beta_{k,m} = Df(x_m)Df(x_{m-1}) \cdots Df(x_k)$ for $k < m$ and $\beta_{k,m} = \mathbf{1}$ for $k \geq m$. Therefore, finding n -cycles by a Newton-Raphson method with multiple shooting requires the inverting of a $[d \times d]$ matrix $\mathbf{1} - Df(x_n)Df(x_{n-1}) \cdots Df(x_1)$.

A16.2 Hybrid Newton-Raphson / relaxation method



Consider a d -dimensional map $x' = f(x)$ with an unstable fixed point x_* . The transformed map is the following one:

$$x' = g(x) = x + \gamma C(f(x) - x),$$

where $\gamma > 0$ and C is a $d \times d$ invertible constant matrix. We note that x_* is also a fixed point of g . Consider the stability matrix at the fixed point x_*

$$A_g = \left. \frac{dg}{dx} \right|_{x=x_*} = \mathbf{1} + \gamma C(A_f - \mathbf{1}).$$

The matrix C is constructed such that the eigenvalues of A_g are of modulus less than one. Assume that A_f is diagonalizable: In the basis of diagonalization, the matrix writes:

$$\tilde{A}_g = \mathbf{1} + \gamma \tilde{C}(\tilde{A}_f - \mathbf{1}),$$

where \tilde{A}_f is diagonal with elements μ_i . We restrict the set of matrices \tilde{C} to diagonal matrices with $\tilde{C}_{ii} = \epsilon_i$ where $\epsilon_i = \pm 1$. Thus \tilde{A}_g is diagonal with eigenvalues $\gamma_i = 1 + \gamma \epsilon_i (\mu_i - 1)$. The choice of γ and ϵ_i is such that $|\gamma_i| < 1$. It is easy to see that if $\text{Re}(\mu_i) < 1$ one has to choose $\epsilon_i = 1$, and if $\text{Re}(\mu_i) > 1$, $\epsilon_i = -1$. If λ is chosen such that

$$0 < \gamma < \min_{i=1, \dots, d} \frac{2|\text{Re}(\mu_i) - 1|}{|\mu_i - 1|^2},$$

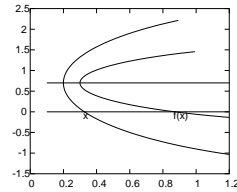
all the eigenvalues of A_g have modulus less than one. The contraction rate at the fixed point for the map g is then $\max_i |1 + \gamma \epsilon_i (\mu_i - 1)|$. If $\text{Re}(\mu_i) = 1$, it is not possible to stabilize x_* by the set of matrices γC .

From the construction of C , we see that 2^d choices of matrices are possible. For example, for 2-dimensional systems, these matrices are

$$C \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

For 2-dimensional dissipative maps, the eigenvalues satisfy $\text{Re}(\mu_1)\text{Re}(\mu_2) \leq \det Df < 1$. The case $(\text{Re}(\mu_1) > 1, \text{Re}(\mu_2) > 1)$ which is stabilized by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ has to be discarded. The minimal set is reduced to three matrices.

Figure A16.1: Illustration of the optimal Poincaré section. The original section $y = 0$ yields a large distance $x - f(x)$ for the Newton iteration. A much better choice is $y = 0.7$.



A16.2.1 Newton method with optimal section



(F. Christiansen)

In some systems it might be hard to find a good starting guess for a fixed point. This can happen, for example, if the topology and/or the symbolic dynamics of the flow is not well understood. By changing the Poincaré section one might get a better initial guess in the sense that x and $f(x)$ are closer together. We illustrate this in figure A16.1. The figure shows a Poincaré section, $y = 0$, an initial guess x , the corresponding $f(x)$ and pieces of the trajectory near these two points.

If Newton iteration does not converge for the initial guess x we might have to work very hard to find a better guess, particularly if this is in a high-dimensional system (high-dimensional in this context might mean a Hamiltonian system with 3 or more degrees of freedom). Clearly, we could easily obtain a much better guess by simply shifting the Poincaré section to $y = 0.7$ where the distance $x - f(x)$ would be much smaller. Naturally, one cannot so easily determine by inspection the best section for a higher dimensional system. Rather, the way to proceed is as follows: We want to have a minimal distance between our initial guess x and its image $f(x)$. We therefore integrate the flow looking for a minimum in the distance $d(t) = |f^t(x) - x|$. $d(t)$ is now a minimum with respect to variations in $f^t(x)$, but not necessarily with respect to x . We therefore integrate x either forward or backward in time. Doing this minimizes d with respect to x , but now it is no longer minimal with respect to $f^t(x)$. We therefore repeat the steps, alternating between correcting x and $f^t(x)$. In most cases this process converges quite rapidly. The result is a trajectory for which the vector $(f(x) - x)$ connecting the two end points is perpendicular to the flow at both points. We can now define a Poincaré section as the hyper-plane that goes through x and is normal to the flow at x , $(x' - x) \cdot v(x) = 0$.

The image $f(x)$ lies in the section. This section is optimal in the sense that a close return on the section is a local minimum of the distance between x and $f^t(x)$. More important, the part of the stability matrix that describes linearization perpendicular to the flow is exactly the stability of the flow in the section when $f(x)$ is close to x . With this method, the Poincaré section changes with each Newton iteration. Should we later want to put the fixed point on a specific Poincaré section, it will only be a matter of moving along the trajectory.