Chapter 31

Pruning dead itineraries

Include sections of the paper with Kai.

RM: Finite and infinite subshifts. How the above stuff gets complicated. Serious pruning here.

The splitting of cycles into the fundamental cycles and the curvature corrections depends on balancing long cycles t_{ab} against their pseudo-trajectory shadows $t_a t_b$. If the $\overline{ab}$ cycle or either of the shadows $\overline{a}$, $\overline{b}$ do not to exist, such curvature cancellation is unbalanced. In a generic dynamical system not every symbol sequence is realized as a physical trajectory; as one looks further and further, one discovers more and more rules which prohibit families of cycles, with unbalanced curvatures of any length. In the above we have used examples for which all possible orbits can be labelled by all possible binary sequences. In this chapter we discuss the cycle expansions for systems with more complicated symbolic dynamics.

Use 1-d inverse iterates to illustrate that pruning implies existence of complex orbits.

Symbolic dynamics of a generic dynamic system is arbitrarily complex; even for the logistic map the grammar is finite only for special parameter values. Our strategy is akin to bounding a real number by a sequence of rational approximants; we converge toward the non–wandering set under investigation by a sequence of self-similar Cantor sets. A "self-similar" Cantor set (in the sense in which we use the word here) is a Cantor set equipped with a *subshift of finite type* symbol dynamics, that is the corresponding grammar can be stated as a finite number of pruning rules, each forbidding a finite subsequence $\_\epsilon_1\epsilon_2\ldots\epsilon_n\_$. Here the notation $\_\epsilon_1\epsilon_2\ldots\epsilon_n\_$ stands for n consecutive symbols ϵ_1, ϵ_2, $\ldots$, ϵ_n, preceeded and followed by arbitrary symbol strings.

The most important lesson of the pruning of the cycle expansions is that prohibition of a finite subsequence unbalances the head of a cycle expansion and increases the number of the fundamental cycles in (14.5). Hence the pruned expansions are expected to start converging only *after* all fundamental cycles have been incorporated - in the last example, the cycles $\overline{1}$, $\overline{10}$, $\overline{10100}$, $\overline{1011100}$. Without cycle expansions, no such crisp and clearcut definition of the fundamental set of scales is available.

Lozi map!bifurcation
bifurcation!Lozi map

A typical Hénon map is not complete. The problem of determining which symbol sequences are absent, or the problem orbit pruning, viewed in the x-y coordinates, appears formidable. It is equivalent to the problem of determining the location of all homoclinic tangencies, or all turnbacks of the Hénon attractor. They are dense on the attractor, and show no obvious self-similar structure in the x-y plane.

However, in the two dimensional Danish representation of the attractor, the pruning problem is easily solved; any periodic orbit which contains certain subsequence of symbols is forbidden. The forbidden sub-sequences are computed from the map, and are a function of its parameters.

We use the Lozi map to illustrate the orbit pruning problem. The Lozi map has the unstable-pruned orbit structure and non-trivial scaling structure of Hénon type, without the subtleties introduced by the coexistence of stable periodic orbits, or worries about the uniqueness of cycle points corresponding to a given symbol sequence. Furthermore, as the Lozi map is linear, it is computationally fast, and the attractor can be covered by hundreds of thousands of periodic points without straining patience of a desktop workstation.

missing fig. **??**

For the Lozi map (3.10) the stable (unstable) manifolds W^s (W^u) are piecewise linear and easily plotted: fig. **??** shows them for parameter values $(a, b) = (1.7, 0.5)$ and $(1.53, 0.5)$. Analysis of the Lozi map is much simpler than that of the Hénon map, as all primary turning points of W^u lie on the line $x = 0$, and the symbol corresponding to a point (x_n, x_{n+1}) is given by the sign of x_{t+1}. An example of a bifurcation of a primary turning point of the Lozi map is sketched in fig. **??**.

KTH: The three turning points bifurcate together, and a jump of the primary turning point is 0.?

Beyond the orbit pruning and the scaling laws for the allowed unstable orbits, the Hénon attractors also own an infinity of attractive orbits coexisting with the strange attractor. We offer heuristic arguments and solid numerical evidence that the coexistence of attractive orbits does not destroy the strange attractor, which is also in this general case described by the two-dimensional danish pastry plot.

31.1 Pruning front

A generic once-folding map does not yield a complete horseshoe; some of the horseshoe pieces might be *pruned*, that is not realized for particular parameter values of the mapping. In 1 dimension, the criterion for whether a given symbolic sequence is realized by a given unimodal map is easily formulated; any orbit that strays to the right of the value computable from the *kneading sequence* (the orbit of the critical point) is pruned. This is a topological statement, independent of a particular unimodal map. Our objective is to generalize this notion to 2-d once-folding maps.

Our strategy is the following: we first construct a 2-d representation of the covering symbolic dynamics of the map. This is a "road map" in which the various sheets of the stable and unstable manifolds are represented by straight sections, and the topology is preserved: the nearby periodic points in the symbol square represent nearby periodic points in the phase space. Next we separate the admissible and the forbidden motions by means of a "pruning front", a boundary between the two kinds of orbits.

We make following assumptions:

(i) *The partition conjecture (Grassberger-Kantz) [13.15]:* the non–wandering set of a once-folding map can be described by a subset of a complete Smale horseshoe, partitioned by the set of primary turning points.

(ii) *The pruning-front conjecture (Cvitanović-Gunaratne-Procaccia) [11.23]:* kneading values of the set of all primary turning points separate the admissible from the forbidden orbits, and there are no other pruning rules.

(iii) *Multimodal map approximation [1.3]:* A 2-dimensional once-folding map can be systematically approximated by a sequence of 1-dimensional n-folding maps.

The intuition behind these conjectures is that the folding induced by a single iteration is the primary folding, and all other folds (turnbacks, homoclinic tangencies) are images or preimages of the primary ones. The asymptotic object is a collection of infinitely many 1-d sheets, and the pruning front is the set of the corresponding kneading sequences, one for each 1-d sheet.

Conjectures (i) and (ii) work for all examples that we have studied, and we are not aware of existence of any once-folding maps that contradict them.

Generic symbolic dynamics is not simple - as a matter of fact, already in one dimension its grammar can be arbitrarily complicated. However, the grammar is finite if the critical point is in the basin of attraction of an attractive periodic point; for example, if the critical point is attracted to a stable 3–cycle, the repeller consists of an isolated $\bar{0}$ fixed point and all orbits built from the two letters $\{1, 10\}$, see exercise ?!.

explain golden mean counting for unimodal maps somewhere

The analogous situation can be attained for a 2–dimensional once–folding map if the parameters of the map adjusted so that the intersection of the backward and forward folds is still transverse, but no longer complete, as in fig. 11.6a. In this particular example the intersections $\_10.1\_$, $\_11.1\_$ have been lost, and consequently any trajectory containing substrings $\_101\_$, $\_111\_$ is pruned. We refer to the left border of this primary pruned region as the *pruning front*; an example of a pruning front is drawn in fig. 11.6(d). The pruning front is a 2–dimensional generalization of the 1–dimensional kneading sequence; the location of each vertical step in the pruning front is the kneading sequence of the corresponding primary turnback of the unstable manifolds. When correct, the pruning front conjecture

three-disk@3-disk!pruning front
pruning front!3-disk

is a complete description of the symbolic dynamics of once-folding maps in the same sense in which the kneading sequence defines the symbolic dynamics of a one–dimensional unimodal map. The intuition behind this conjecture is that the folding induced by a single iteration is the primary folding, and all other folds (turnbacks, homoclinic tangencies) are images or preimages of the primary ones. The topology puts two obvious constraints on the form of a pruning front:

1. The pruning front is symmetric across the horizontal 1/2 line.

2. The pruning front is *monotone* across either half of the symbol square.

This is a consequence of the deterministic foliation; inner folds cannot pierce through the outer folds, and therefore have fewer transverse sections than the outer ones.

31.1 on p. 682

31.2 Hénon map pruning front

Insert table like table 13.4 here for the pruned Hénon map values.

Hénon [3.1] has conjectured that for $(a, b) = (1.4, 0.3)$ the limit cycle of the Hénon map is a *strange attractor*. Its existence has never been proven. While for all practical purposes this is a strange attractor, pruning front techniques that we describe below enable us to find [27.1] stable attractors arbitrarily close by in the parameter space, such as the 13-cycle attractor at $(a, b) = (1.39945219, 0.3)$.

Benedicks and Carleson [31.2] have proven that in the vicinity of $(a, b) = (2, 0)$ strange attractors exist for a Lebesgue measure in the parameter space. PC: recheck the phrasing!

The danish pastry transformation, the simplest Baker's transformation appropriate to Hénon type mappings, yields a binary coordinatization of all possible cycle points.

explain danish pastry here

The symbolic dynamics of once-folding map is given by the danish pastry transformation. (kneading danish pastry). This generates both the longitudinal and transverse alternating binary tree. The longitudinal coordinate is given by the head of a symbolic sequence; the transverse coordinate is given by the tail of the symbolic sequence. The dynamics on this space is given by symbol shift permutations; volume preserving, with 2 expansion and 1/2 contraction.

31.3 3-disk pruning front

31.2 on p. 682

31.3 on p. 683

Commentary

Remark **31.1** What is good about pruning fronts? The rule that everything to one side of the pruning front is forbidden might seem trivial to you, but if you have ever tried to work out symbolic dynamics of some "generic" dynamical systems, you should be struck by its simplicity: instead of pruning a Cantor set embedded within some larger Cantor set, the pruning front cleanly cuts out a *compact* region in the symbol square and that is all - there are no additional pruning rules.

⇓PRELIMINARY

Résumé

⇑PRELIMINARY

References

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[31.5] K.T. Hansen, *Symbolic dynamics of bimodal maps*, in preparation.