local!stability
linear!stability
stability!linear
cycle!stability
cycle!stability
topological!invariant
invariant!topological

Chapter 8

Cycle stability

Topological features of a dynamical system – singularities, periodic orbits, and the ways in which the orbits intertwine – are invariant under a general continuous change of coordinates. Surprisingly, there exist quantities that depend on the notion of metric distance between points, but nevertheless do not change value under a smooth change of coordinates. Local quantities such as the eigenvalues of equilibria and periodic orbits, and global quantities such as Lyapunov exponents, metric entropy, and fractal dimensions are examples of properties of dynamical systems independent of coordinate choice.

We now turn to the first, local class of such invariants, linear stability of periodic orbits of flows and maps. This will give us metric information about local dynamics. If you already know that the eigenvalues of periodic orbits are invariants of a flow, skip this chapter.

fast track:
chapter 9, p. 131

8.1 Stability of periodic orbits

As noted on page 36, a trajectory can be stationary, periodic or aperiodic. For chaotic systems almost all trajectories are aperiodic – nevertheless, equilibria and periodic orbits will turn out to be the key to unraveling chaotic dynamics. Here we note a few of the properties that makes them so precious to a theorist. [1]

add figs

An obvious virtue of periodic orbits is that they are *topological* invariants: a fixed point remains a fixed point for any choice of coordinates, and similarly a periodic orbit remains periodic in any representation of the dynamics. Any re-parametrization of a dynamical system that preserves

[1] Mason: does not jive with relative equilibira in a 3-body problem? This must be rephrased more carefully. PC: will tighten this in the continuous symmetries chapter **??**.

metric!invariant
invariant!metric
multiplier!Floquet
multiplier!characteristic
characteristic!multiplier
Floquet!exponent
exponent!Floquet
give references to templates

its topology has to preserve topological relations between periodic orbits, such as their relative inter-windings and knots. So the mere existence of periodic orbits suffices to partially organize the spatial layout of a non–wandering set. No less important, as we shall now show, is the fact that cycle eigenvalues are *metric* invariants: they determine the relative sizes of neighborhoods in a non–wandering set.

To prove this, we start by noting that due to the multiplicative structure (4.33) of fundamental matrices, the fundamental matrix for the rth repeat of a prime cycle p of period T_p is

$$J^{rT_p}(x) = J^{T_p}(f^{(r-1)T_p}(x)) \cdots J^{T_p}(f^{T_p}(x)) J^{T_p}(x) = (J_p(x))^r \ , \quad (8.1)$$

emphasize - eigenvalues are NOT multiplicative unless on a periodic orbit

where $J_p(x) = J^{T_p}(x)$ is the fundamental matrix for a single traversal of the prime cycle p, $x \in p$ is any point on the cycle, and $f^{rT_p}(x) = x$ as $f^t(x)$ returns to x every multiple of the period T_p. Hence, it suffices to restrict our considerations to the stability of prime cycles.

⇓PRELIMINARY

8.1.1 Better nomenclature

do not like name "stability multipliers", as Λ's are not multiplicative

The linearized Poincaré $[(d-1) \times (d-1)]$ return map of a periodic orbit is called the *monodromy matrix*, and its eigenvalues are called the *Floquet multipliers*. (sometimes *characteristic multipliers* or *roots*).

Floquet theory describes the solutions of linear ordinary differential equations with periodic coefficients, such as the equation of variations (4.2) evaluated on a periodic orbit p,

$$\dot{\delta x} = A(t)\delta x \, , \qquad A(t) = A(x(t)) = A(t+T_p) \, . \quad (8.2)$$

The space of solutions is finite-dimensional and any solution

$$M_p \delta x(x(t)) = \delta x(x(t+T_p))$$

can be expanded in terms of periodic functions $u_j(t) = u_j(t+T_p)$ in the basis

$$u(x(t)) = \sum_j u_j(t) e^{T_p(\mu_{p,j}+i\mu_{p,j})} \mathbf{e}_j(x)$$

spaned by the eigenvectors of the monodromy matrix. If $\Lambda_{p,j}$ is an eigenvalue of M_p and $\mathbf{e}_j(x(t))$ is the corresponding eigenvector, then

$$M_p \mathbf{e}_j(x) = \Lambda_{p,j} \mathbf{e}_j(x) \, .$$

The number $\Lambda_{p,j}$ is called the Floquet multiplier of $\mathbf{e}_j$, and $\lambda_{p,j} + i\theta_{p,j}$ the Floquet or characteristic exponent, where $\Lambda_{p,j} = e^{T_p(\mu_{p,j}+i\nu_{p,j})}$.

⇑PRELIMINARY

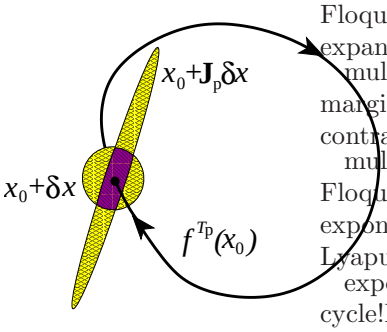

Figure 8.1: For a prime cycle p, fundamental matrix J_p returns an infinitesimal spherical neighborhood of $x_0 \in p$ stretched into an ellipsoid, with overlap ratio along the eigenvector $\mathbf{e}_i$ of $J_p(x)$ given by the eigenvalue $\Lambda_{p,i}$. These ratios are invariant under smooth nonlinear reparametrizations of state space coordinates, and are intrinsic property of cycle p.

Floquet!multipliers
expanding!Floquet multipliers
marginal!stability
contracting!Floquet multipliers
Floquet!exponents
exponent!Floquet
Lyapunov exponent!cycle
cycle!Lyapunov exponent

8.1.2 Fundamental matrix eigenvalues and exponents

We sort the *Floquet multipliers* $\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}$ of the $[d\times d]$ fundamental matrix J_p evaluated on the p-cycle into sets $\{e, m, c\}$

$$\begin{aligned} \text{expanding:} \quad & \{\Lambda_p\}_e &&= \{\Lambda_{p,j} : |\Lambda_{p,j}| > 1\} \\ \text{marginal:} \quad & \{\Lambda_p\}_m &&= \{\Lambda_{p,j} : |\Lambda_{p,j}| = 1\} \\ \text{contracting:} \quad & \{\Lambda_p\}_c &&= \{\Lambda_{p,j} : |\Lambda_{p,j}| < 1\}\,. \end{aligned} \tag{8.3}$$

and denote by Λ_p (no jth eigenvalue index) the product of *expanding* Floquet multipliers

$$\Lambda_p = \prod_e \Lambda_{p,e}\,. \tag{8.4}$$

As J_p is a real matrix, complex eigenvalues always come in complex conjugate pairs, $\Lambda_{p,i+1} = \Lambda^*_{p,i}$, so the product of expanding eigenvalues Λ_p is always real.

Cycle *Floquet exponents* are the stretching/contraction rates per unit time

$$\mu_{p,i} = \frac{1}{T_p} \ln |\Lambda_{p,i}|\ . \tag{8.5}$$

This definition is motivated by the form of the Floquet exponents for the linear dynamical systems, for example (4.15), as well as the fact that exponents so defined can be interpreted as Lyapunov exponents (10.32) evaluated on the prime cycle p. As in the three cases of (8.3), we sort the Floquet exponents $\lambda = \mu \pm \nu$ into three sets

☞ sect. 10.3

$$\begin{aligned} \text{expanding:} \quad & \{\lambda_p\}_e &&= \{\lambda_{p,i} :\ \mu_{p,i} > 0\} \\ \text{marginal:} \quad & \{\lambda_p\}_m &&= \{\lambda_{p,i} :\ \mu_{p,i} = 0\} \\ \text{contracting:} \quad & \{\lambda_p\}_c &&= \{\lambda_{p,i} :\ \mu_{p,i} < 0\}\,. \end{aligned} \tag{8.6}$$

A periodic orbit p of a d-dimensional flow or a map is *stable* if all its Floquet exponents (other than the vanishing longitudinal exponent, to be

stability!window
hyperbolic!flow
flow!hyperbolic
nonhyperbolic!flow
flow!nonhyperbolic
flow!elliptic
critical!point
cycle!superstable
superstable cycle
cycle!stable
stable!cycle
unstable!cycle
☞ sect. 5.3

explained in sect. 8.2.1 below) are strictly negative, $\mu_{p,i} < 0$. The region of system parameter values for which a periodic orbit p is stable is called the *stability window* of p. The set $\mathcal{M}_p$ of initial points that are asymptotically attracted to p as $t \to +\infty$ (for a fixed set of system parameter values) is called the *basin of attraction* of p.

If *all* Floquet exponents (other than the vanishing longitudinal exponent) of *all* periodic orbits of a flow are strictly bounded away from zero, $|\mu_i| \geq \mu_{min} > 0$, the flow is said to be *hyperbolic*. Otherwise the flow is said to be *nonhyperbolic*. In particular, if all $\mu_i = 0$, the orbit is said to be *elliptic*. Such orbits proliferate in Hamiltonian flows. [2]

We often do care about the sign of $\Lambda_{p,i}$ and, if $\Lambda_{p,i}$ is complex, its phase

$$\Lambda_{p,j} = \pm e^{\lambda_{p,j} T_p} = e^{(\mu_{p,j} \pm i\nu_{p,j})T_p} \,. \tag{8.7}$$

☞ sect. 5.2

Keeping track of this by case-by-case enumeration is a self-inflicted, unnecessary nuisance, followed in much of the literature. To avoid this, almost all of our formulas will be stated in terms of the Floquet multipliers Λ_j rather than in the terms of the overall signs, Floquet exponents λ_j and phases θ_j.

Example 8.1 Stability of 1-d map cycles: *The simplest example of cycle stability is afforded by 1-dimensional maps. The stability of a prime cycle p follows from the chain rule (4.39) for stability of the n_pth iterate of the map*

$$\Lambda_p = \frac{d}{dx_0} f^{n_p}(x_0) = \prod_{m=0}^{n_p-1} f'(x_m) \,, \quad x_m = f^m(x_0) \,. \tag{8.8}$$

Λ_p is a property of the cycle, not the initial point, as taking any periodic point in the p cycle as the initial point yields the same result.

A critical point *x_c is a value of x for which the mapping $f(x)$ has vanishing derivative, $f'(x_c) = 0$. For future reference we note that a periodic orbit of a 1-dimensional map is* stable *if*

$$|\Lambda_p| = \left|f'(x_{n_p}) f'(x_{n_p-1}) \cdots f'(x_2) f'(x_1)\right| < 1 \,,$$

and superstable *if the orbit includes a critical point, so that the above product vanishes. For a stable periodic orbit of period n the slope of the nth iterate $f^n(x)$ evaluated on a periodic point x (fixed point of the nth iterate) lies between -1 and 1. If $|\Lambda_p| > 1$, p-cycle is* unstable.

Example 8.2 Stability of cycles for maps: *No matter what method we had used to determine the unstable cycles, the theory to be developed here requires that their Floquet multipliers be evaluated as well. For maps a fundamental matrix is easily*

[2] Mason: for infinite dimensional flows one cannot have λ_{min}; in my thesis have a sequence $|\lambda_i| \to 0$, and all $|\lambda_i| \neq 0$.
PC: I think you are mixing up stability eigenvalues Λ with stability exponents λ in this and the next comment. My λ_{min} is a lower bound on $|\lambda_i|$, not λ_i, so I'm OK unless I have eigenvalues arbitrarily close to 0; but then the flow is not hypebolic

evaluated by picking any cycle point as a starting point, running once around a prime cycle, and multiplying the individual cycle point fundamental matrices according to (4.40). For example, the fundamental matrix M_p for a Hénon map (3.15) prime cycle p of length n_p is given by (4.41),

Henon@Hénon!map!stability
stability!billiards
billiards!stability

$$M_p(x_0) = \prod_{k=n_p}^{1} \begin{pmatrix} -2ax_k & b \\ 1 & 0 \end{pmatrix}, \qquad x_k \in p\,,$$

and the fundamental matrix M_p for a 2-dimensional billiard prime cycle p of length n_p

$$M_p = (-1)^{n_p} \prod_{k=n_p}^{1} \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_k & 1 \end{pmatrix}$$

follows from (6.11). We shall compute Floquet multipliers of Hénon map cycles once we learn how to find their periodic orbits, see exercise 17.11.

add exercise for by-hand fixed point, 2-cycle for Hénon

8.2 Cycle Floquet multipliers are cycle invariants

The 1-dimensional map cycle Floquet multiplier Λ_p is a product of derivatives over all points around the cycle, and is therefore independent of which periodic point is chosen as the initial one. In higher dimensions the form of the fundamental matrix $J_p(x_0)$ in (8.1) does depend on the choice of coordinates and the initial point $x_0 \in p$. Nevertheless, as we shall now show, the cycle *Floquet multipliers* are intrinsic property of a cycle also for multi-dimensional flows. Consider the ith eigenvalue, eigenvector pair $(\Lambda_{p,i}, \mathbf{e}_i)$ computed from J_p evaluated at a cycle point,

$$J_p(x)\mathbf{e}_i(x) = \Lambda_{p,i}\mathbf{e}_i(x)\,, \quad x \in p\,. \tag{8.9}$$

Consider another point on the cycle at time t later, $x' = f^t(x)$ whose fundamental matrix is $J_p(x')$. By the group property (4.33), $J^{T_p+t} = J^{t+T_p}$, and the fundamental matrix at x' can be written either as

$$J^{T_p+t}(x) = J^{T_p}(x')J^t(x) = J_p(x')J^t(x)\,, \quad \text{or} \quad J_p(x')J^t(x) = J^t(x)J_p(x)\,.$$

Multiplying (8.9) by $J^t(x)$, we find that the fundamental matrix evaluated at x' has the same eigenvalue,

$$J_p(x')\mathbf{e}_i(x') = \Lambda_{p,i}\mathbf{e}_i(x')\,, \quad \mathbf{e}_i(x') = J^t(x)\mathbf{e}_i(x)\,, \tag{8.10}$$

but with the eigenvector $\mathbf{e}_i$ transported along the flow $x \to x'$ to $\mathbf{e}_i(x') = J^t(x)\mathbf{e}_i(x)$. Hence, J_p evaluated anywhere along the cycle has the same set of Floquet multipliers $\{\Lambda_{p,1}, \Lambda_{p,2}, \cdots \Lambda_{p,d-1}, 1\}$. As quantities such as $\operatorname{tr} J_p(x)$, $\det J_p(x)$ depend only on the eigenvalues of $J_p(x)$ and not on

invariance!of flows
nonhyperbolic!flow
cycle!marginal stability
marginal stability!cycle
Poincar"'e!return map!cycle
stability!Poincar"'e map cycle

the starting point x, in expressions such as $\det\left(\mathbf{1}-M_p^r(x)\right)$ we may omit reference to any particular cycle point x:

$$\det\left(\mathbf{1}-M_p^r\right)=\det\left(\mathbf{1}-M_p^r(x)\right)\quad \text{for any } x\in p\,. \tag{8.11}$$

We postpone the proof that the cycle Floquet multipliers are smooth conjugacy invariants of the flow to sect. 8.5.

8.2.1 Marginal eigenvalues

The presence of marginal eigenvalues signals either a continuous symmetry of the flow (which one should immediately exploit to simplify the problem), or a non-hyperbolicity of a flow (a source of much pain, hard to avoid).

chapter 21
8.4
page 129

Example 8.3 A periodic orbit of a flow has a marginal eigenvalue: *As $J^t(x)$ transports the velocity field $v(x)$ by (4.7), after a complete period*

$$J_p(x)v(x)=v(x)\,, \tag{8.12}$$

so a periodic orbit of a flow *always has an eigenvector $\mathbf{e}_d(x)=v(x)$ parallel to the local velocity field with the unit eigenvalue*

$$\Lambda_{p,d}=1\,. \tag{8.13}$$

The continuous invariance that gives rise to this marginal eigenvalues is the invariance of a cycle under a translation of its points along the cycle: two points on the cycle (see figure 4.2) initially distance δx apart, $x'(0)-x(0)=\delta x(0)$, are separated by the exactly same δx after a full period T_p. As we shall see in sect. 8.3, this marginal stability direction can be eliminated by cutting the cycle by a Poincaré section and eliminating the continuous flow fundamental matrix in favor of the fundamental matrix of the Poincaré return map.

7.2
page 118
Ole: clarify this — draw a picture.

explain "monodromy"?

If the flow is governed by a time-independent Hamiltonian, the energy is conserved, and that leads to an additional marginal eigenvalue (remember, by symplectic invariance (5.17) real eigenvalues come in pairs). For the collinear helium of sect. 7.3 this marginal eigenvalue is made explicit in appendix D.2.

⇓PRELIMINARY

⇑PRELIMINARY

8.3 Stability of Poincaré map cycles

(R. Paškauskas and P. Cvitanović)

If a continuous flow periodic orbit p pierces the Poincaré section $\mathcal{P}$ once, the section point is a fixed point of the Poincaré return map P with stability (4.45)

$$\hat{J}_{ij}=\left(\delta_{ik}-\frac{v_i\,U_k}{(v\cdot U)}\right)J_{kj}\,, \tag{8.14}$$

with all primes dropped, as the initial and the final points coincide, $x' = f^{T_p}(x) = x$. If the periodic orbit p pierces the set of Poincaré sections $\mathcal{P}$ n times, the same observation applies to the nth iterate of P.

We have already established in (4.46) that the velocity $v(x)$ is a zero-eigenvector of the Poincaré section fundamental matrix, $\hat{J}v = 0$. Consider next $(\Lambda_{p,\alpha}, \mathbf{e}_\alpha)$, the full state space αth (eigenvalue, eigenvector) pair (8.9), evaluated at a cycle point on a Poincaré section,

$$J(x)\mathbf{e}_\alpha(x) = \Lambda_\alpha \mathbf{e}_\alpha(x)\,, \quad x \in \mathcal{P}\,. \tag{8.15}$$

Multiplying (8.14) by $\mathbf{e}_\alpha$ and inserting (8.15), we find that the full state space fundamental matrix and the Poincaré section fundamental matrix $\hat{J}$ has the same eigenvalue

$$\hat{J}(x)\hat{\mathbf{e}}_\alpha(x) = \Lambda_\alpha \hat{\mathbf{e}}_\alpha(x)\,, \quad x \in \mathcal{P}\,, \tag{8.16}$$

where $\hat{\mathbf{e}}_\alpha$ is a projection of the full state space eigenvector onto the Poincaré section:

note: normalization not preserved

$$(\hat{\mathbf{e}}_\alpha)_i = \left(\delta_{ik} - \frac{v_i\, U_k}{(v\cdot U)}\right)(\mathbf{e}_\alpha)_k\,. \tag{8.17}$$

draw figure

Hence, $\hat{J}_p$ evaluated on any Poincaré section point along the cycle p has the same set of Floquet multipliers $\{\Lambda_{p,1}, \Lambda_{p,2}, \cdots \Lambda_{p,d}\}$ as the full state space fundamental matrix J_p.

8.4 Rectification of a 1-dimensional periodic orbit

In sect. 7.4.1 we have constructed the conjugation function for a fixed point. Here we turn to the problem of constructing it for periodic orbits. Each point around the cycle has a differently distorted neighborhood, with differing second and higher order derivatives, so we need to compute a different conjugation function h_a at each cycle point x_a. We expand the map f around each cycle point along the cycle,

$$y_a(\phi) = f_a(\phi) - x_{a+1} = \phi f_{a,1} + \phi^2 f_{a,2} + \dots \tag{8.18}$$

where x_a is a point on the cycle, $f_a(\phi) = f(x_a+\phi)$ is centered on the periodic orbit, and the index k in $f_{a,k}$ refers to the kth order in the expansion (7.20).

For a periodic orbit the conjugation formula (7.19) generalizes to

$$f_a(\phi) = h_{a+1}^{-1}(f_a'(0)h_a(\phi))\,, \qquad a = 1, 2, \cdots, n\,,$$

Floquet!multiplier, metric invariants
metric!invariant!Floquet multiplier

point by point. The conjugationg functions h_a are obtained in the same way as before, by equating coefficients of the expansion (7.20), and assuming that the cycle Floquet multiplier $\Lambda = \prod_{a=0}^{n-1} f'(x_a)$ is not marginal, $|\Lambda| \neq 1$. The explicit expressions for h_a in terms of f are obtained by iterating around the whole cycle,

compute only onece, then propagate around the cycle by iteration?

$$f^n(x_a + \phi) = h_a^{-1}(\Lambda h_a(\phi)) + x_a \,. \tag{8.19}$$

☞ remark 7.2

evaluated at each cycle point a. Again we have the freedom to set $h'_a(0) = 1$ for all a.

8.4.1 Repeats of cycles

We have traded in our initial nonlinear map f for a (locally) linear map Λy and an equally complicated conjugation function h. What is gained by rewriting the map f in terms of the conjugacy function h? Once the neighborhood of a fixed point is linearized, the repeats of it are trivialized; from the conjugation formula (7.20) one can compute the derivatives of a function composed with itself r times:

$$f^r(x) = h^{-1}(\Lambda^r h(x)) \,.$$

One can already discern the form of the expansion for arbitrary repeats; the answer will depend on the conjugacy function $h(x)$ computed for a *single* repeat, and all the dependence on the repeat number will be carried by factors polynomial in Λ^r, a considerable simplification. The beauty of the idea is difficult to gauge at this stage - an appreciation only sets in when one starts computing perturbative corrections, be it in celestial mechanics (where the method was born), be it the quantum or stochastic corrections to "semiclassical" approximations.

8.5 Smooth conjugacies and cycle Floquet multipliers

In sect. 8.2 we have established that for a given flow the cycle Floquet multipliers are intrinsic to a given cycle, independent of the staring point along the cycle. Now we can prove a much stronger statement; cycle Floquet multipliers are *metric invariants* of the flow, the same in *any* representation of the dynamical system.

That the cycle Floquet multipliers are an invariant property of the given dynamical system follows from elementary considerations of sect. 7.1: If the same dynamics is given by a map f in x coordinates, and a map g in the $y = h(x)$ coordinates, then f and g (or any other good representation) are related by (7.2), a reparametrization and a coordinate transformation

$g = h \circ f \circ h^{-1}$. As both f and g are arbitrary representations of the dynamical system, the explicit form of the conjugacy h is of no interest, only the properties invariant under any transformation h are of general import. Furthermore, a good representation should not mutilate the data; h must be a *smooth conjugacy* which maps nearby cycle points of f into nearby cycle points of g. This smoothness guarantees that the cycles are not only topological invariants, but that their linearized neighborhoods are also metrically invariant. For a fixed point $f(x) = x$ of a 1-dimensional map this follows from the chain rule for derivatives,

smooth!conjugacy
conjugacy!smooth
metric!invariant
invariant!metric
conjugacy!invariant
neighborhood

$$\begin{aligned} g'(y) &= h'(f \circ h^{-1}(y)) f'(h^{-1}(y)) \frac{1}{h'(x)} \\ &= h'(x) f'(x) \frac{1}{h'(x)} = f'(x)\,, \end{aligned} \tag{8.20}$$

and the generalization to the Floquet multipliers of periodic orbits of d-dimensional flows is immediate.

Is it? add d-dimensional case

As stability of a flow can always be rewritten as stability of a Poincaré section return map, we find that the Floquet multipliers of any cycle, for a flow or a map in arbitrary dimension, is a metric invariant of the dynamical system.

argue $det \to tr$ can be waste of time
add d-dimensional case

in depth:
appendix D.1, p. 903

7.2
page 118

8.6 There goes the neighborhood

In what follows, our task will be to determine the size of a *neighborhood* of $x(t)$, and that is why we care about the Floquet multipliers, and especially the unstable (expanding) ones. Nearby points aligned along the stable (contracting) directions remain in the neighborhood of the trajectory $x(t) = f^t(x_0)$; the ones to keep an eye on are the points which leave the neighborhood along the unstable directions. The sub-volume $|\mathcal{M}_i| = \prod_i^e \Delta x_i$ of the set of points which get no further away from $f^t(x_0)$ than L, the typical size of the system, is fixed by the condition that $\Delta x_i \Lambda_i = O(L)$ in each expanding direction i. Hence the neighborhood size scales as $\propto 1/|\Lambda_p|$ where Λ_p is the product of expanding eigenvalues (8.4) only; contracting ones play a secondary role. So secondary that even infinitely many of them (for example, the infinity of contracting eigendirections of the spatiotemporal dynamics of sect. 25.1) will not matter.

add a figure of hyperbolic flow
⇓PRELIMINARY
⇑PRELIMINARY
fix

So the physically important information is carried by the expanding sub-volume, not the total volume computed so easily in (4.37). That is also the reason why the dissipative and the Hamiltonian chaotic flows are much more alike than one would have naively expected for "compressible" *vs.* "incompressible" flows. In hyperbolic systems what matters are the

expanding directions. Whether the contracting eigenvalues are inverses of the expanding ones or not is of secondary importance. As long as the number of unstable directions is finite, the same theory applies both to the finite-dimensional ODEs and infinite-dimensional PDEs.

⇓PRELIMINARY

Commentary

Remark 8.1 A resonance condition. In the hyperbolic case there is a resonance condition that must be satisfied: none of the Floquet exponents may be related by ratios of integers. That is, if $\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}$ are the Floquet multipliers of the fundamental matrix, then they are in resonance if there exist integers $n_1, \ldots, n_d$ such that

$$(\Lambda_{p,1})^{n_1}(\Lambda_{p,2})^{n_2}\cdots(\Lambda_{p,d})^{n_d} = 1\,.$$

If there is resonance, then one may get corrections to the basic conjugation formulas in the form of monomials in the variables of the map.

give reference

(R. Mainieri)

⇑PRELIMINARY

Résumé

Periodic orbits play a central role in any invariant characterization of the dynamics, because (a) their existence and inter-relations are a *topological*, coordinate-independent property of the dynamics, and (b) their Floquet multipliers form an infinite set of *metric invariants*: The Floquet multipliers of a periodic orbit remain invariant under any smooth nonlinear change of coordinates $f \to h \circ f \circ h^{-1}$.

We shall show in chapter 11 that extending their local stability eigendirections into stable and unstable manifolds yields important global information about the topological organization of state space.

In hyperbolic systems what matters are the expanding directions. The physically important information is carried by the unstable manifold, and the expanding sub-volume characterized by the product of expanding eigenvalues of J_p. As long as the number of unstable directions is finite, the theory can be applied to flows of arbitrarily high dimension.

References

[8.1] J. Moehlis and K. Josić, "Periodic Orbit," www.scholarpedia.org/article/Periodic_Orbit.

[8.2] G. Floquet, "Sur les equations differentielles lineaires a coefficients periodique," *Ann. Ecole Norm. Ser. 2,* **12**, 47 (1883).