Cantor set
Smale S.!horseshoe

Appendix T

Flotsam

lost Homework Assignements section - retype!

This is a non-appendix which contains text that might be used "somewhere" in the book, or sent to the heap of history.

combine all refs_*.tex files into one at the end of the day (and remember to turn of the light)

1.

2. A whole collection of such 1-d maps useful in illustrating various aspects of dynamics is listed in sect. B.1.

can one get [16.7] from IHES?

3. For a Smale horseshoe forward dynamics keeps doubling the number of unstable direction folds, while the backward dynamics does the same to the stable direction folds. Hence topologically the action of the Smale horseshoes dynamics can be described as a direct product of two unimodal maps, one for the past and one for the future.

   what emerges is a complete binary Cantor set with the Smale horseshoe foliation.

   ☞ sect. 12

   There is a one-to-one relationship between the periodic itineraries and the unstable periodic orbits:

4. All points that do not escape to infinity neither forward nor backward in time are contained in the intersections of the forward and backward folds (figure 12.4a). In particular, a cycle remains a cycle forever, so the set of all periodic orbits is contained within these intersections.

   draw figure

5. The growth rate of this vector is multiplicative along the trajectory

$$\frac{|\eta(t+t')|}{|\eta(0)|} = \frac{|\eta(t+t')|}{|\eta(t)|} \times \frac{|\eta(t)|}{|\eta(0)|} . \tag{T.1}$$

   The evolution of the infinitesimal neighborhood can be represented by the trajectory of a unit vector $u_i(t) = u_i(x_0, \eta(0), t)$ multiplied by the stretching factor $\Lambda^t(x_0, \eta(0)) = |\eta(t)|/|\eta(0)|$. We implement this multiplicative evolution of by adjoining the $(d+1)$-dimensional tangent space $\eta \in T\mathcal{M}$ to the $(d+1)$-dimensional dynamical evolution space $x \in \mathcal{M} \subset \mathbf{R}^{d+1}$.

   crossrefer to sect. I.1

hyperbolic!systems
dynamical!system!hyperbolic
axiom A systems
dynamical!system!axiom A
stability!marginal
trajectory!unstable
trajectory!marginally stable
trajectory!stable

6. For asymptotic times and for almost every initial $(x_0, \eta(0))$, the multiplicative factor (T.1) converges to the leading expanding eigenvalue of the linearized stability matrix for the flow. We shall exploit this in our discussion of the Lyapunov exponents in sect. I.1.

7. particularly strong results exist for Axiom A hyperbolic systems, for which the dynamical zeta functions are meromorphic, and the spectral determinants are entire functions.

8. We shall consider here only purely hyperbolic flows with the topology of a Smale horseshoe. The important conceptual insight of Smale is the realization that for such flows the associated dynamical zeta functions have nice analytic structure. In a more formal setting, such flows are called “Axiom A”, and Ruelle has proven that for expanding analytic maps the dynamical zeta functions are meromorphic, and the spectrum is discrete. This differs very much from the intuition acquired in studies of quantum chaos; there is no “abscissa of absolute convergence” and no “entropy wall”, the exponential proliferation of orbits can be controlled, and the Selberg-type zeta functions are entire and converge everywhere.

Might add some text from the Brasilia 1898 winter school.

9. Under one forward time step (11.21) the entire well-ordered past symbol string flips $w_t \to 1 - w_t$ if the order reversing symbol 1 is moved from the future itinerary S^+ to the past itinerary S^-.

12.5
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10. The strategy outlined here is perhaps too laborious if only a rough estimate of a dimension or a Lyapunov exponent is desired; a simple computer generated average might suffice. However, the labor pays off if classical correlation or quantum resonance spectra are called for. We also hope that the high numerical precision attainable by cycle expansions might enable us to attack higher-dimensional problems where crude averaging is computationally infeasible.

Rosenqvist has also T_p = mean number of bounces. Is there a simple relation?

11. Present work is a physics application of the dynamical systems theory developed in refs. [1.21, 1.24, 1.22, 1.26]; we refer the reader to the above literature for a survey of rigorous results. The emphasis will be on the practical applications of cycle expansions, at some expense to the mathematical rigor. We are interested in the convergence of cycle expansions in generic settings, i.e., situations in which neither the symbolic dynamics is strictly controlled (there is no finite Markov partition), nor is the system uniformly hyperbolic (the phase space is a mixture of stability islands and chaotic regions). Our results will mostly not depend on assumptions about the existence of invariant measures or structural stability of strange sets.

12. In general, depending on how their neighborhoods deform with time, trajectories of a generic dynamical system are qualitatively of three distinct types: they are either asymptotically unstable (positive Lyapunov exponent), asymptotically marginal (vanishing Lyapunov) or asymptotically stable (negative Lyapunov).

13. The presence of marginal eigenvalues signals either an invariance of the flow (which you should immediately exploit to simplify the

problem), or a non-hyperbolicity of a flow (source of much pain, hard to avoid). non-hyperbolicity

☞ chapter 21

The presence of complex numbers should intrigue you because in the definition of the exponential of a matrix we only used real operations. Where did the complex numbers come from?

Jordan form, upper triangular respected, etc

Take two vectors x and y of the phase space $\mathbb{R}^d$, combine them in a vector $w = x + iy$ in $\mathbb{C}^d$, and then extend the action of $\mathbf{A}$ to these complex vectors by $\mathbf{A}w = \mathbf{A}x + i\mathbf{A}y$. The solution $w(t)$ to the complex equation

$$\dot{w} = \mathbf{A}w \tag{T.2}$$

$x(t) = \mathrm{Re}\,(w(t))$ and $y(t) = \mathrm{Im}\,(w(t))$ to the problem (4.9) over the reals.

14. Physically interesting repellers are not piecewise linear, but the above 2–slope approximation gives a rough sketch of the eigenvalue spectrum. Refinements are obtained by replacing the 2nd iterate of the map by four linear segments, and so forth; that is precisely the meaning of the finite cycle length truncations of cycle expansions. The eigenvalues are either real or come in complex pairs - the main point is that for hyperbolic systems we expect them to be exponentially spaced.

RM: What 2–slope?

15. B. Eckhardt [?] credits T. Tél and P. Szepfalusy [?] and H. Fujisaka and M. Inoue [?] for invention of the generalized evolution operators. Tél and Szepfalusy have a brief comment that $\left\langle e^{\alpha(x)} \right\rangle$, our eq. (10.10), could be used to compute escape rates. H. Fujisaka and M. Inoue never refer to Ruelle, but they do indeed write down the Ruelle-Araki operator explicitely.

16. This is even more striking in the case of kinematic dynamo that we shall discuss in sect. **??**. In this case the flow distorts an initial smooth vector field (magnetic field flux lines) into a field whose direction some finite time later varies rapidly across the exponentially shrinking folds. However the eigenfunctions of the evolution operator at large times describe the mean local density of the magnetic flux, with the cancelations of fluxes pointing in opposite directions already taken into account.

17. Periodic point of a given itinerary is the fixed point of the corresponding sequence of maps, for example

$$f_0 \circ f_1 \circ f_1 \circ f_0 \circ f_1(x_{10110}) = x_{10110}\,.$$

18. To derive this formula, separate the Schrödinger equation

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} - V(q) \right) \psi(q,t) = 0$$

Check throughout here. . .

into equations for the density and phase by setting $\psi = \sqrt{\rho}e^{iS/\hbar}$, ρ, S real, and $\rho > 0$. For $\rho \neq 0$ the real part yields

$$\begin{aligned} \dot{q}_i &= \frac{p_i}{m} && \text{(T.3)} \\ \dot{p}_i &= -\frac{\partial}{\partial q_i}\left(V(q) - \frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial q^2}\sqrt{\rho}\right) && \text{(T.4)} \end{aligned}$$

where $p_i = \frac{\partial S}{\partial q_i}$, while the imaginary part yields the mass conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q_i}(\rho \dot{q}_i) = 0$$

Substituting (T.3) we rewrite the mass conservation equation as an equation for evolution of $\kappa = \ln \rho$,

$$\dot{\kappa} = -\frac{1}{m}\frac{\partial^2 S}{\partial q^2}\,.$$

The semiclassical approximation consists in setting $\hbar = 0$ in (T.4)

$$\dot{q}_i = \frac{p_i}{m}\,, \qquad \dot{p}_i = -\frac{\partial}{\partial q_i}V(q)\,, \qquad \dot{\kappa} = -\frac{1}{m}\frac{\partial^2 S}{\partial q^2}\,. \tag{T.5}$$

The first two equations yield the classical trajectory while the third one evolves the local density of the wavefront along the trajectory. To solve it, we need to initialize and evolve

$$M_{ij} = \frac{\partial^2 S}{\partial q_i \partial q_j}$$

recheck this!

This we shall accomplish in sect. **??**. The interpretation is that the contribution of wave packet is centered on the classical orbit, but one also has to keep the track of the local curvature of the wavefront along the trajectory — this is accomplished by keeping track of the cycle fundamental matrix. The result is that wave mechanics is described by purely *classical* evolution (I.4), in the extended space $(q, p, \mathbf{M})$ with the trace (I.5) (for 2-dimensional Hamiltonian flows)

$$\begin{aligned} \mathrm{Tr}\,\mathcal{L}^t &= \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{rh_p}\delta(t - rT_p)}{|\det(1 - \mathbf{J}_p^r)|}\Delta_{p,r}, \\ \Delta_{p,r} &= \int_g d\mathbf{u}\,|\Lambda^{T_p r}(x_p, \mathbf{u})|^{1/2}\delta(\mathbf{u} - R^{T_p r}(x_p, \mathbf{u}))\,, && \text{(T.6)} \end{aligned}$$

and the spectral determinant:

$$F(E) = \exp\left(-\sum_{p,r}\frac{1}{r}\frac{|\Lambda_p|^{r/2}e^{i(S_p(E)/\hbar + \nu_p\pi/2)r}}{(1 - 1/\Lambda_p^r)^2(1 - 1/\Lambda_p^{2r})}\right)\,. \tag{T.7}$$

The final result is very similar in form to the classical spectral determinant (1.16), and is a variant of the *Gutzwiller trace formula*, the seminal idea that is the key to the theory of “Quantum Chaos” as we shall develop it here.

19. [1] What does this formula mean? For a pinball $S_p = \int \mathbf{p}.d\mathbf{q} = \hbar k L_p k$, where L_p is the length of the p cycle (for constant velocity $v = 1$ the cycle period T_p equals the cycle length L_p), and k is the wave number. The simplest conceivable system with exponentially growing errors has only one periodic orbit (for example, there is only one unstable orbit bouncing between two disks in a plane), and the dynamical zeta function (1.9) is simply

write Bohr--Sommerfeld in the zeta fct form here

$$1/\zeta(k) = 1 - \frac{e^{iLk+i\pi m/2}}{\sqrt{\Lambda}} \,, \tag{T.8}$$

where Λ is the expanding eigenvalue of the cycle, L is the cycle length, m is the Maslov index ($m = 4$ for the 2 disk reflections) The resonances $1/\zeta(k_n) = 0$ are given by

$$k_n = \frac{\pi}{L}(m + 2n) - i\lambda/2, \quad n \in Z \,,$$

where $\lambda = \ln|\Lambda|/T$ is the cycle Lyapunov exponent. The real part of the wavenumber k goes through a resonance for each integer number of nodes n in a standing wave between the two disks, and the imaginary part indicates that the resonance is unstable, with the decay rate proportional to the cycle Lyapunov exponent.

20. If t is a continuous parameter, f^t generates a *flow*; if $t = n$ is discrete, f^n is nth iterate of the one-step mapping f. f may (but it does not have to) be invertible. Often it is natural to associate with a given flow a discrete time dynamical system by means of a Poincaré section, a hypersurface in $\mathcal{M}$ that cuts across the flow. For example, instead of continuous pinball trajectory, one can track the successive collisions of the ball with the billiard walls. In this chapter we shall concentrate on such discrete time dynamics and the qualitative information that can be extracted from it.

21. The system has *expansion property* iff each inverse $\psi_{ij} : I_j \to I_i$ (defined when $t_{ij} = 1$) is unique and a contraction.

22. The characteristic polynomial for finite $[N \times N]$ matrix $\mathbf{T}$ is given by

$$\begin{aligned}\det(1 - z\mathbf{T}) &= \sum_{n=0}^{N} (-z)^n \,\mathrm{tr}\,(\wedge^n \mathbf{T}) = 1 + \sum_{n=1}^{N} \frac{(-z)^n}{n!} \mathbf{T}_n \\ &= 1 - z\,\mathrm{tr}\,\mathbf{T} - \frac{z^2}{2}\left((\mathrm{tr}\;\mathbf{T})^2 - \mathrm{tr}\,(\mathbf{T}^2)\right) - \ldots\end{aligned} \tag{T.9}$$

where $\wedge^n \mathbf{T}$ is the n-th exterior power of matrix $\mathbf{T}$ and $\mathbf{T}_n$ is the sum of all possible $[n \times n]$ subdeterminants of $\mathbf{T}$

$$\mathbf{T}_n = \mathrm{tr}\,(\wedge^n \mathbf{T}) = \sum_{i_1,\cdots,i_n=1}^{N} \begin{vmatrix} T_{i_1,i_1} & \cdots & T_{i_1,i_n} \\ \cdots & \cdots & \cdots \\ T_{i_n,i_1} & \cdots & T_{i_n,i_n} \end{vmatrix}$$

[1] Rosenqvist: make his into an exercise later on.

For example,

$$\left(\wedge^2 T\right)_{ij,kl} = \frac{1}{2!}\begin{vmatrix} T_{ik} & T_{jk} \\ T_{il} & T_{jl} \end{vmatrix} = \frac{1}{2!}\left(T_{ik}T_{jl} - T_{jk}T_{il}\right)$$

so $\mathrm{tr}\,(\wedge^2\mathbf{T}) = \sum_{ij}\frac{1}{2!}\left(T_{ii}T_{jj} - T_{ji}T_{ij}\right) = \frac{1}{2!}\left((\mathrm{tr}\ \mathbf{T})^2 - \mathrm{tr}\,(\mathbf{T}^2)\right)$.

The determinant is a sum over all possible permutation cycles of the symbols $i_1, \cdots, i_n$, $1 \le n \le N$.

23. More generally, we shall refer to a symbolic dynamics as an "N-disk" symbolic dynamics if the phase space can be partitioned in N distinct regions such that an orbit starting in a partition can in one step reach all other partitions except itself.

24. **Remark T.1** Multifractals. For reasons that remain mysterious to the authors - perhaps so that Mandelbrot can refer to himself both as the mother of fractals and the grandmother of multifractals - some physics literature referes to any fractal generated by more than one scale as a "multi"-fractal. This usage seems to divide fractals into 2 classes; one consisting essentially of the above Cantor set and the Serapinski gasket, and the second consisting of anything else, including anything of physical interest.

25. Consider a general three-dimensional flow sketched in figure **??**. To be interesting, a flow should be recurrent; otherwise it is merely a transient that we cannot observe for long. If the flow is recurrent, we can cut it by a Poincaré section; if it is a map of a disk domain onto itself, it must have at least one fixed point. Now consider the ways in which the flow can deform the neighborhood of a fixed point. There are essentially two possibilities: the neighborhood can return wrapped around the fixed point (the fixed point is stable or elliptic - see figure **??**a), or squeezed, stretched and folded (the fixed point is unstable or hyperbolic - see figure **??**b).

find these figures!

we ignore the marginal or the parabolic borderline case for now...

26. $$\mathbf{J}_p := \mathbf{J}^{T_p}\,. \qquad \text{(T.10)}$$

27. These approaches explore the homoclinic tangle connected to the central hyperbolic fixed point, but miss structures such as basins of attraction of co-existing attractors. [2]

28. Mathematicians tend to impose invariance by fiat, by declaring that the measure should be such that something or else is "Sup Min" or "Min Sup". Such pronouncements are not very helpful as they tend to be unimplementable computationlly, but we have to live with them at least through to the end of this chapter.

[2] Kai H: The stable manifold of the fixed point misses the "other coexisting structure" but the unstable manifold falls into it. We should maybe discuss more the relationship between the manifolds of the hyperbolic fixed point and the manifolds of the non-wandering set. The union of all crossings between unstable and stable manifolds of all hyperbolic points in the non-wandering set is really the proper thing.

29. Dahlquist:
    The invariant density I talk is

$$\rho(dx) = dx \lim_{t\to\infty} \sum_p \sum_{r=1}^{\infty} \frac{\delta(t - rT_p)}{|\det(1 - J_p^r)|} \int_{x'\in p} \delta(x - x') \frac{dx'}{v}$$

    where v is the speed at x.

30. Note that we have *resummed* the contribution of the cycle p to all times, so truncating the summation up to given p is *not* a finite time $n \le n_p$ approximation.

31. In what follows we shall often absorb z into the transfer operator: $z\mathcal{L} \to \mathcal{L}$, $z^{n_p} t_p \to t_p$.

32. In the final evaluation z is set to $z = 1$, but the organization by powers of z^k is crucial to the convergence of cycle expansions. For example, for 1-d Axiom A mappings $C_k(s) \approx C^{-k^2}$ for any s - this super-exponential convergence is precisely the reason why the variable z was introduced in the first place. The zeros of $F_N(s, 1)$ are determined by standard methods, such as the Newton-Raphson algorithm.

33. As shown in sect. **??**, for Axiom A maps the coefficients in the semiclassical spectral determinant $F(z) = \sum_n C_n z^n$ expansions fall off faster than exponentially, as $C_n \approx \Lambda^{-n^2}$ for 1-d maps, as $C_n \approx \Lambda^{-n^{3/2}}$ for 2-dimensional maps / 3-dimensional flows, and in d dimensions as $\Lambda^{-n^{1+1/d}}$.

34. **Remark T.2** The semiclassical zeta function The semiclassical zeta function corresponds to setting $\Delta_{p,r} = |\Lambda_p^r|^{1/2}(1 - 1/\Lambda_p^r)$, and the "quantum Fredholm determinant"[36.10] is obtained by setting $\Delta_{p,r} = |\Lambda_p^r|^{1/2}$ in (I.11).

35. The Selberg zeta function[40.45] for geodesic flows on surfaces of constant negative curvature is an exception: multiplicativity is guaranteed by the Bowen-Series map [1.22], which reduces the two-dimensional flow to a direct product of one-dimensional maps, and makes it possible to construct the associated transfer operators in terms of one variable[S.48]. probably wrong Bowen-Series reference.

36. **Remark T.3** Flow suspension Formula (24.6) can also be obtained by using the suspension formula in Appendix C.3 of ref. [1.26], with the discrete dynamical system leading from collision to collision with the central disk, and the ceiling function being the time between these hits. probably drop this

37. The notion of Axiom A systems is a mathematical abstraction of 2 and 3.

38. The shadowing cancellations require that a long cycle and the associated pseudocycles (products of shorter cycles following the same symbol sequence) have nearly the same weight *and* a relative minus sign. The first requirement is guaranteed by the hyperbolicity and the smoothness of the flow. The second requirement implies that, contrary

to the "semi-classical" intuition, the cycle expansions are expected to converge for *low eigenvalues*. For sufficiently high wave numbers k the differences in actions $S_p - S_{shadow} = k(L_p - L_{shadow})$ can be of order π or higher, in which case $t_p - t_{shadow} \approx 2t_p$, rather than $t_p - t_{shadow} \approx t_p/|\Lambda_p|$ expected at $k = 0$.

39. As we have seen above, the ζ function reduces to a finite polynomial for piecewise linear mappings, but in general the curvature corrections c_n in (18.5) do not vanish. While the polynomial truncations of the cycle expansions usually already converge well enough, we can improve them by fitting c_2, c_3, $\ldots$, c_N with an exponential $c_n = A(zc)^n$, and summing the tail estimate

$$1/\zeta \approx 1 - \sum_f t_f - \sum_n^N c_n - \frac{A(zc)^{N+1}}{1 - zc} \,. \tag{T.11}$$

could be a multiple pole, could be a complex pair, ...

Such exponential tail estimates and the corresponding poles are expected from Ruelle's relation (**??**) between dynamical zeta functions and spectral determinants, and are indeed observed numerically; this particular estimate works if the leading pole is real. Convergence of dynamical zeta functions cycle expansions can be accelerated by a variety of numerical methods, but both on theoretical grounds and in practice, the preferred alternative is to use spectral determinants instead.

40. EAS saysthis is undoable- Poincaré failed to prove it:
Exercise T.1 Poincaré recurrence *Prove the Poincaré recurrence theorem: Let U be a measure preserving mapping of the phase space into itself and assume that that the measure of the whole phase space is finite. Then, if A is a measurable subset of the phase space, almost every point x of A returns infinitely often to A under the mapping U.*

Hint: Consider the set of points that do not return and its backward iterates.

41. **Remark T.4** Caution! We conclude our discussion of convergence of cycle expansions with a word of caution: as we do not know how quickly the asymptotics should set in, our numerical results can easily be misleading. For example, for a larger disk-disk spacing, pre-asymptotic oscillations are visible in fig. 2?!, and one might mistakenly conclude [18.11] from such data that the semiclassical spectral determinant has a pole. There is no substitute for theorems that establish that appropriate determinants are entire; such oscillations make numerical convergence uncertain already in the simplest 1-dimensional repellers.

42. Description of a non–wandering set in terms of cycles is the most controllable, but also the most laborious approach to exploration of non–wandering sets.

43. M is related to the fundamental matrix (**??**) by

$$M_{ij} = m \frac{\partial}{\partial q_i(t)} \dot{q}_i(t) = m J^{-1}_{kj} \dot{J}_{ik} \tag{T.12}$$

recheck this!

After some algebra, Spiegel gets

$$\dot{M}_{ij} = -M_{ij}^2 + m\frac{\partial}{\partial q_j}\dot{q}_i \tag{T.13}$$

44. For the density of energy levels of a 2-d system the Gutzwiller trace formula reads

$$d(E) = \sum_n \delta(E - E_n) = \overline{d}(E) + d^{osc}(E) \tag{T.14}$$

where $\overline{d}(E)$ is the smoothed part of the level density given by the Thomas-Fermi approximation:

$$\overline{d}(E) = \int \frac{dpdq}{(2\pi\hbar)^2}\delta(E - H(q,p)) \tag{T.15}$$

$d^{osc}(E)$ is the oscillatory part expressed as a sum over all classical periodic orbits with the given energy.

45. Perhaps more surprisingly, (15.16) also yields *quantum* resonances, with the quantum amplitude associated with a given cycle

$$t_p = \frac{1}{\sqrt{\Lambda_p}} e^{\frac{i}{\hbar}S_p(E) - i\pi m_p/2} \,, \tag{T.16}$$

essentially (in a somewhat vague sense) the square root of the classical weight.

46. The semiclassical zeta function

$$\det(\hat{H} - E)_{sc}(E) = \prod_p \prod_{k=0}^{\infty} \left(1 - \frac{e^{-\frac{i}{\hbar}S_p(E) + i\pi m_p/2}}{|\Lambda_p|^{1/2}\Lambda_p^k}\right) \,. \tag{T.17}$$

47. former section: Quantum mechanics I

The second derivative matrix can be decomposed into combinations of dyads and their eigenvalues in the usual way

$$\frac{\partial^2 H}{\partial p \partial p} = \sum_{i=1}^{d} \lambda_i \mathbf{D}_i, \quad \mathbf{D}_i \mathbf{D}_j = \delta_{ij}\mathbf{D}_j. \tag{T.18}$$

The solution close to the singularity can be a linear combination of some of these dyads corresponding to the R singular directions l:

$$\mathbf{M}(t) = \frac{1}{t - t^c} \sum_{l=1}^{R} \frac{1}{\lambda_l} \mathbf{D}_l, \tag{T.19}$$

where R is the number of singular directions. The dynamical systems that we are *really* interested in - for example, smooth bound Hamiltonian potentials - are presumably never really chaotic, and it is still unclear what intuition is more rewarding: are quantum spectra of chaotic dynamics in smooth bound Hamiltonian potentials more like zeros of Riemann zetas or zeros of dynamical zeta functions?

48. $1/\zeta_0$ has a *double* leading pole? The double pole is not as surprising as it might seem at the first glance; indeed, for the semiclassical spectral determinant (15.14) $1/\zeta_0$ must have a double pole in order to cancel the leading double zero of the $(1/\zeta_1)^2$ factor.

49. The phase space for a system of N particles is a $2ND$-dimensional vector space, where D stands for the dimension of the configuration space coordinates $\mathbf{q} = (q_1, q_2, \ldots, q_D)$ and the momenta $\mathbf{p} = (p_1, p_2, \ldots, p_D)$, one (p, q) pair for each particle. The total number ND of the (coordinate, momentum) pairs is called the number of degrees of freedom; if the total energy is conserved, the phase space reduces to the $2ND - 1$ constant energy surface.

50. Our convention is to denote the dimension of the space in which the flow takes place by $d+1$ in order that discrete time dynamics always takes place in d dimensions, in this case by d-dimensional return maps for the Poincaré sections that we shall describe in sect. **??**.

51. When we wish to stress the role of x_0, we shall refer to the trajectory as Some authors indeed denote the trajectories as $x(t; x_0)$ with $x_0 = x(0; x_0)$. to express that the evolution is such that $x_0 \mapsto f^t(xInit)$.

52. In fact, in (**??**) the operator is a partial derivative with respect to time for x_0. The operator $\partial/\partial t$ is taken for fixed x_0.

what is this good for?

will go away

The discussion given here is a conventional one for general flows. When the flow field v_i is not autonomous, then the set of integral curves of v_i at a *fixed* t may differ from the set of trajectories. In that case, the derivative of a function like I following the flow generally consists of two parts as in (**??**). One part is the derivative along the instantaneous trajectory (the level curve at fixed t) and the other is the explicit time derivative.

EAS: Here a reference to a gook like Schutz' Geometry for Physicsts might be suitable.

We are here interested in autonomous flows with v_i independent of time, and that simplifies things for us. The operator $v_i \partial_i$ is the tangent vector to the trajectory with components v_i. When the flow is autonomous, as we are assuming, its tangent vector field is constant in time and its integral curves are coincident with the trajectories. Otherwise, the tangent vector field varies in time and things look more complicated.

53. As it stands the identity (10.14) is correct even if f^t is not an ergodic flow on $\mathcal{M}$. For example, $\mathcal{M}$ could decompose into a series of ergodic components without invalidating the derivations. The decomposition would be reflected in a factorization of the spectral determinant (to be defined in (15.10)).

54. **Remark T.5** Extremization method. The above method was introduced by many people - we have learned it from G. Russberg, F. Steiner and collaborators' hyperbola billiard computations. The convergence rate is really impressive, for the Sinai billiard some 5000 cycles are computed within CPU seconds with rather bad initial guesses.

55. Markov partitions are not canonical, and use of the symbolic dynamics does not respect smoothness. material derivative

56. remark that transients lead to factorization of $\det(1 - zT)$.

57. The total time derivative $\frac{d}{dt} = \partial_t + v_i \partial_i$ is called a material derivative.

58. However, if we are interested in computing the topological entropy (13.4), the determinant is clearly the way to go. If we were trying to find by trial and error all short cycles and extrapolate h from a finite sequence N_1, N_2, N_3, N_4, N_5, $\cdots$, it would take a while before we would get any decent convergence.

59. **Remark T.6** Diffusion induced by one–dimensional maps. (EXTRA:) Statistical mechanicians tend to believe that such complicated behavior is not to be expected in systems with very many degrees of freedom. Indeed, the addition to a large integer dimension of a number smaller than 1 should be as unnoticeable as a microscopic perturbation of a macroscopic quantity. For example, for the Lorentz gas no fractal-like behavior of the conductivity has been detected [24.10]. Simple maps as paradigms of real systems should be used with care. There are features of dynamical systems which are very general, and for which simple maps provide an indispensible source of intuition. Other features might be characteristic of the special model at hand: in the present case, the singular parametric dependence of diffusion for a system with a few degrees of freedom may not be indicative of a similar dependence for a system with very many degrees of freedom.

(P. Cvitanović and L. Rondoni)

60. **Remark T.7** Does the space matter? The exact value of the determinant of the evolution operator depends on the Banach space where we choose to define it. In the space of bounded observables the trace may not be well defined (the sum of the eigenvalues might diverge, for example), and some sort of regularization is needed. The standard method is to use the zeta function regularization (no relation to our dynamical zeta functions), by introducing a factor ν through the zeta function z

$$z(\nu) = \sum_{\alpha=0}^{\infty} e^{-\nu s_\alpha t} .$$

The determinant can then be defined by

$$\ln \det \mathcal{L} = z'(0) .$$

Although this may define a determinant for the operator $\mathcal{L}$, it may not define the right determinant (and later on we will show an example where it does not, see refsect SectStatmech). We prefer to use an alternative method. Rather than trying to regularize we will restrict the action of our operator to a sub-space of the Banach space. If we knew the exact invariant probability measure for our dynamical system we could take this sub-space to be the one-dimensional space just containing our measure, but in general we are not so fortunate. Instead, we consider a sub-space that is known to contain the measure and for which the $\mathcal{L}$ is trace class. This way no physics is discarded. How this is accomplished is the topic of refsect SectStatmech.

RM: This does not yet exist

(Ronnie Mainieri)

61. Equivalently, such graph can replaced by an alphabet. For example, starting with node 0, all possible non-self-intersecting return paths are
$\{0, 1000, 1100(1100)^k 0; \overline{1100}\}$, and any path can be composed of these "letters". As $k = 0, 1, 2, 3, \ldots$, the alphabet is infinite, but complete: any admissible string is a sequence of letters from the alphabet, with no further pruning rules.

62. **Remark T.8** On the validity of cycle averaging formulas. The cycle averaging formulas we have derived above can be viewed as an approximation to the full cycle expansion. Hence, one may be lead to believe that the results cycle averaging formulas produce are not as exact, as those based on full cycle expansions. In reality, there are two aspects to be considered. The first is purely mathematical, and says that rigorous proofs of the validity of both cycle expansions and cycle averaging formulas have been produced only for systems that satisfy the axiom-A conditions. In this respect, cycle averaging formulas and cycle expansions are perfectly equivalent, as both reproduce the correct S.R.B. distributions, in the limit of infinite cycle lengths. The second aspect is more practical in nature, and rests on the observation that periodic orbit expansions of one sort or the other have been succesfully applied in much more general situations than axiom-A. In all these applications, convergence to correct results has been observed independently of the method. However, a key question always was the speed of convergence, and that varies both with the method, and with the problem at hand. In particular, cycle averaging formulas performed comparably to cycle expansions, in the numerical studies of the Lorentz gas (see Chapter 9). On the contrary, full cycle expansions are much better whenever the symbolic dynamics is not affected by pruning. Also, we should note that given any ϵ which is not too large, in the case of a flow, and given the discrete nature of the periods of periodic orbits, there will be many values of τ such that very few, or not orbits at all belong to $P_{\tau,\epsilon}$. However, the problem eases at large periods, as the density of periods in a given window $(\tau, \tau + \epsilon)$ grows without bounds with τ.

(L. Rondoni)

63. **Remark T.9** Cycles and the variational principle. The search for a cycle with a given symbol sequence may be carried out in the following way: fix the sequence $\epsilon_1 \ldots \epsilon_n$, then draw a broken line passing through $q_1 \ldots q_n$ and joining q_n with q_1, where each q_j is a coordinate along the boundary of the disc where the j–th collision takes place (which is determined by the preassigned symbol sequence). Build the (Maupertuis) action functional $L(q_1 \ldots q_n)$ which is just the length of the broken line, and find a minimum of action functional. It may be proved [**?**] that there is just *one* minimum, and this gives you the cycle with the desired symbol sequence if it exists, or a physically impossible orbit (disc crossing) if there is no cycle corresponding to $\epsilon_1 \ldots \epsilon_n$. In practice it may be unfeasible to use this algorithm for searching for long orbits: then a more detailed layout of the symbol plane (along the lines of chapter **??**) has to be taken into account. We also notice that we may employ the C_{6v} symmetry of the elementary cell of the Lorentz gas on the hexagonal lattice to reduce the necessary computational effort: as images of a cycle under rotation and reflection do not have to be computed separately (see chapter 22).

Lamberto Rondoni

64. **Remark T.10** On the singularity of orbital measures. In our derivation

the orbital measures are singular, as they are costructed as linear combinations of delta functions, peaked on the periodic points. Alternatively, one can also view them as non-singular, with mass uniformly spread over nonoverlapping tubes, each one containing a unique periodic orbit. As long as the observables one studies are sufficiently smooth, the equivalence holds, as smooth functions do not probe the phase space thoroughly enough to be sensitive to singularities of measure.

periodic!point!unstable
unstable!periodic point
periodic!point!dense

What we obtained so far is a sequence of singular measures supported on unstable periodic orbits, approaching the natural measure of a map in a "weak" sense. By that, we mean that the sequence of averages (expectation values) of given quantities (observables), calculated using the sequence of orbital measures, approaches the average calculated with the natural measure.

(L. Rondoni)

65. The physical implication of the dominance of the smallest of the cycle eigenvalues is that the higher eigenvalues in the spectrum will be more sensitive to presence of almost stable orbits. In contexts such as the Hénon map "strange attractors" and quasi-ergodic Hamiltonian systems such as the x^2y^2 potential, this means that while leading eigenvalues might converge well (for example, in the computation of the Hausdorff dimension of a non–wandering set), the distribution of high eigenvalues is in principle unknowable, as it depends on the (structurally unstable) homoclinic tangencies and presence of arbitrarily miniscule islands of stability. From this point of view it is not at all clear that we should or can worry deeply about the asymptotic eigenvalue distributions, as is the prevailing trend in studies of quantum chaos. Indeed, from the vantage point of the Selberg product expansions our main concern is the development of accurate cycle expansions for the bottom of quantum spectra, and incorporation of corrections to the semi-classical approximations to the spectrum.

66. **Remark T.11** False zeroes. A. Voros has argued in ref. [36.8] that zeta functions cannot converge where one would like to use them. Things are not all that bad; while various product and series expansions converge only conditionally, the cycle (or cumulant) expansion converges in the region of interest.

67. We see that the phase space partitioning of figure 1.9 is not a peculiarity of pinballs, but a property of a large class of smooth flows.

68. The compression at the bend might lead to local stability, but as all points in this segment escape to infinity under forward iteration, the potential nonhyperbolic region is cut out and thrown away.

69. The unstable periodic orbits are isolated and uniformly sprinkled over the survivors' phase space, as the phase space is tessellated in strips of distinct n-step itineraries, and within each such strip there is a periodic point of period n.

70. the periodic points are dense on the asymptotic repeller

71. Turning this intuition into cold calculation will lead us, in clear physically motivated steps, to almost everything one needs to know about deterministic

volume preservation

chaos: from Smale horseshoes, through Cantor sets, Lyapunov exponents, symbolic dynamics, discrete symmetries, bifurcations, pruning, diffusion, all the way to transfer operators,thermodynamic formalism, the classical and quantum zeta functions, and spectral determinants.

72. While individual trajectories are exceedingly sensitive to noise, the asymptotic density eigenfunctions are robust.

73. The bold claim is that once you understand this, classical ergodicity, wave mechanics and stochastic mechanics will be at your feet, special cases to be worked out at your leisure.

the bad news is that field theories remain out of reach.

74. The Hénon map is a concrete implementation of the Smale horseshoe of sect. 12, a kind of a more realistic, nonlinear baker's map.

75. If the ingoing coordinates are (q_n, p_n), the outgoing coordinates (q_{n+1}, p_{n+1}) refer to the next collision disk. $x \in [0, L] \times (-\pi/2, /\pi/2)$

move exercise 6.8

The natural measure is $dsd \sin\theta$.

76. While various periodic orbit formulas may be formally equivalent, in practice some are vastly preferable to others. Trace formulas, such as (1.14) and the semiclassical Gutzwiller trace formula of chapter **??** are difficult to use for anything other than the leading eigenvalue estimates. As we shall see in chapter 16, not only is the spectral determinant exact, but it is also the tool of preference for evaluation of classical and semiclassical averages in low dimensional chaotic dynamical systems, as it has superior convergence properties (this is illustrated, for example, by table 18.2).

77. If $|\Lambda_e| > 1 > |\Lambda_c|$, the cycle is hyperbolic; if both eigenvalues are contracting the cycle is stable; and if they are both expanding, the cycle is repelling.

78. We shall always (unless explicitely stated otherwise) follow the Einstein convention that the repeated indices are summed over.

79. Physicist's pinball can bounce forever without losing energy.

The only parameter in the system is R/a, the ratio of the center-to-center disk separation R and the disk radius a.

A pinball trajectory is fully determined by specifying (p, q), where q is the position of the collision of the pinball measured as arclength along the reflecting wall, $p = \sin\theta$ is the momentum parallel to the wall, and θ is the angle between the outgoing trajectory and the normal to the wall (see figure 1.7).

Coordinates $x_i = (p_i, q_i)$ are a convenient choice, because they are phase space volume preserving and easy to extract from the pinball trajectory.

The energy is conserved, reflections are specular, so a collision point and angle of incidence suffice to specify a trajectory.

The motion of the pinball is then the motion of a point in a four-dimensional space.

quantum!momentum
momentum
quantum

N	. $F(s)$	$1/\zeta(s)$
1	0.39	0.407
2	0.4105	0.41028
3	0.410338	0.410336
4	0.4103384074	0.4103383
5	0.4103384077696	
6	0.410338407769346482	
7	0.4103384077693464892	
8	0.410338407769346489338468	
9	0.4103384077693464893384613074	
10	0.4103384077693464893384613078192	

Table T.1: 3-disk repeller escape rates for the disk-disk center separation $R = 6$, disk radius $a = 1$, computed from the cycle expansions of the spectral determinant (1.16) and the dynamical zeta function (1.10), as function of the maximal cycle length N. The convergence of the dynamical zeta function is exponential; the convergence of the spectral determinant is super-exponential, with the number of significant digits growing as $N^{3/2}$ with the truncation length.

80. A bar over a finite block of symbols denotes a periodic itinerary with an infinitely repeating basic block; we shall omit the bar whenever it is clear from the context that the trajectory is periodic.

81. The phase space of a generic nonlinear dynamical system is an infinitely interwoven mixture of islands of stability and regions of chaos.

82. In Ruelle's monograph transfer operator technique (or the "Perron-Frobenius theory") and Smale's theory of hyperbolic flows are applied to zeta functions and correlation functions. The hyperbolic case is treated, and the essential spectrum discussed. The Grothendieck theory and Fredholm determinants were introduced in Ruelle's 1989 papers [**?**, 16.7].

can one get from IHES? [16.7]

83. The rapid convergence is illustrated by the number of significant digits of γ computed from truncations of (1.10) to different maximal cycle lengths listed in table T.1.

84. The set of the shortest cycles is drawn in figure 41.5, and a typical comparison of the exact quantum and the cycle expansion eigenenergies is given in table T.2.

recheck -eigenenergies or wave numbers?

85. the quantum momentum of a particle described by (34.1)

try floating table T.2

$$p \sim \psi(q,t)^*(-i\hbar\frac{\partial}{\partial q})\psi(q,t) = \frac{\partial}{\partial q}R(q,t)A^2(q,t) - i\hbar A(q,t)\frac{\partial}{\partial q}A(q,t)$$

remains finite, leading to a well defined classical momentum. Since in general quantum operators are related to classical quantities $C(q,p)$ by $p \to -\hbar\frac{\partial}{\partial q}$ replacement, quantum operators acting on WKB functions also have well defined classical $\hbar \to 0$ limits.

Sune: mention Ehrenfest therem, or reference

M Berry: this is criminal

Remark T.12 A notational convention. Throughout this book the operator ∇ is a shorthand for a gradient if it acts on a scalar function, $(\nabla\phi(x))_i = \partial_i\phi(x)$, and for a divergence if it acts on a vector function, $\nabla\mathbf{v}(x) = \partial_i v_i(x)$.

Planck constant

n	QM	cycles
0	2.903721	2.92825
1	2.145974	2.13562
2	2.061272	2.05923
3	2.033587	2.03288
4	2.021177	2.02085
5	2.014563	2.01439
6	2.010626	2.01052
7	2.008094	2.00802
8	2.006370	2.00632
$\vdots$	$\vdots$	$\vdots$
∞	2.0	2.0

Table T.2: Exact quantum eigenenergies *vs.* the semiclassical eigenenergies obtained from cycle expansions: the parahelium bound state $(L, S = 0)$ series. See chapter **??** for details. (Gregor Tanner)

86. Not that far in future, the classical mechanics textbooks might have a chapter on the Keplerian motions followed by a chapter on chaos; and the quantum mechanics textbooks might have a chapter on the (Keplerian) Bohr hydrogen atom followed by a chapter on the (classically chaotic) helium atom. Here follows our attempt to formulate such "canonical" chapter.

87. Since the wavelength λ is proportional to the Planck constant $\hbar$, this short wavelength semiclassical approximation is often called a small $\hbar$ approximation.

88. $T_p = \frac{d}{dE}S_p(E)$is the classical period of cycle p.

89. Before we do that, a great deal of simplification can be achived by introducing parallel and transverse directions with respect to the direction of the velocity. A small variation dq in q can be decomposed into a parallel displacement $dq_\|$ and transverse displacement vector $dq_\perp$ such, that $dq_\|$ is parallel with the momentum vector $p = \partial_q R(q, q', t) = \partial_q S(q, q', E)$, while $dq_\perp$ is transverse to it. The small variation of R can be written as $dR = \partial_q R(q, q', t) dq_\|$ and we can define the parallel momenta as

$$\frac{\partial R(q, q', t)}{\partial q_\|} = p_\| = |p|.$$

?

The variation does not depend on $dq_\perp$ so we can write

$$\frac{\partial}{\partial q_\perp} R(q, q', t) = p_\perp = 0\,,$$

and similarly for $dq'_\|$, $dq'_\perp$, $p'_\| = -|p'|$ and $p'_\perp = 0$.

The determinant in (34.38) can be written in terms of the new coordinates as

$$\det\left(-\frac{\partial^2 R(q, q', t)}{\partial q_k \partial q'_l}\right) = (-1)^d \det\begin{pmatrix} \frac{\partial^2 R(q,q',t)}{\partial q_\perp \partial q'_\perp} & \frac{\partial^2 R(q,q',t)}{\partial q_\perp \partial q'_\|} \\ \frac{\partial^2 R(q,q',t)}{\partial q_\| \partial q'_\perp} & \frac{\partial^2 R(q,q',t)}{\partial q_\| \partial q'_\|} \end{pmatrix} \quad (T.20)$$

There is an intimate relation between the paralell direction and the time variable, which can be uncovered by taking the $q'_{\|}$ derivative of the Hamilton-Jacobi equation

$$\frac{\partial}{\partial t} R(q,q',t) = -H(q, \frac{\partial}{\partial q} R(q,q',t)). \tag{T.21}$$

We obtain

$$\frac{\partial^2 R(q,q',t)}{\partial q'_{\|} \partial t} = -|\dot{q}| \frac{\partial^2 R(q,q',t)}{\partial q_{\|} \partial q'_{\|}}.$$

Taking the $q'_{\perp}$ derivative of (T.21) yields

$$\frac{\partial^2 R(q,q',t)}{\partial q'_{\perp} \partial t} = -|\dot{q}| \frac{\partial^2 R(q,q',t)}{\partial q_{\|} \partial q'_{\perp}}.$$

With such relations the second derivative matrix of Hamilton's principal function can be rewritten as

$$\det\left(-\frac{\partial^2 R(q,q',t)}{\partial q_k \partial q'_l}\right) = (-1)^d \det \begin{pmatrix} \frac{\partial^2 R(q,q',t)}{\partial q_{\perp} \partial q'_{\perp}} & \frac{1}{\dot{q}'} \frac{\partial^2 R(q,q',t)}{\partial q_{\perp} \partial t} \\ -\frac{1}{\dot{q}} \frac{\partial^2 R(q,q',t)}{\partial t \partial q'_{\perp}} & -\frac{1}{\dot{q}\dot{q}'} \frac{\partial^2 R(q,q',t)}{\partial t^2} \end{pmatrix}.$$

We remove the velocities from the last column and row and get

$$\det\left(-\frac{\partial^2 R(q,q',t)}{\partial q_k \partial q'_l}\right) = \frac{(-1)^d}{|\dot{q}||\dot{q}'|} \det \begin{pmatrix} \frac{\partial^2 R(q,q',t)}{\partial q_{\perp} \partial q'_{\perp}} & \frac{\partial^2 R(q,q',t)}{\partial q_{\perp} \partial t} \\ \frac{\partial^2 R(q,q',t)}{\partial t \partial q'_{\perp}} & \frac{\partial^2 R(q,q',t)}{\partial t^2} \end{pmatrix}. \tag{T.22}$$

Now we show that the second transverse derivative matrix of the Hamilton's principal function coincides with those of the action. The mixed derivative can be calculated by taking the transverse derivatives of the transverse part of the identity (34.40)

$$\frac{\partial^2 R(q,q',t)}{\partial q_{\perp i} \partial q'_{\perp j}} = \frac{\partial^2 S(q,q',E)}{\partial q_{\perp i} \partial q'_{\perp j}} + \frac{\partial^2 S(q,q',E)}{\partial q_{\perp i} \partial E} \frac{\partial E}{\partial q'_{\perp j}}.$$

The last partial derivative can be reexpressed by taking partial derivatives of (34.37) yielding

$$\frac{\partial E}{\partial q'_{\perp j}} = -\frac{\partial^2 R(q,q',t)}{\partial q'_{\perp j} \partial t}.$$

Then taking partial derivatives of (34.40) yields

$$\frac{\partial^2 S(q,q',E)}{\partial q_{\perp i} \partial E} = \frac{\partial^2 R(q,q',t)}{\partial q_{\perp i} \partial t} \frac{\partial t}{\partial E},$$

where

$$\frac{\partial E}{\partial t} = -\frac{\partial^2 R(q,q',t)}{\partial t^2}.$$

Using these relations we finally obtain

$$\frac{\partial^2 S(q,q',E)}{\partial q_{\perp i}\partial q'_{\perp j}} = \frac{\partial^2 R(q,q',t)}{\partial q_{\perp i}\partial q'_{\perp j}} - \frac{\partial^2 R(q,q',t)}{\partial q_{\perp i}\partial t}\frac{\partial^2 R(q,q',t)}{\partial q'_{\perp j}\partial t} \Bigg/ \frac{\partial^2 R(q,q',t)}{\partial t^2} \,.$$

Using the identity of exercise 34.19, the determinant of the second transverse derivative of the action becomes

34.19
page 650

$$\det\left(\frac{\partial^2 S(q,q',E)}{\partial q_{\perp i}\partial q'_{\perp j}}\right) = \frac{1}{\frac{\partial^2}{\partial t^2}R(q,q',t)}\det\left(\begin{array}{cc} \frac{\partial^2 R(q,q',t)}{\partial q_\perp \partial q'_\perp} & \frac{\partial^2 R(q,q',t)}{\partial q_\perp \partial t} \\ \frac{\partial^2 R(q,q',t)}{\partial t \partial q'_\perp} & \frac{\partial^2 R(q,q',t)}{\partial t^2} \end{array}\right) . \quad (T.23)$$

90. say monodromy = one run around

91. The stable/unstable manifolds foliation of the phase space are the essential tool for constructing the Markov partitions of chapters **??** and **??**.

92. Then we shift gears and in sect. A.6 we spin a quantum mechanical tale. As this is only in part a book about quantum mechanics, one could have equally well praised great mathematicians, or pondered non-equilibrium statistical mechanics: appendix A offers one such historical overview of the development of chaotic dynamics.

    mostly of scholarly and human interest only.

    Words are cheap.

    You might want to skip the overture and head straight to the starting line; the book proper starts in sect. 1.3.

    We start out humbly.

93. to set the stage for the formulation of a theory predicting dynamical averages of observables - and that collides with the temptation to present a fuller summary of the theory of dynamical systems.

94. In chapter **??** we started out by counting individual itineraries, and then learned how to put this local information together in order to extract a global property of the system, the growth rate of the total number of trajectories. We shall now traverse the same route again, but this time not just count trajectories but learn how to actually compute them and *measure* their relative importance by measuring sizes of their neighborhoods.

    In chapter **??** we shall assemble together this local and global information and fashion from it the dynamical zeta functions and spectral determinants in chapter 14. They are the weighted generalizations of the topological zeta function of chapter 13.

    In chapters **??** and 13 we learned how to partition the phase space and count the pieces.

95. In sect. 13.1 we have seen that the topological entropy, the asymptotic rate of growth of the number of topologically distinct trajectories, is given by the leading eigenvalue of the transition matrix. It is our

goal now to generalize this observation and relate general asymptotic averages over chaotic dynamics to eigenvalues of evolution operators. As we shall now show, in the case of piecewise-linear approximations to dynamics these operators reduce to finite matrices, but for generic smooth flows, they are infinite-dimensional linear operators, and finding smart ways of computing their eigenvalues requires some thought.

96. Ronnie's text moved from example 2.2:

    The reason we can do that is that it is always possible to convert a flow that depends on time into a flow that does not. The trick is to consider time t one of the dynamic variables and introduce a new variable, say θ, to be the new time. The two will be related by an equation that states that they flow at the same rate. So if we have

$$\dot{x} = v(x,t) \tag{T.24}$$

    we set y to be the vector $\{x,t\}$ and add the equation

$$\frac{d\theta}{dt} = 1 \tag{T.25}$$

    to the vector field v so that $V(y) = \{v(x,t),1\}$ resulting in the autonomous system

$$\dot{y} = V(y)\,. \tag{T.26}$$

97. From chapter 3:

    We shall also find it convenient to define partial stabilities along cycle laps

$$\begin{aligned} f^j(x_a+\phi) &= x_b + h_b(\Lambda_j h_a^{-1}(\phi)) \\ f^k(x_b+\phi) &= x_a + h_a(\Lambda_k h_b^{-1}(\phi)) \end{aligned}$$

    where

$$\Lambda_j = \prod_{c=a}^{b-1} f'(x_c), \quad \Lambda_k = \prod_{c=b}^{a-1} f'(x_c)$$

    and $\Lambda_j\Lambda_k = \Lambda$.

    At this point we note that while in the deterministic case the cycle weight is given by the cycle stability, a quantity invariant under all smooth coordinate transfomations, the higher order corrections are not invariant. The noise is defined with respect to a particular coordinate system, and it has no invariant meaning.

98. **T.0.1 Repeller measures**

large overlap with sect. ??, sect. ??

We shall use the repeller measure (10.15)

$$\mu_i := \frac{e^{n\gamma_n}}{|\Lambda_i|} \tag{T.27}$$

computed on the union of unstable orbits as the working definition of the natural measure for open systems discussed in sect. 10.1.3. μ_i is the measure of the region $\mathcal{M}_i$ associated with the ith periodic point, $\mu_i = \int_{\mathcal{M}_i} d\mu(x)$, so the measure normalization $\int d\mu(x) = 1$ implies

$$\sum_i^{(n)} \mu_i = 1 \ . \tag{T.28}$$

The measure normalization condition (9.3) is satisfied automatically by the definition of the nth level estimate of the escape rate (19.18).

99. The book is organized as follows: in sect. **??** we review the evolution operator formalism for smooth flows. In sect. 16 we explain the theorems that guarantee that spectral determinants for Axiom A systems are entire. In sect. **??** we define and motivate the determinants used in this paper. In chapter 18 we introduce cycle expansions, and give convergence estimates for various expansions. In section B.1 review symbolic dynamics for unimodal maps. In section 12 we sketch the construction of "pruned" Smale horseshoes, In section 12.6 develop the danish pastry, a multisheated, invertible 2-d baker's map and introduce the pruning front.

100. **Remark T.13** Symplectic structure. For hyperbolic flows the symplectic structure plays no dramatic role - the topology and periodic orbits techniques are the same as for volume changing flows. Symplectic structure leads to the pairing off of the stability eigenvalues, so one needs to describe only the expanding directions, the contracting ones are automatic.

101. If we know what happens at two successive collisions we can determine quite easily what happens in between, as the position of a point of reflection together with the outgoing trajectory angle uniquely specifies the trajectory. In sect. 1.4.1 we used this observation to reduce the pinball flow to a map by the Poincaré section method, and associate an iterated mapping to the three-disk flow, a mapping that takes us from one collision to the next.

need figure

102. **Example T.1 Collinear helium, regularized.**

Though very simple in form, the Hamiltonian (5.6) is not the most convenient for numerical investigations of the dynamics of the classical helium system. In the (r_1, r_2) coordinates the potential is singular for $r_i \to 0$ nucleus-electron collisions, with velocity diverging to ∞. These 2-body collisions can be regularized by a rescaling of the time and the coordinates $(r_1, r_2, p_1, p_2) \to (Q_1, Q_2, P_1, P_2)$, in a manner to be described in chapter 41. For the purpose at hand it is sufficient to state the result: In the rescaled coordinates the equations of motion are

$$\dot{P}_1 = 2Q_1\left[2 - \frac{P_2^2}{8} - Q_2^2\left(1 + \frac{Q_2^2}{R^4}\right)\right]; \qquad \dot{Q}_1 = \frac{1}{4}P_1Q_2^2$$
$$\dot{P}_2 = 2Q_2\left[2 - \frac{P_1^2}{8} - Q_1^2\left(1 + \frac{Q_1^2}{R^4}\right)\right]; \qquad \dot{Q}_2 = \frac{1}{4}P_2Q_1^2 . \tag{T.29}$$

where $R = (Q_1^2+Q_2^2)^{1/2}$. These equations look harder to tackle than the harmonic oscillators that you are familiar with from other learned treatises, and indeed they are. But they are also a typical example of kinds of flows that one works with in practice, and the skill required in finding a good re-coordinatization $h(x)$.

flow!incompressible
conservation!phase space volume
Liouville!theorem

PM: Create a problem that illustrates these points.

coarse graining
irreversibility
adjacency matrix

103. If the density itself is a material invariant, combining (**??**) and (35.3) we conclude that $\partial_i v_i = 0$ and $M^t(x_0) = 1$. An example of such incompressible flow is the Hamiltonian flow of sect. 5.2. For incompressible flows the continuity equation (35.3) becomes a statement of conservation of the phase space volume (see sect. 5.2), or the Liouville theorem

$$\partial_t \rho + v_i \partial_i \rho = 0 \,. \tag{T.30}$$

104. The position of the ball is described by a pair of numbers (the spatial coordinates on the plane), and the angle of its velocity vector. As far as Baron Leibniz is concerned, this is a complete description.

105. Just as f^t maps points in the phase space $f : \mathcal{M} \to \mathcal{M}$ and generates flows, the operator $\mathcal{L}^t$ maps densities into densities and generates a semigroup on the space of density functions, satisfying

    (a) $\mathcal{L}^0 = I$

    (b) $\mathcal{L}^t \mathcal{L}^{t'} = \mathcal{L}^{t+t'} \qquad t, t' \geq 0$ (semigroup property)

    (c) $\mathcal{L}^t \rho$ is t-continuous for every density $\rho(x)$, i.e., $\lim_{t\to 0_+} \mathcal{L}^t \rho = \rho$ is pointwise convergent.

EAS: what is the pont of the condition (c)?

106. We have already given an example of how to quantify this instability in sect. 4.1 - for now, a qualitative statement that a flow is *locally unstable* if nearby trajectories separate exponentially with time will suffice.

107. Symbolic dynamics is the coarsest example of coarse graining, the way irreversibility enters chaotic dynamics. The exact trajectory is deterministic, and given an initial point we know, in principle, both its past and its future - its memory is infinite. In contrast, the partitioned phase space is described by the quintessentially probabilistic tools, such as the finite memory Markov graphs.

108. **Remark T.14** Adjacency matrix. If the graph is *not* directed, the corresponding matrix is called the *adjacency matrix*. An adjacency matrix is symmetric and can be diagonalized. The Markov graphs, however, are always directed graphs.

109. The stable (unstable) manifold of the central hyperbolic fixed point (x_1, x_1) can be constructed numerically by starting with a small interval along the local stable (unstable) eigendirection, and iterating the interval n steps backwards (forwards).

110.

transfer!operator
operator!transfer
transfer!matrix
transfer!matrix
Markov!matrix
matrix!transfer
matrix!Markov

T.0.2 Transfer operators

We start by showing that for a piecewise-linear map with a finite Markov partition, the evolution operator is a finite dimensional transfer operator, a straightforward generalization of the topological transition matrices (11.2).

Niels S.: A figure would be helpful here...

Consider as an example of expanding 1-d map $f(x)$, with $|f_s'(x)| > 1$ and monotone on two non-overlapping intervals, a piecewise-linear 2–branch repeller with slopes $\Lambda_0 > 1$ and $\Lambda_1 < -1$:

$$f(x) = \begin{cases} \Lambda_0 x & \text{if } x \in \mathcal{M}_0 = [0, 1/\Lambda_0] \\ \Lambda_1(x-1) & \text{if } x \in \mathcal{M}_1 = [1 + 1/\Lambda_1, 1] \end{cases} . \tag{T.31}$$

As both $f(\mathcal{M}_0)$ and $f(\mathcal{M}_1)$ map onto the entire unit interval $\mathcal{M}$, it follows from (11.4) that the partition into $\{\mathcal{M}_0, \mathcal{M}_1\}$ is a finite Markov partition with the unrestricted 2-letter alphabet $s_n \in \{0, 1\}$, with the transition matrix

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} .$$

The transition matrix T defined in (11.2) counts the allowed transitions, so $T_{dm} = 0$ or 1; however, it is just as easy to keep track of matrix elements T_{dm} that take other ranges of values. Due to the piecewise linearity, the simplest of evolution operators, the Perron-Frobenius operator (**??**) acts on a piecewise constant function $\rho(x) = (\rho_0, \rho_1)$ as a $[2\times2]$ transfer!matrix with matrix elements

$$\begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} \to \mathcal{L}\rho = \begin{pmatrix} \frac{1}{|\Lambda_0|} & \frac{1}{|\Lambda_1|} \\ \frac{1}{|\Lambda_0|} & \frac{1}{|\Lambda_1|} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} . \tag{T.32}$$

13.1
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More generally, T_{dm} is the "penalty" paid for choosing a particular transition, and T is a finite memory *Markov* or *transfer* matrix. Piecewise-linear approximations to dynamical systems yield finite-dimensional transfer operators, provided that the symbolic dynamics is a subshift of finite type, *cf.* sect. 13.4. This is discussed in more detail in appendix B.5.

If we increase the number of steps remembered, the transition matrix grows big quickly, as N-ary dynamics with n-step memory requires an $[N^n \times N^n]$ matrix. As the matrix is very sparse, we shall find it convenient to use the Markov graph representations for T developed in sect. 11.5.1.

T.0.3 Quadratic map as the $b \to 0$ limit of the Hénon map

The non–wandering set of a Hénon map is contained in the intersection of the forward and backward iterated horseshoes (figure **??**). In the $b \to 0$

limit the unstable manifold shrinks to a 1-d arc, folds of the stable manifold stretch off to infinity, and the Hénon map (?!) reduces to the 1-d quadratic map (11.9) depicted in figure **??** a. [3] There are two fixed points $\overline{0}$, $\overline{1}$ in the (x_t, x_{t+1}) plane, with coordinates (x_0, x_0), (x_1, x_1) given by

$$x_0 = \frac{-1-\sqrt{1+4a}}{2a}\,, \quad x_1 = \frac{-1+\sqrt{1+4a}}{2a}\,.$$

The fixed point (x_0, x_0) is always unstable, while for small a the fixed point (x_1, x_1) starts out stable, and goes unstable through a period doubling bifurcation.

For a 1-d map, a point with vanishing derivative is called a *critical point*; in the example at hand, it is the maximum point on the parabola, $x_c = 0$. In one iteration, the neighborhood of a critical point gets folded and squashed into a sharp "turnback"; in the 1-d case there is no confusion as to what is the primary turning point (the critical point), and what are its images. In the general $b \neq 0$ case, the distinction is not so clear, as the folding is soft, and a turnback can be easily confused for a "critical point". What we wish to do here is to make precise the 2-d generalization of the 1-d critical point as a set of "primary turnbacks", i.e., the primary folding points where the stable and unstable manifolds are parallel. In order to accomplish this, we need to first describe the stable and the unstable manifolds of the quadratic map (x_1, x_1) fixed point in some detail.

In the $b \to 0$ limit of the Hénon map the *stable manifold* of the $\overline{1}$ fixpoint stretches out into a set of horizontal lines passing through all of preimages of $\overline{1}$ on the parabola. As the contraction rate is infinite, all points on the stable manifold map onto the parabola in one iteration. The stable manifold is bounded in the vertical direction by the fixed point (x_0, x_0) and its preimage, as the points that start outside the interval $x_0 \leq y \leq f^{-1}(x_0)$ escape, and cannot be preimages of $\overline{1}$. This is clearly seen in figure **??** a.

T.0.4 Scaling potentials

The energy is a quadratic function of momenta, $H = p^2/2m$, so motion at different energies E and E_0 is related by the scaling $p_E \to p_0\sqrt{E/E_0}$ for momenta, $t_E \to t_0\sqrt{E_0/E}$ for times, and

$$S(E) = L\sqrt{2mE} = S(E_0)\sqrt{E/E_0} \tag{T.33}$$

for the actions, where L is the geometrical length of the orbit. These observations will be useful below in the quantum version of the pinball where the (34.15) will combine with $\hbar$ to the relevant quantum variable, the wave number $k = \sqrt{2mE}/\hbar$, (see sect. **??**).

comment somewhere: The eigenvalues of the jacobian transverse to a periodic orbit (see below) are invariant under the above energy rescaling.

[3] Kai H: The version of the logistic map you refer to is $1 - ax^2$ not $Ax(1-x)$ which imply that a=2 not 4 a number of places. The conversion is $a = (A^2 - 2A)/4$.

T.0.5 2-dimensional maps

$$f : R^2 \to R^2$$

fix this could also be repelling, attractive - it is really a saddle.

If eigenvalues $|\Lambda_1|$, $|\Lambda_2| \neq 1$, the fixed point is *hyperbolic*.

Evolution operators

(E.A. Spiegel and P. Cvitanović)

Let us first look at a useful formal restatement of (**??**). This is obtained from the identity

$$\rho(x) = \int_{\mathcal{M}} \delta(y - x)\, \rho(y) dy \tag{T.34}$$

where $\delta(\ \ )$ is the Dirac delta function. Then we transform this integral to the coordinate system based on the initial positions x_0 using the Jacobian to obtain

$$\rho(x(t)) = \int_{\mathcal{M}} \delta\big(x_0 - f^{-t}(x)\big)\, J^t(x_0)\rho(x_0)dx_0\ , \tag{T.35}$$

where we have used the formula $x = f^t(x_0)$.

For the discussion of material invariants, in line with (**??**), we could write the conservation of a material invariant I as

$$\int_{\mathcal{M}} dx\rho(x)I(x) = \int_{\mathcal{M}} dx_0\rho(x_0)I(x_0)\,. \tag{T.36}$$

Then as for ρ, we deduce from this that $I(x) = I(x_0)$. In other words,

$$I(x_0) = \int_{\mathcal{M}} I(y)\delta(y - x_0)\, dy\,. \tag{T.37}$$

where we recall that $x = f^t(x_0)$. We see, that

$$I(x) = \mathcal{L}^t \circ I\,. \tag{T.38}$$

Power spectra

reinsert here the definition of finite Fourier transform!

The power spectrum consists of broad band noise $D(\omega)$ and discrete spectrum $\Delta(\omega)$

$$\left\langle |t\hat{x}(\omega)|^2 \right\rangle \sim t^2 \Delta(\omega) + 2tD(\omega)\,.$$

$D(\omega)$ is nothing else but the diffusion constant [24.6, 24.16] for quantity $t\hat{x}(\omega)$, and $\Delta(\omega)$ is the drift term (factor 2 is the conventional normalization for diffusion in one dimension).

An alternative formula for power spectrum is given by the Kinchine-Wiener relation

$$\left\langle |t\hat{x}(\omega, x)|^2 \right\rangle_x = t \sum_{n=-t+1}^{t-1} (1 - |n|/t) C(n) e^{i\omega n} \qquad \text{(T.39)}$$

where $C(n) = \langle x_t x_{t+n} \rangle_x$ is the space-averaged time correlation function. We have also evaluated $C(n)$ by means of cycle expansions, but the convergence appears to be very slow.

Remark T.15 Power spectra. Refer to refs. [24.6, 24.16] for the diffusion constant for quantity $t\hat{x}(\omega)$,

T.0.6 Pruned symbolic dynamics

If the symbolic dynamics is described by a subshift of a finite type, cycle expansions converge well. An example is given by the tent map $f(x) = 1 - a|x|$, $a = (1 + \sqrt{5})/2$. This value of a corresponds to the $\overline{001}$, $\overline{011}$ 3-cycles bifurcation value, with the symbolic dynamics given by a simple pruning rule; the repeat _00_ is forbidden.

Difficulties can arise if the system is not sufficiently mixing. For example, the tent map $x_{t+1} = 1 - a|x_t|$ for $a = \sqrt{2}$ has two nonoverlaping bands. The system is not mixing, and the power spectrum contains a delta-function peak at $\omega = 1/2$. In symbolic dynamics only sequences having "1" at all odd or at all even places are allowed. Let us focus our attention on the zeta function $1/\zeta$ at the frequency $\omega = 1/2$. The set of cycles with odd periods is empty (except for the fixed point "1", and it also disappears for $a < \sqrt{2}$) and

The straightforward differentiation of $1/\zeta$ does not give correct result: the drift term vanishes and the diffusion constant diverges. The reason is that the mapping f^2 is not mixing and has two symmetric attractors, so the dynamical zeta function is a product of the two zeta functions for these attractors. The probability distribution function for A_t does not tend to a Gaussian hump as $t \to \infty$, but instead to two symmetric humps, drifting away from the origin in opposite directions (that is why the drift term for $1/\zeta$ vanishes). In order to describe the power spectrum for $\omega = 1/2$ correctly, we must restrict the averaging to one hump. This corresponds to considering one of the symmetric attractors of the map f^2.

From this zeta function correct values of discrete and continuous components of the power spectrum at $\omega = 1/2$ are obtained as the drift term and the diffusion constant.

Power spectrum of the Bernoulli map

There exist maps - typically 1-d piecewise-linear maps - for which the natural measure is available in closed form. As for such maps the power spectra are known analytically, we can use them as a benchmarks for tests of cycle expansions. We start with two maps whose symbolic dynamics is described by the full (0,1) binary shift, i.e., all sequences of "0" and "1" are realizable: the Bernoulli shift, and the skew tent map. C. Beck has further results for the Ulam map, based on the Chebyshev polynomials method of ref. [I.24].

$$\left\langle |t\hat{x}(\omega,x)|^2 \right\rangle_x = t \sum_{n=-t+1}^{t-1} (1-|n|/t) C(n) e^{i\omega n} \tag{T.40}$$

where $C(m) = \langle a(x_t) a(x_{t+m}) \rangle$ is the space-averaged time autocorrelation function (**??**).

$$\begin{aligned} D(\omega) &= \frac{1}{2} \frac{\partial^2 Q}{\partial \beta^2}\bigg|_{\beta=0} = \frac{1}{2t} \left\langle |t\hat{x}(\omega)|^2 \right\rangle_x + O\left(e^{tQ_1(0)}\right) \\ \frac{1}{t} \left\langle |t\hat{x}(\omega)|^2 \right\rangle_x &= \int dx \delta(x - f^t(x))\, t |\hat{x}(\omega,x)|^2 \\ &= \sum_p |n_p \hat{x}_p(\omega)|^2 \sum_{r=1}^{\infty} \frac{\delta_{t,n_p r}}{|1-\Lambda_p^r|} \end{aligned} \tag{T.41}$$

For systems with complete binary dynamics

$$\begin{aligned} \frac{1}{t} \left\langle |t\hat{x}(\omega)|^2 \right\rangle_x &= \frac{1}{t} \sum_{\{\epsilon_1\epsilon_2\ldots\epsilon_t\}} \frac{|t\hat{x}(\omega, x_{\{\epsilon_1\epsilon_2\ldots\epsilon_t\}})|^2}{|1-\Lambda_{\{\epsilon_1\epsilon_2\ldots\epsilon_t\}}|} \\ D(\omega) &= \frac{1}{2t} \sum_{\{\epsilon_1\epsilon_2\ldots\epsilon_t\}} \frac{|t\hat{x}(\omega, x_{\{\epsilon_1\epsilon_2\ldots\epsilon_t\}})|^2}{|\Lambda_{\{\epsilon_1\epsilon_2\ldots\epsilon_t\}}|} + O\left(e^{-ct}\right) . \end{aligned} \tag{T.42}$$

T.0.7 Bernoulli shift

For the Bernoulli shift $f(x) = 2x \bmod (1)$, the natural measure is $\mu(x) = 1$, hence

$$C(0) = \left\langle (x_t - \langle x \rangle)^2 \right\rangle = \int_0^1 dx (x - \frac{1}{2})^2 = \frac{1}{12}\,, \tag{T.43}$$

and the correlation function is [I.25]

$$C(m) = \langle (x_t - \langle x \rangle)(x_{t+m} - \langle x \rangle) \rangle = \frac{1}{12} 2^{-m}\,. \tag{T.44}$$

The power spectrum follows from (T.40)

recheck this

$$\begin{aligned} D(\omega) &= \frac{1}{4} \frac{1}{5 - 4\cos 2\pi\omega} \\ D(0) &= \frac{1}{4}\,, \qquad D(\pi) = \frac{1}{36}\,, \qquad D(\pi/2) = \frac{1}{20}\,, \ldots \end{aligned} \tag{T.45}$$

We shall compare this exact solution with the values obtained from cycle expansion.

The periodic points are easily computed:

$$\begin{aligned} x_{\epsilon_1 \epsilon_2 \ldots \epsilon_k} &= .\overline{\epsilon_1 \epsilon_2 \ldots \epsilon_k} = \frac{2^{k-1}\epsilon_1 + 2^{k-2}\epsilon_2 + \ldots + \epsilon_k}{2^k - 1} \\ x_0 &= 0, \quad x_1 = 1, \quad x_{01} = 1/3, \quad x_{10} = 2/3, \\ x_{001} &= 1/7, \quad x_{010} = 2/7, \quad x_{100} = 4/7, \\ x_{011} &= 3/7, \quad x_{110} = 6/7, \quad x_{101} = 5/7, \quad \ldots \end{aligned} \tag{T.46}$$

T.0.8 Trace formula

Frequency $\omega = 0$ - diffusion:
For the Bernoulli map

$$D(0) = \lim_{t\to\infty} \frac{1}{2t} \frac{1}{2^t} \sum_{\{\epsilon_1 \epsilon_2 \ldots \epsilon_t\}} |t\,\hat{x}(0, x_{\{\epsilon_1 \epsilon_2 \ldots \epsilon_t\}})|^2\,. \tag{T.47}$$

The sum over periodic points belonging to cycle p:

$$t\,\hat{x}(0, x_{\{\epsilon_1 \epsilon_2 \ldots \epsilon_t\}}) = \sum_{m=1}^{t} x_{\epsilon_m \epsilon_{m+1} \ldots \epsilon_{m+t-1}} = \sum_{m=1}^{t} \epsilon_m \tag{T.48}$$

Setting $x = 1$ in derivatives of the identity

$$(1+x)^t = \sum_{k=0}^{t} \binom{t}{m} x^m$$

we obtain

$$\begin{aligned} t2^{t-1} &= \sum_{k=1}^{t} \binom{t}{m} m \\ t(t+1)2^{t-2} &= \sum_{k=1}^{t} \binom{t}{m} m^2 \end{aligned} \tag{T.49}$$

$$\frac{1}{2^{t+1}t} \sum_{\{\epsilon_1\epsilon_2\ldots\epsilon_t\}} |\sum_{m=1}^{t} \epsilon_m|^2 = \frac{1}{2^{t+1}t} \sum_{m=1}^{t} \binom{t}{m} m^2 = \frac{1}{8}(1+\frac{1}{t}) \tag{T.50}$$

$$\left.\frac{\partial^2 Q}{\partial \beta^2}\right|_{\beta=0} = \frac{1}{4}\,. \tag{T.51}$$

Frequency $\omega = \pi$:

$$t\,|\hat{x}(\pi, x_{\{\epsilon_1\epsilon_2\ldots\epsilon_t\}})| = |\sum_{m=1}^{t} (-1)^m x_{\epsilon_m\epsilon_{m+1}\ldots\epsilon_{m+t-1}}| = \frac{1}{3}|\sum_{m=1}^{t} (-1)^m \epsilon_m| \tag{T.52}$$

$$\begin{aligned} \sum_{\{\epsilon_1\epsilon_2\ldots\epsilon_t\}} |\sum_{m=1}^{t/2} \epsilon_{2m} - \sum_{m=1}^{t/2} \epsilon_{2m-1}|^2 &= \sum_{m,n=1}^{t/2} \binom{t/2}{m}\binom{t/2}{n}(n-m)^2 \\ &= 2\sum_{m=1}^{t/2} \binom{t/2}{m} \sum_{n=1}^{t/2} \binom{t/2}{n} n^2 - 2\left(\sum_{n=1}^{t/2} \binom{t/2}{n} n\right)^2 \\ &= t\,2^{t-2} \end{aligned} \tag{T.53}$$

For $t = 2, 4, 6, \ldots$ this sum equals 2, 16, 6·16, ….

$$\left.\frac{\partial^2 Q}{\partial \beta^2}\right|_{\beta=0} = \frac{1}{4 \cdot 9}\,. \tag{T.54}$$

T.0.9 $1/\zeta$ function evaluation

The curvature expansions truncate at the two fixed points, so the spectral determinant of the Perron-Frobenius operator is simply

$$F = \left(1 - \frac{z}{|\Lambda_0|} - \frac{z}{|\Lambda_1|}\right)\left(1 - \frac{z}{|\Lambda_0|^2} - \frac{z}{|\Lambda_1|^2}\right)\ldots = \prod_{\alpha=0}^{\infty}\left(1 - \frac{1}{2^\alpha}z\right). \quad (T.55)$$

The leading eigenvalue is 1, as required by the probability conservation. The second eigenvalue controls the exponential fall-off of the 2-point correlations, $\lambda_1 = 1/2$, in agreement with (T.44). The denominator in (24.8) is a frequency independent normalization constant

$$\sum{}'(-1)^k \frac{(n_{p_1} + \cdots + n_{p_k})}{|\Lambda_{p_1} \cdots \Lambda_{p_k}|} = -\frac{1}{2} - \frac{1}{2} - \frac{1}{4}(2 - 1 - 1) + \ldots = -1\,, \quad (T.56)$$

and using (T.48) the cycle expansion (24.8) yields

$$\begin{aligned} 2D(0) &= \frac{0^2}{2} + \frac{1^2}{2} + \frac{1}{4}\left\{\left(\frac{1}{3}+\frac{2}{3}\right)^2 - (0+1)^2\right\} \\ &\quad + \frac{1}{8}\left\{\left(\frac{1}{7}+\frac{2}{7}+\frac{4}{7}\right)^2 - \left(0+\frac{1}{3}+\frac{2}{3}\right)^2\right\} \\ &\quad + \frac{1}{8}\left\{\left(\frac{3}{7}+\frac{6}{7}+\frac{5}{7}\right)^2 - \left(1+\frac{1}{3}+\frac{2}{3}\right)^2\right\} + \ldots = \frac{1}{2} \end{aligned} \quad (T.57)$$

This is simply the ordinary 1-d Brownian walk.

Frequency $\omega = \pi$:
The curvature expansion [18.2] works well for $D(0)$, but already the $D(1/2)$ expansion is not so simple - an infinite sequence has to be summed up:

the last line is a guess, not proven! We need to generalize Markov diagrams to links with phases, to compute all this analytically.

$$\begin{aligned} 2D(1/2) &= +\frac{1}{4}\left(\frac{1}{3}-\frac{2}{3}\right)^2 + \frac{1}{16}\left(\frac{1}{15}-\frac{2}{15}+\frac{4}{15}-\frac{8}{15}\right)^2 \\ &\quad + \frac{1}{16}\left(\frac{3}{15}-\frac{6}{15}+\frac{12}{15}-\frac{9}{15}\right)^2 + \frac{1}{16}\left(\frac{7}{15}-\frac{14}{15}+\frac{13}{15}-\frac{11}{15}\right)^2 + \ldots \\ &= \frac{1}{2^2 3^2} + \frac{1}{2^4}\left(\frac{1}{3^2}+\frac{0}{3^2}+\frac{1}{3^2}\right) + \frac{1}{2^6 3^2}2^2 + \ldots \\ &= \frac{1}{2^2 3^2}\left(\sum_{k=0}^{\infty} 2^{-k}\right) = \frac{1}{18} \end{aligned} \quad (T.58)$$

$$n_p|\hat{x}_p(\pi)| = |\sum_{m=1}^{n_p}(-1)^m x_{\epsilon_m \epsilon_{m+1} \ldots \epsilon_{m+n_p-1}}| = \frac{1}{3}|\sum_{m=1}^{n_p}(-1)^m \epsilon_m| \quad (T.59)$$

T.0.10 Skew tent map

this calculation should be done more analytically?

For the skew tent map

$$f(x) = \begin{cases} ax & \text{if } 0 \le x \le a^{-1}, \\ \frac{a}{a-1}(1-x) & \text{if } a^{-1} \le x \le 1. \end{cases} \tag{T.60}$$

the correlation function was calculated by Grossmann and Thomae [18.4]:

$$C(m) = \langle (x_t - \langle x \rangle)(x_{t+m} - \langle x \rangle) \rangle = \frac{1}{12} \left(\frac{2-a}{a} \right)^m \tag{T.61}$$

The power spectrum is

$$\begin{aligned} D(\omega) &= \frac{1}{6} \frac{a-1}{a(a-2)(1+\cos 2\pi\omega)+2} \\ D(0) &= \frac{1}{12(a-1)}, \qquad D(1/2) = \frac{a-1}{12}, \ldots \\ D(\omega) &= \frac{1}{12} \quad \text{for} \quad a = 2\,. \end{aligned} \tag{T.62}$$

Consider first the zero frequency case, $\omega = 0$. In this case the Ruelle zeta function

$$1/\zeta = (1-t_0)(1-t_1)(1-t_{01})(1-t_{001})... = 1 - t_0 - t_1 - [t_{01} - t_0 t_1]...$$

consists of the fundamental part and curvature corrections, and calculation of diffusion constant is straightforward. Stability of a cycle with $n_{p,1}$ repeats of symbol 1 is $\Lambda_p = a^{n_p}/(1-a)^{n_{p,1}}$. The probability conservation follows trivially from $1/|\Lambda_0| + 1/|\Lambda_1| = 1/a + (a-1)/a = 1$ (all curvatures vanish for this peacewise-linear map), and the denominator in (24.8) is simply -1. For $\beta = 0$ the spectral determinant is given by

$$F = \left(1 - \frac{z}{|\Lambda_0|} - \frac{z}{|\Lambda_1|}\right)\left(1 - \frac{z}{|\Lambda_0|^2} + \frac{z}{|\Lambda_1|^2}\right)\ldots = (1-z)\left(1 - z\frac{2-a}{a}\right)\ldots. \tag{T.63}$$

The leading eigenvalue is 1, by probability conservation; the second eigenvalue controls the exponential falloff of the 2-point correlations, $\lambda_1 = a/(2-a)$, in agreement with (T.61).

The periodic points are easily computed:

$$x_0 = 0, \quad x_1 = \frac{a}{2a-1}, \quad x_{10} = \frac{a^2}{a^2+a-1}, \quad x_{01} = \frac{a}{a^2+a-1}, \quad \ldots \tag{T.64}$$

For $a = 2$ we have:

$$x_0 = 0, \quad x_1 = 2/3, \quad x_{01} = 2/5, \quad x_{10} = 4/5, \quad \ldots \tag{T.65}$$

(the general formula is given in sect. **??**). Even though the power spectrum is given explicitely by (T.62), the cycle expansion does not reproduce this expression in any obvious form; (24.8) yields

these calculations not rechecked.

$$\begin{aligned} 2D(0) &= \left(\frac{a}{2a-1}\right)^2 \frac{a-1}{a} + \left(\frac{a}{a^2+a-1} + \frac{a^2}{a^2+a-1}\right)^2 \frac{a-1}{a^2} \\ &\quad - \left(\frac{a}{2a-1}\right)^2 \frac{a-1}{a^2} + \ldots \end{aligned} \tag{T.66}$$

$$\begin{aligned} A_{01}(1/2) &= x_{01} - x_{10} = \frac{a(a-1)}{a^2+a-1}, \quad \ldots \\ 2D(1/2) &= \frac{(a-1)^3}{(a^2+a-1)^2} + \text{4-cycle contributions} \end{aligned} \tag{T.67}$$

We conclude this section with a numerical evaluation of the power spectrum for the skew tent map (T.60); the result is given in fig. 40.1, with fig. 40.2 illustrating the convergence with maximal cycle length truncations. We have also checked that for the Ruelle zeta functions convergence is exponential, and that for the spectral determinant the convergence is faster than exponential.

track down the figures!

this must be reworked so that for finite Markov graphs, long cycles are re-expressed as the repeats of the fundamental cycles; work out in detail for 2-branch complete binary tent map?.

T.0.11 Tent map cycles

Note that for $a = 2$ the spectrum is flat. This might provide additional checks on our deliriously detailed cycle expansion.

For the symmetric tent map ($a = 2$ in (T.60)) it is convenient to compactify the binary sequences as in sect. F.1.1.

Convergence of cycle expansions

include this somewhere

I heard a statement (made by Ruelle) that Perron-Frobenius operator and the Liouville operator differ only in the type of functions they act on. The L operator acts on the space of square integrable functions, whereas the PF operator acts on some other functions in phase space.

Answer: In my case, ALL operators act on spaces of (piecewise) real analytic functions - eq (4.33) used to have a remark that y^α basis would give continuous spectrum in α - real analiticity restrics the basis to Taylor expansion y^n. I think this is discussed nicely and in much detail in

P. Gaspard, G. Nicolis, A. Provata and S. Tasaki "Spectral signature of the pitchfork bifurcation: Liouville equation approach" *Phys. Rev.* **E 51**, 74 (1995).

Anton V. Andreev:
Can you tell me what these functions are and how they look like? Are they concentrated around periodic orbits etc?

Answer: No - they MUST not be singular - they must be smooth distributions, otherwise spectrum could be anything.

PC 15 sept 1995 I'll try to summarize what I understand the impact of Wirzba's N-disk exact spectra is on various competing determinats:

in the running are:

QM refers to exact quantum mechanics
GV refers to semiclassical zeta function
QC refers to quasiclassical/Madelungian
FD refers to classical Fredholm determinant
CV93 refers to P. Cvitanović and G. Vattay, *Entire Fredholm determinants for evaluation of semi-classical and thermodynamical spectra*, *Phys. Rev. Lett.* **71**, 4138 (1993)
CRR93 refers to P. Cvitanović, P.E. Rosenqvist, H.H. Rugh, and G. Vattay, *A Fredholm determinant for semi-classical quantization*, *CHAOS* **3**, 619 (1993).

Wirzba's work puts the open billiards cycle expansions on firm footing, by clarifying that the exact quantum mechanics for scattering systems requires cumulant form to make sense of what one means by determinant; so it accomplishes the same task for quantum mechanics/ semiclassics that HHRugh proof accomplished for nice classical hyperbolic systems.

The purported virtue of QC over GV is that it is entire, while GV has poles - furthermore, for the first time GV was explicitely given in terms of 4 entire functions, eq (12) of CV93, the derivation requires no saddle points (except throwing away the quantum potential in the step one), and by making the flow classical, we are on terra firma (and, if some day HHR announces that the prrof applies to flows as well, officially rigorous). Andreas wonders why should being entire be so important - even exact QM in 2-dim has a cut, says he.

However, QC carries with it an annoying extra set of resonances, which we noted for the FD and QC in CRR93, Fig. 4. They are harmless, except that fitting them eats up orders in the cycle expansion which would be more profitably used to fish for the real QM resonances, so for short truncations GV does better than QC (even though later on in cycle truncatin length the finite radius of convergence degrades the quality of GV). In other words, in practice QC is worse, rather than better than the old fashioned GV. Yech!

We associated the extra set with the zeros of $F_-(3/2, E)$ in eq (12) of CV93. Unfortunately, the cancellation was not exact (they differed in 4-th decimal (!), see text p. 630 of CRR93, but the "cleaned" spectrum of Fig. 6 is rather like the QM spectrum shifted 1/2 Laypunov down. We did not test the cancellation for the QC case.

Wirzba QM has no such family of resonances, and neither do the short cycle truncations of GV; in other words, numerics suggests that for the quasiclassical case, the $F_+(1/2, E)/F_-(3/2, E)$ cancellation is EXACT. This could be checked by redoing the accurate individual resonance calculation of p. 630 of CRR93 - Per can easily do that numerically. Andreas already says that "the fake resonances of the Vattayvic zeta-function" are coming from the subleading dynamical zeta functions, i.e.,

$$1/\zeta_1 = \prod_p (1 - t_p/\lambda_p)$$

which I believe to be the same statement. If this is true, then we should rethink both the FD and the QC cases - can we cook up operators that are entire and still have no spurious stuff? If it works in QC, than it classically means that maybe noise $\to 0$ is not a trivial limit to the classical Perron-Frobenius or whatever....

Remark T.16 Natural measure for computing pressure? We would like to pretend that (?!), or something like it, is the stat. mech. partition function for this "gas". I do not see it, but here are some tentative, naive and almost surely wrong thoughts on the matter: reread epics... antipodal

If we assume ergodic hypothesis, the time average (**??**) can be rewritten as a a phase space average force exerted on the walls of the billiard by the "gas"

$$\langle F \rangle = \int_\Omega d\hat{\sigma} \cdot \mathbf{p}(\sigma)\rho(\sigma) \,, \qquad \rho(\sigma) = \lim_{t\to\infty} \frac{1}{t} \sum_i^{t_i \le t} \delta(x_i - \sigma) \,. \tag{T.68}$$

where $d\hat{\sigma}$ is a unit normal to the surface element $d\sigma$, and the $\rho(\sigma)$ measure is maybe right in spirit, but probably wrong in detail. In any case, with some fantasy we can imagine that the β parameter in the above is the volume of the billiard boundary, and that the pressure is obtained by varying the volume, something like $pdV = dE$ (force $\cdot$ length = work).

T.0.12 Trace class operators

(A. Wirzba and P. Cvitanović)

For infinite dimensional operators the log det $=$ tr log relation (15.3) is legal if $\mathcal{L}$ is trace class. Operator (or infinite dimensional matrix) A is *trace class* iff

$$\mathrm{tr}\,|A| < \infty\,. \tag{T.69}$$

AW: Thirring, Vol.3 [?]: German ed. p. 169 sect. 3.6.15

If A, B, AB are trace class, then:

$$\begin{aligned} \det(1+A) &= \exp\mathrm{tr}\,\ln(1+A) \\ \det((1+A)(1+B)) &= \det(1+A)\det(1+B)\,. \end{aligned}$$

give counterexamples - what if not trace classs?

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from chapter 23:

Emphasis on the differential rather than the algebraic theory (Lie algebras).

In particular,

$$\int_{g\in G}[dg]\chi_m(g) = \int_{g\in G}[dg]\sum_{i=1}^{d_m} D(g)_{ii} = 0.$$

[PC: so what?]

Here we intend to cure you of the fear of 'infinite-dimensional' dynamical systems. If you are already fearless, you might prefer to skip the chapter on the first reading of the book.
