Exercises

Exercise 20.1 Thermodynamics in higher dimensions *Introduce the time averages of the eigenvalues of the Jacobian*

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log |\Lambda_i^t(x_0)|, \tag{20.23}$$

as a generalization of (10.31).

Show that in higher dimensions Pesin's formula is

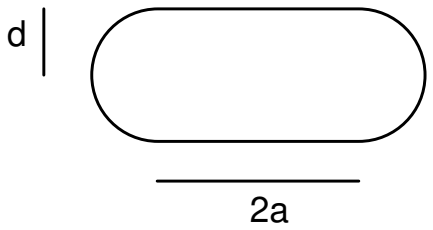
$$K_1 = \sum_i \lambda_i - \gamma, \tag{20.24}$$

where the summation goes for the positive λ_i-s only. (Hint: Use the higher dimensional generalization of (19.10)

$$\mu_i = e^{n\gamma} / |\prod_j \Lambda_{i,j}|,$$

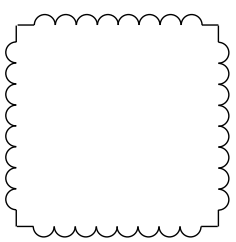
where the product goes for the expanding eigenvalues of the Jacobian of the periodic orbit.

Exercise 20.2 Bunimovich stadium Kolmogorov entropy. *Take for definitiveness $a = 1.6$ and $d = 1$ in the Bunimovich stadium of exercise 6.4,*



estimate the Lyapunov exponent by averaging over a very long trajectory. Biham and Kvale [20.14] estimate the discrete time Lyapunov to $\lambda \approx 1.0 \pm .1$, the continuous time Lyapunov to $\lambda \approx 0.43 \pm .02$, the topological entropy (for their symbolic dynamics) $h \approx 1.15 \pm .03$.

Exercise 20.3 Entropy of rugged-edge billiards. *Take a semi-circle of diameter ε and replace the sides of a unit square by $\lfloor 1/\varepsilon \rfloor$ catenated copies of the semi-circle.*



(a) *Is the billiard ergodic as $\varepsilon \to 0$?*

(b) *(hard) Show that the entropy of the billiard map is*

$$K_1 \to -\frac{2}{\pi}\ln\varepsilon + \text{const}\,,$$

as $\varepsilon \to 0$. *(Hint: do not write return maps.)*

(c) *(harder) Show that when the semi-circles of the Bunimovich stadium are far apart, say* L*, the entropy for the flow decays as*

$$K_1 \to \frac{2\ln L}{\pi L}\,.$$

Exercise 20.4 Two scale map *Compute all those quantities - dimensions, escape rate, entropies, etc. - for the repeller of the one dimensional map*

$$f(x) = \begin{cases} 1+ax & \text{if} \quad x<0, \\ 1-bx & \text{if} \quad x>0. \end{cases} \tag{20.25}$$

where a *and* b *are larger than* 2. *Compute the fractal dimension, plot the pressure and compute the* $f(\alpha)$ *spectrum of singularities. Observe how* K_1 *may be obtained directly from (??).*

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Exercise 20.5 Four scale map *Compute the Rényi entropies and* $g(\lambda)$ *for the four scale map*

$$f(x) = \begin{cases} a_1 x & \text{if} \quad 0<x<b/a_1, \\ (1-b)((x-b/a_1)/(b-b/a_1))+b & \text{if} \quad b/a_1<x<b, \\ a_2(x-b) & \text{if} \quad b<x<b+b/a_2, \\ (1-b)((x-b-b/a_2)/(1-b-b/a_2))+b & \text{if} \quad b+b/a_2<x<1. \end{cases} \tag{20.26}$$

Hint: Calculate the pressure function and use (20.13).

Exercise 20.6 Transfer matrix *Take the unimodal map* $f(x)=\sin(\pi x)$ *of the interval* $I=[0,1]$. *Calculate the four preimages of the intervals* $I_0=[0,1/2]$ *and* $I_1=[1/2,1]$. *Extrapolate* $f(x)$ *with piecewise linear functions on these intervals. Find* a_1, a_2 *and* b *of the previous exercise. Calculate the pressure function of this linear extrapolation. Work out higher level approximations by linearly extrapolating the map on the* 2^n*-th preimages of* I.