Exercises

Exercise 5.1 Complex nonlinear Schrödinger equation. *Consider the complex nonlinear Schrödinger equation in one spatial dimension [5.1]:*

$$i\frac{\partial\phi}{\partial t} + \frac{\partial^2\phi}{\partial x^2} + \beta\phi|\phi|^2 = 0, \quad \beta \neq 0.$$

(a) *Show that the function $\psi : \mathbb{R} \to \mathbf{C}$ defining the traveling wave solution $\phi(x,t) = \psi(x-ct)$ for $c > 0$ satisfies a second-order complex differential equation equivalent to a Hamiltonian system in $\mathbb{R}^4$ relative to the noncanonical symplectic form whose matrix is given by*

$$w_c = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -c \\ 0 & -1 & c & 0 \end{bmatrix}.$$

(b) *Analyze the equilibria of the resulting Hamiltonian system in $\mathbb{R}^4$ and determine their linear stability properties.*

(c) *Let $\psi(s) = e^{ics/2}a(s)$ for a real function $a(s)$ and determine a second order equation for $a(s)$. Show that the resulting equation is Hamiltonian and has heteroclinic orbits for $\beta < 0$. Find them.*

(d) *Find "soliton" solutions for the complex nonlinear Schrödinger equation.*

(Luz V. Vela-Arevalo)

Exercise 5.2 When is a linear transformation canonical?

(a) *Let A be a $n \times n$ invertible matrix. Show that the map $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by $(\mathbf{q}, \mathbf{p}) \mapsto (A\mathbf{q}, (A^{-1})^T\mathbf{p})$ is a canonical transformation.*

(b) *If $\mathbf{R}$ is a rotation in $\mathbb{R}^3$, show that the map $(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{R}\,\mathbf{q}, \mathbf{R}\,\mathbf{p})$ is a canonical transformation.*

(Luz V. Vela-Arevalo)

Exercise 5.3 Cherry's example. *What follows [5.2] is mostly a reading exercise, about a Hamiltonian system that is $linearly\ stable$ but $nonlinearly\ unstable$. Consider the Hamiltonian system on $\mathbb{R}^4$ given by*

$$H = \frac{1}{2}(q_1^2 + p_1^2) - (q_2^2 + p_2^2) + \frac{1}{2}p_2(p_1^2 - q_1^2) - q_1 q_2 p_1.$$

(a) *Show that this system has an equilibrium at the origin, which is linearly stable. (The linearized system consists of two uncoupled oscillators with frequencies in ratios 2:1).*

(b) *Convince yourself that the following is a family of solutions parametrized by a constant τ:*

$$q_1 = -\sqrt{2}\frac{\cos(t-\tau)}{t-\tau}, \quad q_2 = \frac{\cos 2(t-\tau)}{t-\tau},$$
$$p_1 = \sqrt{2}\frac{\sin(t-\tau)}{t-\tau}, \qquad p_2 = \frac{\sin 2(t-\tau)}{t-\tau}.$$

These solutions clearly blow up in finite time; however they start at $t = 0$ at a distance $\sqrt{3}/\tau$ from the origin, so by choosing τ large, we can find solutions starting arbitrarily close to the origin, yet going to infinity in a finite time, so the origin is $nonlinearly\ unstable$.

(Luz V. Vela-Arevalo)

Give the example of $(xy)^2$ elliptic island.