Exercises

three-disk@3-disk!symbolic dynamics
symbolic dynamics!3-disk
map!contracting
map!dissipative
contracting!map
dissipative!map

Exercise 11.1 **Binary symbolic dynamics.** *Verify that the shortest prime binary cycles of the unimodal repeller of figure 11.8 are* $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$. *Compare with table 11.1. Try to sketch them in the graph of the unimodal function* $f(x)$*; compare ordering of the periodic points with figure 11.9. The point is that while overlayed on each other the longer cycles look like a hopeless jumble, the cycle points are clearly and logically ordered by the alternating binary tree.*

Exercise 11.2 **3-disk fundamental domain symbolic dynamics.** *Try to sketch* $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$. *in the fundamental domain, figure 11.6, and interpret the symbols* $\{0,1\}$ *by relating them to topologically distinct types of collisions. Compare with table 11.2. Then try to sketch the location of periodic points in the Poincaré section of the billiard flow. The point of this exercise is that while in the configuration space longer cycles look like a hopeless jumble, in the Poincaré section they are clearly and logically ordered. The Poincaré section is always to be preferred to projections of a flow onto the configuration space coordinates, or any other subset of state space coordinates which does not respect the topological organization of the flow.*

Exercise 11.3 **Generating prime cycles.** *Write a program that generates all binary prime cycles up to given finite length.*

Exercise 11.4 **A contracting baker's map.** *Consider a contracting (or "dissipative") baker's defined in exercise 4.5.*

The symbolic dynamics encoding of trajectories is realized via symbols 0 $(y \leq 1/2)$ *and* 1 $(y > 1/2)$. *Consider the observable* $a(x,y) = x$. *Verify that for any periodic orbit* p $(\epsilon_1 \ldots \epsilon_{n_p})$, $\epsilon_i \in \{0,1\}$

$$A_p = \frac{3}{4} \sum_{j=1}^{n_p} \delta_{j,1} \, .$$

Exercise 11.5 **Reduction of 3-disk symbolic dynamics to binary.**

(a) *Verify that the 3-disk cycles*
$\{\overline{1\,2}, \overline{1\,3}, \overline{2\,3}\}$, $\{\overline{1\,2\,3}, \overline{1\,3\,2}\}$, $\{\overline{12\,13}$ *+ 2 perms.*$\}$,
$\{\overline{121\,232\,313}$ *+ 5 perms.*$\}$, $\{\overline{121\,323}$*+ 2 perms.*$\}$, $\cdots$,
correspond to the fundamental domain cycles $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$ *respectively.*

(b) *Check the reduction for short cycles in table 11.2 by drawing them both in the full 3-disk system and in the fundamental domain, as in figure 11.6.*

(c) *Optional: Can you see how the group elements listed in table 11.2 relate irreducible segments to the fundamental domain periodic orbits?*

kneading value!unimodal
unimodal!kneading value
well ordered symbols!unimodal
unimodal!well ordered symbols
partition!infinite
kneading!sequence
kneading!value
dike map
map!dike

Exercise 11.6 Unimodal map symbolic dynamics. *Show that the tent map point $\gamma(S^+)$ with future itinerary S^+ is given by converting the sequence of s_n's into a binary number by the algorithm (11.11). This follows by inspection from the binary tree of figure 11.9.*

Exercise 11.7 Unimodal map kneading value. *Consider the 1-d quadratic map*

$$f(x) = Ax(1-x)\,, \qquad A = 3.8\,. \tag{11.26}$$

(a) *(easy) Plot (11.26), and the first 4-8 (whatever looks better) iterates of the critical point $x_c = 1/2$.*

(b) *(hard) Draw corresponding intervals of the partition of the unit interval as levels of a Cantor set, as in the symbolic dynamics partition of figure 11.8 (b). Note, however, that some of the intervals of figure 11.8 (b) do not appear in this case - they are* pruned*.*

(c) *(medium) Produce ChaosBook.org quality figure 11.8 (a).*

(d) *(easy) Check numerically that $K = S^+(x_c)$, the itinerary or the "kneading sequence" of the critical point is*

$$K = 10110111101101111010111101111110\ldots$$

The tent map point $\gamma(S^+)$ with future itinerary S^+ is given by converting the sequence of s_n's into a binary number by the algorithm (11.11),

$$\begin{aligned} w_{n+1} &= \begin{cases} w_n & \text{if } s_{n+1}=0 \\ 1-w_n & \text{if } s_{n+1}=1 \end{cases}\,, \qquad w_1 = s_1 \\ \gamma(S^+) &= 0.w_1w_2w_3\ldots = \sum_{n=1}^{\infty} w_n/2^n\,. \end{aligned}$$

(e) *(medium) List the the corresponding kneading value (11.13) sequence $\kappa = \gamma(K)$ to the same number of digits as K.*

(f) *(hard) Plot the missing dike map, figure 11.11, in ChaosBook.org quality, with the same kneading sequence K as $f(x)$. The dike map is obtained by slicing off all $\gamma\left(S^+(x_0)\right) > \kappa$, from the complete tent map figure 11.8 (a), see (11.14).*

How this kneading sequence is converted into a series of pruning rules is a dark art, relegated to sect. 13.6 and appendix F.1.

⇓PRELIMINARY
this might be an obsolete version; if not, use Malloppo text
copy Freddy p.18, 3rd paragraph

Exercise 11.8 One-dimensional repellers. *The simplest example of symbolic dynamics is afforded by 1-d unimodal maps,* ie. *maps with one raising and one decreasing branch. The alphabet consists of two letters, describing the branch on which the iterate is located.*

the quadratic polynomial

$$f(x) = x^2 + c\,, \tag{11.27}$$

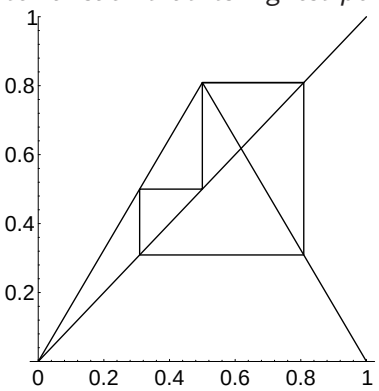
Per's p.29 *the cubic polynomial*

$$f(x) = (1+\epsilon)\mu_c x(1-x^2)\,, \tag{11.28}$$

For $\epsilon > 0$ the map repels, meaning that almost any initial x will iterate out of the unit interval and escape, see figure 11.8.

⇑PRELIMINARY

Exercise 11.9 **"Golden mean" pruned map.** *Consider a symmetrical tent map on the unit interval such that its highest point belongs to a 3-cycle:*



(a) *Find the absolute value Λ for the slope (the two different slopes $\pm\Lambda$ just differ by a sign) where the maximum at $1/2$ is part of a period three orbit, as in the figure.*

(b) *Show that no orbit of this map can visit the region $x > (1+\sqrt{5})/4$ more than once. Verify that once an orbit exceeds $x > (\sqrt{5}-1)/4$, it does not reenter the region $x < (\sqrt{5}-1)/4$.*

(c) *If an orbit is in the interval $(\sqrt{5}-1)/4 < x < 1/2$, where will it be on the next iteration?*

(d) *If the symbolic dynamics is such that for $x < 1/2$ we use the symbol 0 and for $x > 1/2$ we use the symbol 1, show that no periodic orbit will have the substring $_00_$ in it.*

(e) *On the second thought, is there a periodic orbit that violates the above $_00_$ pruning rule?*

For continuation, see exercise 13.6 and exercise 13.8. See also exercise 13.7 and exercise 13.9.

Exercise 11.10 **Binary 3-step transition matrix.** *Construct [8×8] binary 3-step transition matrix analogous to the 2-step transition matrix (11.16). Convince yourself that the number of terms of contributing to $\mathrm{tr}\,T^n$ is independent of the memory length, and that this [$2^m \times 2^m$] trace is well defined in the infinite memory limit $m \to \infty$.*

add somewhere this statement: "Critical points are special - they define the boundary between intervals, ie. interval is split into 0 [left part], x_c [critical point] and 1 [right part]. For the dike map x_c is the whole interval of points along the flat top of the map, but usually it is a point."

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Exercise 11.11 Heavy pruning. *Implement the prunning grammar (12.11), with the pruned blocks*

$$10010, 101, 01001, 01101, 111, 10110,$$

by a method of your own devising, or following the the last example of sect. 11.5 illustrated in figure 13.3. For continuation, see exercise 13.10.

⇑PRELIMINARY