Exercises

Exercise 19.1 **Escape rate of the logistic map.**

(a) *Calculate the fraction of trajectories remaining trapped in the interval* $[0,1]$ *for the logistic map*

$$f(x) = a(1 - 4(x - 0.5)^2), \tag{19.22}$$

and determine the a *dependence of the escape rate* $\gamma(a)$ *numerically.*

(b) *Work out a numerical method for calculating the lengths of intervals of trajectories remaining stuck for* n *iterations of the map.*

(c) *What is your expectation about the* a *dependence near the critical value* $a_c = 1$*?*

Exercise 19.2 **Four scale map decay.** *Compute the second largest eigenvalue of the Perron-Frobenius operator for the four scale map*

$$f(x) = \begin{cases} a_1 x & \text{if } 0 < x < b/a_1, \\ (1-b)((x - b/a_1)/(b - b/a_1)) + b & \text{if } b/a_1 < x < b, \\ a_2(x-b) & \text{if } b < x < b + b/a_2, \\ (1-b)((x - b - b/a_2)/(1 - b - b/a_2)) + b & \text{if } b + b/a_2 < x < 1. \end{cases} \tag{19.23}$$

Exercise 19.3 **Lyapunov exponents for 1-dimensional maps.** *Extend your cycle expansion programs so that the first and the second moments of observables can be computed. Use it to compute the Lyapunov exponent for some or all of the following maps:*

(a) the piecewise-linear flow conserving map, the skew tent map

$$f(x) = \begin{cases} ax & \text{if } 0 \le x \le a^{-1}, \\ \frac{a}{a-1}(1-x) & \text{if } a^{-1} \le x \le 1. \end{cases}$$

(b) the Ulam map $f(x) = 4x(1-x)$

(c) the skew Ulam map

$$f(x) = 0.1218x(1-x)(1-0.6x)$$

with a peak at 0.7.

(d) the repeller of $f(x) = Ax(1-x)$, *for either* $A = 9/2$ *or* $A = 6$ *(this is a continuation of exercise 18.2).*

(e) *for the 2-branch flow conserving map*

$$f_0(x) = \frac{h-p+\sqrt{(h-p)^2+4hx}}{2h}, \qquad x \in [0,p] \tag{19.24}$$
$$f_1(x) = \frac{h+p-1+\sqrt{(h+p-1)^2+4h(x-p)}}{2h}, \qquad x \in [p,1]$$

This is a nonlinear perturbation of ($h = 0$) Bernoulli map (16.6); the first 15 eigenvalues of the Perron-Frobenius operator are listed in ref. [19.1] for $p = 0.8$, $h = 0.1$. Use these parameter values when computing the Lyapunov exponent.

Cases (a) and (b) can be computed analytically; cases (c), (d) and (e) require numerical computation of cycle stabilities. Just to see whether the theory is worth the trouble, also cross check your cycle expansions results for cases (c) and (d) with Lyapunov exponent computed by direct numerical averaging along trajectories of randomly chosen initial points:

(f) *trajectory-trajectory separation (10.27) (hint: rescale δx every so often, to avoid numerical overflows),*

(g) *iterated stability (10.31).*

How good is the numerical accuracy compared with the periodic orbit theory predictions?

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Exercise 19.4 **Sequence of orbital measures for the Ulam map.** *(L. Rondoni)*

As an example, let us develop the sequence of orbital measures for the quadratic map, even if it is not precisely of the form we discussed above. Indeed, this illustrates how some of our assumptions could be relaxed, and still get correct results. Moreover, although for such a simple case it is possible to explicitly find all periodic points up to the desired period, we will only use those that are found by looking at close returns of a generic chaotic trajectory. Consider the Ulam map

$$x_{n+1} = 4x_n(1-x_n) \; . \tag{19.25}$$

This is quadratic, and it has two fixed points, four period two periodic points (including the two repeats of the fixed points) and, in general, 2^n periodic points of period n, as explained in exercise 11.6. How these are structured is given in table 13.1.

make exercise, draw the figure

n	$\langle x \rangle$	$\langle x^2 \rangle$	$\sum N_k$
2	0.62500	0.46875	2
3	0.54167	0.40625	6
4	0.50000	0.37500	12
5	0.50833	0.38125	30
6	0.50354	0.37766	52(54)
7	0.51240	0.38430	120(126)
8	0.51017	0.38263	230(240)
9	0.50914	0.38185	482(504)
10	0.50904	0.38178	940(990)
11	0.50795	0.38097	1962(2046)
12	0.50731	0.38049	3853(4020)
exact	0.5	0.375	

Table 19.1: An approximate cycle expansion using only periodic points of prime orbits, and only those found by a probabilistic exploration of the state space. (In the parenthesis the correct total number of such periodic points, see table 13.2).

The natural measure of the Ulam map (19.25), figure B.2b is explicitly expressed by

need to derive somewhere, refer Ulam

$$d\mu(x) = \frac{dx}{\pi\sqrt{x - x^2}} \,. \tag{19.26}$$

this explicitely shows that periodic points do not have equal weights? But smooth conjugacy to tent map justifies it.

We can now calculate the average values of $\langle x \rangle$ and of $\langle x^2 \rangle$ exactly using the known analytic distribution function (19.26), and also numerically using the cycle expansion. This exercise illustrates well the use of orbital measures to calculate averages. The results are given in table 19.1, where we have used the periodic points of order 2 to 12 which are found by a generic chaotic trajectory, and we have neglected the cycles of a given period which are repeats of cycles of smaller period. This because the contributions of non prime (repeat) cycles become less and less important as the period grows, as one can see also from table 13.1. Notice also, that the number of cycles found numerically differs from the exact number (given in parenthesis for period 6 and larger). All this mimics what actually can be done, when sampling periodic points from the state space of more realistic systems. Indeed, not being able to find all the existing cycles is a common numerical difficulty, linked to their instability. That can easily be remedied for the quadratic map, but in general one does not even know how many cycles of a certain period should be found. Notice the convergence of the method, despite these difficulties.

terrible; compare with the AACII calculation

yes, I note no convergence whatsoever

We have introduced the idea of periodic orbit expansions for simple maps, as an example of a theory of much wider generality. The reason why the method is expected to work in more general situations, is that the relevant attractors can be proven, or have been observed to be hierarchically approximated through sets of progressively longer and longer unstable periodic orbits. Indeed, the weak limit of these measures has been rigorously proved to be the natural measure for Axiom-A systems, independently of the (finite) dimensions of the relevant state spaces.

plot $\log_{10}$ error from table 19.1, numbers are useless. Also plot the error of the correct cycle computation

Remark 19.4 Nonhyperbolicity. In the above example we have succeeded in developing a sequence of orbital measures for the Ulam map, even though the Ulam map does not satisfy the hyperbolicity hypothesis (the critical point is superstable) invoked in the derivation. This illustrates how some of our assumptions could be relaxed, and still get sensible results. To solve this example *exactly* the cyclist way, consult refsect.?!

include the text from ref. [18.3]

(L. Rondoni)

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