Exercises

Ulam map
alternating binary tree
binary!tree, alternating
later recheck for other appearances of the same Ulam equation

Exercise 17.1 Cycles of the Ulam map. *Test your cycle-searching routines by computing a bunch of short cycles and their stabilities for the Ulam map*

$$f(x) = 4x(1-x)\,. \tag{17.21}$$

Exercise 17.2 Cycles stabilities for the Ulam map, exact. *In exercise 17.1 you should have observed that the numerical results for the cycle stability eigenvalues (4.39) are exceptionally simple: the stability eigenvalue of the $x_0 = 0$ fixed point is 4, while the eigenvalue of any other n-cycle is $\pm 2^n$. Prove this. (Hint: the Ulam map can be conjugated to the tent map (11.8). This problem is perhaps too hard, but give it a try - the answer is in many introductory books on nolinear dynamics.)*

⇓PRELIMINARY

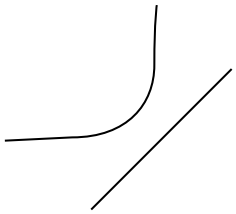
Exercise 17.3 Unimodal map cycles. *Knowledge of the topological coordinate (??) is very useful when searching for periodic orbits. Assume that we have already determined all periodic points $x_a, x_b, \cdots$ of period n, and would like to have a good initial guess for the period $(n+1)$ periodic point x_d with prescribed itinerary $S_d^+ := S^+(x_d)$. It is easy to determine the two closest $\gamma(S_a^+)$, $\gamma(S_b^+)$ that bracket $\gamma(S_d^+)$. If $\gamma(S_a^+) < \gamma(S_d^+) < \gamma(S_b^+)$, then we know that we can restrict the search for x_c into the $x_c \in [x_a, x_b]$ interval. For example, relative ordering of all unimodal map periodic points up to $n = 5$ is given in figure 17.6. Appendix F.1.1 contains further details of the symbolics dynamics for periodic point of unimodal maps.*

⇑PRELIMINARY

Exercise 17.4 Stability of billiard cycles. *Compute stabilities of few simple cycles.*

(a) *A simple scattering billiard is the two-disk billiard. It consists of a disk of radius one centered at the origin and another disk of unit radius located at $L+2$. Find all periodic orbits for this system and compute their stabilities. (You might have done this already in exercise 1.2; at least now you will be able to see where you went wrong when you knew nothing about cycles and their extraction.)*

(b) *Find all periodic orbits and stabilities for a billiard ball bouncing between the diagonal $y = x$ and one of the hyperbola branches $y = 1/x$.*



Exercise 17.5 Cycle stability. *Add to the pinball simulator of exercise 6.1 a routine that evaluates the expanding eigenvalue (4.19) for a given cycle.*

⇓PRELIMINARY
⇑PRELIMINARY

Exercise 17.6 Pinball cycles. *Determine the stability and length of all fundamental domain prime cycles of the binary symbol string lengths up to 5 (*or longer*) for $R : a = 6$ 3-disk pinball.*

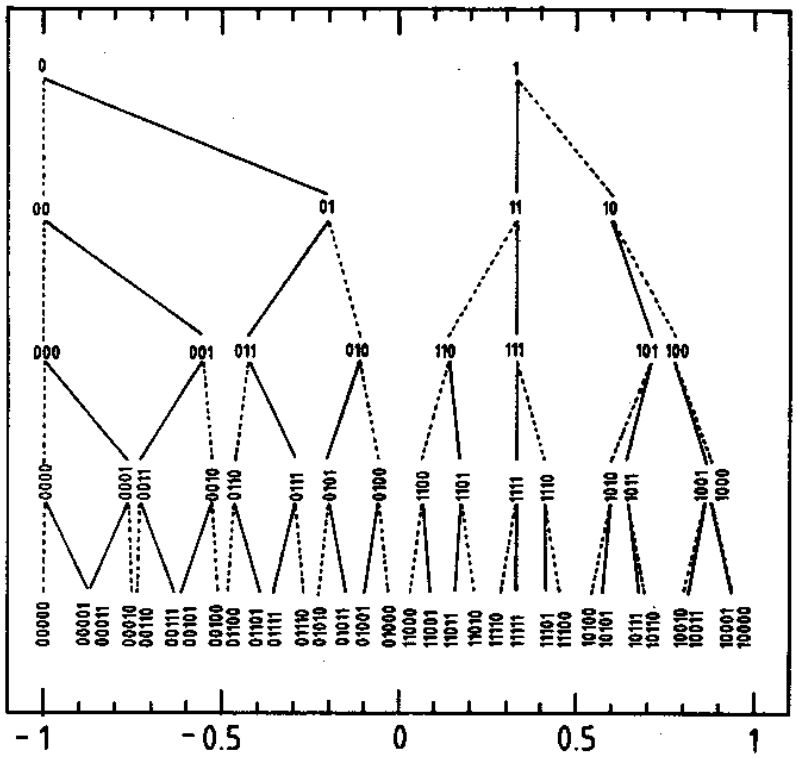


Figure 17.6: The alternating binary tree organization of the periodic points of a unimodal map (in this example, the Ulam map (17.21)). The itinerary of a point is read off the tree by starting at the root and following the branches down to x; relative ordering of points along the x axis is given by the relative ordering of the corresponding nodes. (specify the MAP)

Rossler@Rössler system!cycles
cycles!Rössler system
helium!collinear!stability
helium!collinear!cycles
inverse iteration!Hamiltonian repeller
iteration!inverse!Hamiltonian repeller
periodic!orbit extraction!Hamiltonian repeller
Hamiltonian!repeller, periodic orbits

n_p	p	y_p	z_p	Λ_e
1	1	6.091768319056803	1.299731937639821	-2.4039535318268
2	01	3.915804049621049	3.692833386542665	-3.5120069815161
3	001	2.278281031720258	7.416480984019008	-2.3419235232340
	011	2.932877559129124	5.670805943881501	5.3449081538885
4	0111	3.466758713211455	4.506217531477667	-16.6967406980700
5	01011	4.162798782914948	3.303903338609633	-23.1995830097831
	01111	3.278914359770783	4.890452922955567	36.8863297988981
6	001011	2.122093931936202	7.886172854283211	-6.8576654190825
	010111	4.059210605826523	3.462265228606606	61.6490940089068
	011111	3.361494458061049	4.718206217035575	-92.0825560711089
7	0101011	3.842769382372052	3.815493592299824	77.7611048852412
	0110111	3.025956697151134	5.451444475664179	-95.1838846735358
	0101111	4.102255295518855	3.395643547170646	-142.2379888163439
	0111111	3.327986189581191	4.787462810306583	218.0283602810993

Table 17.1: The Rössler system (2.14): The itinerary p, a periodic point $x_p = (0, y_p, z_p)$ and the expanding eigenvalue Λ_p for all cycles up to the topological length 7. (Joachim Mathiesen)

Exercise 17.7 **Newton-Raphson method.** *Implement the Newton-Raphson method in 2-d and apply it to determination of pinball cycles.*

Exercise 17.8 **Rössler system cycles.** *(continuation of exercise 4.4)*

Determine all cycles up to 5 Poincaré sections returns for the Rössler system (2.14), as well as their stabilities. (Hint: implement (17.16), the multipoint shooting methods for flows; you can cross-check your shortest cycles against the ones listed in table 17.1.)

add hyperlinks to Mathiasen, Simon projects

⇓PRELIMINARY
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Exercise 17.9 **Cycle stability, helium.** *Add to the helium integrator of exercise 2.11 a routine that evaluates the expanding eigenvalue (4.19) for a given cycle.*

Exercise 17.10 **Colinear helium cycles.** *Determine the stability and length of all fundamental domain prime cycles up to symbol sequence length 5 or longer for collinear helium of figure 41.5.*

Exercise 17.11 **Uniqueness of unstable cycles***.** *Prove that there exists only one 3-disk prime cycle for a given finite admissible prime cycle symbol string. Hints: look at the Poincaré section mappings; can you show that there is exponential contraction to a unique periodic point with a given itinerary? Exercise 37.1 might be helpful in this effort.*

Exercise 17.12 **Inverse iteration method for a Hamiltonian repeller.** *Consider the Hénon map (3.15) for area-preserving ("Hamiltonian") parameter value $b = -1$. The coordinates of a periodic orbit of length n_p satisfy the equation*

$$x_{p,i+1} + x_{p,i-1} = 1 - a x_{p,i}^2 \,, \quad i = 1, ..., n_p \,, \tag{17.22}$$

with the periodic boundary condition $x_{p,0} = x_{p,n_p}$. Verify that the itineraries and the stabilities of the short periodic orbits for the Hénon repeller (17.22) at $a = 6$ are as listed in table 17.2.

center of mass

p	Λ_p	$\sum x_{p,i}$
0	$0.71516752438\times10^{1}$	-0.6076252185107
1	$-0.29528463259\times10^{1}$	0.2742918851774
10	$-0.98989794855\times10^{1}$	0.3333333333333
100	$-0.13190727397\times10^{3}$	-0.2060113295833
110	$0.55896964996\times10^{2}$	0.5393446629166
1000	$-0.10443010730\times10^{4}$	-0.8164965809277
1100	$0.57799826989\times10^{4}$	0.0000000000000
1110	$-0.10368832509\times10^{3}$	0.8164965809277
10000	$-0.76065343718\times10^{4}$	-1.4260322065792
11000	$0.44455240007\times10^{4}$	-0.6066540777738
10100	$0.77020248597\times10^{3}$	0.1513755016405
11100	$-0.71068835616\times10^{3}$	0.2484632276044
11010	$-0.58949885284\times10^{3}$	0.8706954728949
11110	$0.39099424812\times10^{3}$	1.0954854155465
100000	$-0.54574527060\times10^{5}$	-2.0341342556665
110000	$0.32222060985\times10^{5}$	-1.2152504370215
101000	$0.51376165109\times10^{4}$	-0.4506624359329
111000	$-0.47846146631\times10^{4}$	-0.3660254037844
110100	$-0.63939998436\times10^{4}$	0.3333333333333
101100	$-0.63939998436\times10^{4}$	0.3333333333333
111100	$0.39019387269\times10^{4}$	0.5485837703548
111010	$0.10949094597\times10^{4}$	1.1514633582661
111110	$-0.10433841694\times10^{4}$	1.3660254037844

Table 17.2: All periodic orbits up to 6 bounces for the Hamiltonian Hénon mapping (17.22) with $a = 6$. Listed are the cycle itinerary, its expanding eigenvalue Λ_p, and its "center of mass". The "center of mass" is listed because it turns out the "center of mass" is often a simple rational or a quadratic irrational.

Hint: *you can use any cycle-searching routine you wish, but for the complete repeller case (all binary sequences are realized), the cycles can be evaluated simply by inverse iteration, using the inverse of (17.22)*

$$x''_{p,i} = S_{p,i}\sqrt{\frac{1-x'_{p,i+1}-x'_{p,i-1}}{a}},\quad i = 1, ..., n_p\,.$$

Here $S_{p,i}$ are the signs of the corresponding cycle point coordinates, $S_{p,i} = x_{p,i}/|x_{p,i}|$.

(G. Vattay)

Exercise 17.13 "Center of mass" puzzle****.** *Why is the "center of mass", listed in table 17.2, a simple rational every so often?*