⇓PRIVATE

DASBUCH

ohne Eigenschaften

Aufchnit I: Deterministiche Chaos

Gone with the Wind School
formerly known as the
Copenhagen School of Cycling

Unsung Heroes, too numerous to list
with a disclaimer by **Professor Gatto Nero**

`BOYSCOUT EDITION`

⇑PRIVATE

ChaosBook.org/version11.9.1, Apr 24 2007 printed June 17, 2007
`ChaosBook.org/boyscouts` comments to: predrag@nbi.dk

⇓PRIVATE
⇑PRIVATE

Contents

Part I: Classical chaos

Contributors . . . xiv
Acknowledgements . . . xvii

1 Overture **1**
1.1 Why ChaosBook? . . . 2
1.2 Chaos ahead . . . 3
1.3 The future as in a mirror . . . 4
1.4 A game of pinball . . . 9
1.5 Chaos for cyclists . . . 14
1.6 Evolution . . . 20
1.7 From chaos to statistical mechanics . . . 23
1.8 A guide to the literature . . . 24
guide to exercises 27 - resumé 28 - references 29 - exercises 32

2 Go with the flow **33**
2.1 Dynamical systems . . . 33
2.2 Flows . . . 37
2.3 Computing trajectories . . . 41
resumé 42 - references 42 - exercises 45

3 Discrete time dynamics **49**
3.1 Poincaré sections . . . 49
3.2 Stroboscopic Poincaré sections . . . 52
3.3 Constructing a Poincaré section . . . 52
3.4 Maps . . . 54
resumé 57 - references 58 - exercises 59

4 Local stability **61**
4.1 Flows transport neighborhoods . . . 61
4.2 Linear flows . . . 64
4.3 Stability of flows . . . 69
4.4 Neighborhood volume . . . 71
4.5 Stability of maps . . . 73
resumé 76 - references 77 - exercises 78

5 Newtonian dynamics **81**
5.1 Hamiltonian flows 81
5.2 Stability of Hamiltonian flows 83
5.3 Symplectic maps 85
5.4 Reversibility 88
resumé 89 - references 89 - exercises 90

6 Billiards **93**
6.1 Billiard dynamics 93
6.2 Stability of billiards 96
6.3 General billiard flows 99
resumé 103 - references 103 - exercises 105

7 Get straight **107**
7.1 Changing coordinates 107
7.2 Rectification of flows 109
7.3 Classical dynamics of collinear helium 110
7.4 Rectification of maps 115
resumé 116 - references 117 - exercises 118

8 Cycle stability **119**
8.1 Stability of periodic orbits 119
8.2 Cycle Floquet multipliers are cycle invariants 123
8.3 Stability of Poincaré map cycles 124
8.4 Rectification of a 1-dimensional periodic orbit 125
8.5 Smooth conjugacies and cycle Floquet multipliers 126
8.6 There goes the neighborhood 127
resumé 128 - references 128 - exercises 129

9 Transporting densities **131**
9.1 Measures 131
9.2 Perron-Frobenius operator 133
9.3 Invariant measures 135
9.4 Density evolution for infinitesimal times 139
9.5 Liouville operator 141
resumé 143 - references 144 - exercises 146

10 Averaging **151**
10.1 Dynamical averaging 151
10.2 Evolution operators 159
10.3 Lyapunov exponents 160
10.4 Why not just run it on a computer? 164
resumé 167 - references 167 - exercises 169

11 Qualitative dynamics, for pedestrians **171**
11.1 Qualitative dynamics 171
11.2 A brief detour; recoding, symmetries, tilings 176
11.3 Stretch and fold 178
11.4 Kneading theory 183
11.5 Markov graphs 185
11.6 Symbolic dynamics, basic notions 188

resumé 192 - references 192 - exercises 195

12 Qualitative dynamics, for cyclists 199
12.1 Going global: Stable/unstable manifolds 200
12.2 Horseshoes 201
12.3 Spatial ordering 204
12.4 Pruning 206
12.5 Pruning dead itineraries 210
12.6 Pruning front 211
12.7 Hénon map pruning front 213
12.8 3-disk pruning front 213
resumé 215 - references 215 - exercises 220

13 Counting 225
13.1 Counting itineraries 225
13.2 Topological trace formula 228
13.3 Determinant of a graph 230
13.4 Topological zeta function 233
13.5 Counting cycles 236
13.6 Infinite partitions 240
13.7 Shadowing 241
resumé 244 - references 245 - exercises 246

14 Trace formulas 253
14.1 Trace of an evolution operator 253
14.2 A trace formula for maps 255
14.3 A trace formula for flows 258
14.4 An asymptotic trace formula 261
resumé 263 - references 264 - exercises 265

15 Spectral determinants 267
15.1 Spectral determinants for maps 267
15.2 Spectral determinant for flows 269
15.3 Dynamical zeta functions 271
15.4 False zeros 275
15.5 Spectral determinants *vs.* dynamical zeta functions 275
15.6 All too many eigenvalues? 278
15.7 Equilibrium points 280
resumé 282 - references 283 - exercises 285

16 Why does it work? 289
16.1 Linear maps: exact spectra 290
16.2 Evolution operator in a matrix representation 294
16.3 Eigenfunctions, an attractive fixed point 297
16.4 Classical Fredholm theory 298
16.5 Analyticity of spectral determinants 300
16.6 Hyperbolic maps 305
16.7 The physics of eigenvalues and eigenfunctions 307
16.8 Troubles ahead 310
resumé 313 - references 314 - exercises 316

17 Fixed points, and how to get them **317**
17.1 Where are the cycles? . . . 318
17.2 One-dimensional mappings . . . 320
17.3 Multipoint shooting method . . . 321
17.4 d-dimensional mappings . . . 324
17.5 Flows . . . 324
17.6 Sundry methods for cycle extraction . . . 328
resumé 330 - references 330 - exercises 332

18 Cycle expansions **337**
18.1 Pseudocycles and shadowing . . . 337
18.2 Construction of cycle expansions . . . 340
18.3 Cycle formulas for dynamical averages . . . 344
18.4 Cycle expansions for finite alphabets . . . 348
18.5 Perturbation expansions in terms of cycles . . . 349
18.6 Stability ordering of cycle expansions . . . 350
18.7 Equilibrium points . . . 353
18.8 Dirichlet series . . . 354
resumé 356 - references 357 - exercises 359

19 Why cycle? **363**
19.1 Escape rates . . . 363
19.2 Natural measure in terms of periodic orbits . . . 366
19.3 Flow conservation sum rules . . . 367
19.4 Correlation functions . . . 368
19.5 The most chaotic of billiards? . . . 370
19.6 Trace formulas *vs.* level sums . . . 371
resumé 374 - references 375 - exercises 376

20 Thermodynamic formalism **381**
20.1 Rényi entropies . . . 381
20.2 Fractal dimensions . . . 386
resumé 389 - references 389 - exercises 391

21 Intermittency **393**
21.1 Intermittency everywhere . . . 394
21.2 Intermittency for pedestrians . . . 397
21.3 Intermittency for cyclists . . . 409
21.4 BER zeta functions . . . 416
resumé 419 - references 420 - exercises 422

22 Discrete symmetries **427**
22.1 Preview . . . 428
22.2 Discrete symmetries . . . 432
22.3 Dynamics in the fundamental domain . . . 434
22.4 Factorizations of dynamical zeta functions . . . 438
22.5 C_2 factorization . . . 440
22.6 C_{3v} factorization: 3-disk game of pinball . . . 442
resumé 446 - references 447 - exercises 449

23 Continuous symmetries **451**
23.1 Compact groups . . . 452
23.2 Continuous symmetries of dynamics . . . 459
23.3 Symmetry reduced trace formula for flows . . . 461
resumé 466 - references 467 - exercises 476

24 Deterministic diffusion **479**
24.1 Diffusion in periodic arrays . . . 480
24.2 Diffusion induced by chains of 1-d maps . . . 484
24.3 Marginal stability and anomalous diffusion . . . 491
24.4 Lorentz gas . . . 495
resumé 501 - references 501 - exercises 503

25 Turbulence? **505**
25.1 Fluttering flame front . . . 506
25.2 Infinite-dimensional flows: Numerics . . . 509
25.3 Visualization . . . 511
25.4 Equilibria of equilibria . . . 512
25.5 Why does a flame front flutter? . . . 514
25.6 Periodic orbits . . . 517
25.7 Intrinsic parametrization . . . 517
25.8 Energy budget . . . 519
resumé 523 - references 524 - exercises 526

26 Universality in transitions to chaos **529**
26.1 Onset of turbulence . . . 530
26.2 Onset of chaos in a numerical experiment . . . 533
26.3 What does all this have to do with fishing? . . . 537
26.4 A universal equation . . . 539
26.5 The unstable manifold . . . 545
26.6 Cookie-cutter's universality . . . 548
resumé 550 - references 550 - exercises 551

27 Complex universality **553**
27.1 Holomorphic dynamics . . . 553
27.2 Mandelbrot cactus . . . 556
27.3 Renormalization and universality . . . 560
resumé 565 - references 565 - exercises 568

28 Irrationally winding **571**
28.1 Mode locking . . . 572
28.2 Local theory: "Golden mean" renormalization . . . 578
28.3 Global theory: Thermodynamic averaging . . . 580
28.4 Hausdorff dimension of irrational windings . . . 582
28.5 Thermodynamics of Farey tree: Farey model . . . 584
resumé 589 - references 589 - exercises 592

29 Periodic orbit sum rules **595**

30 Limbo **597**
30.1 Phase transitions 597
30.2 Scaling, presentation functions 597
resumé 597 -

Part II: Quantum chaos

31 Prologue **601**
31.1 Quantum pinball 602
31.2 Quantization of helium 604
guide to literature 606 - references 606 -

32 Quantum mechanics, briefly **607**
exercises 612

33 WKB quantization **613**
33.1 WKB ansatz 613
33.2 Method of stationary phase 616
33.3 WKB quantization 617
33.4 Beyond the quadratic saddle point 619
33.5 EBK quantization 621
resumé 622 - references 623 - exercises 624

34 Semiclassical evolution **625**
34.1 Hamilton-Jacobi theory 625
34.2 Semiclassical propagator 633
34.3 Semiclassical Green's function 638
resumé 646 - references 646 - exercises 648

35 Noise **651**
35.1 Deterministic transport 652
35.2 Brownian difussion 653
35.3 Weak noise 654
35.4 Weak noise approximation 656
35.5 Straying from the True Path: the cost 657
resumé 660 - references 660 -

36 Semiclassical quantization **663**
36.1 Trace formula 663
36.2 Semiclassical spectral determinant 669
36.3 One-dof systems 670
36.4 Two-dof systems 672
36.5 Semiclassical zeta function 673
36.6 Noisy trace formula 673
resumé 675 - references 676 - exercises 679

37 Relaxation for cyclists **681**
37.1 Fictitious time relaxation 682
37.2 Discrete iteration relaxation method 688
37.3 Least action method 690
37.4 Newton Descent in Loop Space 691
resumé 705 - references 705 - exercises 707

38 Quantum scattering **709**
38.1 Density of states 709
38.2 Quantum mechanical scattering matrix 713
38.3 Krein-Friedel-Lloyd formula 714
38.4 Wigner time delay 718
resumé 720 - references 720 - exercises 723

39 Chaotic multiscattering **725**
39.1 Quantum mechanical scattering matrix 726
39.2 N-scatterer spectral determinant 729
39.3 Semiclassical 1-disk scattering 733
39.4 From quantum cycle to semiclassical cycle 740
39.5 Heisenberg uncertainty 743
resumé 746 -

40 Quantum pinball **749**
40.1 Spectral determinant for two-dimensional systems 751
40.2 Semiclassical zeta function 751
40.3 Three-disk pinball 751
40.4 Classical pinball 754
40.5 Quantum pinball 755
resumé 766 - references 767 -

41 Helium atom **773**
41.1 Classical dynamics of collinear helium 774
41.2 Chaos, symbolic dynamics and periodic orbits 775
41.3 Local coordinates, fundamental matrix 780
41.4 Getting ready 782
41.5 Semiclassical quantization of collinear helium 783
resumé 792 - references 793 - exercises 794

42 Periodic orbit theory of diffraction **797**
42.1 Semiclassical creeping contributions 797
references 799 -

43 Diffraction distraction **801**
43.1 Quantum eavesdropping 801
43.2 An application 807
resumé 814 - references 814 - exercises 816

44 Uniform approximations **819**
44.1 Bifurcations and catastrophes 819
44.2 Periodic orbits near bifurcations 822
44.3 Closed orbits 824
resumé 828 - references 828 - exercises 830

Epilogue **833**

Index **840**

Part III: Appendices on `ChaosBook.org`

A A brief history of chaos **861**
A.1 Chaos is born . . . 861
A.2 Chaos grows up . . . 865
A.3 Chaos with us . . . 866
A.4 Periodic orbit theory . . . 869
A.5 Dynamicist's vision of turbulence . . . 870
A.6 Death of the Old Quantum Theory . . . 873
A.7 Brief history of $\pi/2$ . . . 875
references 879 -

B Maps and billiards **883**
B.1 One-dimensional maps . . . 883
B.2 Linear fractional maps . . . 886
B.3 Billiards with gravity: 2 balls on a line . . . 887
B.4 Wedge billiard . . . 889
B.5 Transfer operators . . . 890

C Fractals and their dimensions **895**
C.1 Fractals . . . 895
C.2 Topological dimension . . . 896
C.3 Fractal dimensions . . . 896
resumé 899 - references 899 -

D Stability of Hamiltonian flows **903**
D.1 Symplectic invariance . . . 903
D.2 Monodromy matrix for Hamiltonian flows . . . 904

E Implementing evolution **907**
E.1 Koopmania . . . 907
E.2 Implementing evolution . . . 909
references 912 - exercises 914

F Symbolic dynamics techniques **915**
F.1 Topological zeta functions for infinite subshifts . . . 915
F.2 Pruned Bernoulli shift . . . 924
F.3 Prime factorization for dynamical itineraries . . . 930
F.4 Möbius inversion formula, ordered sets . . . 934

G Counting itineraries **937**
G.1 Counting curvatures . . . 937
exercises 939

H Finding cycles **941**
H.1 Newton-Raphson method . . . 941
H.2 Hybrid Newton-Raphson / relaxation method . . . 942

I Applications **945**
I.1 Evolution operator for Lyapunov exponents 945
I.2 Advection of vector fields by chaotic flows 950
resumé 954 - references 954 - exercises 956

J Discrete symmetries **957**
J.1 Preliminaries and definitions 957
J.2 Invariant matrices and reducibility 960
J.3 Lattice derivatives 964
J.4 Periodic lattices 966
J.5 Symmetries and evolution operators 972
J.6 C_{4v} factorization 975
J.7 C_{2v} factorization 979
J.8 Hénon map symmetries 981
J.9 Symmetries of the symbol square 982

K Convergence of spectral determinants **983**
K.1 Curvature expansions: geometric picture 983
K.2 On importance of pruning 986
K.3 Ma-the-matical caveats 987
K.4 Estimate of the nth cumulant 988

L Statistical mechanics applications **991**
L.1 Diffusion in sawtooth and cat maps 991
references 997 - exercises 998

M Infinite dimensional operators **1005**
M.1 Matrix-valued functions 1005
M.2 Operator norms 1007
M.3 Trace class and Hilbert-Schmidt class 1008
M.4 Determinants of trace class operators 1010
M.5 Von Koch matrices 1014
M.6 Regularization 1015
references 1017 -

N Statistical mechanics recycled **1019**
N.1 The thermodynamic limit 1019
N.2 Ising models 1021
N.3 Fisher droplet model 1025
N.4 Scaling functions 1030
N.5 Geometrization 1034
resumé 1041 - references 1042 - exercises 1044

O Noise/quantum corrections **1047**
O.1 Periodic orbits as integrable systems 1047
O.2 The Birkhoff normal form 1051
O.3 Bohr-Sommerfeld quantization of periodic orbits 1052
O.4 Quantum calculation of $\hbar$ corrections 1054
references 1061 -

P Quasiclassics, for a general potential **1065**

Q What reviewers say **1069**
Q.1 Niels Bohr . . . 1069
Q.2 Richard P. Feynman . . . 1069
Q.3 Hans Henrik Rugh . . . 1069
Q.4 Divakar Viswanath . . . 1069
Q.5 Benny Lautrup . . . 1069
Q.6 Professor Gatto Nero . . . 1070
Q.7 His Holiness the Pope Gregorius III . . . 1071
Q.8 Coauthors' combined comments . . . 1072

R Solutions **1109**

S Projects **1173**
S.1 Deterministic diffusion, zig-zag map . . . 1175
S.2 Deterministic diffusion, sawtooth map . . . 1182
S.3 Cardioid Billiard . . . 1186
S.4 Ray splitting billiard . . . 1187
S.5 Hopf's last hope . . . 1188
S.6 Hydrogen in external magnetic field - dissociation rate . . . 1188
S.7 Product of random matrices . . . 1188

Bibliography **1193**

T Flotsam **1205**

Copenhagen School
find a way to heroes!unsung label{s:contributors} unsung heroes
⇓PRIVATE

Viele Köche verderben den Brei

⇑PRIVATE

No man but a blockhead ever wrote except for money
Samuel Johnson

This book is a result of collaborative labors of many people over a span of several decades. Coauthors of a chapter or a section are indicated in the byline to the chapter/section title. If you are referring to a specific coauthored section rather than the entire book, cite it as (for example):

> C. Chandre, F.K. Diakonos and P. Schmelcher, section "Discrete cyclist relaxation method", in P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner and G. Vattay, *Chaos: Classical and Quantum* (Niels Bohr Institute, Copenhagen 2005); `ChaosBook.org/version11`.

Chapters without a byline are written by Predrag Cvitanović. Friends whose contributions and ideas were invaluable to us but have not contributed written text to this book, are listed in the acknowledgements.

⇓PRELIMINARY

Professor Gatto Nero

all of the credit

⇑PRELIMINARY

Roberto Artuso

9 Transporting densities 131
14.3 A trace formula for flows 258
19.4 Correlation functions 368
21 Intermittency 393
24 Deterministic diffusion 479
28 Irrationally winding 571
⇓PRELIMINARY
Appendix E: Implementing evolution 907
Appendix L: Diffusion in sawtooth and cat maps 991
⇑PRELIMINARY

Ronnie Mainieri

2 Flows 33
3.3 The Poincaré section of a flow 52
4 Local stability 61
7.1 Understanding flows 109
11.1 Temporal ordering: itineraries 171
⇓PRELIMINARY
?? Thermodynamic formalism for fractals . . . . **??**
?? Generalized everything . . . . **??**
⇑PRELIMINARY
Appendix A: A brief history of chaos 861
Appendix N: Statistical mechanics recycled 1019

Gábor Vattay

20 Thermodynamic formalism 381
34 Semiclassical evolution 625
36 Semiclassical trace formula 663
Appendix O: Noise/quantum corrections 1047

Gregor Tanner

21 Intermittency 393
34 Semiclassical evolution 625
36 Semiclassical trace formula 663
41 The helium atom 773
Appendix D.2: Jacobians of Hamiltonian flows 904
Appendix K.3 Ma-the-matical caveats 987

Ofer Biham

37.1 Cyclists relaxation method 682

Cristel Chandre

37.1 Cyclists relaxation method 682
37.2 Discrete cyclists relaxation methods 688
H.2 Contraction rates 942

Freddy Christiansen

17 Fixed points, and what to do about them 317

Per Dahlqvist

37.3 Orbit length extremization method for billiards 690
21 Intermittency 393
Appendix F.1.1: Periodic points of unimodal maps 921

Carl P. Dettmann

18.6 Stability ordering of cycle expansions 350

Fotis K. Diakonos

37.2 Discrete cyclists relaxation methods 688

G. Bard Ermentrout

Exercise 8.4

Mitchell J. Feigenbaum

Appendix D.1: Symplectic invariance 903

Kai T. Hansen

11.3.1 Unimodal map symbolic dynamics 179
13.6 Topological zeta function for an infinite partition 240
11.4 Kneading theory 183
figures throughout the text

list the figures

Rainer Klages

Figure 24.5

Yueheng Lan

Solutions 1.1, 2.1, 2.2, 2.3, 2.4, 2.5, 10.1, 9.1, 9.2, 9.3, 9.5, 9.7, 9.10, 11.5, 11.2, 11.9, 13.1, 13.2, 13.4, 13.6

Figures 1.9, 11.3, 22.1

Bo Li

Solutions 32.2, 32.1, 33.1

⇓PRIVATE

Domenico Lippolis

44: Uniform approximations 819
Exercise 44.1
Solution 44.1

⇑PRIVATE

Joachim Mathiesen

10.3 Lyapunov exponents 160
Rössler system figures, cycles in chapters 2, 3, 4 and 17

Rytis Paškauskas

4.5.1 Stability of Poincaré return maps 74
8.3 Stability of Poincaré map cycles 124
Exercises 2.9, 3.1, 4.4
Solutions 4.1, 32.1

Adam Prügel-Bennet

Solutions 1.2, 2.11, 6.1, 15.1, 16.3, 37.1, 18.2

Lamberto Rondoni

9 Transporting densities 131
19.2.1 Unstable periodic orbits are dense 366
⇓PRELIMINARY
17.6.1 Cycles from long time series 328
?? Orbital measures . . . . . ??
24.4 Lorentz gas 495
⇑PRELIMINARY

Juri Rolf

Solution 16.3

Per E. Rosenqvist

list the figures
exercises, figures throughout the text

Hans Henrik Rugh

16 Why does it work? 289

Peter Schmelcher

37.2 Discrete cyclists relaxation methods 688

Gábor Simon

Rössler system figures, cycles in chapters 2, 3, 4 and 17

Edward A. Spiegel

2 Flows 33
9 Transporting densities 131
⇓PRELIMINARY
35.1 Deterministic transport 652
⇑PRELIMINARY

Luz V. Vela-Arevalo

5.1 Hamiltonian flows 81
Exercises 5.1, 5.2, 5.3

Niall Whelan

43 Diffraction distraction 801
38 Semiclassical chaotic scattering 709

Andreas Wirzba

38 Semiclassical chaotic scattering 709
⇓PRELIMINARY
42 Periodic orbit theory of diffraction 797
⇑PRELIMINARY
Appendix M: Infinite dimensional operators 1005

Acknowledgements

> I feel I never want to write another book. What's the good! I can eke living on stories and little articles, that don't cost a tithe of the output a book costs. Why write novels any more!
>
> D.H. Lawrence

This book owes its existence to the Niels Bohr Institute's and Nordita's hospitable and nurturing environment, and the private, national and cross-national foundations that have supported the collaborators' research over a span of several decades. P.C. thanks M.J. Feigenbaum of Rockefeller University; D. Ruelle of I.H.E.S., Bures-sur-Yvette; I. Procaccia of the Weizmann Institute; P. Hemmer of University of Trondheim; The Max-Planck Institut für Mathematik, Bonn; J. Lowenstein of New York University; Edificio Celi, Milano; and Fundaçaõ de Faca, Porto Seguro, for the hospitality during various stages of this work, and the Carlsberg Foundation and Glen P. Robinson for support.

please add your acknowledment

The authors gratefully acknowledge collaborations and/or stimulating discussions with E. Aurell, V. Baladi, B. Brenner, A. de Carvalho, D.J. Driebe, B. Eckhardt, M.J. Feigenbaum, J. Frøjland, P. Gaspar, P. Gaspard, J. Guckenheimer, G.H. Gunaratne, P. Grassberger, H. Gutowitz, M. Gutzwiller, K.T. Hansen, P.J. Holmes, T. Janssen, R. Klages, Y. Lan, B. Lauritzen, J. Milnor, M. Nordahl, I. Procaccia, J.M. Robbins, P.E. Rosenqvist, D. Ruelle, G. Russberg, M. Sieber, D. Sullivan, N. Søndergaard, T. Tél, C. Tresser, D. Wintgen, and ?!

scan in, the parrot Gaspar

⇓PRIVATE

⇑PRIVATE

We thank Dorte Glass for typing parts of the manuscript; B. Lautrup and D. Viswanath for comments and corrections to the preliminary versions of this text; the M.A. Porter for lengthening the manuscript by the 2013 definite articles hitherto missing; M.V. Berry for the quotation on page 861 and the remarks quoted on page 875; H. Fogedby for the quotation on page 300; J. Greensite for the quotation on page 5; Ya.B. Pesin for the remarks quoted on page 869; M.A. Porter for the quotations on page 20 and page 869; E.A. Spiegel for quotations on page 1 and page 987; and D. Wojcik for the quotation on page 495.

⇓PRELIMINARY

⇑PRELIMINARY

⇓PRELIMINARY

⇑PRELIMINARY

Fritz Haake's heartfelt lament on page 258 was uttered at the end of the first conference presentation of cycle expansions, in 1988. Joseph Ford introduced himself to the authors of this book by the email quoted on page 601. G.P. Morriss advice to students as how to read the introduction to this book, page 4, was offerred during a 2002 graduate course in Dresden. Benoit B. Mandelbrot's "Feigenbaum's δ as a corollary" remark on page 553 comes from the p. 184 of My Book [27.2]. Kerson Huang's interview of C.N. Yang quoted on page 136 is available on `ChaosBook.org/extras`. L.S. Schulman's remark on our understanding of path integrals, page 654 is taken from chapter 9 of ref. [35.11].

⇓PRELIMINARY

⇑PRELIMINARY

⇓PRELIMINARY

⇑PRELIMINARY

Who is the 3-legged dog reappearing throughout the book? Long ago, when we were innocent and knew not Borel measurable α to Ω sets, P. Cvitanović asked V. Baladi a question about dynamical zeta functions, who then asked J.-P. Eckmann, who then asked D. Ruelle. The answer was transmitted back: "The master says: 'It is holomorphic in a strip' ". Hence His Master's

heroes!unsung
unsung!heroes

⇓PRELIMINARY

Voice logo, and the 3-legged dog is us, still eager to fetch the bone. The answer has made it to the book, though not precisely in His Master's voice. As a matter of fact, the answer *is* the book. We are still chewing on it. Roberto Artuso found the smørrebrød at the Niels Bohr Institute indigestible, so he digested H.M.V.'s wisdom on a strict diet of two Carlsbergs and two pieces of danish pastry for lunch every day, as depicted on the cover.

track the current location of the cover figure

⇑PRELIMINARY

Profound thanks to all the unsung heroes - students and colleagues, too numerous to list here, who have supported this project over many years in many ways, by surviving pilot courses based on this book, by providing invaluable insights, by teaching us, by inspiring us.

⇓PRELIMINARY

Notations used in this book

list all conventions here:

Einstein convention

various kinds of derivatives

replace 'printed date' in the footer by renewcommand{version date}

⇑PRELIMINARY

Chapter 1

Overture

> If I have seen less far than other men it is because I have stood behind giants.
> Edoardo Specchio

Rereading classic theoretical physics textbooks leaves a sense that there are holes large enough to steam a Eurostar train through them. Here we learn about harmonic oscillators and Keplerian ellipses - but where is the chapter on chaotic oscillators, the tumbling Hyperion? We have just quantized hydrogen, where is the chapter on the classical 3-body problem and its implications for quantization of helium? We have learned that an instanton is a solution of field-theoretic equations of motion, but shouldn't a strongly nonlinear field theory have turbulent solutions? How are we to think about systems where things fall apart; the center cannot hold; every trajectory is unstable?

This chapter offers a quick survey of the main topics covered in the book. We start out by making promises - we will right wrongs, no longer shall you suffer the slings and arrows of outrageous Science of Perplexity. We relegate a historical overview of the development of chaotic dynamics to appendix A, and head straight to the starting line: A pinball game is used to motivate and illustrate most of the concepts to be developed in ChaosBook.

Gatto Nero reference

Throughout the book

try to find traffic signs
⇓PRELIMINARY

⇓PRELIMINARY indicates that the text up to ⇑PRELIMINARY is still only a draft, not visible in the web version

⇑PRELIMINARY

indicates that the section requires a hearty stomach and is probably best skipped on first reading

fast track points you where to skip to

tells you where to go for more depth on a particular topic

brain, rat
Poincar“'e, H.

 indicates an exercise that might clarify a point in the text

 indicates that a figure is still missing - you are urged to fetch it

This is a textbook, not a research monograph, and you should be able to follow the thread of the argument without constant excursions to sources. Hence there are no literature references in the text proper, all learned remarks and bibliographical pointers are relegated to the “Commentary” section at the end of each chapter.

1.1 Why ChaosBook?

> It seems sometimes that through a preoccupation with science, we acquire a firmer hold over the vicissitudes of life and meet them with greater calm, but in reality we have done no more than to find a way to escape from our sorrows.
>
> Hermann Minkowski in a letter to David Hilbert

The problem has been with us since Newton’s first frustrating (and unsuccessful) crack at the 3-body problem, lunar dynamics. Nature is rich in systems governed by simple deterministic laws whose asymptotic dynamics are complex beyond belief, systems which are locally unstable (almost) everywhere but globally recurrent. How do we describe their long term dynamics?

The answer turns out to be that we have to evaluate a determinant, take a logarithm. It would hardly merit a learned treatise, were it not for the fact that this determinant that we are to compute is fashioned out of infinitely many infinitely small pieces. The feel is of statistical mechanics, and that is how the problem was solved; in the 1960’s the pieces were counted, and in the 1970’s they were weighted and assembled in a fashion that in beauty and in depth ranks along with thermodynamics, partition functions and path integrals amongst the crown jewels of theoretical physics.

This book is *not* a book about periodic orbits. The red thread throughout the text is the duality between the local, topological, short-time dynamically invariant compact sets and the global long-time dynamics. Chaotic dynamics is generated by the interplay of locally unstable motions, and the interweaving of their global stable and unstable manifolds. These features are robust and accessible in systems as noisy as slices of rat brains. Poincaré, the first to understand deterministic chaos, already said as much (modulo rat brains). Once this topology is understood, a powerful theory yields the observable consequences of chaotic dynamics, such as atomic spectra, transport coefficients, gas pressures.

That is what we will focus on in ChaosBook. The book is a self-contained graduate textbook on classical and quantum chaos. Your professor does not know this material, so you are on your own. We will teach you how to evaluate a determinant, take a logarithm – stuff like that. Ideally, this should take 100 pages or so. Well, we fail - so far we have not found a way to traverse this material in less than a semester, or 200-300 page subset of this text. Nothing can be done about that.

1.2 Chaos ahead

> Things fall apart; the centre cannot hold.
> W.B. Yeats: *The Second Coming*

The study of chaotic dynamics is no recent fashion. It did not start with the widespread use of the personal computer. Chaotic systems have been studied for over 200 years. During this time many have contributed, and the field followed no single line of development; rather one sees many interwoven strands of progress.

In retrospect many triumphs of both classical and quantum physics were a stroke of luck: a few integrable problems, such as the harmonic oscillator and the Kepler problem, though "non-generic," have gotten us very far. The success has lulled us into a habit of expecting simple solutions to simple equations - an expectation tempered by our recently acquired ability to numerically scan the state space of non-integrable dynamical systems. The initial impression might be that all of our analytic tools have failed us, and that the chaotic systems are amenable only to numerical and statistical investigations. Nevertheless, a beautiful theory of deterministic chaos, of predictive quality comparable to that of the traditional perturbation expansions for nearly integrable systems, already exists.

In the traditional approach the integrable motions are used as zeroth-order approximations to physical systems, and weak nonlinearities are then accounted for perturbatively. For strongly nonlinear, non-integrable systems such expansions fail completely; at asymptotic times the dynamics exhibits amazingly rich structure which is not at all apparent in the integrable approximations. However, hidden in this apparent chaos is a rigid skeleton, a self-similar tree of *cycles* (periodic orbits) of increasing lengths. The insight of the modern dynamical systems theory is that the zeroth-order approximations to the harshly chaotic dynamics should be very different from those for the nearly integrable systems: a good starting approximation here is the stretching and folding of baker's dough, rather than the periodic motion of a harmonic oscillator.

So, what is chaos, and what is to be done about it? To get some feeling for how and why unstable cycles come about, we start by playing a game of pinball. The reminder of the chapter is a quick tour through the material covered in ChaosBook. Do not worry if you do not understand every detail at the first reading – the intention is to give you a feeling for the main themes of the book. Details will be filled out later. If you want to get