Epilogue

> Nowadays, whatever the truth of the matter may be (and we will probably never know), the simplest solution is no longer emotionally satisfying. Everything we know about the world militates against it. The concepts of indeterminacy and chaos have filtered down to us from the higher sciences to confirm our nagging suspicions.
>
> L. Sante, "Review of 'American Tabloid' by James Ellroy", *New York Review of Books* (May 11, 1995)

A motion on a strange attractor can be approximated by shadowing long orbits by sequences of nearby shorter periodic orbits. This notion has here been made precise by approximating orbits by prime cycles, and evaluating associated curvatures. A curvature measures the deviation of a long cycle from its approximation by shorter cycles; the smoothness of the dynamical system implies exponential fall-off for (almost) all curvatures. We propose that the theoretical and experimental non–wandering sets be expressed in terms of the symbol sequences of short cycles (a topological characterization of the spatial layout of the non–wandering set) and their eigenvalues (metric structure) ; for example, plotted as fig. ?! or listed as in II, table 1?!. The cycle expansions then offer an efficient method for evaluating periodic orbit averages; accurate estimates can already be obtained from a few fundamental cycles.

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For reasons of clarity we have here motivated the cycle expansions by a simple 1-d repeller of fig. ?!; detailed investigations of a series of low dimensional chaotic systems undertaken in the sequel paper II, as well as the classical and quantum pinball studies of refs. [O.18, 6.3, 1.6], give us some confidence in the general feasibility of the cycle analysis advocated here.

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Cycles as the skeleton of chaos

We wind down this all-too-long treatise by asking: why cycle?

this paragraph is a repeat of parts of intro.tex, remove? rest into DasBuch summary?

We tend to think of a dynamical system as a smooth system whose evolution can be followed by integrating a set of differential equations.

Traditionally one used integrable motions as zeroth-order approximations to physical systems, and accounted for weak nonlinearities perturbatively. However, when the evolution is actually followed through to asymptotic times, one discovers that the strongly nonlinear systems show an amazingly rich structure which is not at all apparent in their formulation in terms of differential equations. In particular, the periodic orbits are important because they form the *skeleton* onto which all trajectories trapped for long times cling. This was already appreciated century ago by H. Poincaré, who, describing in *Les méthodes nouvelles de la méchanique céleste* his discovery of homoclinic tangles, mused that "the complexity of this figure will be striking, and I shall not even try to draw it". Today such drawings are cheap and plentiful; but Poincaré went a step further and, noting that hidden in this apparent chaos is a rigid skeleton, a tree of *cycles* (periodic orbits) of increasing lengths and self-similar structure, suggested that the cycles should be the key to chaotic dynamics.

The zeroth-order approximations to harshly chaotic dynamics are very different from those for the nearly integrable systems: a good starting approximation here is the stretching and kneading of a baker's map, rather than the winding of a harmonic oscillator.

For low dimensional deterministic dynamical systems description in terms of cycles has many virtues:

1. cycle symbol sequences are *topological* invariants: they give the spatial layout of a non–wandering set
2. cycle eigenvalues are *metric* invariants: they give the scale of each piece of a non–wandering set
3. cycles are *dense* on the asymptotic non–wandering set
4. cycles are ordered *hierarchically*: short cycles give good approximations to a non–wandering set, longer cycles only refinements. Errors due to neglecting long cycles can be bounded, and typically fall off exponentially or super-exponentially with the cutoff cycle length
5. cycles are *structurally robust*: for smooth flows eigenvalues of short cycles vary slowly with smooth parameter changes
6. asymptotic averages (such as correlations, escape rates, quantum mechanical eigenstates and other "thermodynamic" averages) can be efficiently computed from short cycles by means of *cycle expansions*

Points 1, 2: That the cycle topology and eigenvalues are invariant properties of dynamical systems follows from elementary considerations. If the same dynamics is given by a map f in one set of coordinates, and a map g in the next, then f and g (or any other good representation) are related by a reparametrization and a coordinate transformation $f = h^{-1} \circ g \circ h$. As both f and g are arbitrary representations of the dynamical system,

the explicit form of the conjugacy h is of no interest, only the properties invariant under any transformation h are of general import. The most obvious invariant properties are topological; a fixed point must be a fixed point in any representation, a trajectory which exactly returns to the initial point (a cycle) must do so in any representation. Furthermore, a good representation should not mutilate the data; h must be a smooth transformation which maps nearby cycle points of f into nearby cycle points of g. This smoothness guarantees that the cycles are not only topological invariants, but that their linearized neighborhoods are also metrically invariant. In particular, the cycle eigenvalues (eigenvalues of the fundamental matrixs $df^n(x)/dx$ of periodic orbits $f^n(x) = x$) are invariant.

Point 5: An important virtue of cycles is their *structural robustness*. Many quantities customarily associated with dynamical systems depend on the notion of "structural stability", i.e., robustness of non–wandering set to small parameter variations.

Still, the sufficiently short unstable cycles are structurally robust in the sense that they are only slightly distorted by such parameter changes, and averages computed using them as a skeleton are insensitive to small deformations of the non–wandering set. In contrast, lack of structural stability wreaks havoc with long time averages such as Lyapunov exponents (see sect. ??), for which there is no guarantee that they converge to the correct asymptotic value in any finite time numerical computation.

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The main recent theoretical advance is **point 4:** we now know how to control the errors due to neglecting longer cycles. As we seen above, even though the number of invariants is infinite (unlike, for example, the number of Casimir invariants for a compact Lie group) the dynamics can be well approximated to any finite accuracy by a small finite set of invariants. The origin of this convergence is geometrical, as we shall see in appendix K.1.2, and for smooth flows the convergence of cycle expansions can even be super-exponential.

The cycle expansions such as (18.5) outperform the pedestrian methods such as extrapolations from the finite cover sums (19.2) for a number of reasons. The cycle expansion is a better averaging procedure than the naive box counting algorithms because the strange attractor is here pieced together in a topologically invariant way from neighborhoods ("space average") rather than explored by a long ergodic trajectory ("time average"). The cycle expansion is co-ordinate and reparametrization invariant - a finite nth level sum (19.2) is not. Cycles are of finite period but infinite duration, so the cycle eigenvalues are already evaluated in the $n \to \infty$ limit, but for the sum (19.2) the limit has to be estimated by numerical extrapolations. And, crucially, the higher terms in the cycle expansion (18.5) are deviations of longer prime cycles from their approximations by shorter cycles. Such combinations vanish exactly in piecewise linear approximations and fall off exponentially for smooth dynamical flows.

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Perhaps it is worth emphasizing again that the Euler product formula (15.16) is an expression for the *exact* transfer operator T. We extract

its eigenvalues directly from (15.16) with no recourse to any explicit (and coordination dependent) eigenfunctions. Our cycle expansions are dominated by short cycles, but that does not mean that we are using finite covers to approximate the set: by resummation that led to (15.16) we have already been lifted to the topologically exact $k \to \infty$ non–wandering set. The approximation will consist in approximating the non–wandering set under investigation by "closeby" Cantor sets with a finite number of already asymptotically exact scales. The finite approximations $\tilde{T}^{(k)}$ were introduced only for reasons of pedagogy: our experience is that computations with the asymptotically exact cycles expression (15.16) are both quicker and of better convergence than computations that go through sequences of finite matrix estimates such as the Markov graph approximations.

However, the cycle expansions are not magic, and they will not converge any better than the more traditional thermodynamic sums *unless the following prerequisites are met*:

1. The essential prerequisite for implementing the above "shadowing" is a good understanding of the symbolic dynamics of the dynamical system; the present formulation requires that the symbolic dynamics be of a finite subshift type (see sect. 12.4). A generic dynamical system is *not* of that type: our strategy is to approach it by a sequence of finite subshift approximants, just as a generic number can be bracketed by a sequence of rational approximants.

2. The weight used in averaging must be multiplicative along the flow, and the flow should be smooth, so that nearby trajectories have nearby weights.

3. Cycle expansions converge only in the hyperbolic phase, i.e., only for averages dominated by the positive entropy of unstable cycles. Marginal fixed points show up indirectly, as power-law corrections and non-analyticities of the ζ functions. If a sequence of fundamental cycles t_f is infinite and accumulating toward marginal stability, the sequence must be summed up in order that the convergence of the cycle expansion be exponential (see sect. ?!).

4. As developed here, the cycle expansions are good only for extracting the leading eigenvalues of transfer operators. If more eigenvalues are needed, techniques for analytically continuing beyond the leading singularities must be developed.

When and if the cycles suffice for the complete characterization (and reconstruction) of a dynamical system is not known, but they do go further toward detailed invariant characterization of low dimensional chaotic dynamical systems than other current methods, and we hope that in the future the data will be presented in terms of cycles rather than "thermodynamic" averages.

We were guided here by the Axiom A intuition developed by Smale, Ruelle, and others: if the dynamical evolution can be cast in terms of an evolution operator multiplicative along the flow, if the corresponding mapping (for ex., return map for a Poincaré section of the flow) is analytic hyperbolic, and if the topology of the repeller is given by a finite Markov partition, then the spectral determinant (15.14) is entire.

The real life challenge are generic dynamical flows, which fit neither schematization. Unfortunately we know of no smooth potential which is both Axiom A, and has bound states. Most systems of interest are *not* of the "Axiom A" category; they are neither purely hyperbolic nor do they have a simple symbolic dynamics grammar. The crucial ingredient for nice analyticity properties of zeta functions is existence of finite grammar (coupled with uniform hyperbolicity). From the hyperbolic dynamics point of view, the Riemann zeta is perhaps the worst possible example; understanding the symbolic dynamics would amount to being able to give a finite grammar definition of all primes. Hyperbolic dynamics suggests that a generic "chaotic" dynamical system should be approached by a sequence of finite grammar approximations[18.2], pretty much as a "generic" number is approached by a sequence of continued fractions.

The procedure described above is one possible way of generating subshifts of finite type. One could have, for example, listed instead all non-vanishing elements of a $[2^4 \times 2^4]$ transition matrix; this yields a larger, non–minimal Markov graph. The Markov graph approach is more compact, but it is still not entirely natural in the context of the present problem: the graphs constructed by the algorithm outlined here are not necessarily minimal, the graphs associated with time reversed dynamics do not reflect this symmetry, and so on. Be it as it may, the method requires only simple string shifts and matches, and is easily implemented on a computer.

The above coarse approximations suffice for establishing the main result of this section, that the coefficients in the cycle expansions of Selberg products fall off faster than exponentially. The spectrum of $\mathcal{L}$ for the piecewise–linear approximate maps is only indicative of the spectrum for the exact nonlinear map; the details are subtle and the reader is referred to ref. [21.23, 16.17] for more careful convergence estimates. In particular, our estimates depended on the assumption that the symbolic dynamics is a subshift of finite type, that the cycle weight is multiplicative along the flow, and that the flow is smooth, so that nearby trajectories have nearby weights.

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In the above we have reviewed the general properties of the cycle expansions; those have been applied to a series of examples of low-dimensional chaos: 1-d strange attractors, the period-doubling repeller, the Hénon-type maps and the mode locking intervals for circle maps. The cycle expansions have also been applied to the irrational windings set of critical circle maps, to the Hamiltonian period-doubling repeller, to a Hamiltonian three-disk game of pinball, to the three-disk quantum scattering resonances and to the extraction of correlation exponents, Feasibility of analysis of experimental

non–wandering set in terms of cycles is discussed in ref. [18.1].

Homework assignment

> "Lo! thy dread empire Chaos is restor'd, Light dies before thy uncreating word; Thy hand, great Anarch, lets the curtain fall, And universal darkness buries all."
>
> Alexander Pope, *The Dunciad*

We conclude cautiously with a homework assignment posed May 22, 1990 (the original due date was May 22, 2000, but alas...):

1. **Topology** Develop optimal sequences ("continued fraction approximants") of finite subshift approximations to generic dynamical systems. Apply to (a) the Hénon map, (b) the Lorenz flow and (c) the Hamiltonian standard map.

2. **Non-hyperbolicity** Incorporate power–law (marginal stability orbits, "intermittency" corrections into cycle expansions. Apply to long-time tails in the Hamiltonian diffusion problem.

3. **Phenomenology** Carry through a convincing analysis of a genuine experimentally extracted data set in terms of periodic orbits.

4. **Invariants** Prove that the scaling functions, or the cycles, or the spectrum of a transfer operator are the maximal set of invariants of an (physically interesting) dynamically generated non–wandering set.

5. **Field theory** Develop a periodic orbit theory of systems with many unstable degrees of freedom. Apply to (a) coupled lattices, (b) cellular automata, (c) neural networks.

6. **Tunneling** Add complex time orbits to quantum mechanical cycle expansions (WKB theory for chaotic systems).

7. **Unitarity** Evaluate corrections to the Gutzwiller semiclassical periodic orbit sums. (a) Show that the zeros (energy eigenvalues) of the appropriate Selberg products are real. (b) Find physically realistic systems for which the "semiclassical" periodic orbit expansions yield the exact quantization.

8. **Atomic spectra** Compute the helium spectrum from periodic orbit expansions (already accomplished by Wintgen and Tanner!).

9. **Symmetries** Include fermions, gauge fields into the periodic orbit theory.

10. **Quantum field theory** Develop quantum theory of systems with infinitely many classically unstable degrees of freedom. Apply to (a) quark confinement (b) early universe (c) the brain.

Conclusion

boredom
Borges, J.L.

> Good-bye. I am leaving because I am bored.
> George Saunders' dying words
>
> Nadie puede escribir un libro. Para Que un libro sea verdaderamente, Se requieren la aurora y el poniente Siglos, armas y el mar que une y separa.
> Jorge Luis Borges El Hacedor, *Ariosto y los arabes*

The buttler did it.