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DASBUCH

ohne Eigenschaften

Aufchnit I: Deterministiche Chaos

Gone with the Wind School
formerly known as the
Copenhagen School of Cycling

Unsung Heroes, too numerous to list
with a disclaimer by **Professor Gatto Nero**

BOYSCOUT EDITION

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> No man but a blockhead ever wrote except for money
> Samuel Johnson

This book is a result of collaborative labors of many people over a span of several decades. Coauthors of a chapter or a section are indicated in the byline to the chapter/section title. If you are referring to a specific coauthored section rather than the entire book, cite it as (for example):

> C. Chandre, F.K. Diakonos and P. Schmelcher, section "Discrete cyclist relaxation method", in P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner and G. Vattay, *Chaos: Classical and Quantum* (Niels Bohr Institute, Copenhagen 2005); `ChaosBook.org/version11`.

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Professor Gatto Nero

all of the credit

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Acknowledgements

> I feel I never want to write another book. What's the good! I can eke living on stories and little articles, that don't cost a tithe of the output a book costs. Why write novels any more!
>
> D.H. Lawrence

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Fritz Haake's heartfelt lament on page 258 was uttered at the end of the first conference presentation of cycle expansions, in 1988. Joseph Ford introduced himself to the authors of this book by the email quoted on page 601. G.P. Morriss advice to students as how to read the introduction to this book, page 4, was offerred during a 2002 graduate course in Dresden. Benoit B. Mandelbrot's "Feigenbaum's δ as a corollary" remark on page 553 comes from the p. 184 of My Book [27.2]. Kerson Huang's interview of C.N. Yang quoted on page 136 is available on `ChaosBook.org/extras`. L.S. Schulman's remark on our understanding of path integrals, page 654 is taken from chapter 9 of ref. [35.11].

Who is the 3-legged dog reappearing throughout the book? Long ago, when we were innocent and knew not Borel measurable α to Ω sets, P. Cvitanović asked V. Baladi a question about dynamical zeta functions, who then asked J.-P. Eckmann, who then asked D. Ruelle. The answer was transmitted back: "The master says: 'It is holomorphic in a strip' ". Hence His Master's

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send in, the parrot Gaspar

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Voice logo, and the 3-legged dog is us, still eager to fetch the bone. The answer has made it to the book, though not precisely in His Master's voice. As a matter of fact, the answer *is* the book. We are still chewing on it. Roberto Artuso found the smørrebrød at the Niels Bohr Institute indigestible, so he digested H.M.V.'s wisdom on a strict diet of two Carlsbergs and two pieces of danish pastry for lunch every day, as depicted on the cover.

Profound thanks to all the unsung heroes - students and colleagues, too numerous to list here, who have supported this project over many years in many ways, by surviving pilot courses based on this book, by providing invaluable insights, by teaching us, by inspiring us.

Notations used in this book

list all conventions here:
Einstein convention
various kinds of derivatives

Chapter 1

Overture

> If I have seen less far than other men it is because I have stood behind giants.
> Edoardo Specchio

Rereading classic theoretical physics textbooks leaves a sense that there are holes large enough to steam a Eurostar train through them. Here we learn about harmonic oscillators and Keplerian ellipses - but where is the chapter on chaotic oscillators, the tumbling Hyperion? We have just quantized hydrogen, where is the chapter on the classical 3-body problem and its implications for quantization of helium? We have learned that an instanton is a solution of field-theoretic equations of motion, but shouldn't a strongly nonlinear field theory have turbulent solutions? How are we to think about systems where things fall apart; the center cannot hold; every trajectory is unstable?

This chapter offers a quick survey of the main topics covered in the book. We start out by making promises - we will right wrongs, no longer shall you suffer the slings and arrows of outrageous Science of Perplexity. We relegate a historical overview of the development of chaotic dynamics to appendix A, and head straight to the starting line: A pinball game is used to motivate and illustrate most of the concepts to be developed in ChaosBook.

Gatto Nero reference

Throughout the book

try to find traffic signs

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indicates that the section requires a hearty stomach and is probably best skipped on first reading

fast track points you where to skip to

tells you where to go for more depth on a particular topic

brain, rat
Poincar"'e, H.

indicates an exercise that might clarify a point in the text

indicates that a figure is still missing - you are urged to fetch it

This is a textbook, not a research monograph, and you should be able to follow the thread of the argument without constant excursions to sources. Hence there are no literature references in the text proper, all learned remarks and bibliographical pointers are relegated to the "Commentary" section at the end of each chapter.

1.1 Why ChaosBook?

> It seems sometimes that through a preoccupation with science, we acquire a firmer hold over the vicissitudes of life and meet them with greater calm, but in reality we have done no more than to find a way to escape from our sorrows.
>
> Hermann Minkowski in a letter to David Hilbert

The problem has been with us since Newton's first frustrating (and unsuccessful) crack at the 3-body problem, lunar dynamics. Nature is rich in systems governed by simple deterministic laws whose asymptotic dynamics are complex beyond belief, systems which are locally unstable (almost) everywhere but globally recurrent. How do we describe their long term dynamics?

The answer turns out to be that we have to evaluate a determinant, take a logarithm. It would hardly merit a learned treatise, were it not for the fact that this determinant that we are to compute is fashioned out of infinitely many infinitely small pieces. The feel is of statistical mechanics, and that is how the problem was solved; in the 1960's the pieces were counted, and in the 1970's they were weighted and assembled in a fashion that in beauty and in depth ranks along with thermodynamics, partition functions and path integrals amongst the crown jewels of theoretical physics.

This book is *not* a book about periodic orbits. The red thread throughout the text is the duality between the local, topological, short-time dynamically invariant compact sets and the global long-time dynamics. Chaotic dynamics is generated by the interplay of locally unstable motions, and the interweaving of their global stable and unstable manifolds. These features are robust and accessible in systems as noisy as slices of rat brains. Poincaré, the first to understand deterministic chaos, already said as much (modulo rat brains). Once this topology is understood, a powerful theory yields the observable consequences of chaotic dynamics, such as atomic spectra, transport coefficients, gas pressures.

That is what we will focus on in ChaosBook. The book is a self-contained graduate textbook on classical and quantum chaos. Your professor

does not know this material, so you are on your own. We will teach you how to evaluate a determinant, take a logarithm – stuff like that. Ideally, this should take 100 pages or so. Well, we fail - so far we have not found a way to traverse this material in less than a semester, or 200-300 page subset of this text. Nothing can be done about that.

1.2 Chaos ahead

> Things fall apart; the centre cannot hold.
> W.B. Yeats: *The Second Coming*

The study of chaotic dynamics is no recent fashion. It did not start with the widespread use of the personal computer. Chaotic systems have been studied for over 200 years. During this time many have contributed, and the field followed no single line of development; rather one sees many interwoven strands of progress.

In retrospect many triumphs of both classical and quantum physics were a stroke of luck: a few integrable problems, such as the harmonic oscillator and the Kepler problem, though "non-generic," have gotten us very far. The success has lulled us into a habit of expecting simple solutions to simple equations - an expectation tempered by our recently acquired ability to numerically scan the state space of non-integrable dynamical systems. The initial impression might be that all of our analytic tools have failed us, and that the chaotic systems are amenable only to numerical and statistical investigations. Nevertheless, a beautiful theory of deterministic chaos, of predictive quality comparable to that of the traditional perturbation expansions for nearly integrable systems, already exists.

In the traditional approach the integrable motions are used as zeroth-order approximations to physical systems, and weak nonlinearities are then accounted for perturbatively. For strongly nonlinear, non-integrable systems such expansions fail completely; at asymptotic times the dynamics exhibits amazingly rich structure which is not at all apparent in the integrable approximations. However, hidden in this apparent chaos is a rigid skeleton, a self-similar tree of *cycles* (periodic orbits) of increasing lengths. The insight of the modern dynamical systems theory is that the zeroth-order approximations to the harshly chaotic dynamics should be very different from those for the nearly integrable systems: a good starting approximation here is the stretching and folding of baker's dough, rather than the periodic motion of a harmonic oscillator.

So, what is chaos, and what is to be done about it? To get some feeling for how and why unstable cycles come about, we start by playing a game of pinball. The reminder of the chapter is a quick tour through the material covered in ChaosBook. Do not worry if you do not understand every detail at the first reading – the intention is to give you a feeling for the main themes of the book. Details will be filled out later. If you want to get

three-disk@3-disk!pinball
Laplace, Pierre-Simon de
Leibniz, Gottfried Wilhelm

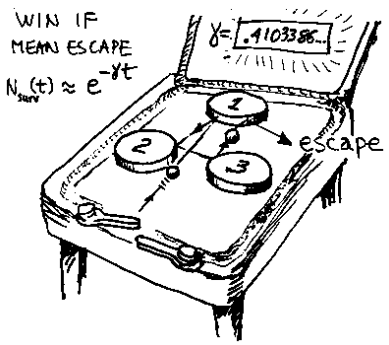


Figure 1.1: A physicist's bare bones game of pinball.

a particular point clarified right now, ☞ on the margin points at the appropriate section.

1.3 The future as in a mirror

> All you need to know about chaos is contained in the introduction of [ChaosBook]. However, in order to understand the introduction you will first have to read the rest of the book.
> Gary Morriss

That deterministic dynamics leads to chaos is no surprise to anyone who has tried pool, billiards or snooker – the game is about beating chaos – so we start our story about what chaos is, and what to do about it, with a game of *pinball*. This might seem a trifle, but the game of pinball is to chaotic dynamics what a pendulum is to integrable systems: thinking clearly about what "chaos" in a game of pinball is will help us tackle more difficult problems, such as computing diffusion constants in deterministic gases, or computing the helium spectrum.

remove "WIN IF ..." in figure 1.1

We all have an intuitive feeling for what a ball does as it bounces among the pinball machine's disks, and only high-school level Euclidean geometry is needed to describe its trajectory. A physicist's pinball game is the game of pinball stripped to its bare essentials: three equidistantly placed reflecting disks in a plane, figure 1.1. A physicist's pinball is free, frictionless, point-like, spin-less, perfectly elastic, and noiseless. Point-like pinballs are shot at the disks from random starting positions and angles; they spend some time bouncing between the disks and then escape.

At the beginning of the 18th century Baron Gottfried Wilhelm Leibniz was confident that given the initial conditions one knew everything a deterministic system would do far into the future. He wrote [1.1], anticipating by a century and a half the oft-quoted Laplace's "Given for one instant an intelligence which could comprehend all the forces by which nature is animated...":

> That everything is brought forth through an established destiny is just as certain as that three times three is nine. [...] If, for example,

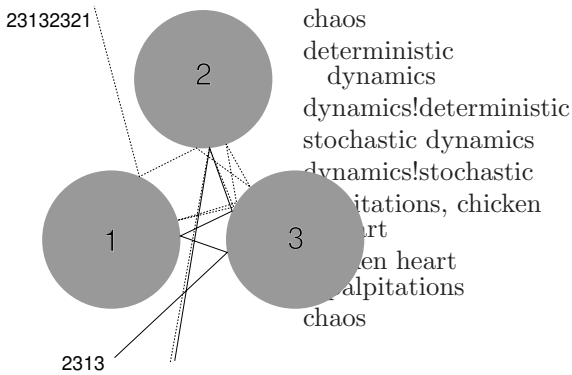


Figure 1.2: Sensitivity to initial conditions: two pinballs that start out very close to each other separate exponentially with time.

> one sphere meets another sphere in free space and if their sizes and their paths and directions before collision are known, we can then foretell and calculate how they will rebound and what course they will take after the impact. Very simple laws are followed which also apply, no matter how many spheres are taken or whether objects are taken other than spheres. From this one sees then that everything proceeds mathematically – that is, infallibly – in the whole wide world, so that if someone could have a sufficient insight into the inner parts of things, and in addition had remembrance and intelligence enough to consider all the circumstances and to take them into account, he would be a prophet and would see the future in the present as in a mirror.

Leibniz chose to illustrate his faith in determinism precisely with the type of physical system that we shall use here as a paradigm of "chaos". His claim is wrong in a deep and subtle way: a state of a physical system can *never* be specified to infinite precision, there is no way to take all the circumstances into account, and a single trajectory cannot be tracked, only a ball of nearby initial points makes physical sense.

1.3.1 What is "chaos"?

> I accept chaos. I am not sure that it accepts me.
> Bob Dylan, *Bringing It All Back Home*

A deterministic system is a system whose present state is *in principle* fully determined by its initial conditions, in contrast to a stochastic system. For a stochastic system the initial conditions determine the future only partially, due to noise, or other external circumstances beyond our control: the present state reflects the past initial conditions plus the particular realization of the noise encountered along the way.

A deterministic system with sufficiently complicated dynamics can fool us into regarding it as a stochastic one; disentangling the deterministic from the stochastic is the main challenge in many real-life settings, from stock markets to palpitations of chicken hearts. So, what is "chaos"?

In a game of pinball, any two trajectories that start out very close to each other separate exponentially with time, and in a finite (and in

sensitivity to initial conditions
initial!conditions!sensitivity to
Lyapunov exponent
Lyapunov time
mixing
entropy!topological
topological!entropy
chaos
mixing
☞ sect. 10.3

practice, a very small) number of bounces their separation $\delta\mathbf{x}(t)$ attains the magnitude of L, the characteristic linear extent of the whole system, figure 1.2. This property of *sensitivity to initial conditions* can be quantified as

$$|\delta\mathbf{x}(t)| \approx e^{\lambda t}|\delta\mathbf{x}(0)|$$

where λ, the mean rate of separation of trajectories of the system, is called the *Lyapunov exponent*. For any finite accuracy $\delta x = |\delta\mathbf{x}(0)|$ of the initial data, the dynamics is predictable only up to a finite *Lyapunov time*

$$T_{\rm Lyap} \approx -\frac{1}{\lambda}\ln|\delta x/L|\,, \tag{1.1}$$

despite the deterministic and, for Baron Leibniz, infallible simple laws that rule the pinball motion.

flip leftsided toSect command?

A positive Lyapunov exponent does not in itself lead to chaos. One could try to play 1- or 2-disk pinball game, but it would not be much of a game; trajectories would only separate, never to meet again. What is also needed is *mixing*, the coming together again and again of trajectories. While locally the nearby trajectories separate, the interesting dynamics is confined to a globally finite region of the state space and thus the separated trajectories are necessarily folded back and can re-approach each other arbitrarily closely, infinitely many times. For the case at hand there are 2^n topologically distinct n bounce trajectories that originate from a given disk. More generally, the number of distinct trajectories with n bounces can be quantified as

$$N(n) \approx e^{hn}$$

☞ sect. 13.1

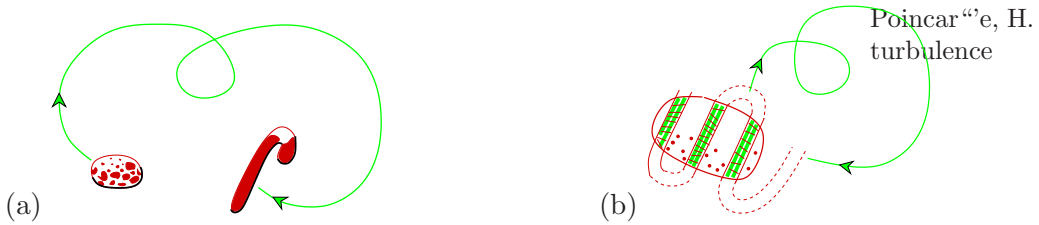
where the *topological entropy* h ($h = \ln 2$ in the case at hand) is the growth rate of the number of topologically distinct trajectories.

☞ sect. 20.1

The appellation "chaos" is a confusing misnomer, as in deterministic dynamics there is no chaos in the everyday sense of the word; everything proceeds mathematically – that is, as Baron Leibniz would have it, infallibly. When a physicist says that a certain system exhibits "chaos," he means that the system obeys deterministic laws of evolution, but that the outcome is highly sensitive to small uncertainties in the specification of the initial state. The word "chaos" has in this context taken on a narrow technical meaning. If a deterministic system is locally unstable (positive Lyapunov exponent) and globally mixing (positive entropy) - figure 1.3 - it is said to be *chaotic*.

double the green line in figure 1.3

While mathematically correct, the definition of chaos as "positive Lyapunov + positive entropy" is useless in practice, as a measurement of these quantities is intrinsically asymptotic and beyond reach for systems observed in nature.



Poincar"'e, H. turbulence

Figure 1.3: Dynamics of a *chaotic* dynamical system is (a) everywhere locally unstable (positive Lyapunov exponent) and (b) globally mixing (positive entropy). (A. Johansen)

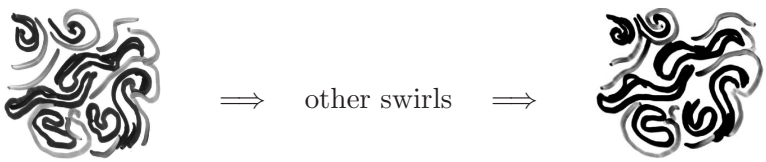
More powerful is Poincaré's vision of chaos as the interplay of local instability (unstable periodic orbits) and global mixing (intertwining of their stable and unstable manifolds). In a chaotic system any open ball of initial conditions, no matter how small, will in finite time overlap with any other finite region and in this sense spread over the extent of the entire asymptotically accessible state space. Once this is grasped, the focus of theory shifts from attempting to predict individual trajectories (which is impossible) to a description of the geometry of the space of possible outcomes, and evaluation of averages over this space. How this is accomplished is what ChaosBook is about.

A definition of "turbulence" is even harder to come by. Intuitively, the word refers to irregular behavior of an infinite-dimensional dynamical system described by deterministic equations of motion - say, a bucket of sloshing water described by the Navier-Stokes equations. But in practice the word "turbulence" tends to refer to messy dynamics which we understand poorly. As soon as a phenomenon is understood better, it is reclaimed and renamed: "a route to chaos", "spatiotemporal chaos", and so on.

☞ chapter 25

In ChaosBook we shall develop a theory of chaotic dynamics for low dimensional attractors visualized as a succession of nearly periodic but unstable motions. In the same spirit, we shall think of turbulence in spatially extended systems in terms of recurrent spatiotemporal patterns. Pictorially, dynamics drives a given spatially extended system (clouds, say) through a repertoire of unstable patterns; as we watch a turbulent system evolve, every so often we catch a glimpse of a familiar pattern:

replace by the color original



For any finite spatial resolution, the system follows approximately for a finite time a pattern belonging to a finite alphabet of admissible patterns, and the long term dynamics can be thought of as a walk through the space of such patterns. In ChaosBook we recast this image into mathematics.

Lyapunov time
chaos!successes
chaos!caveats
dimension!intrisic
degree of freedom
turbulence

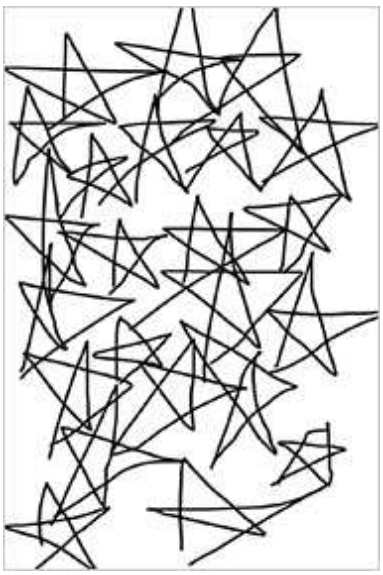


Figure 1.4: Katherine Jones-Smith, "Untitled 5," the drawing used by K. Jones-Smith and R.P. Taylor to test the fractal analysis of Pollock's drip paintings [1.3].

1.3.2 When does "chaos" matter?

> In dismissing Pollock's fractals because of their limited magnification range, Jones-Smith and Mathur would also dismiss half the published investigations of physical fractals.
> Richard P. Taylor [1.4, 1.5]

expand using my lectures

When should we be mindful of chaos? The solar system is "chaotic", yet we have no trouble keeping track of the annual motions of planets. The rule of thumb is this; if the Lyapunov time (1.1) (the time by which a state space region initially comparable in size to the observational accuracy extends across the entire accessible state space) is significantly shorter than the observational time, you need to master the theory that will be developed here. That is why the main successes of the theory are in statistical mechanics, quantum mechanics, and questions of long term stability in celestial mechanics.

In science popularizations too much has been made of the impact of "chaos theory," so a number of caveats are already needed at this point.

At present the theory is in practice applicable only to systems with a low intrinsic *dimension* – the minimum number of coordinates necessary to capture its essential dynamics. If the system is very turbulent (a description of its long time dynamics requires a space of high intrinsic dimension) we are out of luck. Hence insights that the theory offers in elucidating problems of fully developed turbulence, quantum field theory of strong interactions and early cosmology have been modest at best. Even that is a caveat with qualifications. There are applications – such as spatially extended (nonequilibrium) systems, plumber's turbulent pipes, etc. – where the few important degrees of freedom can be isolated and studied profitably by methods to be described here.

☞ chapter 25

fractal!science
dimension!generalized
fractal!aggregates
fractal!probabilistic
fractal!geometry of nature
nature!geometry of
Smale, S.

Thus far the theory has had limited practical success when applied to the very noisy systems so important in the life sciences and in economics. Even though we are often interested in phenomena taking place on time scales much longer than the intrinsic time scale (neuronal interburst intervals, cardiac pulses, etc.), disentangling "chaotic" motions from the environmental noise has been very hard.

In 1980's something happened that might be without parallel; this is an area of science where the advent of cheap computation had actually subtracted from our collective understanding The computer pictures and numerical plots of fractal science of the 1980's have overshadowed the deep insights of the 1970's, and these pictures have since migrated into textbooks. By a regrettable oversight, ChaosBook has none, so "Untitled 5" of figure 1.4 will have to do as the illustration of the power of fractal analysis. Fractal science posits that certain quantities (Lyapunov exponents, generalized dimensions, ...) can be estimated on a computer. While some of the numbers so obtained are indeed mathematically sensible characterizations of fractals, they are in no sense observable and measurable on the length-scales and time-scales dominated by chaotic dynamics.

Even though the experimental evidence for the fractal geometry of nature is circumstantial [1.2], in studies of probabilistically assembled fractal aggregates we know of nothing better than contemplating such quantities. In deterministic systems we can do *much* better.

1.4 A game of pinball

> Formulas hamper the understanding.
> S. Smale

We are now going to get down to the brasstacks. Time to fasten your seatbelts and turn off all electronic devices. But first, a disclaimer: If you understand the rest of this chapter on the first reading, you either do not need this book, or you are delusional. If you do not understand it, it is not because the people who wrote it are smarter than you: the most you can hope for at this stage is to get a flavor of what lies ahead. If a statement in this chapter mystifies/intrigues, fast forward to a section indicated by ☞ on the margin, read only the parts that you feel you need. Of course, we think that you need to learn ALL of it, or otherwise we would not have selected it in the first place.

Confronted with a potentially chaotic dynamical system, we analyze it through a sequence of three distinct stages; I. diagnose, II. count, III. measure. First we determine the intrinsic *dimension* of the system – the minimum number of coordinates necessary to capture its essential dynamics. If the system is very turbulent we are, at present, out of luck. We know only how to deal with the transitional regime between regular motions and chaotic dynamics in a few dimensions. That is still something; even an

symbolic dynamics
dynamics!symbolic
itinerary
pruning
three-disk@3-disk!symbolic dynamics

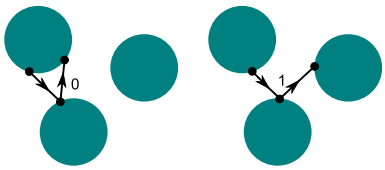


Figure 1.5: Binary labeling of the 3-disk pinball trajectories; a bounce in which the trajectory returns to the preceding disk is labeled 0, and a bounce which results in continuation to the third disk is labeled 1.

infinite-dimensional system such as a burning flame front can turn out to have a very few chaotic degrees of freedom. In this regime the chaotic dynamics is restricted to a space of low dimension, the number of relevant parameters is small, and we can proceed to step II; we *count* and *classify* all possible topologically distinct trajectories of the system into a hierarchy whose successive layers require increased precision and patience on the part of the observer. This we shall do in sect. 1.4.1. If successful, we can proceed with step III: investigate the *weights* of the different pieces of the system.

⇓PRELIMINARY
☞ sect. 25.1
⇑PRELIMINARY
☞ chapter 11
☞ chapter 13

We commence our analysis of the pinball game with steps I, II: diagnose, count. We shall return to step III – measure – in sect. 1.5.

☞ chapter 18

With the game of pinball we are in luck – it is a low dimensional system, free motion in a plane. The motion of a point particle is such that after a collision with one disk it either continues to another disk or it escapes. If we label the three disks by 1, 2 and 3, we can associate every trajectory with an *itinerary*, a sequence of labels indicating the order in which the disks are visited; for example, the two trajectories in figure 1.2 have itineraries _2313_, _23132321_ respectively. The itinerary is finite for a scattering trajectory, coming in from infinity and escaping after a finite number of collisions, infinite for a trapped trajectory, and infinitely repeating for a periodic orbit. Parenthetically, in this subject the words "orbit" and "trajectory" refer to one and the same thing.

1.1
page 32
remember to eliminate figure f_2_02_1

Such labeling is the simplest example of *symbolic dynamics*. As the particle cannot collide two times in succession with the same disk, any two consecutive symbols must differ. This is an example of *pruning*, a rule that forbids certain subsequences of symbols. Deriving pruning rules is in general a difficult problem, but with the game of pinball we are lucky - there are no further pruning rules.

☞ chapter 12
promote sect. 11.6 to chapt 2?

The choice of symbols is in no sense unique. For example, as at each bounce we can either proceed to the next disk or return to the previous disk, the above 3-letter alphabet can be replaced by a binary $\{0, 1\}$ alphabet, figure 1.5. A clever choice of an alphabet will incorporate important features of the dynamics, such as its symmetries.

☞ sect. 11.6

Suppose you wanted to play a good game of pinball, that is, get the pinball to bounce as many times as you possibly can – what would be a winning strategy? The simplest thing would be to try to aim the pinball so it bounces many times between a pair of disks – if you managed to shoot it so it starts out in the periodic orbit bouncing along the line connecting two disk centers, it would stay there forever. Your game would be just as good if you managed to get it to keep bouncing between the three disks

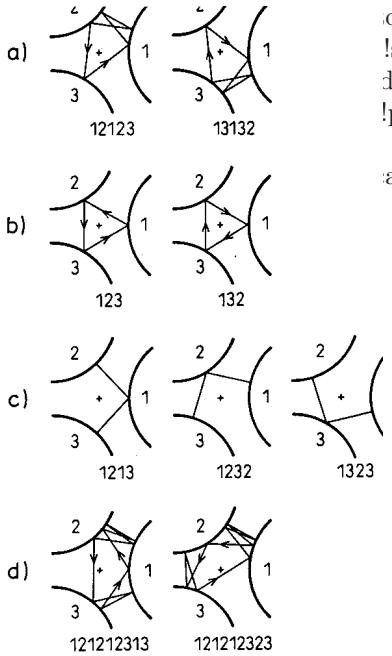


Figure 1.6: Some examples of 3-disk cycles: (a) $\overline{12123}$ and $\overline{13132}$ are mapped into each other by the flip across 1 axis. Similarly (b) $\overline{123}$ and $\overline{132}$ are related by flips, and (c) $\overline{1213}$, $\overline{1232}$ and $\overline{1323}$ by rotations. (d) The cycles $\overline{121212313}$ and $\overline{121212323}$ are related by rotaion *and* time reversal. These symmetries are discussed in more detail in chapter 22. (from ref. [1.6])

forever, or place it on any periodic orbit. The only rub is that any such orbit is *unstable*, so you have to aim very accurately in order to stay close to it for a while. So it is pretty clear that if one is interested in playing well, unstable periodic orbits are important – they form the *skeleton* onto which all trajectories trapped for long times cling. [1]

☞ sect. 43.2

1.4.1 Partitioning with periodic orbits

A trajectory is periodic if it returns to its starting position and momentum. We shall refer to the set of periodic points that belong to a given periodic orbit as a *cycle*.

Short periodic orbits are easily drawn and enumerated - some examples are drawn in figure 1.6 - but it is rather hard to perceive the systematics of orbits from their shapes. In mechanics a trajectory is fully and uniquely specified by its position and momentum at a given instant, and no two distinct state space trajectories can intersect. Their projections onto arbitrary subspaces, however, can and do intersect, in rather unilluminating ways. In the pinball example the problem is that we are looking at the projections of a 4-dimensional state space trajectories onto a 2-dimensional subspace, the configuration space. A clearer picture of the dynamics is obtained by constructing a state space Poincaré section.

narrow text, broaden figure in figure 1.6

Suppose that the pinball has just bounced off disk 1. Depending on its position and outgoing angle, it could proceed to either disk 2 or 3. Not much happens in between the bounces – the ball just travels at constant velocity

[1] Mason: do you discuss "scarring" anywhere? PC: no, I do not find it compelling, neither mathematically nor physically

Poincar"'e!section
section!Poincar"'e
map!return
return map
Birkhoff coordinates
three-disk@3-disk!state space
itinerary
unstable!cycle
cycle!unstable
skeleton of chaos
chaos!skeleton of
periodic!point!unstable
unstable!periodic point

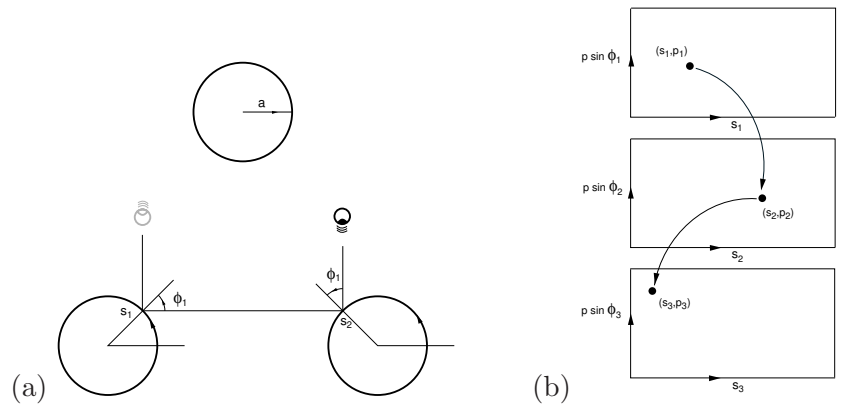
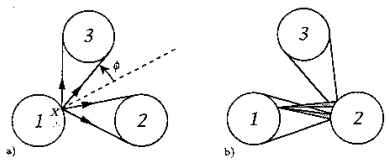


Figure 1.7: (a) The Poincaré section coordinates for the 3-disk game of pinball. (b) Collision sequence $(s_1, p_1) \mapsto (s_2, p_2) \mapsto (s_3, p_3)$ from the boundary of a disk to the boundary of the next disk presented in the Poincaré section coordinates.

Figure 1.8: (a) A trajectory starting out from disk 1 can either hit another disk or escape. (b) Hitting two disks in a sequence requires a much sharper aim. The cones of initial conditions that hit more and more consecutive disks are nested within each other, as in figure 1.9.



along a straight line – so we can reduce the four-dimensional flow to a two-dimensional map f that takes the coordinates of the pinball from one disk edge to another disk edge. Let us state this more precisely: the trajectory just after the moment of impact is defined by marking s_n, the arc-length position of the nth bounce along the billiard wall, and $p_n = p \sin \phi_n$ the momentum component parallel to the billiard wall at the point of impact, figure 1.7. Such a section of a flow is called a *Poincaré section*, and the particular choice of coordinates (due to Birkhoff) is particularly smart, as it conserves the phase space volume. In terms of the Poincaré section, the dynamics is reduced to the *return map* $P : (s_n, p_n) \mapsto (s_{n+1}, p_{n+1})$ from the boundary of a disk to the boundary of the next disk. The explicit form of this map is easily written down, but it is of no importance right now.

fix! put also the "conventional" arrow on particle trajectory in figure 1.7

thicken arrows, fonts in figure 1.7

☞ sect. 6

Next, we mark in the Poincaré section those initial conditions which do not escape in one bounce. There are two strips of survivors, as the trajectories originating from one disk can hit either of the other two disks, or escape without further ado. We label the two strips $\mathcal{M}_0$, $\mathcal{M}_1$. Embedded within them there are four strips $\mathcal{M}_{00}$, $\mathcal{M}_{10}$, $\mathcal{M}_{01}$, $\mathcal{M}_{11}$ of initial conditions that survive for two bounces, and so forth, see figures 1.8 and 1.9. Provided that the disks are sufficiently separated, after n bounces the survivors are divided into 2^n distinct strips: the $\mathcal{M}_i$th strip consists of all points with itinerary $i = s_1 s_2 s_3 \ldots s_n$, $s = \{0, 1\}$. The unstable cycles as a skeleton of chaos are almost visible here: each such patch contains a periodic point $\overline{s_1 s_2 s_3 \ldots s_n}$ with the basic block infinitely repeated. Periodic points are skeletal in the sense that as we look further and further, the strips shrink but the periodic points stay put forever.

Include fig. 15 from LLNN.

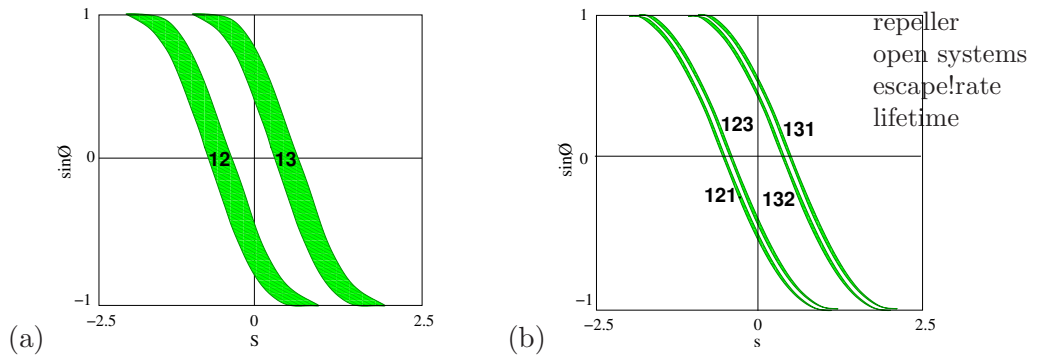


Figure 1.9: The 3-disk game of pinball Poincaré section, trajectories emanating from the disk 1 with x_0 = (arclength, parallel momentum) = (s_0, p_0) , disk radius : center separation ratio a:R = 1:2.5. (a) Strips of initial points $\mathcal{M}_{12}$, $\mathcal{M}_{13}$ which reach disks 2, 3 in one bounce, respectively. (b) Strips of initial points $\mathcal{M}_{121}$, $\mathcal{M}_{131}$ $\mathcal{M}_{132}$ and $\mathcal{M}_{123}$ which reach disks 1, 2, 3 in two bounces, respectively. The Poincaré sections for trajectories originating on the other two disks are obtained by the appropriate relabeling of the strips. (Y. Lan)

We see now why it pays to utilize a symbolic dynamics; it provides a navigation chart through chaotic state space. There exists a unique trajectory for every admissible infinite length itinerary, and a unique itinerary labels every trapped trajectory. For example, the only trajectory labeled by $\overline{12}$ is the 2-cycle bouncing along the line connecting the centers of disks 1 and 2; any other trajectory starting out as 12... either eventually escapes or hits the 3rd disk.

1.4.2 Escape rate

example 10.1

What is a good physical quantity to compute for the game of pinball? Such system, for which almost any trajectory eventually leaves a finite region (the pinball table) never to return, is said to be open, or a *repeller*. The repeller *escape rate* is an eminently measurable quantity. An example of such a measurement would be an unstable molecular or nuclear state which can be well approximated by a classical potential with the possibility of escape in certain directions. In an experiment many projectiles are injected into such a non-confining potential and their mean escape rate is measured, as in figure 1.1. The numerical experiment might consist of injecting the pinball between the disks in some random direction and asking how many times the pinball bounces on the average before it escapes the region between the disks.

1.2
page 32

For a theorist a good game of pinball consists in predicting accurately the asymptotic lifetime (or the escape rate) of the pinball. We now show how periodic orbit theory accomplishes this for us. Each step will be so simple that you can follow even at the cursory pace of this overview, and still the result is surprisingly elegant.

Consider figure 1.9 again. In each bounce the initial conditions get thinned out, yielding twice as many thin strips as at the previous bounce.

survival probability
escape!rate
Poincar"'e, H.

The total area that remains at a given time is the sum of the areas of the strips, so that the fraction of survivors after n bounces, or the *survival probability* is given by

$$\hat{\Gamma}_1 = \frac{|\mathcal{M}_0|}{|\mathcal{M}|} + \frac{|\mathcal{M}_1|}{|\mathcal{M}|}, \qquad \hat{\Gamma}_2 = \frac{|\mathcal{M}_{00}|}{|\mathcal{M}|} + \frac{|\mathcal{M}_{10}|}{|\mathcal{M}|} + \frac{|\mathcal{M}_{01}|}{|\mathcal{M}|} + \frac{|\mathcal{M}_{11}|}{|\mathcal{M}|},$$
$$\hat{\Gamma}_n = \frac{1}{|\mathcal{M}|} \sum_i^{(n)} |\mathcal{M}_i|, \tag{1.2}$$

where i is a label of the ith strip, $|\mathcal{M}|$ is the initial area, and $|\mathcal{M}_i|$ is the area of the ith strip of survivors. $i = 01, 10, 11, \ldots$ is a label, not a binary number. Since at each bounce one routinely loses about the same fraction of trajectories, one expects the sum (1.2) to fall off exponentially with n and tend to the limit

$$\hat{\Gamma}_{n+1}/\hat{\Gamma}_n = e^{-\gamma_n} \to e^{-\gamma}. \tag{1.3}$$

The quantity γ is called the *escape rate* from the repeller.

1.5 Chaos for cyclists

move this to a later section?

provide translation

> Étant données des équations ... et une solution particuliére quelconque de ces équations, on peut toujours trouver une solution périodique (dont la période peut, il est vrai, étre trés longue), telle que la différence entre les deux solutions soit aussi petite qu'on le veut, pendant un temps aussi long qu'on le veut. D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ansi dire, la seule bréche par où nous puissions esseyer de pénétrer dans une place jusqu'ici réputée inabordable.
> H. Poincaré, *Les méthodes nouvelles de la méchanique céleste*

We shall now show that the escape rate γ can be extracted from a highly convergent *exact* expansion by reformulating the sum (1.2) in terms of unstable periodic orbits.

If, when asked what the 3-disk escape rate is for a disk of radius 1, center-center separation 6, velocity 1, you answer that the continuous time escape rate is roughly $\gamma = 0.4103384077693464893384613078192\ldots$, you do not need this book. If you have no clue, hang on.

1.5.1 How big is my neighborhood?

Not only do the periodic points keep track of topological ordering of the strips, but, as we shall now show, they also determine their size.

As a trajectory evolves, it carries along and distorts its infinitesimal neighborhood. Let

$$x(t) = f^t(x_0)$$

denote the trajectory of an initial point $x_0 = x(0)$. Expanding $f^t(x_0 + \delta x_0)$ to linear order, the evolution of the distance to a neighboring trajectory $x_i(t) + \delta x_i(t)$ is given by the fundamental matrix:

$$\delta x_i(t) = \sum_{j=1}^{d} J^t(x_0)_{ij} \delta x_{0j} \,, \qquad J^t(x_0)_{ij} = \frac{\partial x_i(t)}{\partial x_{0j}} \,.$$

A trajectory of a pinball moving on a flat surface is specified by two position coordinates and the direction of motion, so in this case $d = 3$. Evaluation of a cycle fundamental matrix is a long exercise - here we just state the result. The fundamental matrix describes the deformation of an infinitesimal neighborhood of $x(t)$ along the flow; its eigenvectors and eigenvalues give the directions and the corresponding rates of expansion or contraction. The trajectories that start out in an infinitesimal neighborhood are separated along the unstable directions (those whose eigenvalues are greater than unity in magnitude), approach each other along the stable directions (those whose eigenvalues are less than unity in magnitude), and maintain their distance along the marginal directions (those whose eigenvalues equal unity in magnitude). In our game of pinball the beam of neighboring trajectories is defocused along the unstable eigendirection of the fundamental matrix M.

☞ sect. 6.2

As the heights of the strips in figure 1.9 are effectively constant, we can concentrate on their thickness. If the height is $\approx L$, then the area of the ith strip is $\mathcal{M}_i \approx L l_i$ for a strip of width l_i.

Each strip i in figure 1.9 contains a periodic point x_i. The finer the intervals, the smaller the variation in flow across them, so the contribution from the strip of width l_i is well-approximated by the contraction around the periodic point x_i within the interval,

$$l_i = a_i / |\Lambda_i| \,, \tag{1.4}$$

where Λ_i is the unstable eigenvalue of the fundamental matrix $J^t(x_i)$ evaluated at the ith periodic point for $t = T_p$, the full period (due to the low dimensionality, the Jacobian can have at most one unstable eigenvalue). Only the magnitude of this eigenvalue matters, we can disregard its sign. The prefactors a_i reflect the overall size of the system and the particular distribution of starting values of x. As the asymptotic trajectories are strongly mixed by bouncing chaotically around the repeller, we expect their distribution to be insensitive to smooth variations in the distribution of initial points.

☞ sect. 9.3

initial!point
$“xInit$
initial!state $“xInit$
fundamental matrix
unstable!manifold
stable!manifold
marginal!stability
return map
map!return
mixing

hyperbolicity assumption

To proceed with the derivation we need the *hyperbolicity* assumption: for large n the prefactors $a_i \approx O(1)$ are overwhelmed by the exponential growth of Λ_i, so we neglect them. If the hyperbolicity assumption is justified, we can replace $|\mathcal{M}_i| \approx L l_i$ in (1.2) by $1/|\Lambda_i|$ and consider the sum

☞ sect. 14.1.1

$$\Gamma_n = \sum_i^{(n)} 1/|\Lambda_i| \,,$$

where the sum goes over all periodic points of period n. We now define a generating function for sums over all periodic orbits of all lengths:

$$\Gamma(z) = \sum_{n=1}^{\infty} \Gamma_n z^n \,. \tag{1.5}$$

Recall that for large n the nth level sum (1.2) tends to the limit $\Gamma_n \to e^{-n\gamma}$, so the escape rate γ is determined by the smallest $z = e^{\gamma}$ for which (1.5) diverges:

$$\Gamma(z) \approx \sum_{n=1}^{\infty} \left(z e^{-\gamma}\right)^n = \frac{z e^{-\gamma}}{1 - z e^{-\gamma}} \,. \tag{1.6}$$

This is the property of $\Gamma(z)$ that motivated its definition. Next, we devise a formula for (1.5) expressing the escape rate in terms of periodic orbits:

$$\begin{aligned} \Gamma(z) &= \sum_{n=1}^{\infty} z^n \sum_i^{(n)} |\Lambda_i|^{-1} \\ &= \frac{z}{|\Lambda_0|} + \frac{z}{|\Lambda_1|} + \frac{z^2}{|\Lambda_{00}|} + \frac{z^2}{|\Lambda_{01}|} + \frac{z^2}{|\Lambda_{10}|} + \frac{z^2}{|\Lambda_{11}|} \\ &\quad + \frac{z^3}{|\Lambda_{000}|} + \frac{z^3}{|\Lambda_{001}|} + \frac{z^3}{|\Lambda_{010}|} + \frac{z^3}{|\Lambda_{100}|} + \ldots \end{aligned} \tag{1.7}$$

☞ sect. 14.4

For sufficiently small z this sum is convergent. The escape rate γ is now given by the leading pole of (1.6), rather than by a numerical extrapolation of a sequence of γ_n extracted from (1.3). As any finite truncation $n < n_{\rm trunc}$ of (1.7) is a polynomial in z, convergent for any z, finding this pole requires that we know something about Γ_n for any n, and that might be a tall order.

We could now proceed to estimate the location of the leading singularity of $\Gamma(z)$ from finite truncations of (1.7) by methods such as Padé approximants. However, as we shall now show, it pays to first perform a simple resummation that converts this divergence into a *zero* of a related function.

1.5.2 Dynamical zeta function

three-disk@3-disk!prime cycles
zeta function!dynamical
dynamical!zeta function

If a trajectory retraces a *prime* cycle r times, its expanding eigenvalue is Λ_p^r. A prime cycle p is a single traversal of the orbit; its label is a non-repeating symbol string of n_p symbols. There is only one prime cycle for each cyclic permutation class. For example, $p = \overline{0011} = \overline{1001} = \overline{1100} = \overline{0110}$ is prime, but $\overline{0101} = \overline{01}$ is not. By the chain rule for derivatives the stability of a cycle is the same everywhere along the orbit, so each prime cycle of length n_p contributes n_p terms to the sum (1.7). Hence (1.7) can be rewritten as

13.5
page 247
sect. 4.5

$$\Gamma(z) = \sum_p n_p \sum_{r=1}^{\infty} \left(\frac{z^{n_p}}{|\Lambda_p|} \right)^r = \sum_p \frac{n_p t_p}{1-t_p}, \qquad t_p = \frac{z^{n_p}}{|\Lambda_p|} \tag{1.8}$$

where the index p runs through all distinct *prime* cycles. Note that we have resummed the contribution of the cycle p to all times, so truncating the summation up to given p is *not* a finite time $n \le n_p$ approximation, but an asymptotic, *infinite* time estimate based by approximating stabilities of all cycles by a finite number of the shortest cycles and their repeats. The $n_p z^{n_p}$ factors in (1.8) suggest rewriting the sum as a derivative

$$\Gamma(z) = -z \frac{d}{dz} \sum_p \ln(1-t_p) .$$

Hence $\Gamma(z)$ is a logarithmic derivative of the infinite product

$$1/\zeta(z) = \prod_p (1-t_p), \qquad t_p = \frac{z^{n_p}}{|\Lambda_p|} . \tag{1.9}$$

This function is called the *dynamical zeta function*, in analogy to the Riemann zeta function, which motivates the choice of "zeta" in its definition as $1/\zeta(z)$. This is the prototype formula of periodic orbit theory. The zero of $1/\zeta(z)$ is a pole of $\Gamma(z)$, and the problem of estimating the asymptotic escape rates from finite n sums such as (1.2) is now reduced to a study of the zeros of the dynamical zeta function (1.9). The escape rate is related by (1.6) to a divergence of $\Gamma(z)$, and $\Gamma(z)$ diverges whenever $1/\zeta(z)$ has a zero.

sect. 19.1
sect. 15.4

Easy, you say: "Zeros of (1.9) can be read off the formula, a zero

$$z_p = |\Lambda_p|^{1/n_p}$$

for each term in the product. What's the problem?" Dead wrong!

cycle!expansion
shadowing
dynamical system!smooth
smooth!dynamics
shadowing

1.5.3 Cycle expansions

How are formulas such as (1.9) used? We start by computing the lengths and eigenvalues of the shortest cycles. This usually requires some numerical work, such as the Newton's method searches for periodic solutions; we shall assume that the numerics are under control, and that *all* short cycles up to given length have been found. In our pinball example this can be done by elementary geometrical optics. It is very important not to miss any short cycles, as the calculation is as accurate as the shortest cycle dropped – including cycles longer than the shortest omitted does not improve the accuracy (unless exponentially many more cycles are included). The result of such numerics is a table of the shortest cycles, their periods and their stabilities.

☞ chapter 17

☞ sect. 37.3

Now expand the infinite product (1.9), grouping together the terms of the same total symbol string length

$$\begin{aligned} 1/\zeta &= (1-t_0)(1-t_1)(1-t_{10})(1-t_{100})\cdots \\ &= 1-t_0-t_1-[t_{10}-t_1t_0]-[(t_{100}-t_{10}t_0)+(t_{101}-t_{10}t_1)] \\ &\quad -[(t_{1000}-t_0t_{100})+(t_{1110}-t_1t_{110}) \\ &\quad +(t_{1001}-t_1t_{001}-t_{101}t_0+t_{10}t_0t_1)]-\ldots \end{aligned} \tag{1.10}$$

The virtue of the expansion is that the sum of all terms of the same total length n (grouped in brackets above) is a number that is exponentially smaller than a typical term in the sum, for geometrical reasons we explain in the next section.

☞ chapter 18

☞ sect. 18.1

The calculation is now straightforward. We substitute a finite set of the eigenvalues and lengths of the shortest prime cycles into the cycle expansion (1.10), and obtain a polynomial approximation to $1/\zeta$. We then vary z in (1.9) and determine the escape rate γ by finding the smallest $z = e^{\gamma}$ for which (1.10) vanishes.

back to physics here: explain escape = Lyap-entropy

1.5.4 Shadowing

When you actually start computing this escape rate, you will find out that the convergence is very impressive: only three input numbers (the two fixed points $\overline{0}$, $\overline{1}$ and the 2-cycle $\overline{10}$) already yield the pinball escape rate to 3-4 significant digits! We have omitted an infinity of unstable cycles; so why does approximating the dynamics by a finite number of the shortest cycle eigenvalues work so well?

☞ sect. 18.2.2

The convergence of cycle expansions of dynamical zeta functions is a consequence of the smoothness and analyticity of the underlying flow. Intuitively, one can understand the convergence in terms of the geometrical picture sketched in figure 1.10; the key observation is that the long orbits are *shadowed* by sequences of shorter orbits.

dynamical system!smooth
smooth!dynamics

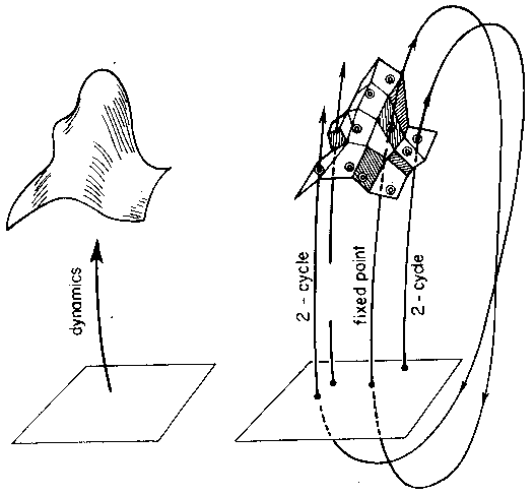


Figure 1.10: Approximation to (a) a smooth dynamics by (b) the skeleton of periodic points, together with their linearized neighborhoods. Indicated are segments of two 1-cycles and a 2-cycle that alternates between the neighborhoods of the two 1-cycles, shadowing first one of the two 1-cycles, and then the other.

A typical term in (1.10) is a difference of a long cycle $\{ab\}$ minus its shadowing approximation by shorter cycles $\{a\}$ and $\{b\}$

$$t_{ab} - t_a t_b = t_{ab}(1 - t_a t_b/t_{ab}) = t_{ab}\left(1 - \left|\frac{\Lambda_{ab}}{\Lambda_a \Lambda_b}\right|\right) , \tag{1.11}$$

where a and b are symbol sequences of the two shorter cycles. If all orbits are weighted equally ($t_p = z^{n_p}$), such combinations cancel exactly; if orbits of similar symbolic dynamics have similar weights, the weights in such combinations almost cancel.

This can be understood in the context of the pinball game as follows. Consider orbits $\overline{0}$, $\overline{1}$ and $\overline{01}$. The first corresponds to bouncing between any two disks while the second corresponds to bouncing successively around all three, tracing out an equilateral triangle. The cycle $\overline{01}$ starts at one disk, say disk 2. It then bounces from disk 3 back to disk 2 then bounces from disk 1 back to disk 2 and so on, so its itinerary is $\overline{2321}$. In terms of the bounce types shown in figure 1.5, the trajectory is alternating between 0 and 1. The incoming and outgoing angles when it executes these bounces are very close to the corresponding angles for 0 and 1 cycles. Also the distances traversed between bounces are similar so that the 2-cycle expanding eigenvalue Λ_{01} is close in magnitude to the product of the 1-cycle eigenvalues $\Lambda_0 \Lambda_1$.

To understand this on a more general level, try to visualize the partition of a chaotic dynamical system's state space in terms of cycle neighborhoods as a tessellation of the dynamical system, with smooth flow approximated by its periodic orbit skeleton, each "face" centered on a periodic point, and the scale of the "face" determined by the linearization of the flow around the periodic point, figure 1.10.

The orbits that follow the same symbolic dynamics, such as $\{ab\}$ and a

Dirac delta function
evolution!operator

"pseudo orbit" $\{a\}\{b\}$, lie close to each other in state space; long shadowing pairs have to start out exponentially close to beat the exponential growth in separation with time. If the weights associated with the orbits are multiplicative along the flow (for example, by the chain rule for products of derivatives) and the flow is smooth, the term in parenthesis in (1.11) falls off exponentially with the cycle length, and therefore the curvature expansions are expected to be highly convergent.

☞ chapter 16

1.6 Evolution

The above derivation of the dynamical zeta function formula for the escape rate has one shortcoming; it estimates the fraction of survivors as a function of the number of pinball bounces, but the physically interesting quantity is the escape rate measured in units of continuous time. For continuous time flows, the escape rate (1.2) is generalized as follows. Define a finite state space region $\mathcal{M}$ such that a trajectory that exits $\mathcal{M}$ never reenters. For example, any pinball that falls of the edge of a pinball table in figure 1.1 is gone forever. Start with a uniform distribution of initial points. The fraction of initial x whose trajectories remain within $\mathcal{M}$ at time t is expected to decay exponentially

$$\Gamma(t) = \frac{\int_{\mathcal{M}} dxdy\, \delta(y - f^t(x))}{\int_{\mathcal{M}} dx} \;\to\; e^{-\gamma t}\,.$$

The integral over x starts a trajectory at every $x \in \mathcal{M}$. The integral over y tests whether this trajectory is still in $\mathcal{M}$ at time t. The kernel of this integral

$$\mathcal{L}^t(y,x) = \delta\big(y - f^t(x)\big) \tag{1.12}$$

is the Dirac delta function, as for a deterministic flow the initial point x maps into a unique point y at time t. For discrete time, $f^n(x)$ is the nth iterate of the map f. For continuous flows, $f^t(x)$ is the trajectory of the initial point x, and it is appropriate to express the finite time kernel $\mathcal{L}^t$ in terms of a generator of infinitesimal time translations

$$\mathcal{L}^t = e^{t\mathcal{A}}\,,$$

☞ sect. 9.5

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very much in the way the quantum evolution is generated by the Hamiltonian H, the generator of infinitesimal time quantum transformations.

As the kernel $\mathcal{L}$ is the key to everything that follows, we shall give it a name, and refer to it and its generalizations as the *evolution operator* for a d-dimensional map or a d-dimensional flow.

periodic!point
exponential proliferation
self-similar

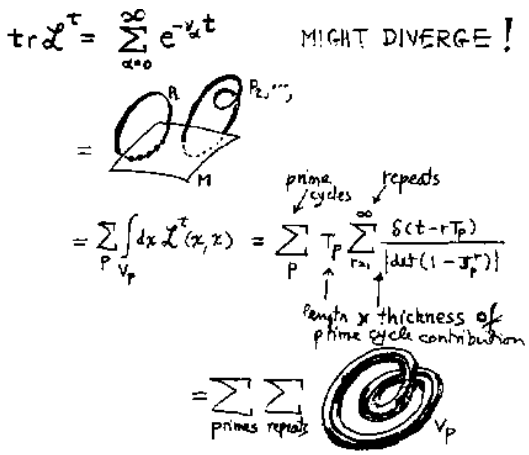

Figure 1.11: The trace of an evolution operator is concentrated in tubes around prime cycles, of length T_p and thickness $1/|\Lambda_p|^r$ for the rth repetition of the prime cycle p.

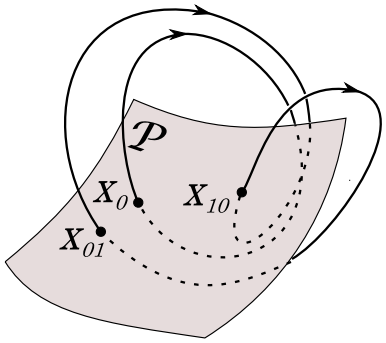

Figure 1.12: Replaces first part of figure 1.11. Tubes still to be drawn

The number of periodic points increases exponentially with the cycle length (in the case at hand, as 2^n). As we have already seen, this exponential proliferation of cycles is not as dangerous as it might seem; as a matter of fact, all our computations will be carried out in the $n \to \infty$ limit. Though a quick look at chaotic dynamics might reveal it to be complex beyond belief, it is still generated by a simple deterministic law, and with some luck and insight, our labeling of possible motions will reflect this simplicity. If the rule that gets us from one level of the classification hierarchy to the next does not depend strongly on the level, the resulting hierarchy is approximately self-similar. We now turn such approximate self-similarity to our advantage, by turning it into an operation, the action of the evolution operator, whose iteration encodes the self-similarity.

1.6.1 Trace formula

> In physics, when we do not understand something, we give it a name.
>
> Matthias Neubert

⇓PRELIMINARY

⇑PRELIMINARY

Recasting dynamics in terms of evolution operators changes everything. So far our formulation has been heuristic, but in the evolution operator

remove top 2 lines of figure 1.11

Poisson!resummation
Laplace!transform
trace!formula!classical

formalism the escape rate and any other dynamical average are given by exact formulas, extracted from the spectra of evolution operators. The key tools are *trace formulas* and *spectral determinants*.

The trace of an operator is given by the sum of its eigenvalues. The explicit expression (1.12) for $\mathcal{L}^t(x,y)$ enables us to evaluate the trace. Identify y with x and integrate x over the whole state space. The result is an expression for $\operatorname{tr}\mathcal{L}^t$ as a sum over neighborhoods of prime cycles p and their repetitions

☞ sect. 14.3

$$\operatorname{tr}\mathcal{L}^t = \sum_p T_p \sum_{r=1}^{\infty} \frac{\delta(t-rT_p)}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|}\ . \tag{1.13}$$

This formula has a simple geometrical interpretation sketched in figure 1.11. After the rth return to a Poincaré section, the initial tube $\mathcal{M}_p$ has been stretched out along the expanding eigendirections, with the overlap with the initial volume given by $1/\left|\det\left(\mathbf{1}-M_p^r\right)\right| \to 1/|\Lambda_p|$, the same weight we obtained heuristically in sect. 1.5.1.

The "spiky" sum (1.13) is disquieting in the way reminiscent of the Poisson resummation formulas of Fourier analysis; the left-hand side is the smooth eigenvalue sum $\operatorname{tr} e^{\mathcal{A}t} = \sum e^{s_\alpha t}$, while the right-hand side equals zero everywhere except for the set $t = rT_p$. [2] A Laplace transform smooths the sum over Dirac delta functions in cycle periods and yields the *trace formula* for the eigenspectrum $s_0, s_1, \cdots$ of the classical evolution operator:

$$\begin{aligned}\int_{0_+}^{\infty} dt\, e^{-st}\operatorname{tr}\mathcal{L}^t &= \operatorname{tr}\frac{1}{s-\mathcal{A}} = \\ \sum_{\alpha=0}^{\infty}\frac{1}{s-s_\alpha} &= \sum_p T_p\sum_{r=1}^{\infty}\frac{e^{r(\beta\cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|}\,.\end{aligned} \tag{1.14}$$

☞ sect. 14.1

The beauty of trace formulas lies in the fact that everything on the right-hand-side – prime cycles p, their periods T_p and the stability eigenvalues of M_p – is an invariant property of the flow, independent of any coordinate choice.

1.6.2 Spectral determinant

The eigenvalues of a linear operator are given by the zeros of the appropriate determinant. One way to evaluate determinants is to expand them in terms of traces, using the identities

4.1
page 78

$$\begin{aligned}\ln\det(s-\mathcal{A}) &= \operatorname{tr}\ln(s-\mathcal{A}) \\ \frac{d}{ds}\ln\det(s-\mathcal{A}) &= \operatorname{tr}\frac{1}{s-\mathcal{A}}\,,\end{aligned} \tag{1.15}$$

[2] Whelan: Inconsistent notation of eigenvalues compared to the previous figure.

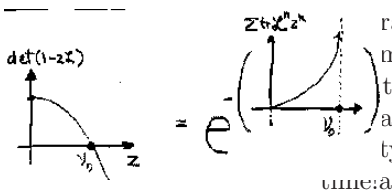

Figure 1.13: Spectral determinant is preferable to the trace as it vanishes smoothly at the leading eigenvalue, while the trace formula diverges.

ral!determinant
minant!spectral
tical mechanics
anics!statistical
ty!evolution
time:arrow of
irreversibility

and integrating over s. In this way the *spectral determinant* of an evolution operator becomes related to the traces that we have just computed:

chapter 15

$$\det(s-\mathcal{A}) = \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{-sT_p r}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|}\right) \ . \tag{1.16}$$

The $1/r$ factor is due to the s integration, leading to the replacement $T_p \to T_p/rT_p$ in the periodic orbit expansion (1.14).

The motivation for recasting the eigenvalue problem in this form is sketched in figure 1.13; exponentiation improves analyticity and trades in a divergence of the trace sum for a zero of the spectral determinant. In this way we have retaced the heuristic derivation of the divergent sum (1.6) and the dynamical zeta function (1.9), but this time with no approximations: formula (1.16) is *exact*. The computation of the zeros of $\det(s-\mathcal{A})$ proceeds very much like the computations of sect. 1.5.3.

sect. 15.5

1.7 From chaos to statistical mechanics

find a Boltzman suicide quote

The replacement of dynamics of individual trajectories by evolution operators which propagate densities feels like a bit of mathematical voodoo. Actually, something very radical has taken place. Consider a chaotic flow, such as the stirring of red and white paint by some deterministic machine. *If* we were able to track individual trajectories, the fluid would forever remain a striated combination of pure white and pure red; there would be no pink. What is more, if we reversed the stirring, we would return to the perfect white/red separation. However, that cannot be – in a very few turns of the stirring stick the thickness of the layers goes from centimeters to Ångströms, and the result is irreversibly pink.

Understanding the distinction between evolution of individual trajectories and the evolution of the densities of trajectories is key to understanding statistical mechanics – this is the conceptual basis of the second law of thermodynamics, and the origin of irreversibility of the arrow of time for deterministic systems with time-reversible equations of motion: reversibility is attainable for distributions whose measure in the space of density functions goes exponentially to zero with time.

By going to a description in terms of the asymptotic time evolution operators we give up tracking individual trajectories for long times, by trading

Boltzmann, L
stosszahlansatz
hyperbolic!non-
Roux!Henri
☞ chapter 24

it in for a very effective description of the asymptotic trajectory densities. This will enable us, for example, to give exact formulas for transport coefficients such as the diffusion constants without *any* probabilistic assumptions (in contrast to the *stosszahlansatz* of Boltzmann).

A century ago it seemed reasonable to assume that statistical mechanics applies only to systems with very many degrees of freedom. More recent is the realization that much of statistical mechanics follows from chaotic dynamics, and already at the level of a few degrees of freedom the evolution of densities is irreversible. Furthermore, the theory that we shall develop here generalizes notions of "measure" and "averaging" to systems far from equilibrium, and transports us into regions hitherto inaccessible with the tools of equilibrium statistical mechanics.

The concepts of equilibrium statistical mechanics do help us, however, to understand the ways in which the simple-minded periodic orbit theory falters. A non-hyperbolicity of the dynamics manifests itself in power-law correlations and even "phase transitions".

☞ chapter 21
⇓PRELIMINARY
☞ sect. 30.1
⇑PRELIMINARY
add neurosciences section

1.8 A guide to the literature

> But the power of instruction is seldom of much efficacy, except in those happy dispositions where it is almost superfluous.
>
> Gibbon

This text aims to bridge the gap between the physics and mathematics dynamical systems literature. The intended audience is Henri Roux, the perfect physics graduate student with a theoretical bent who does not believe anything he is told. As a complementary presentation we recommend Gaspard's monograph [1.8] which covers much of the same ground in a highly readable and scholarly manner.

As far as the prerequisites are concerned - ChaosBook is not an introduction to nonlinear dynamics. Nonlinear science requires a one semester basic course (advanced undergraduate or first year graduate). A good start is the textbook by Strogatz [1.9], an introduction to flows, fixed points, manifolds, bifurcations. It is the most accessible introduction to nonlinear dynamics - a book on differential equations in nonlinear disguise, and its broadly chosen examples and many exercises make it a favorite with students. It is not strong on chaos. There the textbook of Alligood, Sauer and Yorke [1.10] is preferable: an elegant introduction to maps, chaos, period doubling, symbolic dynamics, fractals, dimensions - a good companion to ChaosBook. Introductions more comfortable to physicists are the textbooks by Ott [1.12], with the baker's map used to illustrate many key techniques in analysis of chaotic systems, and by Tél and M. Gruiz [1.11], where chaotic dynamics is introduced through classical mechanics. They are perhaps harder than the above two as first books on nonlinear dynamics. Sprott's

Henri Roux: "You do not do *bifurcations*?" PC: "No, we do not do bifurcations here. You should already know all about bifurcations."

textbook [1.13] is a very useful compendium of the '70s and onward "chaos" literature which we, in the spirit of promises made in sect. 1.1, tend to pass over in silence.

chaos!deterministic
spatiotemporal dynamics
dynamics!spatiotemporal
chaos!quantum

[3]

An introductory course should give students skills in qualitative and numerical analysis of dynamical systems for short times (trajectories, fixed points, bifurcations) and familiarize them with Cantor sets and symbolic dynamics for chaotic systems. A good introduction to numerical experimentation with physically realistic systems is Tufillaro, Abbott, and Reilly [1.14]. Korsch and Jodl [1.15] and Nusse and Yorke [1.16] also emphasize hands-on approach to dynamics. With this, and a graduate level-exposure to statistical mechanics, partial differential equations and quantum mechanics, the stage is set for any of the one-semester advanced courses based on ChaosBook. The courses taught so far start out with the introductory chapters on qualitative dynamics, symbolic dynamics and flows, and then continue in different directions:

Deterministic chaos. Chaotic averaging, evolution operators, trace formulas, zeta functions, cycle expansions, Lyapunov exponents, billiards, transport coefficients, thermodynamic formalism, period doubling, renormalization operators.

A graduate level introduction to statistical mechanics from the dynamical point view is given by Dorfman [1.31]; the Gaspard monograph [1.8] covers the same ground in more depth. Driebe monograph [1.32] offers a nice introduction to the problem of irreversibility in dynamics. The role of "chaos" in statistical mechanics is critically dissected by Bricmont in his highly readable essay *"Science of Chaos or Chaos in Science?"* [1.33]. [4]

Spatiotemporal dynamical systems. Partial differential equations for dissipative systems, weak amplitude expansions, normal forms, symmetries and bifurcations, pseudospectral methods, spatiotemporal chaos.

☞ chapter 31

Quantum chaos. Semiclassical propagators, density of states, trace formulas, semiclassical spectral determinants, billiards, semiclassical helium, diffraction, creeping, tunneling, higher-order $\hbar$ corrections. For more on this topic, hop to the quantum chaos introduction, chapter 31.

This book puts more emphasis on periodic orbit theory than any other current nonlinear dynamics texbook. The role of unstable periodic orbits was already fully appreciated by Poincaré [1.17, 1.18], who noted that hidden in the apparent chaos is a rigid skeleton, a tree of *cycles* (periodic orbits) of increasing lengths and self-similar structure, and suggested that the cycles should be the key to chaotic dynamics. Periodic orbits have been at core of much of the mathematical work on the theory of the classical and

[3] Mason: Guckenheimer-Holmes? Lichtenberg-Lieberman? PC: I have not found them useful

[4] Whelan: There is a neat little book by Ruelle on the same topic. Unfortunately I gave it away so can't give the reference. PC: I have it, but have only skimmed through

Smale, S.
Sinai, Ya.
Bowen, R.
Ruelle, D.
Diffusion Limited Aggregates

quantum dynamical systems ever since. We refer the reader to the reprint selection [1.19] for an overview of some of that literature.

1.8.1 What is not in ChaosBook

This book offers a breach into a domain hitherto reputed unreachable, a domain traditionally traversed only by mathematical physicists and pure mathematicians. What distinguishes it from pure mathematics is the insistence on computability and numerical convergence of methods offered. A rigorous proof, the end of the story as far as a mathematician is concerned, might state that in a given setting, for times in excess of 10^{32} years, turbulent dynamics settles onto an attractor of dimension less than 600. Such a theorem is of a little use for a working scientist, especially if a numerical experiment indicates that within the span of the best simulation the dynamics seems to have settled on a (transient?) attractor of dimension less than 3.

point to quote on 14?

If you find ChaosBook not rigorous enough, you should turn to the mathematics literature. The most extensive reference is the treatise by Katok and Hasselblatt [1.20], an impressive compendium of modern dynamical systems theory. The fundamental papers in this field, all still valuable reading, are Smale [1.21], Bowen [1.22] and Sinai [1.24]. Sinai's paper is prescient and offers a vision and a program that ties together dynamical systems and statistical mechanics. It is written for readers versed in statistical mechanics. For a dynamical systems exposition, consult Anosov and Sinai [1.23]. Markov partitions were introduced by Sinai in ref. [1.25]. The classical text (though certainly not an easy read) on the subject of dynamical zeta functions is Ruelle's *Statistical Mechanics, Thermodynamic Formalism* [1.26]. In Ruelle's monograph transfer operator technique (or the "Perron-Frobenius theory") and Smale's theory of hyperbolic flows are applied to zeta functions and correlation functions. The status of the theory from Ruelle's point of view is compactly summarized in his 1995 Pisa lectures [1.27]. Further excellent mathematical references on thermodynamic formalism are Parry and Pollicott's monograph [1.28] with emphasis on the symbolic dynamics aspects of the formalism, and Baladi's clear and compact reviews of the theory of dynamical zeta functions [1.29, 1.30].

ChaosBook resolutely skirts number-theoretical magic such as spaces of constant negative curvature, Poincaré tilings, modular domains, Selberg Zeta functions, Riemann hypothesis, ... Why? While this beautiful mathematics has been very inspirational, especially in studies of quantum chaos, almost no powerful method in its repertoire survives a transplant to a physical system that you are likely to care about.

ChaosBook skirts mathematics and empirical practice of fractal analysis, such as Hausdorff and fractal dimensions. Addison's introduction to fractal dimensions [1.35] offers a well-motivated entry into this field. While in studies of probabilistically assembled fractals such as Diffusion Limited Aggregates (DLA) better measures of "complexity" are lacking, for deterministic systems there are much better, physically motivated and experimentally

measurable quantities (escape rates, diffusion coefficients, spectrum of helium, ...) that we focus on here. brain, rat

If you were wandering while reading this introduction "what's up with rat brains?", the answer is yes indeed, there is a line of research in neuronal dynamics that focuses on possible unstable periodic states, described for example in ref. [1.36, 1.37, 1.38, 1.39].

A guide to exercises

> God can afford to make mistakes. So can Dada!
> Dadaist Manifesto

The essence of this subject is incommunicable in print; the only way to develop intuition about chaotic dynamics is by computing, and the reader is urged to try to work through the essential exercises. As not to fragment the text, the exercises are indicated by text margin boxes such as the one on this margin, and collected at the end of each chapter. The problems that you should do have **underlined titles**. The rest (**smaller type**) are optional. Difficult problems are marked by any number of *** stars. If you solve one of those, it is probably worth a publication. By the end of a (two-semester) course you should have completed at least three small projects: (a) compute everything for a one-dimensional repeller, (b) compute escape rate for a 3-disk game of pinball, (c) compute a part of the quantum 3-disk game of pinball, or the helium spectrum, or if you are interested in statistical rather than the quantum mechanics, compute a transport coefficient. The essential steps are:

18.2 page 359

- **Dynamics**
  1. count prime cycles, exercise 1.1, exercise 11.1, exercise 11.5
  2. pinball simulator, exercise 6.1, exercise 17.5
  3. pinball stability, exercise 8.1, exercise 17.5
  4. pinball periodic orbits, exercise 17.7, exercise 17.6
  5. helium integrator, exercise 2.11, exercise 17.9
  6. helium periodic orbits, exercise 41.4, exercise 17.10

⇓PRELIMINARY ⇑PRELIMINARY

- **Averaging, numerical**
  1. pinball escape rate, exercise 10.4
  2. Lyapunov exponent, exercise 20.2 , or pressure, exercise ?? or exercise ??.

⇓PRELIMINARY ⇑PRELIMINARY

- **Averaging, periodic orbits**
  1. cycle expansions, exercise 18.1, exercise 18.2
  2. pinball escape rate, exercise 18.5, exercise 18.6

curvature!expansion
dynamical
system!smooth
smooth!dynamics
add more
⇑PRELIMINARY
⇓PRELIMINARY
⇑PRELIMINARY

put her a link toSect{s:contributors}

3. cycle expansions for averages, exercise 18.1, exercise 19.3
4. cycle expansions for diffusion, exercise 24.1
5. pruning, Markov graphs, exercise 13.7
6. desymmetrization exercise 22.1
7. intermittency, phase transitions, exercise 21.8
8. semiclassical quantization exercise 38.3
9. ortho-, para-helium, lowest eigenenergies exercise 41.7

make appendix into exercise
⇓PRELIMINARY

Solutions for some of the problems are given in appendix R. A clean solution, a pretty figure, or a nice exercise that you contribute to ChaosBook will be gratefully acknowledged. Often going through a solution is more instructive than reading the chapter that problem is supposed to illustrate.

Résumé

This text is an exposition of the best of all possible theories of deterministic chaos, and the strategy is: 1) count, 2) weigh, 3) add up.

In a chaotic system any open ball of initial conditions, no matter how small, will spread over the entire accessible state space. Hence the theory focuses on describing the geometry of the space of possible outcomes, and evaluating averages over this space, rather than attempting the impossible: precise prediction of individual trajectories. The dynamics of densities of trajectories is described in terms of evolution operators. In the evolution operator formalism the dynamical averages are given by exact formulas, extracted from the spectra of evolution operators. The key tools are *trace formulas* and *spectral determinants*.

The theory of evaluation of the spectra of evolution operators presented here is based on the observation that the motion in dynamical systems of few degrees of freedom is often organized around a few *fundamental* cycles. These short cycles capture the skeletal topology of the motion on a strange attractor/repeller in the sense that any long orbit can approximately be pieced together from the nearby periodic orbits of finite length. This notion is made precise by approximating orbits by prime cycles, and evaluating the associated curvatures. A curvature measures the deviation of a longer cycle from its approximation by shorter cycles; smoothness and the local instability of the flow implies exponential (or faster) fall-off for (almost) all curvatures. Cycle expansions offer an efficient method for evaluating classical and quantum observables.

The critical step in the derivation of the dynamical zeta function was the hyperbolicity assumption, i.e., the assumption of exponential shrinkage of all strips of the pinball repeller. By dropping the a_i prefactors in (1.4), we have given up on any possibility of recovering the precise distribution of starting x (which should anyhow be impossible due to the exponential

growth of errors), but in exchange we gain an effective description of the asymptotic behavior of the system. The pleasant surprise of cycle expansions (1.9) is that the infinite time behavior of an unstable system is as easy to determine as the short time behavior.

averaging
weight!multiplicative

To keep the exposition simple we have here illustrated the utility of cycles and their curvatures by a pinball game, but topics covered in ChaosBook – unstable flows, Poincaré sections, Smale horseshoes, symbolic dynamics, pruning, discrete symmetries, periodic orbits, averaging over chaotic sets, evolution operators, dynamical zeta functions, spectral determinants, cycle expansions, quantum trace formulas, zeta functions, and so on to the semiclassical quantization of helium – should give the reader some confidence in the broad sway of the theory. The formalism should work for any average over any chaotic set which satisfies two conditions:

1. the weight associated with the observable under consideration is multiplicative along the trajectory,

2. the set is organized in such a way that the nearby points in the symbolic dynamics have nearby weights.

The theory is applicable to evaluation of a broad class of quantities characterizing chaotic systems, such as the escape rates, Lyapunov exponents, transport coefficients and quantum eigenvalues. A big surprise is that the quantum mechanics of systems classically chaotic is very much like the classical mechanics of chaotic systems; both are described by zeta functions and cycle expansions of the same form, with the same dependence on the topology of the classical flow.

⇓PRELIMINARY

This success points to the most serious problem in classical chaos, namely the problem of Mandelbrot. At time of writing this text, there is still no solution in sight.

⇑PRELIMINARY

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complete Leibniz ref.

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Exercises

three-disk@3-disk!symbolic dynamics
Check the IOP exercises format...
symbolic dynamics!3-disk
Sensitivity!initial conditions

Exercise 1.1 **3-disk symbolic dynamics.** *As periodic trajectories will turn out to be our main tool to breach deep into the realm of chaos, it pays to start familiarizing oneself with them now by sketching and counting the few shortest prime cycles (we return to this in sect. 13.4). Show that the 3-disk pinball has $3 \cdot 2^n$ itineraries of length n. List periodic orbits of lengths 2, 3, 4, 5, $\cdots$. Verify that the shortest 3-disk prime cycles are 12, 13, 23, 123, 132, 1213, 1232, 1323, 12123, $\cdots$. Try to sketch them.*

Exercise 1.2 **Sensitivity to initial conditions.** *Assume that two pinball trajectories start out parallel, but separated by 1 Ångström, and the disks are of radius $a = 1$ cm and center-to-center separation $R = 6$ cm. Try to estimate in how many bounces the separation will grow to the size of system (assuming that the trajectories have been picked so they remain trapped for at least that long). Estimate the Who's* Pinball Wizard *'s typical score (number of bounces) in a game without cheating, by hook or crook (by the end of chapter 18 you should be in position to make very accurate estimates).*

dynamical!system

Chapter 2

Go with the flow

Poetry is what is lost in translation.
Robert Frost

(R. Mainieri, P. Cvitanović and E.A. Spiegel) [1]

this chapter essentially finished

We start out with a recapitulation of the basic notions of dynamics. Our aim is narrow; we keep the exposition focused on prerequisites to the applications to be developed in this text. We assume that the reader is familiar with dynamics on the level of the introductory texts mentioned in sect. 1.8, and concentrate here on developing intuition about what a dynamical system can do. It will be a coarse brush sketch - a full description of all possible behaviors of dynamical systems is beyond human ken. Anyway, for a novice there is no shortcut through this lengthy detour; a sophisticated traveler might prefer to skip this well-trodden territory and embark upon the journey at chapter 9.

fast track:
chapter 9, p. 131

2.1 Dynamical systems

to Benny: "mirror" aludes to the Leibnitz quote

In a dynamical system we observe the world as a function of time. We express our observations as numbers and record how they change with time; given sufficiently detailed information and understanding of the underlying natural laws, we see the future in the present as in a mirror. The motion of the planets against the celestial firmament provides an example. Against the daily motion of the stars from East to West, the planets distinguish themselves by moving among the fixed stars. Ancients discovered that by knowing a sequence of planet's positions - latitudes and longitudes - its future position could be predicted.

sect. 1.3

[1] Lautrup: improve language twrd end of chapter - too "kort for hovedet"

state space
phase space
representative point
deterministic!dynamics
dynamical!system!deterministic
state space
dynamical!system
dynamical!system!smooth
smooth!dynamics
flow
iteration
map

For the solar system, tracking the latitude and longitude in the celestial sphere suffices to completely specify the planet's apparent motion. All possible values for positions and velocities of the planets form the *phase space* of the system. More generally, a state of a physical system, at a given instant in time, can be represented by a single point in an abstract space called *state space* or *phase space* $\mathcal{M}$. As the system changes, so does the *representative point* in state space. We refer to the evolution of such points as *dynamics*, and the function f^t which specifies where the representative point is at time t as the *evolution rule*.

If there is a definite rule f that tells us how this representative point moves in $\mathcal{M}$, the system is said to be deterministic. For a deterministic dynamical system, the evolution rule takes one point of the state space and maps it into exactly one point. However, this is not always possible. For example, knowing the temperature today is not enough to predict the temperature tomorrow; knowing the value of a stock today will not determine its value tomorrow. The state space can be enlarged, in the hope that in a sufficiently large state space it is possible to determine an evolution rule, so we imagine that knowing the state of the atmosphere, measured over many points over the entire planet should be sufficient to determine the temperature tomorrow. Even that is not quite true, and we are less hopeful when it comes to stocks.

For a deterministic system almost every point has a unique future, so trajectories cannot intersect. We say "almost" because there might exist a set of measure zero (tips of wedges, cusps, etc.) for which a trajectory is not defined. We may think such sets a nuisance, but it is quite the contrary - they will enable us to partition state space, so that the dynamics can be better understood.

☞ chapter 12

Locally, the state space $\mathcal{M}$ looks like $\mathbb{R}^d$, meaning that d numbers are sufficient to determine what will happen next. Globally, it may be a more complicated manifold formed by patching together several pieces of $\mathbb{R}^d$, forming a torus, a cylinder, or some other geometric object. When we need to stress that the dimension d of $\mathcal{M}$ is greater than one, we may refer to the point $x \in \mathcal{M}$ as x_i where $i = 1, 2, 3, \ldots, d$. The evolution rule $f^t : \mathcal{M} \to \mathcal{M}$ tells us where a point x is in $\mathcal{M}$ after a time interval t. The pair $(\mathcal{M}, f)$ constitute a *dynamical system*.

The dynamical systems we will be studying are smooth. This is expressed mathematically by saying that the evolution rule f^t can be differentiated as many times as needed. Its action on a point x is sometimes indicated by $f(x, t)$ to remind us that f is really a function of two variables: the time and a point in state space. Note that time is relative rather than absolute, so only the time interval is necessary. This follows from the fact that a point in state space completely determines all future evolution, and it is not necessary to know anything else. The time parameter can be a real variable ($t \in \mathbb{R}$), in which case the evolution is called a *flow*, or an integer ($t \in \mathbb{Z}$), in which case the evolution advances in discrete steps in time, given by *iteration* of a *map*. Actually, the evolution parameter need not

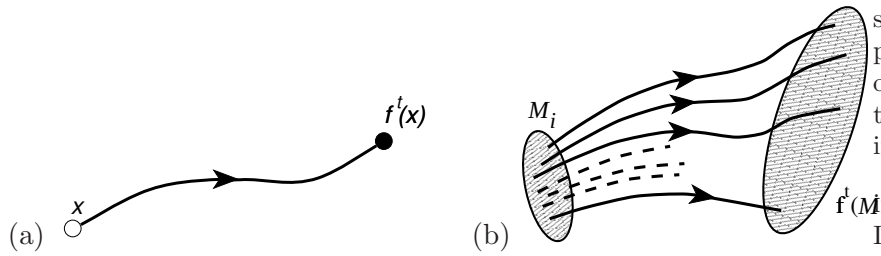


Figure 2.1: (a) A trajectory traced out by the evolution rule f^t. Starting from the state space point x, after a time t, the point is at $f^t(x)$. (b) The evolution rule f^t can be used to map a region $\mathcal{M}_i$ of the state space into the region $f^t(\mathcal{M}_i)$.

spatial!profile
profile!spatial
orbit
trajectory
initial!point
$"xInit$
initial!state $"xInit$
Lagrangian!coordinates
Eulerian!coordinates
orbit
wandering point
point!wandering

be the physical time; for example, a time-stationary solution of a partial differential equation is parametrized by spatial variables. In such situations one talks of a "spatial profile" rather than a "flow".

Nature provides us with innumerable dynamical systems. They manifest themselves through their trajectories: given an initial point x_0, the evolution rule traces out a sequence of points $x(t) = f^t(x_0)$, the *trajectory* through the point $x_0 = x(0)$. A trajectory is parameterized by the time t and thus belongs to $(f^t(x_0), t) \in \mathcal{M} \times \mathbb{R}$. By extension, we can also talk of the evolution of a region $\mathcal{M}_i$ of the state space: just apply f^t to every point in $\mathcal{M}_i$ to obtain a new region $f^t(\mathcal{M}_i)$, as in figure 2.1.

2.1
page 45

Because f^t is a single-valued function, any point of the trajectory can be used to label the trajectory. If we mark the trajectory by its inital point x_0, we are describing it in the *Lagrangian coordinates*. We can regard the transport of the material point at $t = 0$ to its current point $x(t) = f^t(x_0)$ as a coordinate transformation from the Lagrangian coordinates to the *Eulerian coordinates*.

The subset of points in $\mathcal{M}$ that belong to the (possibly infinite) trajectory of a given point x_0 is called the *orbit* of x_0; we shall talk about forward orbits, backward orbits, periodic orbits, etc.. For a flow, an orbit is a smooth continuous curve; for a map, it is a sequence of points.

What are the possible trajectories? This is a grand question, and there are many answers, chapters to follow offering some. Here is the first attempt to classify all possible trajectories:

stationary:	$f^t(x) = x$	for all t
periodic:	$f^t(x) = f^{t+T_p}(x)$	for a given minimum period T_p
aperiodic:	$f^t(x) \neq f^{t'}(x)$	for all $t \neq t'$.

a little picture here of the three trajectory types.

The ancients tried to make sense of all dynamics in terms of periodic motions; epicycles, integrable systems. The embarassing truth is that for a generic dynamical systems almost all motions are aperiodic. So we refine the classification by dividing aperiodic motions into two subtypes: those that wander off, and those that keep coming back.

transient
periodic!orbit
orbit!periodic
quasiperiodicity
torus
incommesurate
recurrence
point!non-wandering
non-wandering set
contracting!flow
flow!contracting
attractor
Lyapunov functional
functional!Lyapunov
basin of attraction
attractor!basin
attractor!strange
strange!attractor

A point $x \in \mathcal{M}$ is called a *wandering point*, if there exists an open neighborhood $\mathcal{M}_0$ of x to which the trajectory never returns

$$f^t(x) \notin \mathcal{M}_0 \qquad \text{for all} \quad t > t_{\min}\,. \tag{2.1}$$

In physics literature, the dynamics of such state is often referred to as *transient*.

A *periodic orbit* (or a *cycle*) corresponds to a trajectory that returns exactly to the initial point in a finite time. Periodic orbits form a very small subset of the state space, in the same sense that rational numbers are a set of zero measure on the unit interval.

Periodic orbits and stationary points are the simplest examples of "non-wandering" invariant sets preserved by dynamics. Dynamics can also preserve higher-dimensional smooth compact invariant manifolds; most commonly encountered are the M-dimensional tori of Hamiltonian dynamics, with notion of periodic motion generalized to quasiperiodic (superposition of M incommesurate frequencies) motion on a smooth torus.

draw a quasi periodic motion on a torus

For times much longer than a typical "turnover" time, it makes sense to relax the notion of exact (quasi)periodicity, and replace it by the notion of *recurrence*. A point is *recurrent* or *non-wandering* if for any open neighborhood $\mathcal{M}_0$ of x and any time t_{min} there exists a later time t, such that

$$f^t(x) \in \mathcal{M}_0\,. \tag{2.2}$$

In other words, the trajectory of a non-wandering point reenters the neighborhood $\mathcal{M}_0$ infinitely often. We shall denote by Ω the *non–wandering set* of f, i.e., the union of all the non-wandering points of $\mathcal{M}$. The set Ω, the non–wandering set of f, is the key to understanding the long-time behavior of a dynamical system; all calculations undertaken here will be carried out on non–wandering sets.

So much about individual trajectories. What about clouds of initial points? If there exists a connected state space volume that maps into itself under forward evolution (and you can prove that by the method of Lyapunov functionals, or several other methods available in the literature), the flow is globally contracting onto a subset of $\mathcal{M}$ which we shall refer to as the *attractor*. The attractor may be unique, or there can coexist any number of distinct attracting sets, each with its own *basin of attraction*, the set of all points that fall into the attractor under foward evolution. The attractor can be a fixed point, a periodic orbit, aperiodic, or any combination of the above. The most interesting case is that of an aperiodic recurrent attractor, to which we shall refer loosely as a *strange attractor*. We say 'loosely', as will soon become apparent that diagnosing and proving existence of a genuine, card-carrying strange attractor is a highly nontrivial undertaking. [2]

☞ example 2.2

[2] Mason: do you distinguish strange chaotic attractor strange non-chaotic attractor, in spirit of Merry Landia? PC: no.

Conversely, if we can enclose the non–wandering set Ω by a connected state space volume $\mathcal{M}_0$ and then show that almost all points within $\mathcal{M}_0$, but not in Ω, eventually exit $\mathcal{M}_0$, we refer to the non–wandering set Ω as a *repeller*. An example of a repeller is not hard to come by - the pinball game of sect. 1.3 is a simple chaotic repeller.

repeller
flow
irreversible!dynamics
dynamics!irreversible
semigroup!dynamical
group!dynamical
Lyapunov time
reversible!dynamics
dynamics!reversible

It would seem, having said that the periodic points are so exceptional that almost all non-wandering points are aperiodic, that we have given up the ancients' fixation on periodic motions. Nothing could be further from truth. As longer and longer cycles approximate more and more accurately finite segments of aperiodic trajectories, we shall establish control over non–wandering sets by defining them as the closures of the union of all periodic points.

Before we can work out an example of a non–wandering set and get a better grip on what chaotic motion might look like, we need to ponder flows in a little more depth.

2.2 Flows

> There is no beauty without some strangeness.
> William Blake

crosscheck Blake quote

A *flow* is a continuous-time dynamical system. The evolution rule f^t is a family of mappings of $\mathcal{M} \to \mathcal{M}$ parameterized by $t \in \mathbb{R}$. Because t represents a time interval, any family of mappings that forms an evolution rule must satisfy:

2.2
page 45

this excludes billiards?

(a) $f^0(x) = x$ (in 0 time there is no motion)

(b) $f^t(f^{t'}(x)) = f^{t+t'}(x)$ (the evolution law is the same at all times)

(c) the mapping $(x,t) \mapsto f^t(x)$ from $\mathcal{M} \times \mathbb{R}$ into $\mathcal{M}$ is continuous.

The family of mappings $f^t(x)$ thus forms a continuous (forward semi-) group. Why "semi-"group? It may fail to form a group if the dynamics is not reversible, and the rule $f^t(x)$ cannot be used to rerun the dynamics backwards in time, with negative t; with no reversibility, we cannot define the inverse $f^{-t}(f^t(x)) = f^0(x) = x$, in which case the family of mappings $f^t(x)$ does not form a group. In exceedingly many situations of interest - for times beyond the Lyapunov time, for asymptotic attractors, for dissipative partial differential equations, for systems with noise, for non-invertible maps - the dynamics cannot be run backwards in time, hence, the circumspect emphasis on *semi*groups. On the other hand, there are many settings of physical interest, where dynamics is reversible (such as finite-dimensional Hamiltonian flows), and where the family of evolution maps f^t does form a group.

⇓PRELIMINARY
☞ sect. 25.1
⇑PRELIMINARY

vector field
tangent!bundle
Duffing oscillator

For infinitesimal times, flows can be defined by differential equations. We write a trajectory as

$$x(t+\tau) = f^{t+\tau}(x_0) = f(f(x_0,t),\tau) \tag{2.3}$$

and express the time derivative of a trajectory at point $x(t)$,

$$\left.\frac{dx}{d\tau}\right|_{\tau=0} = \partial_\tau f(f(x_0,t),\tau)|_{\tau=0} = \dot{x}(t)\,. \tag{2.4}$$

2.3
page 45

as the time derivative of the evolution rule, a vector evaluated at the same point. By considering all possible trajectories, we obtain the vector $\dot{x}(t)$ at any point $x \in \mathcal{M}$. This *vector field* is a (generalized) velocity field:

$$v(x) = \dot{x}(t)\,. \tag{2.5}$$

Newton's laws, Lagrange's method, or Hamilton's method are all familiar procedures for obtaining a set of differential equations for the vector field $v(x)$ that describes the evolution of a mechanical system. Equations of mechanics may appear different in form from (2.5), as they are often involve higher time derivatives, but an equation that is second or higher order in time can always be rewritten as a set of first order equations.

We are concerned here with a much larger world of general flows, mechanical or not, defined by a time-independent vector field (2.5). At each point of the state space a vector indicates the local direction in which the orbit evolves. The length of the vector $|v(x)|$ is proportional to the speed at the point x, and the direction and length of $v(x)$ changes from point to point. When the state space is a complicated manifold embedded in $\mathbb{R}^d$, one can no longer think of the vector field as being embedded in the state space. Instead, we have to imagine that each point x of state space has a different tangent plane $T\mathcal{M}_x$ attached to it. The vector field lives in the union of all these tangent planes, a space called the *tangent bundle* $\mathbf{T}\mathcal{M}$.

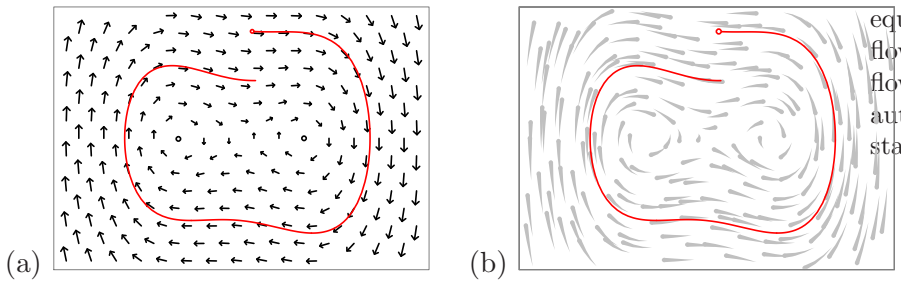
Example 2.1 A two-dimensional vector field $v(x)$**:** *A simple example of a flow is afforded by the unforced Duffing system*

$$\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= -0.15\,y(t) + x(t) - x(t)^3 \end{aligned} \tag{2.6}$$

plotted in figure 2.2. The velocity vectors are drawn superimposed over the configuration coordinates $(x(t), y(t))$ *of state space* $\mathcal{M}$*, but they belong to a different space, the* tangent bundle $\mathbf{T}\mathcal{M}$*.*

2.4
page 45

$$\text{If} \qquad v(x_*) = 0\,, \tag{2.7}$$



(a) (b)

equilibrium!point
flow!autonomous
flow!stationary
autonomous flow
stationary!flow

Figure 2.2: (a) The two-dimensional vector field for the Duffing system (2.6), together with a short trajectory segment. (b) The flow lines. Each "comet" represents the same time interval of a trajectory, starting at the tail and ending at the head. The longer the comet, the faster the flow in that region.

x_* is an *equilibrium point* (also referred to as a *stationary*, *fixed*, *critical*, *stagnation* point, *zero* of v, or *steady state*), and the trajectory remains forever stuck at x_*. Otherwise the trajectory passing through x_0 at time $t = 0$ can be obtained by integrating the equations (2.5):

$$x(t) = f^t(x_0) = x_0 + \int_0^t d\tau\, v(x(\tau))\,, \qquad x(0) = x_0\,. \tag{2.8}$$

We shall consider here only *autonomous* flows, i.e., flows for which the velocity field v_i is *stationary*, not explicitly dependent on time. A non-autonomous system

$$\frac{dy}{d\tau} = w(y,\tau)\,, \tag{2.9}$$

can always be converted into a system where time does not appear explicitly. To do so, extend ("suspend") state space to be $(d+1)$-dimensional by defining $x = \{y, \tau\}$, with a stationary vector field

$$v(x) = \begin{bmatrix} w(y,\tau) \\ 1 \end{bmatrix}\,. \tag{2.10}$$

The new flow $\dot{x} = v(x)$ is autonomous, and the trajectory $y(\tau)$ can be read off $x(t)$ by ignoring the last component of x.

2.5
page 46

Example 2.2 A flow with a strange attractor: *The Duffing flow of figure 2.2 is bit of a bore - every trajectory ends up in one of the two attractive equilibrium points. Let's construct a flow that does not die out, but exhibits a recurrent dynamics. Start with a harmonic oscillator*

$$\dot{x} = -y\,, \qquad \dot{y} = x\,. \tag{2.11}$$

The solutions are re^{it}, re^{-it}, and the whole x-y plane rotates with constant angular velocity $\theta = 1$, period $T = 2\pi$. Now make the system unstable by adding

$$\dot{x} = -y\,, \qquad \dot{y} = x + ay\,, \qquad a > 0\,. \tag{2.12}$$

Rossler@Rössler system
contracting!flow
flow!contracting
attractor!strange
strange!attractor

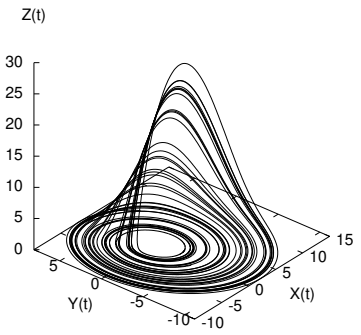


Figure 2.3: A trajectory of the Rössler flow at time $t = 250$. (G. Simon)

The plane is still rotating with constant angular velocity, but trajectories are now spiraling out. Any flow in the plane either escapes, falls into an attracting equilibrium point, or converges to a limit cycle. Richer dynamics requires at least one more dimension. In order to prevent the trajectory from escaping to ∞, kick it into 3rd dimension when x reaches some value c by adding

$$\dot{z} = b + z(x - c)\,, \qquad c > 0\,. \tag{2.13}$$

As x crosses c, z shoots upwards exponentially, $z \simeq e^{(x-c)t}$. In order to bring it back, start decreasing x by modifying its equation to

$$\dot{x} = -y - z\,.$$

Large z drives the trajectory toward $x = 0$; there the exponential contraction by e^{-ct} kicks in, and the trajectory drops back toward the x-y plane. This frequently studied example of an autonomous flow is called the Rössler system *(for definitiveness we fix the parameters a, b, c in what follows):*

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\,, \qquad a = b = 0.2\,, \quad c = 5.7\,. \end{aligned} \tag{2.14}$$

2.9 page 46

The system is as simple as they get - it would be linear, were it not for the sole bilinear term zx. Even for so "simple" a system the nature of long-time solutions is far from obvious.

There are two repelling equilibrium points:

$$\begin{aligned} (x^-, y^-, z^-) &= (\,0.0070, -0.0351, 0.0351\,) \\ (x^+, y^+, z^+) &= (\,5.6929, -28.464, 28.464\,) \end{aligned} \tag{2.15}$$

One is close to the origin by construction - the other, some distance away, must exist because the equilibrium has a 2nd-order nonlinearity.

refer also to figure 11.7
change to sect 3.2

To see what other solutions look like we need to resort to numerical integration. A typical numerically integrated long-time trajectory is sketched in figure 2.3. As we shall show in sect. 4.1, for this flow any finite volume of initial conditions shrinks with time, so the flow is contracting. *Trajectories that start out sufficiently close to the origin seem to converge to a* strange attractor. *We say "seem", as there exists no proof that such an attractor is asymptotically aperiodic - it might well be that what we see is but a long transient on a way to an attractive periodic orbit. For now, accept that figure 2.3 and similar figures in what follows are examples of "strange attractors". (continued in exercise 2.9 and example 3.3)* *(Rytis Paškauskas)*

3.6 page 60

3

Duffing system
Rossler@Rössler system

fast track:
chapter 3, p. 49

2.3 Computing trajectories

> On two occasions I have been asked [by members of Parliament], 'Pray, Mr. Babbage, if you put into the machine wrong figures, will the right answers come out?' I am not able rightly to apprehend the kind of confusion of ideas that could provoke such a question.
> Charles Babbage

RM: expand

You have not learned dynamics unless you know how to integrate numerically whatever dynamical equations you face. Sooner or later, you need to implement some finite time-step prescription for integration of the equations of motion (2.5). The simplest is the Euler integrator which advances the trajectory by $\delta\tau \times$ velocity at each time step:

add figure

$$x_i \to x_i + v_i(x)\delta\tau\,. \tag{2.16}$$

This might suffice to get you started, but as soon as you need higher numerical accuracy, you will need something better. There are many excellent reference texts and computer programs that can help you learn how to solve differential equations numerically using sophisticated numerical tools, such as pseudo-spectral methods or implicit methods. If you are interested in Hamiltonian flows you might want to implement a symplectic integrator of type discussed in appendix E.2.1. If a "sophisticated" integration routine takes days and gobbles up terabits of memory, you are using brain-damaged high level software. Try writing a few lines of your own Runge-Kutta code in some mundane everyday language. While you absolutely need to master the requisite numerical methods, this is neither the time nor the place to expound upon them; how you learn them is your business. And if you have developed some nice routines for solving problems in this text or can point another student to some, let us know.

⇓PRELIMINARY

perhaps quote Numerical Recepies

⇑PRELIMINARY

2.7 page 46
2.8 page 46
2.10 page 47
2.11 page 47

In chapter 25 we shall dispose of the fear of "infinite-dimensional" dynamical systems - you might prefer to skip the section on first reading.

⇓PRELIMINARY
⇑PRELIMINARY

Commentary

Remark 2.1 Model ODE and PDE systems. The Duffing system (2.6) arises in the study of electronic circuits [2.2]. Rössler system was introduced in ref. [2.3] as a

⇓PRELIMINARY
⇑PRELIMINARY

[3] Mason: Warwick Tucker has proven rigorously via interval arithmetic that the

chaos!diagnostics
frequency analysis

simplified set of equations describing no particular physical system, but capturing the essence of chaos in a simplest imaginable smooth flow. Otto Rössler, a man of classical education, was inspired in this quest by that rarely cited grandfather of chaos, Anaxagoras (456 B.C.). This, and references to earlier work can be found in refs. [2.5, 2.7]. We recommend in particular the inimitable Abraham and Shaw illustrated classic [2.6] for its beautiful sketches of the Rössler and many other flows.

move this to Hamiltonian chapter

Remark 2.2 Diagnosing chaos. In sect. 1.3.1 we have stated that a deterministic system exhibits "chaos" if its dynamics is locally unstable (positive Lyapunov exponent) and globally mixing (positive entropy). In sect. 10.3 we shall define Lyapunov exponents, and discuss their evaluation, but already at this point it would be handy to have a few quick numerical methods to diagnose chaotic dynamics. Laskar's *frequency analysis* method [2.11] is useful for extracting quasi-periodic and weakly chaotic regions of state space in Hamiltonian dynamics with many degrees of freedom. For references to several other numerical methods, see ref. [2.12].

Remark 2.3 Dynamical systems software: J.D. Meiss [2.9] has many years maintained *Sci.nonlinear FAQ* which is now in part superseded by the SIAM Dynamical Systems website `www.dynamicalsystems.org`. The website glossary contains most of Meiss's FAQ plus new ones, and a up-to-date software list [2.10], with links to DSTool, xpp, AUTO, etc.. Springer on-line *Encyclopaedia of Mathematics* maintains links to Dynamical systems software packages on eom.springer.de/D/d130210.htm.

dig up Guckenheimer SIAM reference

Résumé

use in the maps.tex instead

Chaotic dynamics with a low-dimensional attractor can be visualized as a succession of nearly periodic but unstable motions. In the same spirit, turbulence in spatially extended systems can be described in terms of recurrent spatiotemporal patterns. Pictorially, dynamics drives a given spatially extended system through a repertoire of unstable patterns; as we watch a turbulent system evolve, every so often we catch a glimpse of a familiar pattern. For any finite spatial resolution and finite time the system follows approximately a pattern belonging to a finite repertoire of possible patterns, and the long-term dynamics can be thought of as a walk through the space of such patterns. Recasting this image into mathematics is the subject of this book.

Lorentz attractor is strange for the original parameters, and a long periodic orbit for the slightly different parameters. PC: find the reference

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FIND good Duffing oscillator ref

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move to another chapter

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The Lorenz model is investigated by means of the Carleman embedding. Attention is focused on the analytical properties of the Laplace transform of a trajectory. It is shown that beyond the turbulent threshold, finite linear embedding violates basic analytical requirements

[2.23] W.H. Steeb, *"Embedding of nonlinear finite dimensional systems in linear infinite dimensional systems and Bose operators"*, *Hadronic Journal* **6** 68, (1983).
Nonlinear autonomous systems x=f(x), where f is a polynomial in x, can be embedded into an infinite linear system with a technique described by Carleman (1932). The author demonstrates that the infinite linear system can be written with the help of Bose annihilation and creation operators

[2.24] K. Kowalski, *"Hilbert space description of classical dynamical systems"*, *Physica* **A 145**, 408 (1987).
Classical dynamical systems can be described by a Hilbert space, Schrödinger-like equation with the Hamiltonian (a non-Hermitian one) expressed in terms of Bose creation and annihilation operators. The presented formalism includes the Carleman embedding as the special case.

[2.25] K. Kowalski, *"Nonlinear dynamical systems and classical orthogonal polynomials"*, *J. Math. Phys.* **38**, 2483 (1997).
Nonlinear dynamical systems with analytic nonlinearities can be brought down to the abstract Schrödinger equation a in Hilbert space with a boson Hamiltonian. The Fourier coefficients of the expansion of solutions to the Schrödinger equation in the particular occupation number representation are expressed by means of the classical orthogonal polynomials. The introduced formalism amounts to a generalization of the classical methods for linearization of nonlinear differential equations such as the Carleman embedding technique and Koopman approach.

⇑PRELIMINARY

Exercises

evolution!group
group!evolution
ordinary differential equations!almost
differential equations!ordinary!almost
gradient!system
dynamical!system!gradient
potential!problems

The problems that you should do have **underlined titles**. The rest (**smaller type**) are optional. Difficult problems are marked by any number of *** stars.

Exercise 2.1 Trajectories do not intersect. *A trajectory in the state space $\mathcal{M}$ is the set of points one gets by evolving $x \in \mathcal{M}$ forwards and backwards in time:*

$$C_x = \{y \in \mathcal{M} : f^t(x) = y \text{ for } t \in \mathbb{R}\}\,.$$

Show that if two trajectories intersect, then they are the same curve.

Exercise 2.2 Evolution as a group. *The trajectory evolution f^t is a one-parameter (semi-)group where*

$$f^{t+s} = f^t \circ f^s\,.$$

Show that it is a commutative (semi-)group.

In this case, the commutative character of the (semi-)group of evolution functions comes from the commutative character of the time parameter under addition. Can you think of any other (semi-)group replacing time?

Exercise 2.3 Almost ODE's.

(a) *Consider the point x on $\mathbb{R}$ evolving according $\dot{x} = e^{\dot{x}}$. Is this an ordinary differential equation?*

(b) *Is $\dot{x} = x(x(t))$ an ordinary differential equation?*

(c) *What about $\dot{x} = x(t+1)$?*

Exercise 2.4 All equilibrium points are fixed points. *Show that a point of a vector field v where the velocity is zero is a fixed point of the dynamics f^t.*

4

Exercise 2.5 Gradient systems. *Gradient systems (or "potential problems") are a simple class of dynamical systems for which the velocity field is given by the gradient of an auxiliary function, the "potential" ϕ*

$$\dot{x} = -\nabla\phi(x)$$

where $x \in \mathbb{R}^d$, and ϕ is a function from that space to the reals $\mathbb{R}$.

(a) *Show that the velocity of the particle is in the direction of most rapid decrease of the function ϕ.*

(b) *Show that all extrema of ϕ are fixed points of the flow.*

three-disk@3-disk!state space
repeller!3-disk
state space!3-disk
Runge-Kutta integration
integration!Runge-Kutta
Rossler@Rössler system
↑PRELIMINARY
Rossler@Rössler system!equilibria
equilibrium!Rössler system

(c) *Show that it takes an infinite amount of time for the system to reach an equilibrium point.*

(d) *Show that there are no periodic orbits in gradient systems.*

↓PRELIMINARY

Exercise 2.6 3-disk repeller fit. *Show by numerical comparison that the $R{:}a = 6{:}1$ 3-disk repeller has a rather thin repeller, and can be roughly fit with $a \approx 20$ in (??).*

Exercise 2.7 Runge-Kutta integration. *Implement the fourth-order Runge-Kutta integration formula (see, for example, ref. [2.8]) for $\dot{x} = v(x)$:*

$$\begin{aligned} x_{n+1} &= x_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(\delta\tau^5) \\ k_1 &= \delta\tau v(x_n)\,, \qquad k_2 = \delta\tau v(x_n + k_1/2) \\ k_3 &= \delta\tau v(x_n + k_2/2)\,, \quad k_4 = \delta\tau v(x_n + k_3)\,. \end{aligned} \tag{2.17}$$

If you already know your Runge-Kutta, program what you believe to be a better numerical integration routine, and explain what is better about it.

Exercise 2.8 Rössler system. *Use the result of exercise 2.7 or some other integration routine to integrate numerically the Rössler system (2.14). Does the result look like a "strange attractor"?*

Exercise 2.9 Equilibria of the Rössler system.

(a) *Find all equilibrium points (x^q, y^q, z^q) of the Rössler system (2.14). How many are there?*

(b) *Assume that $b = a$. As we shall see, some surprisingly large, adn surprisingly small numbers arise in this system. In order to understand their size, introduce parameters*

really $(c, \epsilon, D, p^{\pm})$?

$$\epsilon = a/c\,, \qquad D = 1 - 4\epsilon^2\,, \qquad p^{\pm} = (1 \pm \sqrt{D})/2\,. \tag{2.18}$$

Express all the equilibria in terms of $(c, \epsilon, D, p^{\pm})$. Expand equilibria to the first order in ϵ. Note that it makes sense because for $a = b = 0.2$, $c = 5.7$ in (2.14), $\epsilon \approx 0.03$.

(continued as exercise 3.1)

(Rytis Paškauskas)

Exercise 2.10 Can you integrate me? *Integrating equations numerically is not for the faint of heart. It is not always possible to establish that a set of nonlinear ordinary differential equations has a solution for all times and there are many cases were the solution only exists for a limited time interval, as, for example, for the equation $\dot{x} = x^2\,, \qquad x(0) = 1\,.$*

[4] Mason: Make up Poincaré-Benedixon and Lasaller's theorem exercises

(a) *For what times do solutions of* *helium!collinear*

$$\dot{x} = x(x(t))$$

exist? Do you need a numerical routine to answer this question?

(b) *Let's test the integrator you wrote in exercise 2.7. The equation* $\ddot{x} = -x$ *with initial conditions* $x(0) = 2$ *and* $\dot{x} = 0$ *has as solution* $x(t) = e^{-t}(1 + e^{2t})$. *Can your integrator reproduce this solution for the interval* $t \in [0, 10]$*? Check you solution by plotting the error as compared to the exact result.*

(c) *Now we will try something a little harder. The equation is going to be third order*

$$\dddot{x} + 0.6\ddot{x} + \dot{x} - |x| + 1 = 0\,,$$

which can be checked - numerically - to be chaotic. As initial conditions we will always use $\ddot{x}(0) = \dot{x}(0) = x(0) = 0$. *Can you reproduce the result* $x(12) = 0.8462071873$ *(all digits are significant)? Even though the equation being integrated is chaotic, the time intervals are not long enough for the exponential separation of trajectories to be noticeble (the exponential growth factor is* ≈ 2.4*).*

(d) *Determine the time interval for which the solution of* $\dot{x} = x^2, x(0) = 1$ *exists.*

RM: Check this Lyapunov

Exercise 2.11 **Classical collinear helium dynamics.** *In order to apply periodic orbit theory to quantization of helium we shall need to compute classical periodic orbits of the helium system. In this exercise we commence their evaluation for the collinear helium atom (5.6)*

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{Z}{r_1} - \frac{Z}{r_2} + \frac{1}{r_1 + r_2}\,.$$

The nuclear charge for helium is $Z = 2$. *Colinear helium has only 3 degrees of freedom and the dynamics can be visualized as a motion in the* (r_1, r_2), $r_i \geq 0$ *quadrant. In* (r_1, r_2)*-coordinates the potential is singular for* $r_i \to 0$ *nucleus-electron collisions. These 2-body collisions can be regularized by rescaling the coordinates, with details given in sect. 7.3. In the transformed coordinates* (x_1, x_2, p_1, p_2) *the Hamiltonian equations of motion take the form*

$$\begin{aligned}
\dot{P}_1 &= 2Q_1\left[2 - \frac{P_2^2}{8} - Q_2^2(1 + \frac{Q_2^2}{R^4})\right]; \qquad \dot{Q}_1 = \frac{1}{4}P_1 Q_2^2 \\
\dot{P}_2 &= 2Q_2\left[2 - \frac{P_1^2}{8} - Q_1^2(1 + \frac{Q_1^2}{R^4})\right]; \qquad \dot{Q}_2 = \frac{1}{4}P_2 Q_1^2\,. \qquad (2.19)
\end{aligned}$$

where $R = (Q_1^2 + Q_2^2)^{1/2}$.

(a) *Integrate the equations of motion by the fourth order Runge-Kutta computer routine of exercise 2.7 (or whatever integration routine you like). A convenient way to visualize the 3-*d *state space orbit is by projecting it onto the 2-dimensional* $(r_1(t), r_2(t))$ *plane.*

(Gregor Tanner, Per Rosenqvist)

map
strobe method
Poincar"'e!section
section!Poincar"'e
map!return
Poincar"'e!return map
first return function

Chapter 3

Discrete time dynamics

Do it again!
Isabelle, age 3

(R. Mainieri and P. Cvitanović)

index whole chapter as "map"

The time parameter in the sect. 2.1 definition of a dynamical system can be either continuous or discrete. Discrete time dynamical systems arise naturally from flows; one can observe the flow at fixed time intervals (by strobing it), or one can record the coordinates of the flow when a special event happens (the Poincaré section method). This triggering event can be as simple as vanishing of one of the coordinates, or as complicated as the flow cutting through a curved hypersurface.

redraw figure 3.3

3.1 Poincaré sections

Successive trajectory intersections with a *Poincaré section*, a $(d-1)$-dimensional hypersurface or a set of hypersurfaces $\mathcal{P}$ embedded in the d-dimensional state space $\mathcal{M}$, define the *Poincaré return map* $P(x)$, a $(d-1)$-dimensional map of form

draw figure

$$x' = P(x) = f^{\tau(x)}(x)\,, \qquad x', x \in \mathcal{P}\,. \tag{3.1}$$

Here the *first return function* $\tau(x)$ - sometimes referred to as the *ceiling function* - is the time of flight to the next section for a trajectory starting at x. The choice of the section hypersurface $\mathcal{P}$ is altogether arbitrary. It is rarely possible to define a single section that cuts across all trajectories. In practice one often needs only a local section - a finite hypersurface of codimension 1 volume intersected by a ray of trajectories near to the trajectory of interest. The hypersurface can be specified implicitly through a function $U(x)$ that is zero whenever a point x is on the Poincaré section,

$$x \in \mathcal{P} \quad \text{iff} \quad U(x)\,. \tag{3.2}$$

transversality!condition
map!return
Poincar"'e!return map
Poincaré!section!hyperplane

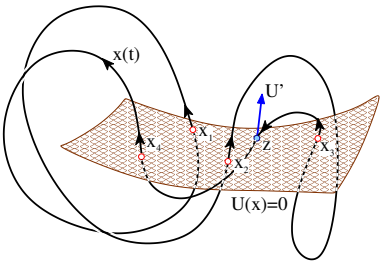

Figure 3.1: A $x(t)$ trajectory that intersects a Poincaré section $\mathcal{P}$ at times t_1, t_2, t_3, t_4, and closes a cycle (x_1, x_2, x_3, x_4), $x_k = x(t_k) \in \mathcal{P}$ of topological length 4 with respect to this section. Note that the intersections are not normal to the section, and that the crossing z does not count, as it in the wrong direction.

The gradient of $U(x)$ evaluated at $x \in \mathcal{P}$ serves a two-fold function. First, the flow should pierce the hypersurface $\mathcal{P}$, rather than being tangent to it. A nearby point $x + \delta x$ is in the hypersurface $\mathcal{P}$ if $U(x + \delta x) = 0$. A nearby point on the trajectory is given by $\delta x = v\delta t$, so a traversal is ensured by the *transversality condition*

$$(v\cdot\partial U) = \sum_{j=1}^{d} v_j(x)\partial_j U(x) \neq 0\,, \quad \partial_j U(x) = \frac{d}{dx_j}U(x)\,, \quad x \in \mathcal{P}\,. \tag{3.3}$$

Second, the gradient $\partial_j U$ defines the orientation of the hypersurface $\mathcal{P}$. The flow is oriented as well, and a periodic orbit can pierce $\mathcal{P}$ twice, traversing it in either direction, as in figure 3.1. Hence the definition of Poincaré return map $P(x)$ needs to be supplemented with the orientation condition

draw figure

$$\begin{aligned} x_{n+1} = P(x_n)\,, \qquad & U(x_{n+1}) = U(x_n) = 0\,, \quad n \in \mathbb{Z}^+ \\ & \sum_{j=1}^{d} v_j(x_n)\partial_j U(x_n) \;>\; 0\,. \end{aligned} \tag{3.4}$$

In this way the continus time t flow $f^t(x)$ is reduced to a discrete time n sequence x_n of successive *oriented* trajectory traversals of $\mathcal{P}$.

With a sufficiently clever choice of a Poincaré section or a set of sections, any orbit of interest intersects a section. Depending on the application, one might need to convert the discrete time n back to the continuous flow time. This is accomplished by adding up the first return function times $\tau(x_n)$, with the accumulated flight time given by

$$t_{n+1} = t_n + \tau(x_n)\,, \qquad t_0 = 0\,, \qquad x_n \in \mathcal{P}\,. \tag{3.5}$$

Other quantities integrated along the trajectory can be defined in a similar manner, and will need to be evaluated in the process of evaluating dynamical averages.

☞ chapter 10

A few examples may help visualize this.

Example 3.1 Hyperplane $\mathcal{P}$: *The simplest choice of a Poincaré section is a plane specified by a point (located at the tip of the vector r_0) and a direction vector a perpendicular to the plane. A point x is in this plane if it satisfies the condition*

$$U(x) = (x - r_0)\cdot a = 0\,. \tag{3.6}$$

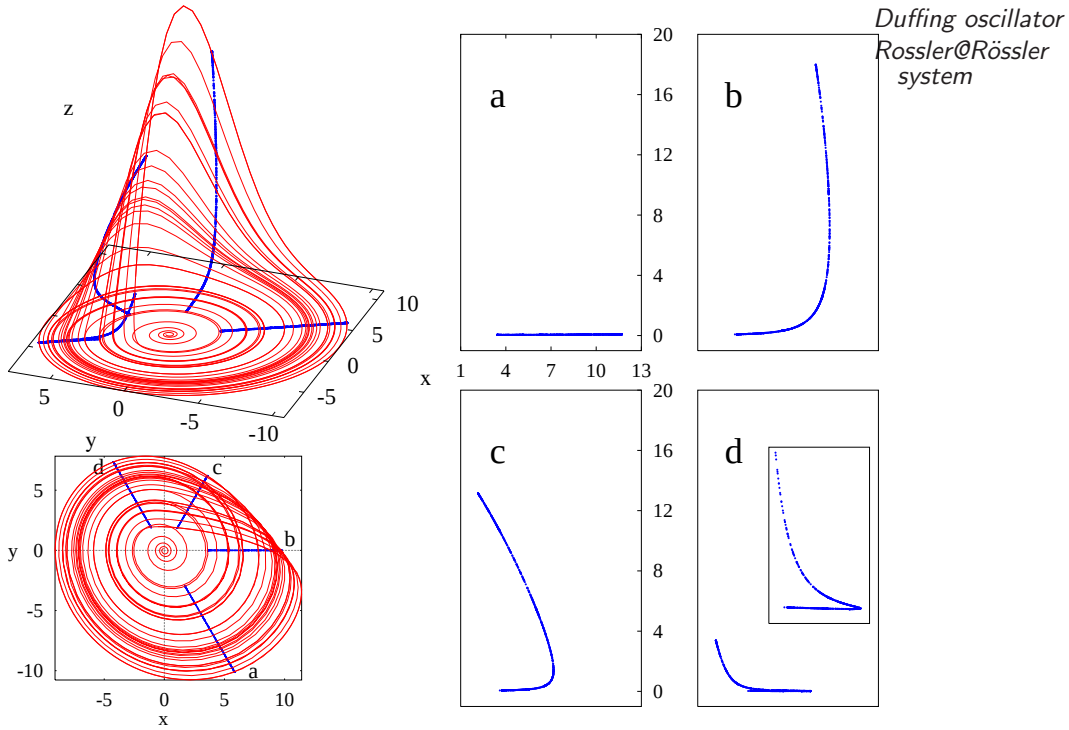

Figure 3.2: (Right:) a sequence of Poincaré sections of the Rössler strange attractor, defined by planes through the z axis, oriented at angles (a) -60^o (b) 0^o, (c) 60^o, (d) 120^o, in the x-y plane. (Left:) side and x-y plane view of a typical trajectory with Poincaré sections superimposed. (Rytis Paškauskas)

Consider a circular periodic orbit centered at r_0, but not lying in $\mathcal{P}$. It pierces the hyperplane twice; the $(v \cdot a) > 0$ traversal orientation condition (3.4) ensures that the first return time is the full period of the cycle.

Example 3.2 Pendulum: *The phase space of a simple pendulum is 2-dimensional: momentum on the vertical axis and position on the horizontal axis. We choose the Poincaré section to be the positive horizontal axis. Now imagine what happens as a point traces a trajectory through this phase space. As long as the motion is oscillatory, in the pendulum all orbits are loops, so any trajectory will periodically intersect the line, that is the Poincaré section, at one point.*

Consider next a pendulum with friction, such as the unforced Duffing system plotted in figure 2.2. Now every trajectory is an inward spiral, and the trajectory will intersect the Poincaré section $y = 0$ at a series of points that get closer and closer to either of the equilibrium points; the Duffing oscillator at rest.

Motion of a pendulum is so simple that you can sketch it yourself on a piece of paper. The next example (as well as example 25.3) offers a better illustration of the utility of visualization of dynamics by means of Poincaré sections.

⇓PRELIMINARY
⇑PRELIMINARY

Example 3.3 Rössler flow: *Consider figure 2.3, a typical trajectory of the 3-dimensional Rössler flow (2.14). It wraps around the z axis, so a good choice for a Poincaré section is a plane passing through the z axis. A sequence of such Poincaré*

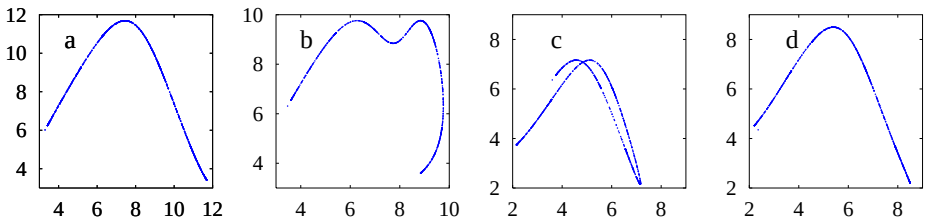

Figure 3.3: Return maps for the $R_n \to R_{n+1}$ radial distance Poincaré sections of figure 3.2. (Rytis Paškauskas)

sections placed radially at increasing angles with respect to the x axis, figure 3.2, illustrates the "stretch & fold" action of the Rössler flow. To orient yourself, compare this with figure 2.3, and note the different z-axis scales. Figure 3.2 assembles these sections into a series of snapshots of the flow. A line segment $[A, B]$, traversing the width of the attractor, starts out close to the x-y plane, and after the stretching (a) → (b) followed by the folding (c) → (d), the folded segment returns close to the x-y plane strongly compressed. In one Poincaré return the $[A, B]$ interval is stretched, folded and mapped onto itself, so the flow is expanding. *It is also* mixing, *as in one Poincaré return the point C from the interior of the attractor is mapped into the outer edge, while the edge point B lands in the interior.*

Once a particular Poincaré section is picked, we can also exhibit the return map (3.1), as in figure 3.3. Cases (a) and (d) are examples of nice 1-to-1 return maps. However, (b) and (c) appear multimodal and non-invertible, artifacts of projection of a 2-d return map $(R_n, z_n) \to (R_{n+1}, z_{n+1})$ onto a 1-dimensional subspace $R_n \to R_{n+1}$. (continued in example 4.1)

remember unstable manifold return map!

fast track:
sect. 3.4, p. 54

The above examples illustrate why a Poincaré section gives a more informative snapshot of the flow than the full flow portrait. For example, while the full flow portrait of the Rössler flow figure 2.3 gives us no sense of the thickness of the attractor, we see clearly in the figure **??** Poincaré sections that even though the return map is 2-d → 2-d, the flow contraction is so strong that for all practical purposes it renders the return map 1-dimensional.

point here to the Rössler equilibrium example chapter 4

⇓PRELIMINARY

3.2 Stroboscopic Poincaré sections

not sure this is worth describing

⇑PRELIMINARY

3.3 Constructing a Poincaré section

For almost any flow of physical interest a Poincaré section is not available in analytic form. We describe here a numerical method for determining a Poincaré section.

remark 3.1

Consider the system (2.5) of ordinary differential equations in the vector variable $x = (x_1, x_2, \ldots, x_d)$

$$\frac{dx_i}{dt} = v_i(x,t)\,, \tag{3.7}$$

where the flow velocity v is a vector function of the position in state space x and the time t. In general v cannot be integrated analytically and we will have to resort to numerical integration to determine the trajectories of the system. Our task is to determine the points at which the numerically integrated trajectory traverses a given hypersurface. The hypersurface will be specified implicitly through a function $U(x)$ that is zero whenever a point x is on the Poincaré section, such as the hyperplane (3.6).

If we use a tiny step size in our numerical integrator, we can observe the value of U as we integrate; its sign will change as the trajectory crosses the hypersurface. The problem with this method is that we have to use a very small integration time step. In order to actually land on the Poincaré section one might try to interpolate the intersection point from the two trajectory points on either side of the hypersurface. However, there is a better way.

Let t_a be the time just before U changes sign, and t_b the time just after it changes sign. The method for landing exactly on the Poincaré section will be to convert one of the space coordinates into an integration variable for the part of the trajectory between t_a and t_b. Using

$$\frac{dx_k}{dx_1}\frac{dx_1}{dt} = \frac{dx_k}{dx_1}v_1(x,t) = v_k(x,t) \tag{3.8}$$

we can rewrite the equations of motion (3.7) as

$$\frac{dt}{dx_1} = \frac{1}{v_1}\,,\ \cdots\,,\ \frac{dx_d}{dx_1} = \frac{v_d}{v_1}\,. \tag{3.9}$$

Now we use x_1 as the "time" in the integration routine and integrate it from $x_1(t_a)$ to the value of x_1 on the hypersurface, which can be found from the hypersurface intersection condition (3.6). The quantity x_1 need not be perpendicular to the Poincaré section; any x_i can be chosen as the integration variable, privided the x_i-axis is not parallel to the Poincaré section at the trajectory intersection point. If the section crossing is transverse (see (3.3)), v_1 cannot vanish in the short segment bracketed by the integration step preceeding the section, and the point on the Poincaré section.

Example 3.4 Computation of Rössler flow Poincaré sections. *Poincaré sections of figure 3.2 are defined by the fixing angle $U(x) = \theta - \theta_0 = 0$. Convert Rössler equation (2.14) to cylindrical coordinates:*

$$\begin{aligned}
\dot r &= v_r = -z\cos\theta + \arcsin^2\theta \\
\dot\theta &= v_\theta = 1 + \frac{z}{r}\sin\theta + \frac{a}{2}\sin 2\theta \\
\dot z &= v_z = b + z(r\cos\theta - c)
\end{aligned} \tag{3.10}$$

trajectory!discrete
map!iteration
iteration!map
orbit

For parameter values (2.14), and (x_0, y_0, z_0) sufficently far away from the inner equilibrium, θ increases monotonically.

$$\frac{dr}{d\theta} = v_r/v_\theta\,,\ \frac{dt}{d\theta} = 1/v_\theta\,,\ \frac{dz}{d\theta} = v_z/v_\theta \tag{3.11}$$

to $\theta = \theta_0$, and then continue integration in (x, y, z) coordinates. (Radford Mitchell, Jr.)

[1]

3.4 Maps

define "map" vs "mapping"

Though we have motivated discrete time dynamics by considering sections of a continuous flow, there are many settings in which dynamics is inherently discrete, and naturally described by repeated iterations of the same map

rewrite

$$f : \mathcal{M} \to \mathcal{M}\,,$$

or sequences of consecutive applications of a finite set of maps,

$$\{f_A, f_B, \dots f_Z\} : \mathcal{M} \to \mathcal{M}\,, \tag{3.12}$$

for example maps relating different sections among a set of Poincaré sections. The discrete "time" is then an integer, the number of applications of a map. As writing out formulas involving repeated applications of a set of maps explicitly can be awkward, we streamline the notation by denoting a map composition by "$\circ$"

$$f_Z(\cdots f_B(f_A(x)))\cdots) = f_Z \circ \cdots f_B \circ f_A(x)\,, \tag{3.13}$$

and the nth iterate of map f by

$$f^n(x) = f \circ f^{n-1}(x) = f\left(f^{n-1}(x)\right)\,, \qquad f^0(x) = x\,.$$

ref to figure 2.1 or better?
☞ sect. 2.1

The *trajectory* of x is the set of points

$$\left\{x, f(x), f^2(x), \dots, f^n(x)\right\}\,,$$

and the *orbit* of x is the subset of all points of $\mathcal{M}$ that can be reached by iterations of f. For example, the orbit of x_1 in figure 3.4 is the 4-cycle (x_1, x_2, x_3, x_4).

[1] Mason: nonlinear ∞-dimensional systems: compare PDEs, coupled map lattices, CLMs etc.; they run the gamut from everything (time,space, field variable) continuous to everything discrete; can then discuss when each setting is appropriate. PC: I am all for it, as long as it makes ChaosBook shorter.

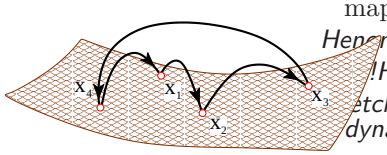

Figure 3.4: A flow $x(t)$ of figure 3.1 represented by a Poincaré return map that maps points in the Poincaré section $\mathcal{P}$ as $x_{n+1} = f(x_n)$. In this example the orbit of x_1 consists of the four cycle points (x_1, x_2, x_3, x_4)

Poincar"'e!return map!polynomial
Hen@Hénon!map
!Hénon
tch & fold dynamics

The functional form of such Poincaré return maps P as figure 3.3 can be approximated by tabulating the results of integration of the flow from x to the first Poincaré section return for many $x \in \mathcal{P}$, and constructing a function that interpolates through these points. If we find a good approximation to $P(x)$, we can get rid of numerical integration altogether, by replacing the continuous time trajectory $f^t(x)$ by iteration of the Poincaré return map $P(x)$. Constructing accurate $P(x)$ for a given flow can be tricky, but we can already learn much from approximate Poincaré return maps. Multinomial approximations [2]

add exercise

refer to Wisdom's approach in remarks

$$P_k(x) = a_k + \sum_{j=1}^{d} b_{kj} x_j + \sum_{i,j=1}^{d} c_{kij} x_i x_j + \dots , \qquad x \in \mathcal{P} \tag{3.14}$$

to Poincaré return maps

$$\begin{pmatrix} x_{1,n+1} \\ x_{2,n+1} \\ \dots \\ x_{d,n+1} \end{pmatrix} = \begin{pmatrix} P_1(x_n) \\ P_2(x_n) \\ \dots \\ P_d(x_n) \end{pmatrix} , \qquad x_n, x_{n+1} \in \mathcal{P}$$

motivate the study of model mappings of the plane, such as the Hénon map.

Example 3.5 Hénon map: *The map*

$$\begin{aligned} x_{n+1} &= 1 - a x_n^2 + b y_n \\ y_{n+1} &= x_n \end{aligned} \tag{3.15}$$

is a nonlinear 2-dimensional map most frequently employed in testing various hunches about chaotic dynamics. The Hénon map is sometimes written as a 2-step recurrence relation

$$x_{n+1} = 1 - a x_n^2 + b x_{n-1} . \tag{3.16}$$

An n-step recurrence relation is the discrete-time analogue of an nth order differential equation, and it can always be replaced by a set of n 1-step recurrence relations.

The Hénon map is the simplest map that captures the "stretch & fold" dynamics of return maps such as Rössler's, figure 3.2. It can be obtained by a truncation of a polynomial approximation (3.14) to a Poincaré return map to second order, or, more cleverly, by a rectification of a return map to the normal form (??).

⇓PRELIMINARY
⇑PRELIMINARY

[2] Mason: I can contribute some nice Poincaré map figures

Lozi map
map!Lozi
quadratic map
map!quadratic
injective
surjective
Bourbaki, N.

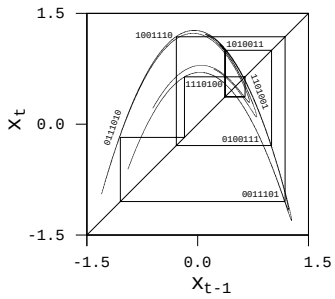

Figure 3.5: The strange attractor and an unstable period 7 cycle of the Hénon map (3.15) with $a = 1.4$, $b = 0.3$. The periodic points in the cycle are connected to guide the eye. (K.T. Hansen [1.7])

A quick sketch of the long-time dynamics of such a mapping (an example is depicted in figure 3.5), is obtained by picking an arbitrary starting point and iterating (3.15) on a computer. We plot here the dynamics in the (x_n, x_{n+1}) plane, rather than in the (x_n, y_n) plane, because we think of the Hénon map as a model return map $x_n \to x_{n+1}$. As we shall soon see, periodic orbits will be key to understanding the long-time dynamics, so we also plot a typical periodic orbit of such a system, in this case an unstable period 7 cycle. Numerical determination of such cycles will be explained in sect. 37.1 , and the cycle point labels $0111010, 1110100, \cdots$ in sect. 12.2.

3.5
page 60

need figure of Lozi strange attractor

Example 3.6 Lozi map: *Another example frequently employed is the* Lozi map, *a linear, "tent map" version of the Hénon map given by*

$$\begin{aligned} x_{n+1} &= 1 - a|x_n| + by_n \\ y_{n+1} &= x_n \,. \end{aligned} \tag{3.17}$$

Though not realistic as an approximation to a smooth flow, the Lozi map is a very helpful tool for developing intuition about the topology of a large class of maps of the "stretch & fold" type.

⇓PRELIMINARY
example 12.1
⇑PRELIMINARY

replace figure 3.5 by one without 7-cycle, move this fig to a symbolic dynamics chapter

What we get by iterating such maps is - at least qualitatively - not unlike what we get from Poincaré section of flows such as the Rössler flow figures 3.3 and **??** For an arbitrary initial point this process might converge to a stable limit cycle, to a strange attractor, to a false attractor (due to roundoff errors), or diverge. In other words, mindless iteration is essentially uncontrollable, and we will need to resort to more thoughtful explorations. As we shall explain in due course below, strategies for systematic exploration rely on stable/unstable manifolds, periodic points, saddle-straddle methods and so on.

3.6
page 60

that we shall discuss in chapters 12 and 17

motivate ala Henon

Example 3.7 Parabola: *One iteration of the Hénon map stretches and folds a region of the (x, y) plane centered around the origin. The parameter a controls the amount of stretching, while the parameter b controls the thickness of the folded image through the "1-step memory" term bx_{n-1} in (3.16). In figure 3.5 the parameter b is rather large, $b = 0.3$, so the attractor is rather thick, with the transverse fractal structure clearly visible. For vanishingly small b the Hénon map reduces to the 1-dimensional*

use Biham figures

quadratic map

$$x_{n+1} = 1 - ax_n^2 \,. \tag{3.18}$$

By setting $b = 0$ we lose determinism, as on reals the inverse of map (3.18) has two preimages $\{x_{n-1}^+, x_{n-1}^-\}$ for most x_n. If Bourbaki is your native dialect: the Hénon map is injective *or one-to-one, but the quadratic map is* surjective *or many-to-one. Still, this 1-dimensional approximation is very instructive.*

3.7
60

Poincaré!section!Hénon trick
Henon@Hénon!map
bifurcation!saddle-node
saddle-node!bifurcation
Lorenz, E.N.
Pomeau, Y.
Henon@Hénon, M.
Mira, C.
Lozi map
map!Lozi
Misiurewicz, M.
Young, L.-S.

As we shall see in sect. 11.3.1, an understanding of 1-dimensional dynamics is indeed the essential prerequisite to unravelling the qualitative dynamics of many higher-dimensional dynamical systems. For this reason many expositions of the theory of dynamical systems commence with a study of 1-dimensional maps. We prefer to stick to flows, as that is where the physics is.

☞ appendix J.8

insert here Tanner's mixed phase space, standard map

Commentary

Remark 3.1 Determining a Poincaré section. The idea of changing the integration variable from time to one of the coordinates, although simple, avoids the alternative of having to interpolate the numerical solution to determine the intersection. The trick described in sect. 3.3 is due to Hénon [3.4, 3.5, 3.6].

Remark 3.2 Hénon, Lozi maps. The Hénon map is of no particular physical import in and of itself - its significace lies in the fact that it is a minimal normal form for modeling flows near a saddle-node bifurcation, and that it is a prototype of the stretching and folding dynamics that leads to deterministic chaos. It is generic in the sense that it can exhibit arbitrarily complicated symbolic dynamics and mixtures of hyperbolic and non–hyperbolic behaviors. Its construction was motivated by the best known early example of "deterministic chaos", the Lorenz equation [2.1]. Y. Pomeau's studies of the Lorenz attractor on an analog computer, and his insights into its stretching and folding dynamics motivated Hénon [3.1] to introduce the Hénon map in 1976. Hénon's and Lorenz's original papers can be found in reprint collections refs. [3.2, 3.3]. They are a pleasure to read, and are still the best introduction to the physics motivating such models. A detailed description of the dynamics of the Hénon map is given by Mira and coworkers [3.7], as well as very many other authors.

The Lozi map [3.9] is particularly convenient in investigating the symbolic dynamics of 2-d mappings. Both the Lorenz and Lozi systems are uniformly smooth systems with singularities. The continuity of measure for the Lozi map was proven by M. Misiurewicz [3.10], and the existence of the SRB measure was established by L.-S. Young.

☞ sect. 9.1

find L.-S. Young reference, genuflect to Carlesonm

Résumé

⇓PRELIMINARY

link to Dahlqvist-Russberg paper

⇑PRELIMINARY

In recurrent dynamics a trajectory exits a region in state space and then reenters it infinitely often, with a finite mean return time. If the orbit

is periodic, it returns after a full period. So, on average, nothing much really happens along the trajectory – what is important is behavior of neighboring trajectories transverse to the flow. This observation motivates a replacament of the continuous time flow by iterative mapping, the Poincaré return map.

The visualization of strange attractors is greatly facilitated by a felicitous choice of Poincaré sections, and the reduction of flows to Poincaré return maps. This observation motivates in turn the study of discrete-time dynamical systems generated by iterations of maps.

A particularly natural application of the Poincaré section method is the reduction of a billiard flow to a boundary-to-boundary return map, described in chapter 6 below. As we shall show in chapter 7, further simplification of a Poincaré return map, or any nonlinear map, can be attained through rectifying these maps locally by means of smooth conjugacies.

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Exercises

Rossler@Rössler system
helium!collinear

Exercise 3.1 **Poincaré sections of the Rössler flow.** *(continuation of exercise 2.9) Calculate numerically a Poincaré section (or several Poincaré sections) of the Rössler flow. As the Rössler flow state space is 3-dimensional, the flow maps onto a 2-dimensional Poincaré section. Do you see that in your numerical results? How good an approximation would a replacement of the return map for this section by a 1-dimensional map be? More precisely, estimate the thickness of the strange attractor. (continued as exercise 4.4)*

(Rytis Paškauskas)

Exercise 3.2 **A return Poincaré map for the Rössler flow.** *(continuation of exercise 3.1) That Poincaré return maps of figure 3.3 appear multimodal and non-invertible is an artifact of projections of a 2-dimensional return map $(R_n, z_n) \to (R_{n+1}, z_{n+1})$ onto a 1-dimensional subspace $R_n \to R_{n+1}$.*

Construct a genuine $s_{n+1} = f(s_n)$ return map by parametrazing points on a Poincaré section of the attractor figure 3.2 by a Euclidean length s computed curvilinearly along the attractor section.

This is best done (using methods to be developed in what follows) by a continuation of the unstable manifold of the 1-cycle embedded in the strange attractor, figure 17.1 (b).

(Predrag Cvitanović)

Exercise 3.3 **Arbitrary Poincaré sections.** *We will generalize the construction of Poincaré sections so that they can have any shape, as specified by the equation $U(x) = 0$.*

(a) *Start by modifying your integrator so that you can change the coordinates once you get near the Poincaré section. You can do this easily by writing the equations as*

$$\frac{dx_k}{ds} = \kappa f_k \,, \tag{3.19}$$

with $dt/ds = \kappa$, and choosing κ to be 1 or $1/f_1$. This allows one to switch between t and x_1 as the integration "time."

(b) *Introduce an extra dimension x_{n+1} into your system and set*

$$x_{n+1} = U(x) \,. \tag{3.20}$$

How can this be used to find a Poincaré section?

Exercise 3.4 **Classical collinear helium dynamics.** *(continuation of exercise 2.11)*

Make a Poincaré surface of section by plotting (r_1, p_1) whenever $r_2 = 0$: Note that for $r_2 = 0$, p_2 is already determined by (5.6). Compare your results with figure 41.3 (b).

(Gregor Tanner, Per Rosenqvist)

Henon@Hénon!map!fixed points
Henon@Hénon!attractor
attractor!Hénon
fixed point!maps
map!fixed point

Exercise 3.5 **Hénon map fixed points.** *Show that the two fixed points* (x_0, x_0), (x_1, x_1) *of the Hénon map (3.15) are given by*

$$\begin{aligned} x_0 &= \frac{-(1-b) - \sqrt{(1-b)^2 + 4a}}{2a}\,, \\ x_1 &= \frac{-(1-b) + \sqrt{(1-b)^2 + 4a}}{2a}\,. \end{aligned} \tag{3.21}$$

Exercise 3.6 **How strange is the Hénon attractor?**

(a) *Iterate numerically some 100,000 times or so the Hénon map*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 - ax^2 + y \\ bx \end{bmatrix}$$

for $a = 1.4$, $b = 0.3$. *Would you describe the result as a "strange attractor"? Why?*

(b) *Now check how robust the Hénon attractor is by iterating a slightly different Hénon map, with* $a = 1.39945219$, $b = 0.3$. *Keep at it until the "strange" attracttor vanishes like the smile of the Chesire cat. What replaces it? Would you describe the result as a "strange attractor"? Do you still have confidence in your own claim for the part (a) of this exercise?*

Give the example of $(xy)^2$ elliptic island.

Exercise 3.7 **Fixed points of maps.** *A continuous function* F *is a contraction of the unit interval if it maps the interval inside itself.*

(a) *Use the continuity of* F *to show that a one-dimensional contraction* F *of the interval* $[0, 1]$ *has at least one fixed point.*

(b) *In a uniform (hyperbolic) contraction the slope of* F *is always smaller than one,* $|F'| < 1$. *Is the composition of uniform contractions a contraction? Is it uniform?*

local!stability
linear!stability
stability!linear
neighborhood
equation!of variations

Chapter 4

Local stability

(R. Mainieri and P. Cvitanović)

So far we have concentrated on description of the trajectory of a single initial point. Our next task is to define and determine the size of a *neighborhood* of $x(t)$. We shall do this by assuming that the flow is locally smooth, and describe the local geometry of the neighborood by studying the flow linearized around $x(t)$. Nearby points aligned along the stable (contracting) directions remain in the neighborhood of the trajectory $x(t) = f^t(x_0)$; the ones to keep an eye on are the points which leave the neighborhood along the unstable directions. As we shall demonstrate in chapter 14, in hyperbolic systems what matters are the expanding directions. The reprecussion are far-reaching: As long as the number of unstable directions is finite, the same theory applies to finite-dimensional ODEs, state space volume preserving Hamiltonian flows, and dissipative, volume contracting infinite-dimensional PDEs.

4.1 Flows transport neighborhoods

As a swarm of representative points moves along, it carries along and distorts neighborhoods, as sketched in figure 2.1 (b). The deformation of an infinitesimal neighborhood is best understood by considering a trajectory originating near $x_0 = x(0)$ with an initial infinitesimal displacement $\delta x(0)$, and letting the flow transport the displacement $\delta x(t)$ along the trajectory $x(x_0, t) = f^t(x_0)$. The system of linear *equations of variations* for the displacement of the infinitesimally close neighbor $x + \delta x$ follows from the flow equations (2.5) by Taylor expanding to linear order

$$\dot{x}_i + \dot{\delta x}_i = v_i(x + \delta x) \approx v_i(x) + \sum_j \frac{\partial v_i}{\partial x_j} \delta x_j \,.$$

The infinitesimal displacement δx is thus transported along the trajectory $x(x_0, t)$, with time variation given by

might want to cite Arnold [7.1]

tangent!space
tangent!bundle
matrix!stability
stability!matrix
initial!point
$"xInit$
fundamental matrix
linearized!flow
flow!linearized
replace A_{ij} by J_{ij}? and J_{ij}?

$$\frac{d}{dt}\delta x_i(x_0,t) = \sum_j \left.\frac{\partial v_i(x)}{\partial x_j}\right|_{x=x(x_0,t)} \delta x_j(x_0,t)\,. \tag{4.1}$$

As both the displacement and the trajectory depend on the initial point x_0 and the time t, we shall often abbreviate the notation to $x(x_0,t) \to x(t) \to x$, $\delta x_i(x_0,t) \to \delta x_i(t) \to \delta x$ in what follows. Taken together, the set of equations

$$\dot{x}_i = v_i(x)\,, \quad \dot{\delta x}_i = \sum_j A_{ij}(x)\delta x_j \tag{4.2}$$

fix notation: $(\mathbf{T})\mathcal{M}$?

draw infinitesim. displc., explain what you mean by shearing

governs the dynamics in the tangent bundle $(x,\delta x) \in \mathbf{T}\mathcal{M}$ obtained by adjoining the d-dimensional tangent space $\delta x \in \mathbf{T}_x\mathcal{M}$ to every point $x \in \mathcal{M}$ in the d-dimensional state space $\mathcal{M} \subset \mathbb{R}^d$. The *stability matrix*

$$A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j} \tag{4.3}$$

describes the instantaneous rate of shearing of the infinitesimal neighborhood of $x(t)$ by the flow.

Example 4.1 Rössler flow, linearized: *For the Rössler flow (2.14) the stability matrix is*

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-c \end{pmatrix}\,. \tag{4.4}$$

Taylor expanding a *finite time* flow to linear order,

$$f_i^t(x_0+\delta x) = f_i^t(x_0) + \sum_j \frac{\partial f_i^t(x_0)}{\partial x_{0j}}\delta x_j + \cdots\,, \tag{4.5}$$

draw finite displc.

one finds that the linearized neighborhood is transported by

$$\delta x(t) = J^t(x_0)\delta x(0)\,, \qquad J_{ij}^t(x_0) = \left.\frac{\partial x_i(t)}{\partial x_j}\right|_{x=x_0}\,. \tag{4.6}$$

add a section on eigenvalues of matrices

This Jacobian matrix has inherited the name *fundamental solution matrix* or simply *fundamental matrix* from the theory of linear ODEs. It describes the deformation of an infinitesimal neighborhood at finite time t in the co-moving frame of $x(t)$.

As this is a deformation in the linear approximation, one can think of it as a linear deformation of an infinitesimal sphere enveloping x_0 into an

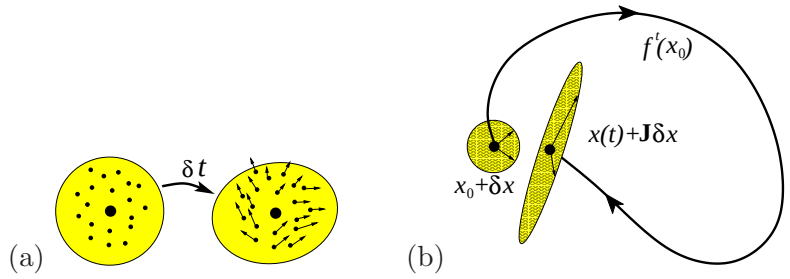

Figure 4.1: (a) A swarm of neighboring points of $x(t)$ is instantaneously sheared by the action of the stability matrix A - a bit hard to draw. (b) The fundamental matrix J^t maps an infinitesimal spherical neighborhood of x_0 into an ellipsoidal neighborhood finite time t later.

eigen-direction
stability!neutral
neutral!stability
stability!indifferent
indifferent!stability
Lagrangian!coordinates
Eulerian!coordinates
cycle!marginal stability
marginal stability!cycle

ellipsoid around $x(t)$, described by the eigenvectors and eigenvalues of the fundamental matrix of the linearized flow, figure 4.1. Nearby trajectories separate along the *unstable directions*, approach each other along the *stable directions*, and change their distance along the *marginal directions* at a rate slower than exponential, corresponding to the eigenvalues of the fundamental matrix with magnitude larger than, smaller than, or equal 1. In the literature adjectives *neutral* or *indifferent* are often used instead of "marginal", (attracting) stable directions are sometimes called "asymptotically stable", and so on.

One of the eigen-directions is what one might expect, the direction of the flow itself. To see that, consider two initial points along a trajectory separated by infinitesimal flight time δt: $\delta x(0) = f^{\delta t}(x_0) - x_0 = v(x_0)\delta t$. By the semigroup property of the flow, $f^{t+\delta t} = f^{\delta t+t}$. Expanding both sides of $f^t(f^{\delta t}(x_0)) = f^{\delta t}(f^t(x_0))$, keeping the leading term in δt, and using the definition of the fundamental matrix (4.6), we observe that $J^t(x_0)$ transports the velocity vector at x_0 to the velocity vector at $x(t)$ at time t:

$$v(x(t)) = J^t(x_0)\, v(x_0)\,. \tag{4.7}$$

In nomenclature of page 62, the fundamental matrix maps the initial, Lagrangian coordinate frame into the current, Eulerian coordinate frame.

The velocity at point $x(t)$ in general does not point in the same direction as the velocity at point x_0, so this is not an eigenvalue condition for J^t; the fundamental matrix computed for an arbitrary segment of an arbitrary trajectory has no invariant meaning.

As the eigenvalues of finite time J^t have invariant meaning only for periodic orbits, we postpone their interpretation to chapter 8. However, already at this stage we see that if the orbit is periodic, $x(T_p) = x(0)$, at any point along cycle p the velocity v is an eigenvector of the fundamental matrix $J_p = J^{T_p}$ with a unit eigenvalue,

$$J_p(x)\, v(x) = v(x)\,, \qquad x \in p\,. \tag{4.8}$$

linear!flows
flow!linear

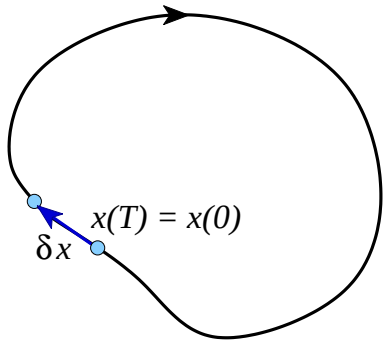

Figure 4.2: For a periodic orbit p, any two points along the cycle are mapped into themselves after one cycle period T, hence $\delta x(0) = v(x_0)\delta t$ is mapped into itself by the cycle fundamental matrix J_p.

Two successive points along the cycle separated by $\delta x(0)$ have the same separation after a completed period $\delta x(T_p) = \delta x(0)$, hence eigenvalue 1 (see figure 4.2).

As we started by assuming that we know the equations of motion, from (4.3) we also know stability matrix A, the instantaneous rate of shear of an infinitesimal neighborhood $\delta x_i(t)$ of the trajectory $x(t)$. What we do not know is the finite time deformation (4.6). Our next task is to relate the stability matrix A to fundamental matrix J^t.

We are interested in smooth, differentiable flows. If a flow is smooth, in a sufficiently small neighborhood it is essentially linear. Hence the next section, which might seem an embarrassment (what is a section on *linear* flows doing in a book on *non*linear dynamics?), offers a firm stepping stone on the way to understanding nonlinear flows. If you know your eigenvalues and eigenvectors, you may prefer to fast forward here.

fast track:
sect. 4.3, p. 69

4.2 Linear flows

> Diagonalizing the matrix: that's the key to the whole thing.
> Governor Arnold Schwarzenegger

Linear fields are the simplest vector fields. Described by linear differential equations which can be solved explicitly, with solutions that are good for all times. The state space for linear differential equations is $\mathcal{M} = \mathbb{R}^d$, and the equations of motion (2.5) are written in terms of a vector x and a constant stability matrix A as

$$\dot{x} = v(x) = Ax\,. \tag{4.9}$$

Solving this equation means finding the state space trajectory

$$x(t) = (x_1(t), x_2(t), \ldots, x_d(t))$$

passing through the point x_0.

trajectory

If $x(t)$ is a solution with $x(0) = x_0$ and $y(t)$ another solution with $y(0) = y_0$, then the linear combination $ax(t) + by(t)$ with $a, b \in \mathbb{R}$ is also a solution, but now starting at the point $ax_0 + by_0$. At any instant in time, the space of solutions is a d-dimensional vector space, which means that one can find a basis of d linearly independent solutions. How do we solve the linear differential equation (4.9)? If instead of a matrix equation we have a scalar one, $\dot{x} = \lambda x$, with a a real number, then the solution is

Euler!formula
exponential!of a matrix
matrix!exponential

$$x(t) = e^{t\lambda} x_0 \,. \tag{4.10}$$

In order to solve the d-dimensional matrix case, it is helpful to rederive the solution (4.10) by studying what happens for a short time step δt. If at time $t = 0$ the position is $x(0)$, then

$$\frac{x(\delta t) - x(0)}{\delta t} = \lambda x(0) \,, \tag{4.11}$$

which we iterate m times to obtain Euler's formula for compounding interest

$$x(t) \approx \left(1 + \frac{t}{m}\lambda\right)^m x(0) \,. \tag{4.12}$$

⇓PRELIMINARY
??
page ??
⇑PRELIMINARY

The term in parentheses acts on the initial condition $x(0)$ and evolves it to $x(t)$ by taking m small time steps $\delta t = t/m$. As $m \to \infty$, the term in parentheses converges to $e^{t\lambda}$. Consider now the matrix version of equation (4.11):

$$\frac{x(\delta t) - x(0)}{\delta t} = A x(0) \,. \tag{4.13}$$

A representative point x is now a vector in $\mathbb{R}^d$ acted on by the matrix A, as in (4.9). Denoting by $\mathbf{1}$ the identity matrix, and repeating the steps (4.11) and (4.12) we obtain Euler's formula for the exponential of a matrix:

$$x(t) = \lim_{m\to\infty} \left(\mathbf{1} + \frac{t}{m}A\right)^m x(0) = e^{tA} x(0) \,. \tag{4.14}$$

We will use this expression as the definition of the exponential of a matrix.

How do we compute the exponential (4.14)?

in depth:
appendix M.2, p. 1007

fast track:
sect. 4.3, p. 69

stability!eigenvalue
characteristic!exponent
characteristic!value

Example 4.2 Fundamental matrix eigenvalues, diagonalizable case: *Should we be so lucky that A happens to be a diagonal matrix with real eigenvalues $(\mu_1, \mu_2, \ldots, \mu_d)$, the exponential is simply*

$$J^t = e^{tA} = \begin{pmatrix} e^{t\mu_1} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e^{t\mu_d} \end{pmatrix} . \tag{4.15}$$

Next, suppose that A is diagonalizable and that U is a nonsingular matrix that brings it to a diagonal form $A_D = U^{-1}AU$. The transformation U is a linear coordinate transformation which rotates, skews, and possibly flips the coordinate axis of the vector space. Then J can also be brought to a diagonal form (insert factors $\mathbf{1} = UU^{-1}$ between the steps of the product (4.14)):

4.2
page 78

$$J^t = e^{tA} = U e^{tA_D} U^{-1} . \tag{4.16}$$

The action of both A and J is very simple; the axes of orthogonal coordinate system where A is diagonal are also the eigen-directions of both A and J^t, and under the flow the neighborhood is deformed by a multiplication by an eigenvalue factor for each coordinate axis.

Throughout this text the symbol Λ_k will always denote the kth *eigenvalue* (in literature sometimes referred to as the *multiplier*) of the finite time fundamental matrix J^t. Symbol λ_k will be reserved for the kth *characteristic exponent* or *characteristic value*, with real part μ_k and ν_k the kth phase

$$\Lambda_k = e^{t\lambda_k} = e^{t(\mu_k + i\nu_k)} . \tag{4.17}$$

The $J^t(x_0)$ depends on the initial point x_0 and the elapsed time t. For notational brevity we tend to omit this dependence, but in general

$$\Lambda_k = \Lambda_k(x_0, t) , \ \lambda_k = \lambda_k(x_0, t) , \ \nu_k = \nu_k(x_0, t) , \cdots ,$$

depend on both the trajectory traversed and the choice of coordinates. Conventionally we label eigenvalues Λ in decreasing order

$$|\Lambda_1| \geq |\Lambda_2| \geq \ldots \geq |\Lambda_d|$$

Since $|\Lambda_j| = e^{t\mu_j}$, this is the same as

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_d$$

sect. 8.1.2

As A has only real entries, it will in general have either real eigenvalues, or complex conjugate pairs of eigenvalues. That is not surprising, but also the corresponding eigenvectors can be either real or complex. All coordinates used in defining the flow are real numbers, so what is the meaning of a *complex* eigenvector?

explicit formulas
fundamental
ixs, eigenvalues
the Notebooks,
KTH article.

Example 4.3 Complex eigenvalues: *To develop some intuition about that, let us work out the behavior for the simplest nontrivial case, the 2–d/ case*

characteristic equation
in/out nodes
inward/outward spirals
center
saddle

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} . \tag{4.18}$$

The eigenvalues λ_1, λ_2 of A are the roots

$$\lambda_{1,2} = \frac{1}{2}\left(\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4 \det A}\right) \tag{4.19}$$

of the characteristic equation

$$\det (A - z\mathbf{1}) = (\lambda_1 - z)(\lambda_2 - z) \, = \, 0 \,, \tag{4.20}$$

$$\left\| \begin{array}{cc} A_{11} - z & A_{12} \\ A_{21} & A_{22} - z \end{array} \right\| = z^2 - (A_{11} + A_{22})\, z + (A_{11}A_{22} - A_{12}A_{21}) \ .$$

For the distinct eigenvalues case $\lambda_1 \neq \lambda_2$, he eigenvectors are obtained by applying to an arbitrary vector $x \in \mathbb{R}^2$ the projection operators

$$P_1 = \frac{A - \lambda_2 \mathbf{1}}{\lambda_1 - \lambda_2} \,, \qquad P_2 = \frac{A - \lambda_1 \mathbf{1}}{\lambda_2 - \lambda_1} \,. \tag{4.21}$$

The qualitative behavior of e^A for real eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ differs from the case that they form a complex conjugate pair,

$$\lambda_1 = \mu_1 + i\nu_1 \,, \quad \lambda_2 = \lambda_1^* = \mu_1 - i\nu_1 \,.$$

These two possibilities are refined further into sub-cases depending on the signs of the real part. The matrix might have only one eigenvector, or two linearly independent eigenvectors, which may or may not be orthogonal. Along each of these directions the motion is of the form $x_k \exp(t\lambda_k)$. If the exponent λ_k is positive, then the component x_k will grow; if the exponent λ_k is negative, it will shrink.

generic example

align figure 4.3

We sketch the full set of possibilities in figure 4.3 (a), and work out in detail the case when A can be brought to the diagonal form, and the case of degenerate eigenvalues.

add mathworld list?

Example 4.4 Complex eigenvalues, diagonal: *If A can be brought to the diagonal form, the solution (4.14) to the differential equation (4.9) can be written either as*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{t\mu_1} & 0 \\ 0 & e^{t\mu_2} \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} , \tag{4.22}$$

or

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{t\mu} \begin{pmatrix} e^{it\nu} & 0 \\ 0 & e^{-it\nu} \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} . \tag{4.23}$$

In the case $\mu_1 > 0$, $\mu_2 < 0$, x_1 grows exponentially with time, and x_2 contracts exponentially. This behavior, called a saddle, *is sketched in figure 4.3 (b), as are the remaining possibilities: in/out nodes, inward/outward spirals, and the center. Spirals arise from taking a real part of the action of J^t on a complex eigenvector. The magnitude of $|x(t)|$ diverges exponentially when $\mu > 0$, and contracts toward 0 when the $\mu < 0$, whereas the imaginary phase ν controls its oscillations.*

Spirals figure 4.3

refer to streamlines of figure 4.3 (b)

singular value decomposition
eigenvalue
Lagrangian!coordinates
Eulerian!coordinates
flow!stability
stability!flow

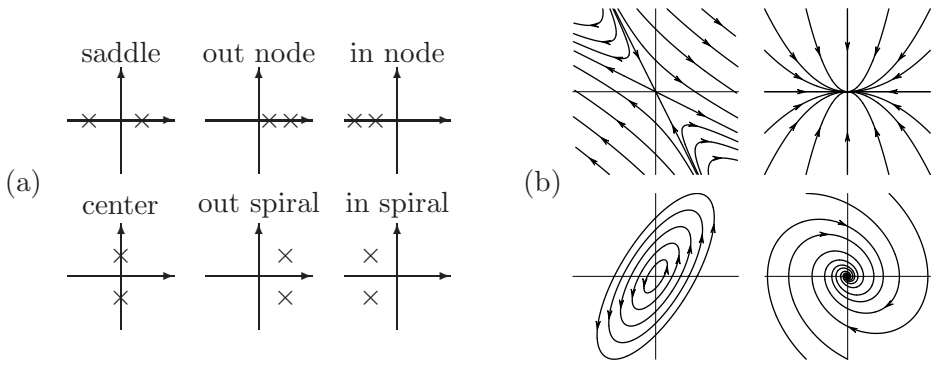

Figure 4.3: (a) Qualitatively distinct types of exponents of a $[2\times 2]$ fundamental matrix. (b) Streamlines for several typical 2-dimensional flows: saddle (hyperbolic), in node (attracting), center (elliptic), in spiral.

⇓PRELIMINARY
finish example 4.5

Example 4.5 Degenerate eigenvalues: *The simplest nontrivial example of a matrix with a degenerate pair of eigenvalues is*

$$A = \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix} . \tag{4.24}$$

The qualitative behavior of e^A for eigenvalue $\mu \in \mathbb{R}$ will differ [PC COMPLETE]

The matrix might have only one eigenvector, or two linearly independent eigenvectors, which may or may not be orthogonal.

⇑PRELIMINARY

In general J^t is neither diagonal, nor diagonalizable, nor constant along the trajectory. As any matrix, J^t can also be expressed in the singular value decomposition form

⇓PRELIMINARY
☞ remark 4.2
⇑PRELIMINARY

$$J = UDV^T$$

give 2-d example of ellipsoid explicitelly

where D is diagonal, and U, V are orthogonal matrices. The diagonal elements Λ_1, Λ_2, ..., Λ_d of D are the eigenvalues.

rows of **V**?

Under the action of the flow an infinitesimally small ball of initial points is deformed into an ellipsoid: $|\Lambda_i|$ is the relative stretching of the ith principal axis of the ellipsoid, the columns of the matrix V are the principal axes e_i of stretching in the Lagrangian coordinate frame, and the orthogonal matrix U gives the orientation of the ellipse in the Eulerian coordinates.

define Lagrangian=co-moving, Eulerian=external coordinate frames

Now that we have some feeling for the qualitative behavior of eigenvectors and eigenvalues of linear flows, we are ready to return to the nonlinear case.

4.3 Stability of flows

Frøyland's

equilibrium!point
Rossler@Rössler system!equilibria
equilibrium!Rössler system

How do you determine the eigenvalues of the finite time local deformation J^t for a general nonlinear smooth flow? The fundamental matrix is computed by integrating the equations of variations (4.2)

$$x(t) = f^t(x_0)\,, \quad \delta x(x_0,t) \;=\; J^t(x_0)\delta x(x_0,0)\,. \tag{4.25}$$

The equations are linear, so we should be able to integrate them - but in order to make sense of the answer, we derive it step by step.

4.3.1 Stability of equilibria

For a start, consider the case where x is an equilibrium point (2.7). Expanding around the equilibrium point x_*, using the fact that the stability matrix $A = A(x_*)$ in (4.2) is constant, and integrating,

$$f^t(x) = x_* + e^{At}(x - x_*) + \cdots\,, \tag{4.26}$$

we verify that the simple formula (4.14) applies also to the fundamental matrix of an equilibrium point,

$$J^t(x_*) = e^{At}\,, \qquad A = A(x_*)\,. \tag{4.27}$$

Example 4.6 Stability of equilibria of the Rössler flow. *The Rösler system (2.14) has two equilibrium points (2.15), the inner equilibrium (x^-, y^-, z^-), and the outer equilibrium point (x^+, y^+, z^+). Together with their exponents (eigenvalues of the stability matrix) the two equilibria now yield quite detailed information about the flow. Figure 4.4 shows two trajectories which start in the neighborhood of the "+" equilibrium point. Trajectories to the right of the outer equilibrium point "+" escape, and those to the left spiral toward the inner equilibrium point "−", where they seem to wander chaoticaly for all times. The stable manifold of outer equilibrium point thus serves as a attraction basin boundary. Consider now the eigenvalues of the two equilibria*

2.9 page 46

4.4 page 78

$$\begin{array}{ll} (\mu_1^-, \mu_2^- \pm i\,\nu_2^-) = & (-5.686, \qquad\quad 0.0970 \pm i\,0.9951) \\ (\mu_1^+, \mu_2^+ \pm i\,\nu_2^+) = & (0.1929, \quad -4.596\times 10^{-6} \pm i\,5.428) \end{array} \tag{4.28}$$

Outer equilibrium: *The $\mu_2^+ \pm i\,\nu_2^+$ complex eigenvalue pair implies that that neighborhood of the outer equilibrium point rotates with angular period $T_+ \approx |2\pi/\nu_2^+| = 1.1575$. The multiplier by which a trajectory that starts near the "+" equilibrium point contracts in the stable manifold plane is the excrutiatingly slow $\Lambda_2^+ \approx \exp(\mu_2^+ T_+) = 0.9999947$ per rotation. For each period the point of the stable manifold moves away along the unstable eigen-direction by factor $\Lambda_1^+ \approx \exp(\mu_1^+ T_+) = 1.2497$. Hence the slow spiraling on both sides of the "+" equilibrium point.*

Euler!limit
Euler!product

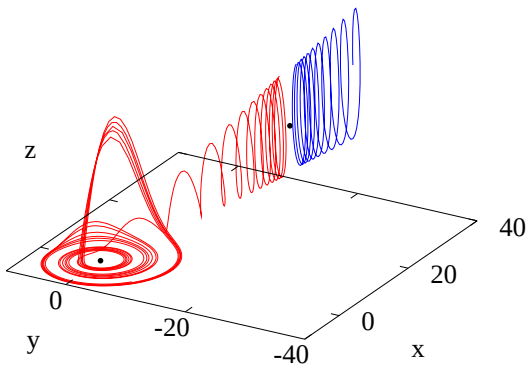

Figure 4.4: Two trajectories of the Rössler flow initiated in the neighborhood of the "+" or "outer" equilibrium point (2.15). (R. Paškauskas)

Inner equilibrium: *The $\mu_2^- \pm i\,\nu_2^-$ complex eigenvalue pair tells us that neighborhood of the "−" equilibrium point rotates with angular period $T_- \approx |2\pi/\nu_2^-| = 6.313$, slightly faster than the harmonic oscillator estimate in (2.11). The multiplier by which a trajectory that starts near the "−" equilibrium point spirals away per one rotation is $\Lambda_{radial} \approx \exp(\mu_2^- T_-) = 1.84$. The μ_1^- eigenvalue is essentially the z expansion correcting parameter c introduced in (2.13). For each Poincaré section return, the trajectory is contracted into the stable manifold by the amazing factor of $\Lambda_1 \approx \exp(\mu_1^- T_-) = 10^{-15.6}$ (!).*

Suppose you start with a 1 mm *interval pointing in the Λ_1 eigen-direction. After one Poincaré return the interval is of order of* 10^{-4} fermi, *the furthest we will get into subnuclear structure in this book. Of course, from the mathematical point of view, the flow is reversible, and the Poincaré return map is invertible. (Rytis Paškauskas)*

4.3.2 Stability of trajectories

Next, consider the case of a general, non-stationary trajectory $x(t)$. The exponential of a constant matrix can be defined either by its Taylor series expansion, or in terms of the Euler limit (4.14):

☞ appendix M.1

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \tag{4.29}$$

$$= \lim_{m\to\infty} \left(\mathbf{1} + \frac{t}{m} A\right)^m . \tag{4.30}$$

Taylor expanding is fine if A is a constant matrix. However, only the second, tax-accountant's discrete step definition of an exponential is appropriate for the task at hand, as for a dynamical system the local rate of neighborhood distortion $A(x)$ depends on where we are along the trajectory. The linearized neighborhood is multiplicatively deformed along the flow, and the m discrete time step approximation to J^t is therefore given by a generalization of the Euler product (4.30):

$$J^t = \lim_{m\to\infty} \prod_{n=m}^{1} (\mathbf{1} + \delta t A(x_n)) = \lim_{m\to\infty} \prod_{n=m}^{1} e^{\delta t\, A(x_n)} \tag{4.31}$$

$$= \lim_{m\to\infty} e^{\delta t\, A(x_n)} e^{\delta t\, A(x_{m-1})} \cdots e^{\delta t\, A(x_2)} e^{\delta t\, A(x_1)} ,$$

where $\delta t = (t - t_0)/m$, and $x_n = x(t_0 + n\delta t)$. Slightly perverse indexing of the products indicates that in our convention the successive infinitesimal deformation are applied by multiplying from the left. The two formulas for J^t agree to leading order in δt, and the $m \to \infty$ limit of this procedure is the integral

time!ordered integration

need integration exercises for (4.32)

☞ appendix E

$$J^t_{ij}(x_0) = \left[\mathbf{T} e^{\int_0^t d\tau A(x(\tau))}\right]_{ij} , \tag{4.32}$$

where $\mathbf{T}$ stands for time-ordered integration, *defined* as the continuum limit of the successive left multiplications (4.31). This formula for J is the main result of this chapter.

☞ appendix I.1

It is evident from the time-ordered product structure (4.31) that the fundamental matrices are multiplicative along the flow,

$$J^{t+t'}(x) = J^{t'}(x')J^t(x), \qquad \text{where } x' = f^t(x)\,. \tag{4.33}$$

The formula (4.31) is a matrix generalization of the crude Euler integrator (2.16) and is neither smart not accurate. Much better numerical accuracy is obtained by the following observation: To linear order in δt,

$$J^{t+\delta t} - J^t = AJ^t\,\delta t + O\left(\delta t^2\right) ,$$

so the fundamental matrix itself satisfies the linearized equation (4.1)

$$\frac{d}{dt}J^t(x) = A(x)\,J^t(x)\,, \quad \text{initial condition } \; J^0(x) = \mathbf{1}\,. \tag{4.34}$$

left, right eigenvectors

Given a numerical routine for integrating the equations of motion, evaluation of the fundamental matrix requires minimal additional programming effort; one simply extends the d-dimensional integration routine and integrates concurrently with $f^t(x)$ the d^2 elements of $J^t(x)$.

The qualifier "simply" is perhaps too glib. Integration will work for short finite times, but for exponentially unstable flows one quickly runs into numerical over- and/or underflow problems, so further thought will have to go into implementating this calculation.

in depth:
sect. 10.3, p. 160

4.4 Neighborhood volume

☞ sect. 10.3
☞ remark 10.3

Consider an small state space volume $\Delta V = d^d x$ centered around the point x_0 at time $t = 0$. The volume $\Delta V'$ around the point $x' = x(t)$ time t later

Jacobian
Lyapunov exponent

is

$$\Delta V' = \frac{\Delta V'}{\Delta V}\,\Delta V = \left|\det \frac{\partial x'}{\partial x}\right| \Delta V = \left|\det\, J(x)^t\right| \Delta V\,, \tag{4.35}$$

so the Jacobian $\det J$ is the ratio of the initial and the final volumes. The Jacobian of the flow is easily evaluated.

explain that eigenvalue 1 along the flow requires "gauge fixing", choice of arbitrary starting point

4.1
page 78

Consider $\det J^t(x_0) = \prod_{i=1}^d \Lambda_i(x_0,t)$, the product of the Floquet multipliers. We shall refer to this determinant as the *Jacobian* of the flow. By means of the time-ordered product (4.31) and the identity $\ln\det M = \mathrm{tr}\,\ln M$ the Jacobian is given by

$$\det J^t(x_0) = \exp\left[\int_0^t d\tau\, \mathrm{tr}\,\mathbf{A}(x(\tau))\right] = \exp\left[\int_0^t d\tau\, \partial_i v_i(x(\tau))\right] . \tag{4.36}$$

(Here, as elsewhere in this book, a repeated index implies summation.) As the divergence $\partial_i v_i$ is a scalar quantity, the integral in the exponent needs *no time ordering*. All we need to do is evaluate the time average

$$\begin{aligned}\overline{\partial_i v_i} &= \lim_{t\to\infty}\frac{1}{t}\int_0^t d\tau \sum_{i=1}^d A_{ii}(x(\tau)) \\ &= \frac{1}{t}\ln\left|\prod_{i=1}^d \Lambda_i(x_0,t)\right| = \sum_{i=1}^d \lambda_i(x_0,t)\end{aligned} \tag{4.37}$$

along the trajectory. If the flow is not singular (for example, the trajectory does not run head-on into the Coulomb $1/r$ singularity), the stability matrix elements are bounded everywhere, $|A_{ij}| < M$, and so is the trace $\sum_i A_{ii}$. The time integral in (4.36) grows at most linearly with t, hence $\overline{\partial_i v_i}$ is bounded for all times, and numerical estimates of the $t\to\infty$ limit in (4.37) are not marred by any blowups.

⇓PRIVATE

Example 4.7 Rössler flow state space contraction:

⇑PRIVATE

Even if we were to insist on extracting $\overline{\partial_i v_i}$ from (4.31) by first multiplying fundamental matrices along the flow, and then taking the logarithm, we can avoid exponential blowups in J^t by using the multiplicative structure (4.33), $\det J^{t'+t}(x_0) = \det J^{t'}(x')\det J^t(x_0)$ to restart with $J^0(x') = \mathbf{1}$ whenever the eigenvalues of $J^t(x_0)$ start getting out of hand. In numerical evaluations of Lyapunov exponents, $\lambda_i = \lim_{t\to\infty}\lambda_i(x_0,t)$, the sum rule (4.37) can serve as a helpful check on the accuracy of the computation.

sect. 10.3

The divergence $\partial_i v_i$ is an important characterization of the flow - it describes the behavior of a state space volume in the infinitesimal neighborhood of the trajectory. If $\partial_i v_i < 0$, the flow is *locally contracting*, and the

trajectory might be falling into an attractor. If $\partial_i v_i(x) < 0$, for all $x \in \mathcal{M}$, the flow is *globally* contracting, and the dimension of the attractor is necessarily smaller than the dimension of state space $\mathcal{M}$. If $\partial_i v_i = 0$, the flow preserves state space volume and $\det J^t = \mathbf{1}$. A flow with this property is called *incompressible*. An important class of such flows are the Hamiltonian flows considered in sect. 5.2.

contracting!flow
flow!contracting
incompressible flow
flow!incompressible
Roux!Henri
map!stability
stability!maps
monodromy matrix

But before we can get to that, Henri Roux, the perfect student always on alert, pipes up. He does not like our definition of the fundamental matrix in terms of the time-ordered exponential (4.32). Depending on the signs of Floquet exponents, the left hand side of (4.36) can be either positive or negative. But the right hand side is an exponential of a real number, and that can only be positive. What gives? As we shall see much later on in this text, in discussion of topological indices arising in semiclassical quantization, this is not at all a dumb question.

☞ sect. 33.2

⇓PRELIMINARY

☞ sect. 34.3.2

in depth:
appendix M.1, p. 1005

in depth:
appendix D.1, p. 903

⇑PRELIMINARY
⇓PRELIMINARY
⇑PRELIMINARY

4.5 Stability of maps

The transformation of an infinitesimal neighborhood of a trajectory under the iteration of a map follows from Taylor expanding the iterated mapping at *discrete* time n to linear order, as in (4.5). The linearized neighborhood is transported by the fundamental matrix evaluated at a discrete set of times $n = 1, 2, \ldots$,

$$M^n_{ij}(x_0) = \left.\frac{\partial f^n_i(x)}{\partial x_j}\right|_{x=x_0} . \tag{4.38}$$

We shall refer to this Jacobian matrix as the *monodromy* matrix, in order to include the case where the map is a Poincaré return map for a flow. Derivative notation $J^t(x_0) \to Df^t(x_0)$ is frequently employed in the literature. In this book Λ_k denotes the kth *eigenvalue* of the finite time monodromy matrix $M^n(x_0)$, and μ_k the real part of kth *eigen-exponent*

$$|\Lambda| = e^{n\lambda} , \qquad \Lambda_\pm = e^{n(\mu \pm i\nu)} .$$

For complex eigenvalue pairs the phase ν describes rotational motion in the plane defined by the corresponding pair of eigenvectors.

fix this: " with period $T = 2\pi/\nu$."

Example 4.8 Stability of a 1-dimensional map: *Consider a 1-d map $f(x)$. The chain rule yields the stability of the nth iterate*

$$\Lambda(x_0, n) = \frac{d}{dx} f^n(x_0) = \prod_{m=0}^{n-1} f'(x_m) , \quad x_m = f^m(x_0) . \tag{4.39}$$

Henon@Hénon!map!stability
Poincar"'e!return map!stability
stability!Poincar"'e return map

The 1-step product formula for the stability of the nth iterate of a d-dimensional map

$$\begin{aligned} M^n(x_0) &= M(x_{n-1}) \cdots M(x_1) M(x_0)\,, \\ M(x)_{kl} &= \frac{\partial}{\partial x_l} f_k(x)\,, \quad x_m = f^m(x_0) \end{aligned} \tag{4.40}$$

follows from the chain rule for matrix derivatives

$$\frac{\partial}{\partial x_i} f_j(f(x)) = \sum_{k=1}^{d} \frac{\partial}{\partial y_k} f_j(y)\Big|_{y=f(x)} \frac{\partial}{\partial x_i} f_k(x)\,.$$

If you prefer to think of a discrete time dynamics as a sequence of Poincaré section returns, then (4.40) follows from (4.33): fundamental matrices are multiplicative along the flow.

10.1 page 169

Example 4.9 Hénon map monodromy matrix: *For the Hénon map (3.15) the monodromy matrix for the nth iterate of the map is*

$$M^n(x_0) = \prod_{m=n}^{1} \begin{pmatrix} -2a x_m & b \\ 1 & 0 \end{pmatrix}\,, \qquad x_m = f_1^m(x_0, y_0)\,. \tag{4.41}$$

The determinant of the Hénon one time step monodromy matrix (4.41) is constant,

$$\det M = \Lambda_1 \Lambda_2 = -b \tag{4.42}$$

so in this case only one eigenvalue $\Lambda_1 = -b/\Lambda_2$ needs to be determined. This is not an accident; a constant Jacobian was one of desiderata that led Hénon to construct a map of this particular form.

explain Henon = normal form in conjug.tex

insert Rossler discussion here

fast track: chapter 5, p. 81

4.5.1 Stability of Poincaré return maps

(R. Paškauskas and P. Cvitanović)

We now relate the linear stability of the Poincaré return map $P : \mathcal{P} \to \mathcal{P}$ defined in sect. 3.1 to the stability of the continuous time flow in the full state space.

The hypersurface $\mathcal{P}$ can be specified implicitly through a function $U(x)$ that is zero whenever a point x is on the Poincaré section. A nearby point $x + \delta x$ is in the hypersurface $\mathcal{P}$ if $U(x + \delta x) = 0$, and the same is true for

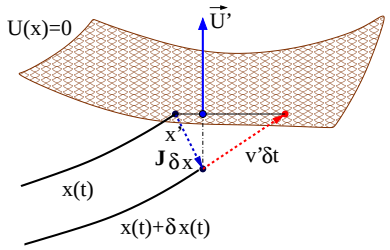

Figure 4.5: If $x(t)$ intersects the Poincaré section $\mathcal{P}$ at time τ, the nearby $x(t) + \delta x(t)$ trajectory intersects it time $\tau+\delta t$ later. As $(U' \cdot v'\delta t) = -(U' \cdot J\,\delta x)$, the difference in arrival times is given by $\delta t = -(U' \cdot J\,\delta x)/(U' \cdot v')$.

variations around the first return point $x' = x(\tau)$, so expanding $U(x')$ to linear order in δx leads to the condition

ce U_i by normal ne surface n_i = , $x \in \mathcal{P}$

$$\sum_{i=1}^{d+1} \frac{\partial U(x')}{\partial x_i} \left.\frac{dx'_i}{dx_j}\right|_{\mathcal{P}} = 0\,. \tag{4.43}$$

In what follows U_i is the gradient of U defined in (3.3), unprimed quantities refer to the starting point $x = x_0 \in \mathcal{P}$, $v = v(x_0)$, and the primed quantities to the first return: $x' = x(\tau)$, $v' = v(x')$, $U' = U(x')$. For brevity we shall also denote the full state space fundamental matrix at the first return by $J = J^\tau(x_0)$. Both the first return x' and the time of flight to the next Poincaré section $\tau(x)$ depend on the starting point x, so the fundamental matrix

replace $\hat{J}$ by $\mathbf{M}$ when monodromy gets defined

$$\hat{J}(x)_{ij} = \left.\frac{dx'_i}{dx_j}\right|_{\mathcal{P}} \tag{4.44}$$

with both initial and the final variation constrained to the Poincaré section hypersurface $\mathcal{P}$ is related to the continuous flow fundamental matrix by

$$\left.\frac{dx'_i}{dx_j}\right|_{\mathcal{P}} = \frac{\partial x'_i}{\partial x_j} + \frac{dx'_i}{d\tau}\frac{d\tau}{dx_j} = J_{ij} + v'_i\frac{d\tau}{dx_j}\,.$$

The return time variation $d\tau/dx$, figure 4.5, is eliminated by substituting this expression into the constraint (4.43),

$$0 = \partial_i U'\, J_{ij} + (v' \cdot \partial U')\frac{d\tau}{dx_j}\,,$$

redraw figure

yielding the projection of the full space $(d+1)$-dimensional fundamental matrix to the Poincaré map d-dimensional fundamental matrix:

$$\hat{J}_{ij} = \left(\delta_{ik} - \frac{v'_i\,\partial_k U'}{(v' \cdot \partial U')}\right) J_{kj}\,. \tag{4.45}$$

Substituting (4.7) we verify that the initial velocity $v(x)$ is a zero-eigenvector of $\hat{J}$

$$\hat{J}v = 0\,, \tag{4.46}$$

linear!flows
flow!linear
later on - comment on Jacobi, constant energy → unit eigenvalue
singular value decomposition
time!ordered integration

so the Poincaré section eliminates variations parallel to v, and $\hat{J}$ is a rank d matrix, i.e., one less than the dimension of the continuous time flow.

Commentary

Remark 4.1 Linear flows. The theory of linear flows and their stability is only sketched in sect. 4.2. They are presented at length in many textbooks. We liked the discussion in chapter 1 of Perko [4.1] and chapters 3 and 5 of Glendinning [4.2]. The nomenclature is a bit confusing. Sometimes A, the stability matrix (4.3) which describes the instantaneous shear of the trajectory point $x(x_0, t)$ is refered to as the "Jacobian matrix", a particularly unfortunate usage when one considers linearized stability of an equilibrium point (4.27). What Jacobi had in mind in his 1841 fundamental paper [4.3] on the determinants today known as "jacobians" were transformations between different coordinate frames. More unfortunate still is refering to $J^t = e^{tA}$ as an "evolution operator", which here (see sect. 10.2) refers to something althogether different. In this book fundamental matrix J^t always refers to (4.6), the linearized deformation after a finite time t, either for a continuous time flow, or a discrete time mapping.

⇓PRELIMINARY

Remark 4.2 Singular value decomposition of fundamental matrix. For a discussion of the singular value decomposition (SVD) of fundamental matrix, see for example Thiffeault [4.4].

⇑PRELIMINARY

Résumé

A neighborhood of a trajectory deforms as it is transported by a flow. In the linear approximation, the stability matrix A describes the shearing/-compression/expansion of an infinitesimal neighborhood in an infinitesimal time step. The deformation after a finite time t is described by the fundamental matrix

$$J^t(x_0) = \mathbf{T} e^{\int_0^t d\tau A(x(\tau))} ,$$

where $\mathbf{T}$ stands for the time-ordered integration, defined multiplicatively along the trajectory. For discrete time maps this is multiplication by time step fundamental matrix M along the n points x_0, x_1, $x_2, \ldots, x_{n-1}$ on the trajectory of x_0,

$$M^n(x_0) = M(x_{n-1}) M(x_{n-2}) \cdots M(x_1) M(x_0) ,$$

with $M(x)$ the single discrete time step fundamental matrix. In this book Λ_k denotes the kth *eigenvalue* of the finite time fundamental matrix $J^t(x_0)$, and μ_k the real part of kth *eigen-exponent*

$$|\Lambda| = e^{n\lambda}\,, \qquad \Lambda_\pm = e^{n(\mu\pm i\nu)}\,.$$

For complex eigenvalue pairs the phase ν describes rotational motion in the plane defined by the corresponding pair of eigenvectors.

fix this: " with period $T = 2\pi/\nu$."

The eigenvalues and eigen-directions of the fundamental matrix describe the deformation of an initial infinitesimal sphere of neighboring trajectories into an ellipsoid a finite time t later. Nearby trajectories separate exponentially along unstable directions, approach each other along stable directions, and change slowly (algebraically) their distance along marginal directions. The fundamental matrix J^t is in general neither symmetric, nor diagonalizable by a rotation, nor do its (left or right) eigenvectors define an orthonormal coordinate frame. Furthermore, although the fundamental matrices are multiplicative along the flow, in dimensions higher than one their eigenvalues in general are not. This lack of multiplicativity has important repercussions for both classical and quantum dynamics.

☞ appendix I.1

⇓PRIVATE

☞ chapter 40

⇑PRIVATE

add useful refs

References

[4.1] L. Perko, *Differential Equations and Dynamical Systems* (Springer-Verlag, New York 1991).

[4.2] P. Glendinning, *Stability, Instability, and Chaos* (Cambridge Univ. Press, Cambridge 1994).

[4.3] C. G. J. Jacobi, "De functionibus alternantibus earumque divisione per productum e differentiis elementorum conflatum," in *Collected Works*, Vol. 22, 439; *J. Reine Angew. Math. (Crelle)* (1841).

[4.4] J.-L. Thiffeault, *Physica* **D 172**, 139 (2002); `nlin.CD/0101012`

contraction!state space!Rossler@Rössler
Rossler@Rössler system

Exercises

Exercise 4.1 Trace-log of a matrix. *Prove that*

$$\det\, M = e^{\mathrm{tr}\,\ln M}\,.$$

for an arbitrary finite dimensional matrix M.

move to a better place - like count.tex - it is not used in this chapter

Exercise 4.2 Stability, diagonal case. *Verify the relation (4.16)*

$$J^t = e^{t\mathbf{A}} = \mathbf{U}^{-1} e^{t\mathbf{A}_D}\mathbf{U}\,, \quad \textit{where} \quad \mathbf{A}_D = \mathbf{U}\mathbf{A}\mathbf{U}^{-1}\,.$$

Exercise 4.3 State space volume contraction in Rössler flow.

(a) *Compute the Rössler flow volume contraction rate at the equilibria.*

(b) *Study numerically the instantaneous $\partial_i v_i$ along a typical trajectory on the Rössler attractor; color-code the points on the trajectory by the sign (and perhaps the magnitude) of $\partial_i v_i$. If you see regions of local expansion, explain them.*

(c) *Compute numerically the average contraction rate (4.37) along a typical trajectory on the Rössler attractor.*

(d) *(optional) color-code the points on the trajectory by the sign (and perhaps the magnitude) of $\partial_i v_i - \overline{\partial_i v_i}$.*

(e) *Argue on basis of your results that this attractor is of dimension smaller than the state space $d = 3$.*

(f) *(optional) Start some trajectories on the escape side of the outer equilibrium, color-code the points on the trajectory. Is the flow volume contracting?*

Exercise 4.4 Topology of the Rössler flow. *(continuation of exercise 3.1)*

(a) *Show that equation $|\det(A-\lambda\mathbf{1})| = 0$ for Rössler system in the notation of exercise 2.18 can be written as*

$$\lambda^3 + \lambda^2 c\,(p^{\mp} - \epsilon) + \lambda(p^{\pm}/\epsilon + 1 - c^2\epsilon p^{\mp}) \mp c\sqrt{D} = 0 \qquad (4.47)$$

(b) *Solve (4.47) for eigenvalues $\lambda^{\pm}$ for each equilibrium as an expansion in powers of ϵ. Derive*

$$\begin{array}{l} \lambda_1^- = -c + \epsilon c/(c^2+1) + o(\epsilon) \\ \lambda_2^- = \epsilon c^3/[2(c^2+1)] + o(\epsilon^2) \\ \theta_2^- = 1 + \epsilon/[2(c^2+1)] + o(\epsilon) \\ \lambda_1^+ = c\epsilon(1-\epsilon) + o(\epsilon^3) \\ \lambda_2^+ = -\epsilon^5 c^2/2 + o(\epsilon^6) \\ \theta_2^+ = \sqrt{1+1/\epsilon}\,(1+o(\epsilon)) \end{array} \qquad (4.48)$$

Compare with exact eigenvalues. What are dynamical implications of the extravagant value of λ_1^-?

(continued as exercise 17.8)

(Rytis Paškauskas)

map!contracting
map!dissipative
contracting!map
dissipative!map

Exercise 4.5 A contracting baker's map. *Consider a contracting (or "dissipative") baker's map, acting on a unit square* $[0,1]^2 = [0,1] \times [0,1]$, *defined by*

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n/3 \\ 2y_n \end{pmatrix} \quad y_n \le 1/2$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n/3 + 1/2 \\ 2y_n - 1 \end{pmatrix} \quad y_n > 1/2\,.$$

This map shrinks strips by a factor of 1/3 in the x*-direction, and then stretches (and folds) them by a factor of 2 in the* y*-direction.*

By how much does the state space volume contract for one iteration of the map?

⇓PRELIMINARY

(continued in exercise 11.4)

⇑PRELIMINARY

Newtonian dynamics
flow!Hamiltonian
degree of freedom

Chapter 5

Newtonian dynamics

You might think that the strangeness of contracting flows, flows such as the Rössler flow of figure 2.3 is of concern only to chemists; real physicists do Hamiltonians, right? Not at all - while it is easier to visualize aperiodic dynamics when a flow is contracting onto a lower-dimensional attracting set, there are plenty examples of chaotic flows that do preserve the full symplectic invariance of Hamiltonian dynamics. The truth is, the whole story started with Poincaré's restricted 3-body problem, a realization that chaos rules also in general (non-Hamiltonian) flows came much later.

Here we briefly review parts of classical dynamics that we will need later on; symplectic invariance, canononical transformations, and stability of Hamiltonian flows. We discuss billiard dynamics in some detail in chapter 6.

5.1 Hamiltonian flows

(P. Cvitanović and L.V. Vela-Arevalo)

An important class of flows are Hamiltonian flows, given by a Hamiltonian $H(q,p)$ together with the Hamilton's equations of motion

☞ appendix D

revert to (q,p) order throughtout

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}\,, \tag{5.1}$$

with the $2D$ phase space coordinates x split into the configuration space coordinates and the conjugate momenta of a Hamiltonian system with D degrees of freedom (dof):

☞ sect. 34.1.1

$$x = (\mathbf{q},\mathbf{p})\,, \qquad \mathbf{q} = (q_1, q_2, \ldots, q_D)\,, \qquad \mathbf{p} = (p_1, p_2, \ldots, p_D)\,. \tag{5.2}$$

The energy, or the value of the Hamiltonian function at the state space

level set
Duffing!undamped
integrable system
helium!collinear

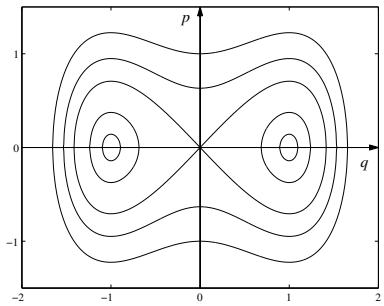

Figure 5.1: Phase plane of the unforced, undamped Duffing oscillator. The trajectories lie on level sets of the Hamiltonian (5.4).

point $x = (\mathbf{q}, \mathbf{p})$ is constant along the trajectory $x(t)$,

$$\begin{aligned} \frac{d}{dt} H(\mathbf{q}(t), \mathbf{p}(t)) &= \frac{\partial H}{\partial q_i}\dot{q}_i(t) + \frac{\partial H}{\partial p_i}\dot{p}_i(t) \\ &= \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} = 0\,, \end{aligned} \tag{5.3}$$

so the trajectories lie on surfaces of constant energy, or *level sets* of the Hamiltonian $\{(q,p) : H(q,p) = E\}$. For 1-dof Hamiltonian systems this is basically the whole story.

Example 5.1 Unforced undamped Duffing oscillator: *When the damping term is removed from the Duffing oscillator (2.6), the system can be written in Hamiltonian form with the Hamiltonian*

$$H(q,p) = \frac{p^2}{2} - \frac{q^2}{2} + \frac{q^4}{4}. \tag{5.4}$$

This is a 1-dof Hamiltonian system, with a 2-dimensional state space, the plane (q,p). *The Hamilton's equations (5.1) are*

$$\dot{q} = p\,, \qquad \dot{p} = q - q^3\,. \tag{5.5}$$

For 1-dof systems, the "surfaces" of constant energy (5.3) are simply curves in the phase plane (q,p), *and the dynamics is very simple: the curves of constant energy* are *the trajectories, as shown in figure 5.1.*

trajectories need arrows, label font too small in figure 5.1

Thus all 1-dof systems are *integrable*, in the sense that the entire phase plane is foliated by curves of constant energy, either periodic – as is the case for the harmonic oscillator (a "bound state") – or open (a "scattering trajectory"). Add one more degree of freedom, and chaos breaks loose.

☞ example 7.1

Example 5.2 Collinear helium: *In chapter 41, we shall apply the periodic orbit theory to the quantization of helium. In particular, we will study* collinear helium, *a doubly charged nucleus with two electrons arranged on a line, an electron on each side of the nucleus. The Hamiltonian for this system is* [1]

☞ chapter 41

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{2}{r_1} - \frac{2}{r_2} + \frac{1}{r_1 + r_2}\,. \tag{5.6}$$

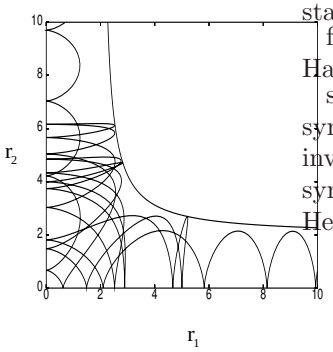

Figure 5.2: A typical collinear helium trajectory in the $r_1 - r_2$ plane; the trajectory enters along the r_1-axis and then, like almost every other trajectory, after a few bounces escapes to infinity, in this case along the r_2-axis.

stability!Hamiltonian flows
Hamiltonian!flows, stability
symplectic!invariance
invariance!symplectic
symplectic!form
Hessian matrix

Collinear helium has 2 dof, and thus a 4-dimensional phase space $\mathcal{M}$, which energy conservation reduces to 3 dimensions. The dynamics can be projected onto the 2-dimensional configuration plane, the (r_1, r_2), $r_i \geq 0$ quadrant, figure 5.2. It looks messy, and, indeed, it will turn out to be no less chaotic than a pinball bouncing between three disks. As always, a Poincaré section will be more informative than this rather arbitrary projection of the flow.

5.2 Stability of Hamiltonian flows

Hamiltonian flows offer an illustration of the ways in which an invariance of equations of motion can affect the dynamics. In the case at hand, the *symplectic invariance* will reduce the number of independent stability eigenvalues by a factor of 2 or 4.

5.2.1 Canonical transformations

The equations of motion for a time-independent, D-dof Hamiltonian (5.1) can be written

$$\dot{x}_i = \omega_{ij} H_j(x)\,, \qquad \omega = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}\,, \quad H_j(x) = \frac{\partial}{\partial x_j} H(x)\,, \tag{5.7}$$

use O de Almeida book argument here?

where $x = (\mathbf{q}, \mathbf{p}) \in \mathcal{M}$ is a phase space point, $H_k = \partial_k H$ is the column vector of partial derivatives of H, $\mathbf{I}$ is the $[D \times D]$ unit matrix, and ω the $[2D \times 2D]$ *symplectic* the form

$$\omega^T = -\omega\,, \qquad \omega^2 = -\mathbf{1}\,. \tag{5.8}$$

The linearized motion in the vicinity $x + \delta x$ of a phase space trajectory $x(t) = (\mathbf{q}(t), \mathbf{p}(t))$ is described by the fundamental matrix (4.25). For Hamiltonian flows the stability matrix (4.34) takes form

$$\frac{d}{dt} M^t(x) = A(x) M^t(x)\,, \qquad A_{ij}(x) = \omega_{ik}\, H_{kj}(x)\,, \tag{5.9}$$

[1] Mason: mention briefly "transition state"

symplectic!transformation
canonical!transformation
Lie group!orthogonal
orthogonal!Lie group
symplectic!Lie algebra
Lie algebra!symplectic
canonical transformation
symplectic!map

where the matrix of second derivatives $H_{kn} = \partial_k \partial_n H$ is called the *Hessian matrix*. From the symmetry of H_{kn} it follows that

$$A^T \omega + \omega A = 0\,. \tag{5.10}$$

This is the defining property for infinitesimal generators of *symplectic* (or canonical) transformations, transformations which leave the symplectic form ω invariant.

More explicitely: just as the rotation group $O(d)$ is the Lie group of all matrix transformations which preserve a bilinear symmetric invariant $x^2 = x_i \delta_{ij} x_j$, i.e., the length squared, the symplectic group $Sp(2D)$ is defined as the Lie group of all matrix transformations $x' = Mx$ which preserve a bilinear *antisymmetric* invariant $x_i \omega_{ij} y_j$. The symplectic Lie algebra $sp(2D)$ follows by writing $M = \exp(\delta t A)$ and linearizing $M = 1 + \delta t A$. This yields (5.10) as the defining property of infinitesimal symplectic transformations.

Consider now a smooth nonlinear change of variables of form $y = h(x)$, and the Hamiltonian as the function of the new phase space variables, $K(x) = H(h(x))$. Rewriting (5.7) as a function of y and employing the chain rule we find

$$\begin{aligned} \partial K(x) &= (\partial h)^T\, \partial H(h(x)) = (\partial h)^T\, (-\omega\, \dot{y}) \\ &= -(\partial h)^T\, \omega\, (\partial h)\, \dot{x}\,, \end{aligned} \tag{5.11}$$

where $(\partial h)_{kn} = \partial h_k / \partial x_n$. Consider the case that ∂h, the linearization of h, is a symplectic transformation, i.e., a map that preserves the symplectic form,

$$(\partial h)^T\, \omega\, (\partial h) = \omega\,. \tag{5.12}$$

In that case $K(x)$ also induces a Hamiltonian flow $\dot{x}_i = \omega_{ij}\, K_j(x)$, and h is called a *canonical transformation*. We care about canonical transformations for two reasons. First (and this is a dark art), if the canonical transformation h is very cleverly chosen, the flow in new coordinates might be considerably simpler than the original flow. Second, Hamiltonian flows themselves are a prime example of canonical transformations.

☞ example 7.1

Example 5.3 Hamiltonian flows are canonical: *For Hamiltonian flows it follows from (5.10) that* $\frac{d}{dt}\left(M^T \omega M\right) = 0$, *and since at the initial time* $M^0(x_0) = \mathbf{1}$, M *is a symplectic transformation* $M^T \omega M = \omega$. *This equality is valid for all times, so a Hamiltonian flow* $f^t(x)$ *is a canonical transformation, with the linearization* $\partial_x f^t(x)$ *a symplectic transformation (5.12):*

$$M^T \omega M = \omega\,. \tag{5.13}$$

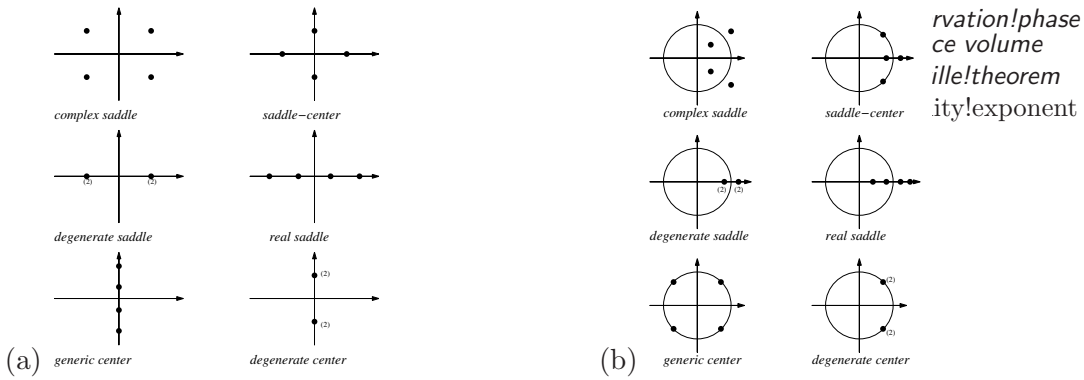

Figure 5.3: (a) Stability exponents of a Hamiltonian equilibrium point, 2-dof. (b) Stability of a symplectic map in $\mathbb{R}^4$.

For notational brevity here we have suppressed the dependence on time and the initial point, $M = M^t(x_0)$. By elementary properties of determinants it follows from (5.13) that Hamiltonian flows are phase space volume preserving:

$$\det M = 1\,. \tag{5.14}$$

5.2.2 Stability of equilibria of Hamiltonian flows

☞ sect. 4.3.1

For an equilibrium point x_q the stability matrix A is constant. Its eigenvalues describe the linear stability of the equilibrium point. In the case of Hamiltonian flows, from (5.10) it follows that the characteristic polynomial of A for an equilibrium x_q satisfies

$$\begin{aligned}\det(A-\lambda\mathbf{1}) &= \det(\omega^{-1}(A-\lambda\mathbf{1})\omega) = \det(-\omega A\omega-\lambda\mathbf{1}) \\ &= -\det(A^T+\lambda\mathbf{1}) = -\det(A+\lambda\mathbf{1})\,. \end{aligned} \tag{5.15}$$

A is the matrix (5.10) with real matrix elements, so its eigenvalues (the stability exponents of (4.27)) are either real or come in complex pairs. Symplectic invariance implies in addition that if λ is an eigenvalue, then $-\lambda$, λ^* and $-\lambda^*$ are also eigenvalues. Distinct symmetry classes of the stability exponents of an equilibrium point in a 2-dof system are displayed in figure 5.3 (a). It is worth noting that while the linear stability of equilibria in a Hamiltonian system always respects this symmetry, the nonlinear stability can be completely different.

5.3 page 91

need to replace figure 4.3 (a) by edited version of figure 5.3 (a)

Luz, give references in a remark

5.3 Symplectic maps

A stability eigenvalue $\Lambda = \Lambda(x_0, t)$ associated to a trajectory is an eigenvalue of the fundamental matrix M. The transpose M^T and the inverse M^{-1} are related by

$$M^{-1} = -\omega M^T \omega\,, \tag{5.16}$$

loxodromic
conservation!phase space volume
Liouville!theorem
elliptic!stability
residue!stability
stability!residue
hyperbolic!flow
flow!hyperbolic
inverse hyperbolic flow
flow!inverse hyperbolic
Greene's residue criterion
residue!Greene's
Henon@Hénon!map!Hamiltonian
map!Hénon!Hamiltonian
map!Hamiltonian!Hénon
Hamiltonian!Hénon map
symplectic!Hénon map
area preserving!Hénon map
Henon@Hénon!map

so the characteristic polynomial satisfies

$$\begin{aligned}\det(M-\Lambda\mathbf{1}) &= \det(M^T-\Lambda\mathbf{1}) = \det(-\omega M^T\omega-\Lambda\mathbf{1})\\ &= \det(M^{-1}-\Lambda\mathbf{1}) = \det(M^{-1})\det(\mathbf{1}-\Lambda M)\\ &= \Lambda^{2D}\det(M-\Lambda^{-1}\mathbf{1})\,. \qquad (5.17)\end{aligned}$$

Hence if Λ is an eigenvalue of M, so are $1/\Lambda$, Λ^* and $1/\Lambda^*$. Real (non-marginal, $|\Lambda|\neq 1$) eigenvalues always come paired as Λ, $1/\Lambda$. The Liouville conservation of phase space volumes (5.14) is an immediate consequence of this pairing up of eigenvalues. The complex eigenvalues come in pairs Λ, Λ^*, $|\Lambda|=1$, or in loxodromic quartets Λ, $1/\Lambda$, Λ^* and $1/\Lambda^*$. These possibilities are illustrated in figure 5.3 (b).

Example 5.4 2-dimensional symplectic maps: *In the 2-dimensional case the eigenvalues (8.3) depend only on* $\mathrm{tr}\,M^t$

$$\Lambda_{1,2}=\frac{1}{2}\left(\mathrm{tr}\,M^t\pm\sqrt{(\mathrm{tr}\,M^t-2)(\mathrm{tr}\,M^t+2)}\right)\,. \qquad (5.18)$$

The trajectory is elliptic if the stability residue $|\mathrm{tr}\,M^t|-2\leq 0$, *with complex eigenvalues* $\Lambda_1=e^{i\theta t}$, $\Lambda_2=\Lambda_1^*=e^{-i\theta t}$. *If* $|\mathrm{tr}\,M^t|-2>0$, λ *is real, and the trajectory is either*

$$\text{hyperbolic}\qquad \Lambda_1=e^{\lambda t}\,,\quad \Lambda_2=e^{-\lambda t}\,,\ \textit{or} \qquad (5.19)$$

$$\text{inverse hyperbolic}\qquad \Lambda_1=-e^{\lambda t}\,,\quad \Lambda_2=-e^{-\lambda t}\,. \qquad (5.20)$$

⇓PRIVATE

explain Greene's residue

Greene's residue criterion states

(i) hyperbolic (positive eigenvalues) if $r<O$

(ii) elliptic (complex eigenvalues) if $O<r<l$

(iii) inverse hyperbolic (negative eigenvalues) if $r>1$.

⇑PRIVATE

Example 5.5 Hamiltonian Hénon map, reversibility: *By (4.42) the Hénon map (3.15) for* $b=-1$ *value is the simplest 2-d orientation preserving area-preserving map, often studied to better understand topology and symmetries of Poincaré sections of 2 dof Hamiltonian flows. We find it convenient to multiply (3.16) by* a *and absorb the* a *factor into* x *in order to bring the Hénon map for the* $b=-1$ *parameter value into the form*

$$x_{i+1}+x_{i-1}=a-x_i^2\,,\quad i=1,...,n_p\,, \qquad (5.21)$$

The 2-dimensional Hénon map *for* $b=-1$ *parameter value*

$$\begin{aligned}x_{n+1} &= a-x_n^2-y_n\\ y_{n+1} &= x_n\,. \qquad (5.22)\end{aligned}$$

is Hamiltonian (symplectic) in the sense that it preserves area in the $[x,y]$ *plane.*

For definitiveness, in numerical calculations in examples to follow we shall fix (arbitrarily) the stretching parameter value to $a=6$, *a value large enough to guarantee that all roots of* $0=f^n(x)-x$ *(periodic points) are real.*

link to next exer

5.3.1 Poincaré invariants

conservation!phase space volume
Liouville!theorem
Poincar"'e!invariants
Stokes theorem

Let C a region in the phase space and $V(0)$ its volume. Denoting the flow of the Hamiltonian system by $f^t(x)$, the volume of C after a time t is $V(t) = f^t(C)$, and using (5.14) we derive the *Liouville theorem*:

$$\begin{aligned} V(t) &= \int_{f^t(C)} dx = \int_C \left\| \frac{\partial f^t(x')}{\partial x} \right\| dx' \\ & \int_C \det(M) dx' = \int_C dx' = V(0)\,, \end{aligned} \tag{5.23}$$

Hamiltonian flows preserve phase space volumes.

The symplectic structure of Hamilton's equations buys us much more than the "incompressibility," or the phase space volume conservation. Consider the symplectic product of two infinitesimal vectors

$$\begin{aligned} (\delta x, \delta\hat{x}) &= \delta x^T \omega \delta\hat{x} = \delta p_i \delta\hat{q}_i - \delta q_i \delta\hat{p}_i \\ &= \sum_{i=1}^{D} \{\text{oriented area in the } (q_i, p_i) \text{ plane}\}\,. \end{aligned} \tag{5.24}$$

Time t later we have

$$(\delta x', \delta\hat{x}') = \delta x^T M^T \omega M \delta\hat{x} = \delta x^T \omega \delta\hat{x}\,.$$

This has the following geometrical meaning. We imagine there is a reference phase space point. We then define two other points infinitesimally close so that the vectors δx and $\delta\hat{x}$ describe their displacements relative to the reference point. Under the dynamics, the three points are mapped to three new points which are still infinitesimally close to one another. The meaning of the above expression is that the area of the parallelopiped spanned by the three final points is the same as that spanned by the inital points. The integral (Stokes theorem) version of this infinitesimal area invariance states that for Hamiltonian flows the D oriented areas $\mathcal{V}_i$ bounded by D loops $\Omega\mathcal{V}_i$, one per each (q_i, p_i) plane, are separately conserved:

$$\int_{\mathcal{V}} dp \wedge dq = \oint_{\Omega\mathcal{V}} p \cdot dq = \text{invariant}\,. \tag{5.25}$$

Morally a Hamiltonian flow is really D-dimensional, even though its phase space is $2D$-dimensional. Hence for Hamiltonian flows one emphasizes D, the number of the degrees of freedom.

in depth:
appendix D.1, p. 903

⇓PRIVATE

Henon@Hénon!map!reversible
map!reversible!Hénon
reversible!Hénon map

5.4 Reversibility

Example 5.6 Hamiltonian Hénon map, reversibility: *The Hénon map (5.22) is reversible, with its inverse interchanging the roles of x and y:*

$$\begin{aligned} x_{n-1} &= y_n \\ y_{n-1} &= a - y_n^2 - x_n \,, \end{aligned} \tag{5.26}$$

hence the dynamics is symmetric in the $[x, y]$ plane: a trajectory maps into a trajectory under the flip across the $x = y$ diagonal

$$\begin{pmatrix} y \\ x \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5.27}$$

and *time reversal. The reversor R is orientation reversing, $\det[\partial R] = -1$, and is an involution, $R^2 = \mathbb{1}$. In other words, the Hamiltonian Hénon map is conjugate to its inverse $f \circ R = R \circ f^{-1}$, and can be factored into a pair of orientation reversing involutions, $f = (fR) \circ R = T \circ R$, with*

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ a - x^2 - y \end{pmatrix} \,. \tag{5.28}$$

⇓PRELIMINARY

Equivalently, writing $f = S \circ (Sf) = S \circ U$, the reversor

$$U \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - y^2 - x \\ y \end{pmatrix} \tag{5.29}$$

factorizes the Hénon map as $f = ST$.

⇑PRELIMINARY
⇑PRIVATE

Commentary

> In theory there is no difference between theory and practice. In practice there is.
> Yogi Berra

Remark 5.1 Hamiltonian dynamics literature.

If you are reading this book, in theory you already know everything that is in this chapter. In practice you do not. Try this: Put your right hand on your heart and say: "I understand why nature preferes symplectic geometry". Honest? We make an attempt in sect. 34.1. Out there there are about 2 centuries of accumulated literature on Hamilton, Lagrange, Jacobi etc. formulation of mechanics, some of it excellent. In context of what we will need here, we make a very subjective recommendation - we enjoyed reading Percival and Richards [7.10] and Ozorio de Almeida [7.11].

Remark 5.2 Symplectic. The term symplectic - Greek for twining or plaiting together - was introduced into mathematics by Hermann Weyl. "Canonical" lineage is church-doctrinal: Greek "kanon", referring to a reed used for measurement, came to mean in Latin a rule or a standard.

aps move
rk A.4 here,
it reformulates
-Somm. (A.4)

residue!stability
stability!residue
symplectic!invariance
invariance!symplectic

Remark 5.3 The sign convention of ω. The overall sign of ω, the symplectic invariant in (5.7), is set by the convention that the Hamilton's principal function (for energy conserving flows) is given by $R(q,q',t) = \int_q^{q'} p_i dq_i - Et$. With this sign convention the action along a classical path is minimal, and the kinetic energy of a free particle is positive.

Remark 5.4 Symmetries of the symbol square. For a more detailed discussion of symmetry lines see refs. [17.4, 3.7, 12.47]. In discussion of example 5.6, we have followed refs. [5.3, 5.4, 5.5].

⇓PRELIMINARY
⇑PRELIMINARY
improve references
⇓PRELIMINARY

Remark 5.5 Stability residue. Stability residue was introduced by Greene [5.7].

Résumé

Hamiltonian flow $\dot{x}_i = \omega_{ij} H_j(x)$, $\omega_{mn} = -\omega_{nm}$ is canonical in the sense that $x' = f^t(x)$ is a transformation whose fundamental matrix $J = \partial x'/\partial x$ preserves the symplectic invariant $M^T \omega M = \omega$. Infinitesimal symplectic transformations are generated by $A_{ij} = \omega_{ik} H_{kj}$ satisfying $A^T\omega + \omega A = 0$. This implies that A are elements of the symplectic algebra $sp(2D)$, and that the $2D$-dimensional flow preserves D oriented infinitesimal areas $\delta x^T \omega \delta\hat{x} = \delta p_i \delta\hat{q}_i - \delta q_i \delta\hat{p}_i$. The Liouville phase space volume conservation is a consequences of this general invariance.

⇑PRELIMINARY

References

[5.1] J.E. Marsden and T.S. Ratiu, *Introduction to Mechanics and Symmetry* (Springer, New York, 1994).

complete ref. [5.2]

[5.2] Cherry (1959), (1968).

[5.3] D.G. Sterling, H.R. Dullin and J.D. Meiss, "Homoclinic Bifurcations for the Hénon Map," *Physica D* **134**, 153 (1999); `chao-dyn/9904019`.

[5.4] H.R. Dullin, J.D. Meiss and D.G. Sterling, "Symbolic Codes for Rotational Orbits," `nlin.CD/0408015`.

[5.5] A. Gómez and J.D. Meiss, "Reversible Polynomial Automorphisms of the Plane: the Involutory Case," *Phys. Lett. A* **312**, 49 (2003); `nlin.CD/0209055`.

[5.6] J.M. Greene, *J. Math. Phys.* **20**, 1183 (1979).

[5.7] J.M. Greene, "Two-Dimensional Measure-Preserving Mappings", *J. Math. Phys.* **9**, 760 (1968)

Exercises

Exercise 5.1 Complex nonlinear Schrödinger equation. *Consider the complex nonlinear Schrödinger equation in one spatial dimension [5.1]:*

$$i\frac{\partial\phi}{\partial t} + \frac{\partial^2\phi}{\partial x^2} + \beta\phi|\phi|^2 = 0, \quad \beta \neq 0.$$

(a) *Show that the function $\psi : \mathbb{R} \to \mathbf{C}$ defining the traveling wave solution $\phi(x,t) = \psi(x-ct)$ for $c > 0$ satisfies a second-order complex differential equation equivalent to a Hamiltonian system in $\mathbb{R}^4$ relative to the noncanonical symplectic form whose matrix is given by*

$$w_c = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -c \\ 0 & -1 & c & 0 \end{bmatrix}.$$

(b) *Analyze the equilibria of the resulting Hamiltonian system in $\mathbb{R}^4$ and determine their linear stability properties.*

(c) *Let $\psi(s) = e^{ics/2}a(s)$ for a real function $a(s)$ and determine a second order equation for $a(s)$. Show that the resulting equation is Hamiltonian and has heteroclinic orbits for $\beta < 0$. Find them.*

(d) *Find "soliton" solutions for the complex nonlinear Schrödinger equation.*

(Luz V. Vela-Arevalo)

Exercise 5.2 When is a linear transformation canonical?

(a) *Let A be a $n \times n$ invertible matrix. Show that the map $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by $(\mathbf{q}, \mathbf{p}) \mapsto (A\mathbf{q}, (A^{-1})^T\mathbf{p})$ is a canonical transformation.*

(b) *If $\mathbf{R}$ is a rotation in $\mathbb{R}^3$, show that the map $(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{R}\,\mathbf{q}, \mathbf{R}\,\mathbf{p})$ is a canonical transformation.*

(Luz V. Vela-Arevalo)

Exercise 5.3 Cherry's example. *What follows [5.2] is mostly a reading exercise, about a Hamiltonian system that is $linearly\ stable$ but $nonlinearly\ unstable$. Consider the Hamiltonian system on $\mathbb{R}^4$ given by*

$$H = \frac{1}{2}(q_1^2 + p_1^2) - (q_2^2 + p_2^2) + \frac{1}{2}p_2(p_1^2 - q_1^2) - q_1 q_2 p_1.$$

(a) *Show that this system has an equilibrium at the origin, which is linearly stable. (The linearized system consists of two uncoupled oscillators with frequencies in ratios 2:1).*

(b) *Convince yourself that the following is a family of solutions parametrized by a constant τ:*

$$q_1 = -\sqrt{2}\frac{\cos(t-\tau)}{t-\tau}, \quad q_2 = \frac{\cos 2(t-\tau)}{t-\tau},$$
$$p_1 = \sqrt{2}\frac{\sin(t-\tau)}{t-\tau}, \qquad p_2 = \frac{\sin 2(t-\tau)}{t-\tau}.$$

These solutions clearly blow up in finite time; however they start at $t = 0$ at a distance $\sqrt{3}/\tau$ from the origin, so by choosing τ large, we can find solutions starting arbitrarily close to the origin, yet going to infinity in a finite time, so the origin is $nonlinearly\ unstable$.

(Luz V. Vela-Arevalo)

Give the example of $(xy)^2$ elliptic island.

billiards
stadium billiard
billiard!stadium
specular reflection
section!Poincar"'e
Poincar"'e!section
Poincar"'e!section!3-disk
three-disk@3-disk!geometry

Chapter 6

Billiards

The dynamics that we have the best intuitive grasp on, and find easiest to grapple with both numerically and conceptually, is the dynamics of billiards. For billiards, discrete time is altogether natural; a particle moving through a billiard suffers a sequence of instantaneous kicks, and executes simple motion inbetween, so there is no need to contrive a Poincaré section. We have already used this system in sect. 1.3 as the intuitively most accessible example of chaos. Here we define billiard dynamics more precisely, anticipating the applications to come.

Do in detail the disk scattering dynamics; LiversageFig.1

6.1 Billiard dynamics

A billiard is defined by a connected region $Q \subset \mathbb{R}^D$, with boundary $\partial Q \subset \mathbb{R}^{D-1}$ separating Q from its complement $\mathbb{R}^D \setminus Q$. The region Q can consist of one compact, finite volume component (in which case the billiard phase space is bounded, as for the stadium billiard figure 6.1), or can be infinite in extent, with its complement $\mathbb{R}^D \setminus Q$ consisting of one or several finite or infinite volume components (in which case the phase space is open, as for the 3-disk pinball game figure 1.1). In what follows we shall most often restrict our attention to *planar billiards*. A point particle of mass m and momentum $p_n = mv_n$ moves freely within the billiard, along a straight line, until it encounters the boundary. There it reflects specularly (*specular* = mirrorlike), with no change in the tangential component of momentum, and instantaneous reversal of the momentum component normal to the boundary,

put arrows everywhere on figure 6.1, denote ball position by (s_n, ϕ_n, τ_n)

check OED

$$p' = p - 2(p \cdot \hat{n})\hat{n}\,, \tag{6.1}$$

with $\hat{n}$ the unit vector normal to the boundary ∂Q at the collision point. The angle of incidence equals the angle of reflection, as illustrated in figure 6.2. A billiard is a Hamiltonian system with a $2D$-dimensional phase space $x = (q, p)$ and potential $V(q) = 0$ for $q \in Q$, $V(q) = \infty$ for $q \in \partial Q$.

Smilansky teaches: generating function; canonincal transformations; action angle variables. Problem: prove reflectin law by minimization of action

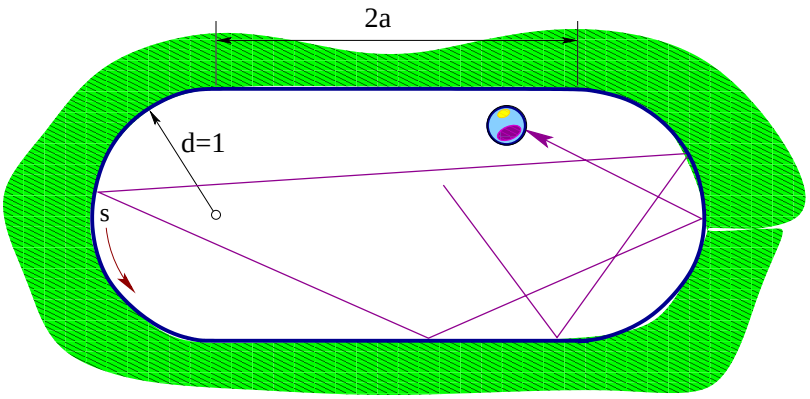

Figure 6.1: The stadium billiard is a 2-dimensional domain bounded by two semi-circles of radius $d = 1$ connected by two straight walls of length $2a$. At the points where the straight walls meet the semi-circles, the curvature of the border changes discontinuously; these are the only singular points of the flow. The length a is the only parameter.

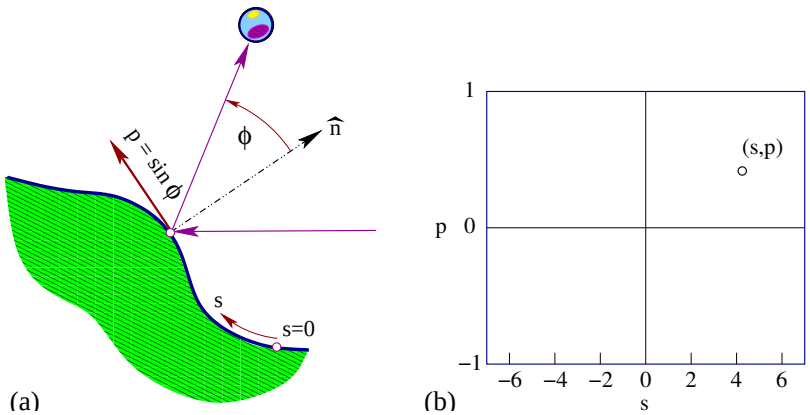

Figure 6.2: (a) A planar billiard trajectory is fixed by specifying the perimeter length parametrized by s and the outgoing trajectory angle ϕ, both measured counterclockwise with respect to the outward normal $\hat{n}$. (b) The Birkhoff phase space coordinate pair (s, p) fully specifies the trajectory, where $p = |p| \sin\phi$ is the momentum component tangential to the boundary (and we set $|p| = 1$ whenever convenient).

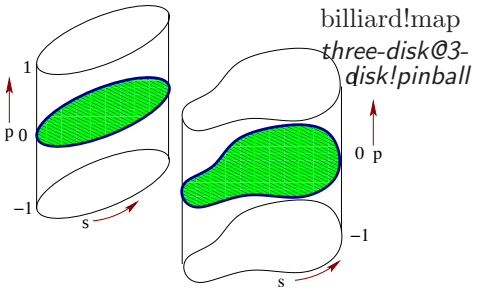

Figure 6.3: In $D = 2$ the billiard Poincaré section is a cylinder, with the parallel momentum p ranging over $p \in \{-1, 1\}$, and with the s coordinate is cyclic along each connected component of ∂Q. The rectangle figure 6.2 (b) is such cylinder unfolded, with periodic boundary conditions glueing together the left and the right edge of the rectangle.

billiard!map
three-disk@3-disk!pinball

A billiard flow has a natural Poincaré section defined by Birkhoff coordinates s_n, the arc length position of the nth bounce measured along the billiard boundary, and $p_n = |p| \sin \phi_n$, the momentum component parallel to the boundary, where ϕ_n is the angle between the outgoing trajectory and the normal to the boundary. We measure both the arc length s, and the parallel momentum p counterclockwise relative to the outward normal (see figure 6.2 as well as figure 1.7 (a)). In $D = 2$, the Poincaré section is a cylinder (topologically an annulus), figure 6.3, where the parallel momentum p ranges for $-|p|$ to $|p|$, and the s coordinate is cyclic along each connected component of ∂Q. The volume in the full phase space is preserved by the Liouville theorem (5.23). The Birkhoff coordinates $x = (s, p) \in \mathcal{P}$, are the natural choice, because with them the the Poincaré return map preserves the phase space volume in the (s, p) parametrized Poincaré section (a perfectly good coordinate set (s, ϕ) does not do that).

label the two areas Q_1, Q_2 in figure 6.3. Draw corresponding rectangles?

6.8
page 106
sect. 6.2

Without loss of generality we set $m = |v| = |p| = 1$. Poincaré section condition eliminates one dimension, and the energy conservation $|p| = 1$ eliminates another, so the Poincaré section return map P is $(2D-2)$-dimensional.

The dynamics is given by the Poincaré return map

$$P : (s_n, p_n) \mapsto (s_{n+1}, p_{n+1}) \tag{6.2}$$

from the nth collision to the $(n+1)$st collision. [1] The discrete time dynamics map P is equivalent to the Hamiltonian flow (5.1) in the sense that both describe the same full trajectory. Let t_n denate the instant of nth collision. Then the position of the pinball $\in Q$ at time $t_n + \tau \leq t_{n+1}$ is given by $2D-2$ Poincaré section coordinates $(s_n, p_n) \in \mathcal{P}$ together with τ, the distance reached by the pinball along the nth section of its trajectory.

replace the reference to figure 1.7 (a) by ?

emphasize that the discrete/continuous time averages are the same

add figures from my papers

Example 6.1 3-disk game of pinball: *In case of bounces off a circular disk, the position coordinate $s = r\theta$ is given by angle $\theta \in [0, 2\pi]$. For example, for the 3-disk game of pinball of figure 1.5 and figure 1.7 we have two types of collisions:*

$$P_0 : \begin{cases} \phi' = -\phi + 2 \arcsin p \\ p' = -p + \frac{a}{R} \sin \phi' \end{cases} \qquad \textit{back-reflection} \tag{6.3}$$

6.1
page 105

replace this by a smarter formula!

[1] Mason: Poincare sections cannot handle flows like Shilnikov. Mention exceptional case, like when the section cannot decipher the type of the trajectory

stability!billiards
billiards!stability
Sinai-Bunimovich curvature
curvature!Sinai-Bunimovich

$$P_1 : \begin{cases} \phi' = \phi - 2\arcsin p + 2\pi/3 \\ p' = p - \frac{a}{R}\sin\phi' \end{cases} \qquad \textit{reflect to 3rd disk}\,. \tag{6.4}$$

Here a = radius of a disk, and R = center-to-center separation. Actually, as in this example we are computing intersections of circles and straight lines, nothing more than high-school geometry is required. There is no need to compute arcsin*'s either - one only needs to compute a square root per each reflection, and the simulations can be very fast.*

Trajectory of the pinball in the 3-disk billiard is generated by a series of P_0's and P_1's. At each step on has to check whether the trajectory intersects the desired disk (and no disk inbetween). With minor modifications, the above formulas are valid for any smooth billiard as long as we replace a by the local curvature of the boundary at the point of collision.

rewrite this section in the way we really compute - this impact parameter stuff is not the right thing!

6.2 page 105

6.2 Stability of billiards

We turn next to the question of local stability of discrete time billiard systems. Infinitesimal equations of variations (4.2) do not apply, but the multiplicative structure (4.33) of the finite-time fundamental matrices does. As they are more physical than most maps studied by dynamicists, let us work out the billiard stability in some detail.

On the face of it, a plane billiard phase space is 4-dimensional. However, one dimension can be eliminated by energy conservation, and the other by the fact that the magnitude of the velocity is constant. We shall now show how going to a local frame of motion leads to a [2×2] fundamental matrix. In sect. 6.2.1 we show that due to the symplectic invariance the situation is even simpler; the stability of a 2-dimensional billiard flow is given by a single number, the Sinai-Bunimovich curvature.

⇓PRELIMINARY

⇑PRELIMINARY

Consider a 2-dimensional billiard with phase space coordinates $x = (q_1, q_2, p_1, p_2)$. Let t_k be the instant of the kth collision of the pinball with the billiard boundary, and $t_k^{\pm} = t_k \pm \epsilon$, ϵ positive and infinitesimal. With the mass and the velocity equal to 1, the momentum direction can be specified by angle θ: $x = (q_1, q_2, \sin\theta, \cos\theta)$. Now parametrize the 2-$d$ neighborhood of a trajectory segment by $\delta x = (\delta z, \delta\theta)$, where

⇓PRELIMINARY
draw the figure!
⇑PRELIMINARY

$$\delta z = \delta q_1 \cos\theta - \delta q_2 \sin\theta\,, \tag{6.5}$$

$\delta\theta$ is the variation in the direction of the pinball motion. Due to energy conservation, there is no need to keep track of $\delta q_{\parallel}$, variation along the flow, as that remains constant. $(\delta q_1, \delta q_2)$ is the coordinate variation transverse to the kth segment of the flow, figure 6.4. From the Hamilton's equations of motion for a free particle, $dq_i/dt = p_i$, $dp_i/dt = 0$, we obtain the equations of motion (4.1) for the linearized neighborhood

⇓PRIVATE
⇑PRIVATE

$$\frac{d}{dt}\delta\theta = 0, \quad \frac{d}{dt}\delta z = \delta\theta\,. \tag{6.6}$$

Figure 6.4: Variations in the phase space coordinates of a pinball between the $(k-1)$th and the kth collision. (a) δq_k variation away from the direction of the flow. (b) δz_k angular variation tranverse to the direction of the flow. (c) $\delta q_{\|}$ variation in the direction of the flow is conserved by the flow.

Let $\delta\theta_k = \delta\theta(t_k^+)$ and $\delta z_k = \delta z(t_k^+)$ be the local coordinates immediately after the kth collision, and $\delta\theta_k^- = \delta\theta(t_k^-)$, $\delta z_k^- = \delta z(t_k^-)$ immediately before. Integrating the free flight from t_{k-1}^+ to t_k^- we obtain

draw figure

$$\begin{aligned} \delta z_k^- &= \delta z_{k-1} + \tau_k \delta\theta_{k-1}\,, \qquad \tau_k = t_k - t_{k-1} \\ \delta\theta_k^- &= \delta\theta_{k-1}\,, \end{aligned} \tag{6.7}$$

and the stability matrix (4.32) for the kth free flight segment is

$$M_T(x_k) = \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix}\,. \tag{6.8}$$

At incidence angle ϕ_k (the angle between the outgoing particle and the outgoing normal to the billiard edge), the incoming transverse variation δz_k^- projects onto an arc on the billiard boundary of length $\delta z_k^-/\cos\phi_k$. The corresponding incidence angle variation $\delta\phi_k = \delta z_k^-/\rho_k\cos\phi_k$, ρ_k = local radius of curvature, increases the angular spread to

doodle a little picture here - then the relation becomes obvious, it is a "footprint" of flashlight shining on a surface at an angle.

$$\begin{aligned} \delta z_k &= -\delta z_k^- \\ \delta\theta_k &= -\delta\theta_k^- - \frac{2}{\rho_k\cos\phi_k}\delta z_k^-\,, \end{aligned} \tag{6.9}$$

so the fundamental matrix associated with the reflection is

Tanner changed signs here, Predrag changed them back

$$M_R(x_k) = -\begin{pmatrix} 1 & 0 \\ r_k & 1 \end{pmatrix}\,, \qquad r_k = \frac{2}{\rho_k\cos\phi_k}\,. \tag{6.10}$$

The full fundamental matrix for n_p consecutive bounces describes a beam of trajectories defocused by M_T along the free flight (the τ_k terms below) and defocused/refocused at reflections by M_R (the r_k terms below)

replace n_p fundamental domain - count symmetry wall reflections

$$M_p = (-1)^{n_p}\prod_{k=n_p}^{1}\begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ r_k & 1 \end{pmatrix}\,, \tag{6.11}$$

volume preservation
Sinai-Bunimovich curvature
curvature!Sinai-Bunimovich

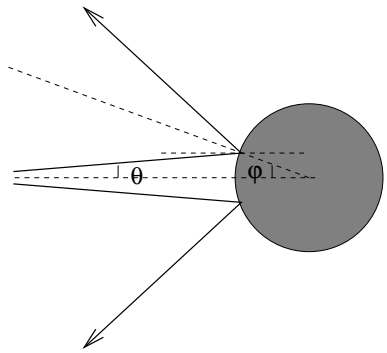

Figure 6.5: Defocusing of a beam of nearby trajectories at a billiard collision. (A. Wirzba)

6.4 page 106

where τ_k is the flight time of the kth free-flight segment of the orbit, $r_k = 2/\rho_k \cos\phi_k$ is the defocusing due to the kth reflection, and ρ_k is the radius of curvature of the billiard boundary at the kth scattering point (for our 3-disk game of pinball, $\rho = 1$). As the billiard dynamics is phase space volume preserving, $\det M = 1$, and the eigenvalues are given by (5.18).

repalce figure 6.5

This is still another example of the fundamental matrix chain rule (4.40) for discrete time systems, rather similar to the Hénon map stability (4.41). Stability of every flight segment or reflection taken alone is a shear with two unit eigenvalues,

$$\det M_T = \det \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix}, \quad \det M_R = \det \begin{pmatrix} 1 & 0 \\ r_k & 1 \end{pmatrix}, \tag{6.12}$$

but acting in concert in the intervowen sequence (6.11) they can lead to a hyperbolic deformation of the infinitesimal neighborhood of a billiard trajectory.

8.1 page 129

As a concrete application, consider the 3-disk pinball system of sect. 1.3. Analytic expressions for the lengths and eigenvalues of $\overline{0}$, $\overline{1}$ and $\overline{10}$ cycles follow from elementary geometrical considerations. Longer cycles require numerical evaluation by methods such as those described in chapter 17.

8.2 page 129
6.3 page 105
chapter 17

explain that Liouville holds, but phase space is lost thorugh escape

add the billiard mean-time of flight section from Sune article

⇓PRELIMINARY

need to draw a figure of a horocycle!

6.2.1 Sinai-Bunimovich curvatures

Imagine a set of projectiles leaving a point (q_1, q_2) in all directions, parametrized by angle θ; they generate a "horocycle" in the configuration space, a set of all points reached by time t, see figure 6.6. A $\delta\theta$ wedge of angles stretches into a horocycle arc $\delta z = t\delta\theta$, and $\kappa = \delta\theta/\delta z = 1/t$ is the local curvature of the horocycle.

Eqs. (6.7) and (6.9) can be rewritten as

$$\frac{\delta z_k^-}{\delta\theta_k^-} = \frac{\delta z_{k-1}}{\delta\theta_{k-1}} + \tau_k$$

billiard!general

Figure 6.6: A horocycle.

$$\frac{\delta\theta_k}{\delta z_k} = \frac{\delta\theta_k^-}{\delta z_k^-} + \frac{2}{\rho_k \cos\phi_k}\,, \tag{6.13}$$

leading to the continued fraction recursion for the curvature κ immediately after the kth bounce

$$\kappa_k = r_k + \frac{1}{\tau_k - \frac{1}{\kappa_{k-1}}}\,, \qquad \kappa_k = -\frac{\delta\theta_k}{\delta z_k}\,. \tag{6.14}$$

Somehow this came out backwards in time?

add exercise relating M and κ

⇑PRELIMINARY

⇓PRIVATE

6.3 General billiard flows

Consider a trajectory $x(t)$ obtained by integrating vector field $v(x)$

$$x_k^- = T(x_{k-1}) = x_{k-1} + \int_{t_{k-1}^+}^{t_k^-} d\tau\, v(x(\tau))\,. \tag{6.15}$$

Here $t_k^+ = t_k + \epsilon$ denotes the time infinitesimally after the instant of the kth kick, $t_k^- = t_k - \epsilon$ the time infinitesimally before it, and $x_k^- = x(t_k^-)$, $x_k = x(t_k^+)$. The trajectory is smooth almost all of the time, except for a sequence of instantaneous kicks (such as hard wall billiard colisions),

$$x_k = R(x_k^-)\,. \tag{6.16}$$

Map T is the integral (6.15) of the smooth flow evolution between two consecutive kicks, from t_{k-1}^+ to t_k^-. R describes the instantaneous kick change of velocity at time t_k^- to velocity at t_k^+. The discrete "time" k is an integer, the number of applications of the $R \circ T$ map, where we streamline the notation by denoting a map composition by "$\circ$"

$$x_k = R \circ T \cdots R \circ T(x_0) = (R \circ T)^k(x_0)\,. \tag{6.17}$$

restitution coefficient
section!Poincaré
Poincaré!section
billiard!map
stability!billiards
billiards!stability

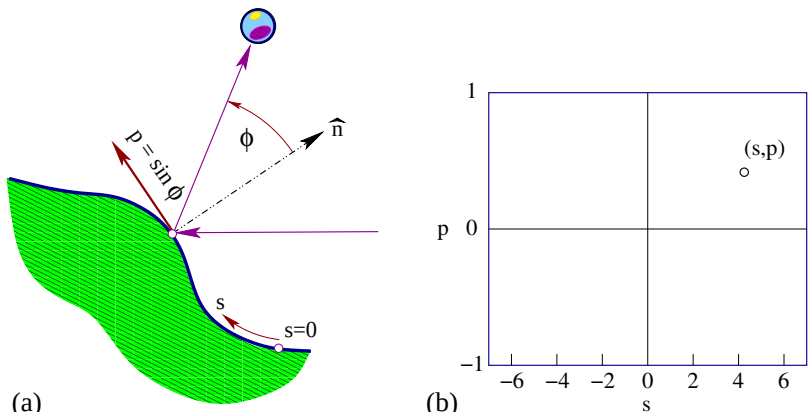

Figure 6.7: (a) A trajectory of a general billiard is fixed by specifying the perimeter wall arclength, the outgoing wall-normal momentum, and the outgoing wall-parallel momentum $(s, p_\parallel, p_\perp)$. For $\gamma < 1$ the ball loses its bounce, $\phi^+ < \phi^-$. [REDRAW] (b) [REPLACE by a perspective 3-d section.]

is "Consider a $D = 2$ *planar billiard*" needed?

Example 6.2 When the ball loses its bounce. *A point particle with position and momentum $x(t) = (q(t), p(t))$ moves within the billiard, along a given integrable trajectory (free flight, gravity, constant electric or magnetic field, harmonic oscillator, ...) until it encounters a boundary. There it reflects following the Newton's law of restitution, with no change in the tangential component of momentum, and instantaneous reversal of the momentum component normal to the boundary, in the $p, \hat{n}$ plane:*

$$R : \begin{cases} q = q^- \\ p = p^- - (1+\gamma)(p^- \cdot \hat{n})\hat{n} \end{cases} , \qquad 0 < \gamma \leq 1 , \tag{6.18}$$

with $\hat{n}$ the unit vector normal to the billiard boundary ∂Q at the collision point, and γ the restitution coefficient.

To avoid confusion: for Hamiltonian flight segments, the flow field v in (6.15) combines the Newtonian velocities and accelerations, $v(x) = (\dot{q}(x), \dot{p}(x))$.

Due to the collisional loss of energy, an inelastic billiard is not a Hamiltonian system. For $\gamma < 1$ the phase space volume is not preserved, the Liouville theorem (5.23) does not apply, and sustaning motion requires an external driving mechanism that can compensate for the collisional energy loss. Poincaré section condition eliminates one dimension, hence a 3-d Poincaré section for a plane billiard flow can be defined by marking s_k, the arc length position of the kth kick measured along the billiard boundary, and the outgoing wall-normal and wall-parallel momenta $(s_k, p_{\perp,k}, p_{\parallel,k})$. Alternatively, on can can keep track of the wall-parallel momentum $p_{\parallel,k}$ and the energy shell $E_k = p_k^2/2m + V(q_k)$ that the particle is on after kth billiard wall reflection.

We turn next to the question of local stability of a general billiard flow.

If $x(t)$ reaches the wall at time t_k, a neighboring trajectory (see figure 4.5) reflects off the billiard wall time δt later, following the reflection law (6.18):

$$x(t_k^+ + \delta t) + \delta x(t_k^+ + \delta t) = R(x(t_k^- + \delta t) + \delta x(t_k^- + \delta t)) .$$

To linear order

$$x_k^+ + \frac{dx(t_k^+)}{dt}\delta t + \delta x_k^+ \;=\; R\left(x_k^- + \frac{dx(t_k^-)}{dt}\delta t + \delta x_k^-\right)$$

$$= \quad x_k^+ + \frac{\partial R(x_k^-)}{\partial x}(\delta x_k^- + v_k^- \delta t)\,,.$$

volume contraction

Hence

$$\delta x_k^+ = M_R(x_k^-)\delta x_k^- - \left(v_k^+ - M_R(x_k^-)v_k^-\right)\delta t \tag{6.19}$$

where

$$M_R(x_k^-) = \frac{\partial R(x_k^-)}{\partial x}\,, \qquad v_k^+ = v(x_k^+)\,, \quad v_k^- = v(x_k^-)\,.$$

If you wish, you can compute the extra flight time δt by elementary geometry, from figure 4.5, as in (4.45). However, $M_R(x^-)$ is a fundamental matrix like any other; you can think of it as a limit of a smooth flow where the trajectory is strongly deflected over a time interval 2ϵ, $\epsilon \to 0$. According to (4.7), a fundamental matrix transports the velocity vector at x_0 to the velocity vector at $x(t)$ at time t:

$$v(x(t)) = J^t(x_0)\, v(x_0)\,,$$

hence the prefactor of δt in (6.19) *vanishes*. Another way to think of this is as in sect. 4.5.1, where we have proven that the spectrum of transverse eigenvalues of a fundamental matrix for a continuous flow is independent of the choice of a transverse Poincaré sections, with different choices differing by such δt's contributions along the direction of the flow.

The cycle fundamental matrix for n_p consecutive bounces describes a beam of trajectories defocused by M_T along the flight segments and defocused/refocused at kicks by M_R

$$M_p = \prod_{k=n_p}^{1} M_R(x_{k+1}^-)M_T(x_k)\,. \tag{6.20}$$

To summarize: we have shown that the time delay contributes only along the direction of the flow, and by (4.7) the longitudinal part does not contribute computation of stability eigenvalues of a cycle.

⇓PRELIMINARY

Guys, what am I missing? If you do no believe me, why does the Hamiltonian linearized stability for particles bouncing in gravity not require this δt term? There are accelerations...

The generalized billiard dynamics is not phase space volume preserving; for impact normal to the billiard wall $\det M = \gamma^2$.

fix this

⇑PRELIMINARY

 ⇑PRIVATE

billiards
baker's map
three-disk@3-disk!pinball
intermittency
bifurcation!generic
renormalization
smooth!potential
three-disk@3-disk!pinball
symbolic!dynamics!at a bifurcation
three-dimensional@3-dimensional sphere!scattering
scattering!3-dimensional spheres

Commentary

start with
(convex),
Bunimovich (conc

Remark 6.1 Billiards. The 3-disk game of pinball is to chaotic dynamics what a pendulum is to integrable systems; the simplest physical example that captures the essence of chaos. Another contender for the title of the "harmonic oscillator of chaos" is the baker's map which is used as the red thread through Ott's introduction to chaotic dynamics [1.12]. The baker's map is the simplest reversible dynamical system which is hyperbolic and has positive entropy. We will not have much use for the baker's map here, as due to its piecewise linearity it is so nongeneric that it misses all of the subtleties of cycle expansions curvature corrections that will be central to this treatise.

That the 3-disk game of pinball is a quintessential example of deterministic chaos appears to have been first noted by B. Eckhardt [6.1]. The model was studied in depth classically, semiclassically and quantum mechanically by P. Gaspard and S.A. Rice [6.2], and used by P. Cvitanović and B. Eckhardt [6.3] to demonstrate applicability of cycle expansions to quantum mechanical problems. It has been used to study the higher order $\hbar$ corrections to the Gutzwiller quantization by P. Gaspard and D. Alonso Ramirez [6.4], construct semiclassical evolution operators and entire spectral determinants by P. Cvitanović and G. Vattay [6.5], and incorporate the diffraction effects into the periodic orbit theory by G. Vattay, A. Wirzba and P.E. Rosenqvist [6.6]. The full quantum mechanics and semiclassics of scattering systems is developed here in the 3-disk scattering context in chapter 38. Gaspard's monograph [1.8], which we warmly recommend, utilizies the 3-disk system in much more depth than will be attained here. For further links check `ChaosBook.org`.

☞ chapter 18

A pinball game does miss a number of important aspects of chaotic dynamics: generic bifurcations in smooth flows, the interplay between regions of stability and regions of chaos, intermittency phenomena, and the renormalization theory of the "border of order" between these regions. To study these we shall have to face up to much harder challenge, dynamics of smooth flows.

Nevertheless, pinball scattering is relevant to smooth potentials. The game of pinball may be thought of as the infinite potential wall limit of a smooth potential, and pinball symbolic dynamics can serve as a *covering* symbolic dynamics in smooth potentials. One may start with the infinite wall limit and adiabatically relax an unstable cycle onto the corresponding one for the potential under investigation. If things go well, the cycle will remain unstable and isolated, no new orbits (unaccounted for by the pinball symbolic dynamics) will be born, and the lost orbits will be accounted for by a set of pruning rules. The validity of this adiabatic approach has to be checked carefully in each application, as things can easily go wrong; for example, near a bifurcation the same naive symbol string assignments can refer to a whole island of distinct periodic orbits.

☞ sect. 37.1

Remark 6.2 Further reading. The chapter 1 of Gaspard monograph [1.8] is recommended reading if you are interested in Hamiltonian flows, and billiards in particular. A. Wirzba has generalized the stability analysis of sect. 6.2 to scattering off 3-dimensional spheres (follow the links in `ChaosBook.org/extras`). A clear discussion of linear stability for the general d-dimensional case is given in Gaspard [1.8], sect. 1.4.

⇓PRELIMINARY

find Knauff scattering reference

Knauff's scattering problem is surprising in this context, as it is Anosov flow that resembles a billiard.

⇑PRELIMINARY
⇓PRIVATE

Remark 6.3 Impact oscillators. Relate sect. 6.3 to the elastic impact oscillator literature, refs. [6.11, 6.12, 6.13, 6.14].

Remark 6.4 Restitution coefficients. (From Ref. [6.15]): In the normal direction, the restitution laws mostly used are those given by Newton and by Poisson. Newton's is a purely kinematical law, Poisson's is dynamical and Begin-Boulanger uses energy consideration.

The coefficient of restitution given by Newton considers the normal relative velocities pre and post-collision.

The coefficient of restitution given by Poisson is the ratio between the normal impulse in the expansion phase and the normal impulse in the compression phase. This coefficient of restitution takes into consideration the dynamics of the system in the virtual process of collision.

The coefficient of restitution given by Beghin-Boulanger considers the relation between the kinetic energies in the expansion phase and compression phases. This coefficient of restitution considers the exchange of energy during the virtual process of collision.

When friction is not considered all three coefficients of restitution are equivalent (ref. [3] in Ref. [6.15]).

γ_t = coefficient of tangential restitution

⇑PRIVATE

Résumé

A particulary natural application of the Poincaré section method is the reduction of a billiard flow to a boundary-to-boundary return map.

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⇑PRELIMINARY

Exercises

three-disk@3-disk!simulator
pinball!simulator
three-disk@3-disk!simulator

maybe use different periods for bounces and continuous time escape rates?

Exercise 6.1 **A pinball simulator.** *Implement the disk → disk maps to compute a trajectory of a pinball for a given starting point, and a given $R{:}a$ = (center-to-center distance):(disk radius) ratio for a 3-disk system. As this requires only computation of intersections of lines and circles together with specular reflections, implementation should be within reach of a high-school student. Please start working on this program now; it will be continually expanded in chapters to come, incorporating the Jacobian calculations, Newton root–finding, and so on.*

Fast code will use elementary geometry (only one $\sqrt{\cdots}$ per iteration, rest are multiplications) and eschew trigonometric functions. Provide a graphic display of the trajectories and of the Poincaré section iterates. To be able to compare with the numerical results of coming chapters, work with $R{:}a = 6$ and/or 2.5 values. Draw the correct versions of figure 1.9 or figure 11.4 for $R{:}a = 2.5$ and/or 6.

Exercise 6.2 **Trapped orbits.** *Shoot 100,000 trajectories from one of the disks, and trace out the strips of figure 1.9 for various $R{:}a$ by color coding the initial points in the Poincaré section by the number of bounces preceeding their escape. Try also $R{:}a = 6{:}1$, though that might be too thin and require some magnification. The initial conditions can be randomly chosen, but need not - actually a clearer picture is obtained by systematic scan through regions of interest.*

Include fig. 15 from LLNN.

Exercise 6.3 **Pinball stability.** *Add to your exercise 6.1 pinball simulator a routine that computes the the [2×2] Jacobian matrix. To be able to compare with the numerical results of coming chapters, work with $R{:}a = 6$ and/or 2.5 values.*

Exercise 6.4 **Stadium billiard.** *Consider the* Bunimovich stadium *[6.8, 6.9] defined in figure 6.1. The fundamental matrix associated with the reflection is given by (6.10). Here we take $\rho_k = -1$ for the semicircle sections of the boundary, and $\cos\phi_k$ remains constant for all bounces in a rotation sequence. The time of flight between two semicircle bounces is $\tau_k = 2\cos\phi_k$. The fundamental matrix of one semicircle reflection folowed by the flight to the next bounce is*

$$\mathbf{J} = (-1)\begin{pmatrix} 1 & 2\cos\phi_k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2/\cos\phi_k & 1 \end{pmatrix} = (-1)\begin{pmatrix} -3 & 2\cos\phi_k \\ 2/\cos\phi_k & 1 \end{pmatrix}.$$

*A shift must always be followed by $k = 1, 2, 3, \cdots$ bounces along a semicircle, hence the natural symbolic dynamics for this problem is n-ary, with the corresponding fundamental matrix given by shear (*ie. *the eigenvalues remain equal to 1 throughout the whole rotation), and k bounces inside a circle lead to*

$$\mathbf{J}^k = (-1)^k\begin{pmatrix} -2k-1 & 2k\cos\phi \\ 2k/\cos\phi & 2k-1 \end{pmatrix}. \tag{6.21}$$

compare to Jonas

three-disk@3-disk!simulator
three-disk@3-disk!phase space
repeller!3-disk
phase space!3-disk

The fundamental matrix of a cycle p of length n_p is given by

$$\mathbf{J}_p = (-1)^{\sum n_k} \prod_{k=1}^{n_p} \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ n_k r_k & 1 \end{pmatrix} . \tag{6.22}$$

whole formula wrong

append Jonas re here

Find Woytkowski's purely geometric description

Adopt your pinball simulator to the Bunimovich stadium.

⇓PRELIMINARY

Exercise 6.5 Sinai-Bunimovich curvature. *Make an exercise relating κ and* **J**.

⇑PRELIMINARY

Exercise 6.6 A test of your pinball simulator. *Test your exercise 6.3 pinball simulator by computing numerically cycle stabilities by tracking distances to nearby orbits. Compare your result with the exact analytic formulas of exercise 8.1 and 8.2.*

⇓PRELIMINARY

Exercise 6.7 3-disk repeller fit. *Show by numerical comparison that the $R{:}a = 6{:}1$ 3-disk repeller has a rather thin repeller, and can be roughly fit with $a \approx 20$ in (??).*

⇑PRELIMINARY

Exercise 6.8 Birkhoff coordinates. *Prove that the Birkhoff coordinates are phase space volume preserving. Hint: compute the determinant of (6.11).*

migrate the hint into solutions

Chapter 7

Get straight

> We owe it to a book to withhold judgment until we reach page 100.
> Henrietta McNutt, George Johnson's seventh-grade English teacher

A Hamiltonian system is said to be "integrable" if one can find a change of coordinates to an action-angle coordinate frame where the phase space dynamics is described by motion on circles, one circle for each degree of freedom. In the same spirit, a natural description of a hyperbolic, unstable flow would be attained if one found a change of coordinates into a frame where the stable/unstable manifolds are straight lines, and the flow is along hyperbolas. Achieving this globally for anything but a handful of contrived examples is too much to hope for. Still, as we shall now show, we can make some headway on straightening out the flow locally.

Even though such nonlinear coordinate transformations are very important, especially in celestial mechanics, we shall not necessarily use them much in what follows, so you can safely skip this chapter on the first reading. Except, perhaps, you might want to convince yourself that cycle stabilities are indeed metric invariants of flows (sect. 8.5), and you might like transformations that turn a Keplerian ellipse into a harmonic oscillator (example 7.2) and regularize the 2-body Coulomb collisions (sect. 7.3) in classical helium.

fast track:
chapter 9, p. 131

7.1 Changing coordinates

Problems are handed down to us in many shapes and forms, and they are not always expressed in the most convenient way. In order to simplify a given problem, one may stretch, rotate, bend and mix the coordinates, but in doing so, the vector field will also change. The vector field lives in a

change!of coordinates
coordinate!change
smooth!conjugacy
conjugacy!smooth
time!as parametrization

(hyper)plane tangent to state space and changing the coordinates of state space affects the coordinates of the tangent space as well, in a way that we will now describe.

Denote by h the *conjugation function* which maps the coordinates of the initial state space $\mathcal{M}$ into the reparametrized state space $\mathcal{M}' = h(\mathcal{M})$, with a point $x \in \mathcal{M}$ related to a point $y \in \mathcal{M}'$ by

$$y = h(x) = (y_1(x), y_2(x), \ldots, y_d(x)) .$$

The change of coordinates must be one-to-one and span both $\mathcal{M}$ and $\mathcal{M}'$, so given any point y we can go back to $x = h^{-1}(y)$. For smooth flows the reparametrized dynamics should support the same number of derivatives as the initial one. If h is a (piecewise) analytic function, we refer to h as a *smooth conjugacy*.

The evolution rule $g^t(y_0)$ on $\mathcal{M}'$ can be computed from the evolution rule $f^t(x_0)$ on $\mathcal{M}$ by taking the initial point $y_0 \in \mathcal{M}'$, going back to $\mathcal{M}$, evolving, and then mapping the final point $x(t)$ back to $\mathcal{M}'$:

$$y(t) = g^t(y_0) = h \circ f^t \circ h^{-1}(y_0) . \tag{7.1}$$

Here "$\circ$" stands for functional composition $h \circ f(x) = h(f(x))$, so (7.1) is a shorthand for $y(t) = h(f^t(h^{-1}(y_0)))$.

The vector field $\dot{x} = v(x)$ in $\mathcal{M}$, locally tangent to the flow f^t, is related to the flow by differentiation (2.4) along the trajectory. The vector field $\dot{y} = w(y)$ in $\mathcal{M}'$, locally tangent to g^t follows by the chain rule:

do this in maps.tex as well

$$\begin{aligned} w(y) &= \left.\frac{dg^t}{dt}(y)\right|_{t=0} = \left.\frac{d}{dt}\left(h \circ f^t \circ h^{-1}(y)\right)\right|_{t=0} \\ &= h'(h^{-1}(y))v(h^{-1}(y)) = h'(x)v(x) . \end{aligned} \tag{7.2}$$

7.1
page 118

With the indices reinstated, this stands for

$$w_i(y) = \frac{\partial h_i(x)}{\partial x_j} v_j(x) , \qquad y_i = h_i(x) . \tag{7.3}$$

Imagine that the state space is a rubber sheet with the flow lines drawn on it. A coordinate change h corresponds to pulling and tugging on the rubber sheet smoothly, without cutting, glueing, or self-intersections of the distorted rubber sheet. Trajectories that are closed loops in $\mathcal{M}$ will remain closed loops in the new manifold $\mathcal{M}'$, but their shapes will change. Globally h deforms the rubber sheet in a highly nonlinear manner, but locally it simply rescales and shears the tangent field by $\partial_j h_i$, hence the simple transformation law (7.2) for the velocity fields.

The time itself is a parametrization of points along flow lines, and it can also be reparametrized, $s = s(t)$, with the attendent modification of (7.2). An example is the 2-body collision regularization of the helium Hamiltonian (5.6), to be undertaken in sect. 7.3 below.

rectification!flows
vector fields!singularities
coordinate!change
conjugacy

7.2 Rectification of flows

A profitable way to exploit invariance of dynamics under smooth conjugacies is to use it to pick out the simplest possible representative of an equivalence class. In general and globally these are just words, as we have no clue how to pick such "canonical" representative, but for smooth flows we can always do it locally and for sufficiently short time, by appealing to the *rectification theorem*, a fundamental theorem of ordinary differential equations. The theorem assures us that there exists a solution (at least for a short time interval) and what the solution looks like. The rectification theorem holds in the neighborhood of points of the vector field $v(x)$ that are not singular, that is, everywhere except for the equilibrium points (2.7), and points at which v is infinite. According to the theorem, in a small neighborhood of a non-singular point there exists a change of coordinates $y = h(x)$ such that $\dot{x} = v(x)$ in the new, *canonical* coordinates takes form

$$\begin{aligned} &\dot{y}_1 = \dot{y}_2 = \cdots = \dot{y}_{d-1} = 0 \\ &\dot{y}_1 = d\,, \end{aligned} \tag{7.4}$$

with unit velocity flow along y_1, and no flow along any of the remaining directions. This is an example of a one-parameter Lie group of transformations, with finite time τ action

$$\begin{aligned} y_i' &= y_i\,, i = 1, 2, \ldots, d-1 \\ y_d' &= y_d + \tau\,. \end{aligned}$$

expand this into discussion of normal forms

Example 7.1 Harmonic oscillator, rectified: *As a simple example of global rectification of a flow consider the harmonic oscillator*

$$\dot{q} = p\,, \qquad \dot{p} = -q\,. \tag{7.5}$$

The trajectories $x(t) = (q(t), p(t))$ just go around the origin, so a fair guess is that the system would have a simpler representation in polar coordinates $y = (r, \theta)$:

$$h^{-1} : \left\{ \begin{array}{lcl} q &=& h_1^{-1}(r,\theta) = r\cos\theta \\ p &=& h_2^{-1}(r,\theta) = r\sin\theta \end{array} \right. . \tag{7.6}$$

The fundamental matrix of the transformation is

$$h' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\dfrac{\sin\theta}{r} & -\dfrac{\cos\theta}{r} \end{bmatrix} \tag{7.7}$$

three body@3-body problem

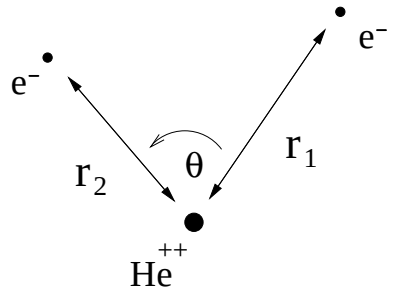

Figure 7.1: Coordinates for the helium three body problem in the plane.

resulting in (7.2) of rectified form

8.4 page 129

$$\dot{r} = 0\,, \qquad \dot{\theta} = -1\,. \tag{7.8}$$

In the new coordinates the radial coordinate r is constant, and the angular coordinate θ wraps around a cylinder with constant angular velocity. There is a subtle point in this change of coordinates: the domain of the map h^{-1} is not the plane $\mathbb{R}^2$, but rather the plane minus the origin. We had mapped a plane into a cylinder, and coordinate transformations should not change the topology of the space in which the dynamics takes place; the coordinate transformation is not defined on the equilibrium point $x = (0, 0)$, or $r = 0$.

7.3 Classical dynamics of collinear helium

(G. Tanner)

So far much has been said about 1-dimensional maps, game of pinball and other curious but rather idealized dynamical systems. If you have become impatient and started wondering what good are the methods learned so far in solving real life physical problems, good news are here. We will apply here concepts of nonlinear dynamics to nothing less than the helium, a dreaded three-body Coulomb problem.

Can we really jump from three static disks directly to three charged particles moving under the influence of their mutually attracting or repelling forces? It turns out, we can, but we have to do it with care. The full problem is indeed not accessible in all its detail, but we are able to analyze a somewhat simpler subsystem – collinear helium. This system plays an important role in the classical and quantum dynamics of the full three-body problem.

The classical helium system consists of two electrons of mass m_e and charge $-e$ moving about a positively charged nucleus of mass m_{he} and charge $+2e$.

The helium electron-nucleus mass ratio $m_{he}/m_e = 1836$ is so large that we may work in the infinite nucleus mass approximation $m_{he} = \infty$, fixing the nucleus at the origin. Finite nucleus mass effects can be taken into account without any substantial difficulty. We are now left with two

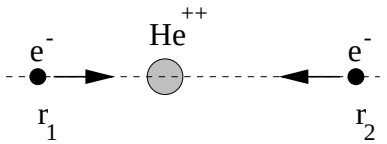

Figure 7.2: Collinear helium, with the two electrons on opposite sides of the nucleus.

electrons moving in three spatial dimensions around the origin. The total angular momentum of the combined electron system is still conserved. In the special case of angular momentum $L = 0$, the electrons move in a fixed plane containing the nucleus. The three body problem can then be written in terms of three independent coordinates only, the electron-nucleus distances r_1 and r_2 and the inter-electron angle Θ, see figure 7.1.

41.2 page 794

This looks like something we can lay our hands on; the problem has been reduced to three degrees of freedom, six phase space coordinates in all, and the total energy is conserved. But let us go one step further; the electrons are attracted by the nucleus but repelled by each other. They will tend to stay as far away from each other as possible, preferably on opposite sides of the nucleus. It is thus worth having a closer look at the situation where the three particles are all on a line with the nucleus being somewhere between the two electrons. If we, in addition, let the electrons have momenta pointing towards the nucleus as in figure 7.2, then there is no force acting on the electrons perpendicular to the common interparticle axis. That is, if we start the classical system on the dynamical subspace $\Theta = \pi$, $\frac{d}{dt}\Theta = 0$, the three particles will remain in this *collinear configuration* for all times.

7.3.1 Scaling

In what follows we will restrict the dynamics to this collinear subspace. It is a system of two degrees of freedom with the Hamiltonian

$$H = \frac{1}{2m_e}\left(p_1^2 + p_2^2\right) - \frac{2e^2}{r_1} - \frac{2e^2}{r_2} + \frac{e^2}{r_1 + r_2} = E\,, \tag{7.9}$$

where E is the total energy. As the dynamics is restricted to the fixed energy shell, the four phase space coordinates are not independent; the energy shell dependence can be made explicit by writing $(r_1, r_2, p_1, p_2) \to (r_1(E), r_2(E), p_1(E), p_2(E))$.

We will first consider the dependence of the dynamics on the energy E. A simple analysis of potential versus kinetic energy tells us that if the energy is positive both electrons can escape to $r_i \to \infty$, $i = 1, 2$. More interestingly, a single electron can still escape even if E is negative, carrying away an unlimited amount of kinetic energy, as the total energy of the remaining inner electron has no lower bound. Not only that, but one electron *will* escape eventually for almost all starting conditions. The overall dynamics thus depends critically on whether $E > 0$ or $E < 0$. But how does the dynamics change otherwise with varying energy? Fortunately,

regularization
Kustaanheimo-Stiefel transformation

not at all. Helium dynamics remains invariant under a change of energy up to a simple scaling transformation; a solution of the equations of motion at a fixed energy $E_0 = -1$ can be transformed into a solution at an arbitrary energy $E < 0$ by scaling the coordinates as

$$r_i(E) = \frac{e^2}{(-E)}\, r_i, \quad p_i(E) = \sqrt{-m_e E}\, p_i, \quad i = 1, 2\,,$$

together with a time transformation $t(E) = e^2 m_e^{1/2} (-E)^{-3/2}\, t$. We include the electron mass and charge in the scaling transformation in order to obtain a non–dimensionalized Hamiltonian of the form

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} - \frac{2}{r_1} - \frac{2}{r_2} + \frac{1}{r_1 + r_2} = -1\,. \tag{7.10}$$

The case of negative energies chosen here is the most interesting one for us. It exhibits chaos, unstable periodic orbits and is responsible for the bound states and resonances of the quantum problem treated in sect. 41.5.

7.3.2 Regularization of two–body collisions

Next, we have a closer look at the singularities in the Hamiltonian (7.10). Whenever two bodies come close to each other, accelerations become large, numerical routines require lots of small steps, and numerical precision suffers. No numerical routine will get us through the singularity itself, and in collinear helium electrons have no option but to collide with the nucleus. Hence a *regularization* of the differential equations of motions is a necessary prerequisite to any numerical work on such problems, both in celestial mechanics (where a spaceship executes close approaches both at the start and its destiantion) and in quantum mechanics (where much of semiclassical physics is dominated by returning classical orbits that probe the quantum wave function at the nucleus).

There is a fundamental difference between two–body collisions $r_1 = 0$ or $r_2 = 0$, and the triple collision $r_1 = r_2 = 0$. Two–body collisions can be regularized, with the singularities in equations of motion removed by a suitable coordinate transformation together with a time transformation preserving the Hamiltonian structure of the equations. Such regularization is not possible for the triple collision, and solutions of the differential equations can not be continued through the singularity at the origin. As we shall see, the chaos in collinear helium originates from this singularity of triple collisions.

A regularization of the two–body collisions is achieved by means of the Kustaanheimo–Stiefel (KS) transformation, which consists of a coordinate dependent time transformation which stretches the time scale near the origin, and a canonical transformation of the phase space coordinates. In

order to motivate the method, we apply it first to the 1-dimensional Kepler problem

remark 41.1

$$H = \frac{1}{2}p^2 - \frac{2}{x} = E\,. \tag{7.11}$$

Example 7.2 Keplerian ellipse, rectified: *To warm up, consider the $E = 0$ case, starting at $x = 0$ at $t = 0$. Even though the equations of motion are singular at the intial point, we can immediately integrate*

$$\frac{1}{2}\dot{x}^2 - \frac{2}{x} = 0$$

by means of separation of variables [1]

$$\sqrt{x}dx = \sqrt{2}dt\,, \qquad x = (3t)^{\frac{2}{3}}\,, \tag{7.12}$$

and observe that the solution is not singular. The aim of regularization is to compensate for the infinite acceleration at the origin by introducing a fictitious time, in terms of which the passage through the origin is smooth.

A time transformation $dt = f(q,p)d\tau$ for a system described by a Hamiltonian $H(q,p) = E$ leaves the Hamiltonian structure of the equations of motion unaltered, if the Hamiltonian itself is transformed into $\mathcal{H}(q,p) = f(q,p)(H(q,p) - E)$. For the 1–dimensional Coulomb problem with (7.11) we choose the time transformation $dt = xd\tau$ which lifts the $|x| \to 0$ singularity in (7.11) and leads to a new Hamiltonian

$$\mathcal{H} = \frac{1}{2}xp^2 - 2 - Ex = 0. \tag{7.13}$$

The solution (7.12) is now parametrized by the fictitous time $d\tau$ through a pair of equations [2]

$$x = \tau^2\,, \qquad t = \frac{1}{3}\tau^3\,.$$

make a problem, recheck this

The equations of motion are, however, still singular as $x \to 0$:

$$\frac{d^2x}{d\tau^2} = -\frac{1}{2x}\frac{dx}{d\tau} + xE\,.$$

Appearance of the square root in (7.12) now suggests a canonical transformation of form

$$x = Q^2\,, \quad p = \frac{P}{2Q} \tag{7.14}$$

which maps the Kepler problem into that of a harmonic oscillator with Hamiltonian

$$H(Q,P) = \frac{1}{8}P^2 - EQ^2 = 2, \tag{7.15}$$

with all singularities completely removed.

[1] Tanner: Luz corrected Gregor's $x = (9/2)^{\frac{1}{3}}\, t^{\frac{2}{3}}$

[2] Tanner: Luz corrected Gregor's $x = \frac{1}{2}\tau^2\,, t = \frac{1}{6}\tau^3$

We now apply this method to collinear helium. The basic idea is that one seeks a higher-dimensional generalization of the "square root removal" trick (7.14), by introducing a new vector Q with property $r = |Q|^2$. In this simple 1-dimensional example the KS transformation can be implemented by

$$r_1 = Q_1^2\,, \qquad r_2 = Q_2^2\,, \qquad p_1 = \frac{P_1}{2Q_1}\,, \qquad p_2 = \frac{P_2}{2Q_2} \tag{7.16}$$

41.1
page 794

and reparametrization of time by $d\tau = dt/r_1 r_2$. The singular behavior in the original momenta at r_1 or $r_2 = 0$ is again compensated by stretching the time scale at these points. The Hamiltonian structure of the equations of motions with respect to the new time τ is conserved, if we consider the Hamiltonian [3]

$$H_{ko} = \frac{1}{8}(Q_2^2 P_1^2 + Q_1^2 P_2^2) - 2R_{12}^2 + Q_1^2 Q_2^2(-E + 1/R_{12}^2) = 0 \tag{7.17}$$

with $R_{12} = (Q_1^2 + Q_2^2)^{1/2}$, and we will take $E = -1$ in what follows. The equations of motion now have the form

$$\begin{aligned} \dot{P}_1 &= 2Q_1\left[2 - \frac{P_2^2}{8} - Q_2^2\left(1 + \frac{Q_2^2}{R_{12}^4}\right)\right]; \qquad \dot{Q}_1 = \frac{1}{4}P_1 Q_2^2 \\ \dot{P}_2 &= 2Q_2\left[2 - \frac{P_1^2}{8} - Q_1^2\left(1 + \frac{Q_1^2}{R_{12}^4}\right)\right]; \qquad \dot{Q}_2 = \frac{1}{4}P_2 Q_1^2. \end{aligned} \tag{7.18}$$

Individual electron–nucleus collisions at $r_1 = Q_1^2 = 0$ or $r_2 = Q_2^2 = 0$ no longer pose a problem to a numerical integration routine. The equations (7.18) are singular only at the triple collision $R_{12} = 0$, i.e., when both electrons hit the nucleus at the same time.

The new coordinates and the Hamiltonian (7.17) are very useful when calculating trajectories for collinear helium; they are, however, less intuitive as a visualization of the three-body dynamics. We will therefore refer to the old coordinates r_1, r_2 when discussing the dynamics and the periodic orbits.

To summarize, we have brought a 3-body problem into a form where the 2-body collisions have been transformed away, and the phase space trajectories computable numerically. To appreciate the full beauty of what has been attained, you have to fast-forward to chapter 41; we are already "almost" ready to quantize helium by semiclassical methods.

fast track:
chapter 8, p. 119

[3] Tanner: Luz recoverd E in Gregor's $\left(1 + 1/R_{12}^2\right)$

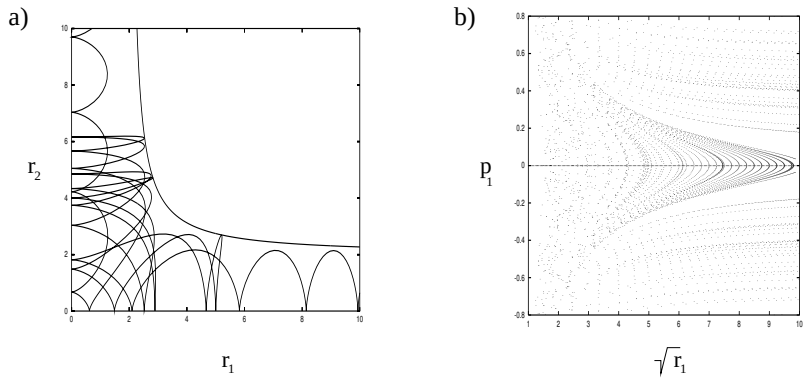

Figure 7.3: (a) A typical trajectory in the $r_1 - r_2$ plane; the trajectory enters here along the r_1 axis and escapes to infinity along the r_2 axis; (b) Poincaré map (r_2=0) for collinear helium. Strong chaos prevails for small r_1 near the nucleus.

rectification!maps

7.4 Rectification of maps

write up as a continuation of (7.1)

In sect. 7.2 we had argued that nonlinear coordinate transformations can be profitably employed to simplify the representation of a flow. We shall now apply the same idea to nonlinear maps, and determine a smooth nonlinear change of coordinates that flattens out the vicinity of a fixed point and makes the map *linear* in an open neighborhood. In its simplest form the idea can be implemented only for an isolated nondegenerate fixed point (otherwise are needed in the normal form expansion around the point), and only in a finite neigborhood of a point, as the conjugating function in general has a finite radius of convergence. In sect. 8.4 we will extend the method to periodic orbits.

use my overhead here

mention period doubling operator spectrum

⇓PRELIMINARY

☞ sect. 44.1

⇑PRELIMINARY

7.4.1 Rectification of a fixed point in one dimension

remark that h^{-1} is entire if f is?

7.2 page 118

Consider a 1-dimensional map $x_{n+1} = f(x_n)$ with a fixed point at $x = 0$, with stability $\Lambda = f'(0)$. If $|\Lambda| \neq 1$, one can determine term-by-term the power series for a smooth conjugation $h(x)$ centered at the fixed point, $h(0) = 0$, that flattens out the neighborhood of the fixed point

$$f(x) = h^{-1}(\Lambda h(x)) \tag{7.19}$$

h here inverse(!) of (2.16)

and replaces the nonlinear map $f(x)$ by a *linear* map $y_{n+1} = \Lambda y_n$.

To compute the conjugation h we use the functional equation $h^{-1}(\Lambda x) = f(h^{-1}(x))$ and the expansions

$$\begin{aligned} f(x) &= \Lambda x + x^2 f_2 + x^3 f_3 + \dots \\ h^{-1}(x) &= x + x^2 h_2 + x^3 h_3 + \dots \ . \end{aligned} \tag{7.20}$$

normal form
form!normal

Equating the coefficients of x^k on both sides of the functional equation yields h_k order by order as a function of f_2, $f_3, \ldots$. If $h(x)$ is a conjugation, so is any scaling $h(bx)$ of the function for a real number b. Hence the value of $h'(0)$ is not determined by the functional equation (7.19); it is convenient to set $h'(0) = 1$.

The algebra is not particularly illuminating and best left to computers. In any case, for the time being we will not use much beyond the first, linear term in these expansions.

Here we have assumed $\Lambda \neq 1$. If the fixed point has first k–1 derivatives vanishing, the conjugacy is to the kth *normal form*.

⇓PRELIMINARY
☞ remark 8.1
⇑PRELIMINARY

In several dimensions, Λ is replaced by the Jacobian matrix, and one has to check that the eigenvalues M are non-resonant, that is, there is no integer linear relation between the stability exponents (8.5).

Commentary

Remark 7.1 Rectification of flows. See Section 2.2.5 of ref. [7.12] for a pedagogical introduction to smooth coordinate reparametrizations. Explicit examples of transformations into cannonical coordinates for a group of scalings and a group of rotations are worked out.

Remark 7.2 Rectification of maps. The methods outlined above are standard in the analysis of fixed points and construction of normal forms for bifurcations, see for example ref. [1.20, 7.2, 7.4, 7.5, 7.6, 7.7, 7.8, 7.9, 3.8]. The geometry underlying such methods is pretty, and we enjoyed reading, for example, Percival and Richards [7.10], chaps. 2 and 4 of Ozorio de Almeida's monograph [7.11], and, as always, Arnol'd [7.1].

decide where to cite Arnold

Recursive formulas for evaluation of derivatives needed to evaluate (7.20) are given, for example, in Appendix A of ref. [9.5].

Résumé

Dynamics $(\mathcal{M}, f)$ is invariant under the group of all smooth conjugacies

$$(\mathcal{M}, f) \;\to\; (\mathcal{M}', g) = (h(\mathcal{M}), h \circ f \circ h^{-1})\,.$$

This invariance can be used to (i) find a simplified representation for the flow and (ii) identify a set of invariants, numbers computed within a particular choice of $(\mathcal{M}, f)$, but invariant under all $\mathcal{M} \to h(\mathcal{M})$ smooth conjugacies.

The $2D$-dimensional phase space of an integrable Hamiltonian system of D degrees of freedom is fully foliated by D-tori. In the same spirit, for

a uniformly hyperbolic, chaotic dynamical system one would like to change into a coordinate frame where the stable/unstable manifolds form a set of transversally interesecting hyper-planes, with the flow everywhere locally hyperbolic. That cannot be achieved in general: Fully globally integrable and fully globally chaotic flows are a very small subset of all possible flows, a "set of measure zero" in the world of all dynamical systems.

equivalence!of dynamical systems
dynamical!systems!equivalent

What we *really* care about is developping invariant notions of what a given dynamical system is. The totality of smooth one-to-one nonlinear coordinate transformations h which map all trajectories of a given dynamical system $(\mathcal{M}, f^t)$ onto all trajectories of dynamical systems $(\mathcal{M}', g^t)$ gives us a huge equivalence class, much larger than the equivalence classes familiar from the theory of linear transformations, such as the rotation group $O(d)$ or the Galilean group of all rotations and translations in $\mathbb{R}^d$. In the theory of Lie groups, the full invariant specification of an object is given by a finite set of Casimir invariants. What a good full set of invariants for a group of general nonlinear smooth conjugacies might be is not known, but the set of all periodic orbits and their stability eigenvalues will turn out to be a good start.

RM: add something about nearby systems and structural stability

use commented out text in the conjug.tex

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coordinate!transformations
transformation!coordinate
smooth!conjugacy
conjugacy!smooth
three-disk@3-disk!phase space
repeller!3-disk
phase space!3-disk

Exercises

Exercise 7.1 Coordinate transformations. *Changing coordinates is conceptually simple, but can become confusing when carried out in detail. The difficulty arises from confusing functional relationships, such as $x(t) = h^{-1}(y(t))$ with numerical relationships, such as $w(y) = h'(x)v(x)$. Working through an example will clear this up.*

(a) *The differential equation in the $\mathcal{M}$ space is $\dot{x} = \{2x_1, x_2\}$ and the change of coordinates from $\mathcal{M}$ to $\mathcal{M}'$ is $h(x_1, x_2) = \{2x_1 + x_2, x_1 - x_2\}$. Solve for $x(t)$. Find h^{-1}.*

(b) *Show that in the transformed space $\mathcal{M}'$, the differential equation is*

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5y_1 + 2y_2 \\ y_1 + 4y_2 \end{bmatrix} . \tag{7.21}$$

Solve this system. Does it match the solution in the $\mathcal{M}$ space?

Exercise 7.2 Linearization for maps. *Let $f : C \to C$ be a map from the complex numbers into themselves, with a fixed point at the origin and analytic there. By manipulating power series, find the first few terms of the map h that conjugates f to αz, that is,*

$$f(z) = h^{-1}(\alpha h(z)) .$$

There are conditions on the derivative of f at the origin to assure that the conjugation is always possible. Can you formulate these conditions by examining the series?

(difficulty: medium)

⇓PRELIMINARY

Exercise 7.3 3-disk repeller fit. *Show by numerical comparison that the $R{:}a = 6{:}1$ 3-disk repeller has a rather thin repeller, and can be roughly fit with $a \approx 20$ in (??).*

⇑PRELIMINARY

local!stability
linear!stability
stability!linear
cycle!stability
cycle!stability
topological!invariant
invariant!topological

Chapter 8

Cycle stability

Topological features of a dynamical system – singularities, periodic orbits, and the ways in which the orbits intertwine – are invariant under a general continuous change of coordinates. Surprisingly, there exist quantities that depend on the notion of metric distance between points, but nevertheless do not change value under a smooth change of coordinates. Local quantities such as the eigenvalues of equilibria and periodic orbits, and global quantities such as Lyapunov exponents, metric entropy, and fractal dimensions are examples of properties of dynamical systems independent of coordinate choice.

We now turn to the first, local class of such invariants, linear stability of periodic orbits of flows and maps. This will give us metric information about local dynamics. If you already know that the eigenvalues of periodic orbits are invariants of a flow, skip this chapter.

fast track:
chapter 9, p. 131

8.1 Stability of periodic orbits

As noted on page 36, a trajectory can be stationary, periodic or aperiodic. For chaotic systems almost all trajectories are aperiodic – nevertheless, equilibria and periodic orbits will turn out to be the key to unraveling chaotic dynamics. Here we note a few of the properties that makes them so precious to a theorist. [1]

add figs

An obvious virtue of periodic orbits is that they are *topological* invariants: a fixed point remains a fixed point for any choice of coordinates, and similarly a periodic orbit remains periodic in any representation of the dynamics. Any re-parametrization of a dynamical system that preserves

[1] Mason: does not jive with relative equilibira in a 3-body problem? This must be rephrased more carefully. PC: will tighten this in the continuous symmetries chapter **??**.

metric!invariant
invariant!metric
multiplier!Floquet
multiplier!characteristic
characteristic!multiplier
Floquet!exponent
exponent!Floquet
give references to templates

its topology has to preserve topological relations between periodic orbits, such as their relative inter-windings and knots. So the mere existence of periodic orbits suffices to partially organize the spatial layout of a non–wandering set. No less important, as we shall now show, is the fact that cycle eigenvalues are *metric* invariants: they determine the relative sizes of neighborhoods in a non–wandering set.

To prove this, we start by noting that due to the multiplicative structure (4.33) of fundamental matrices, the fundamental matrix for the rth repeat of a prime cycle p of period T_p is

$$J^{rT_p}(x) = J^{T_p}(f^{(r-1)T_p}(x)) \cdots J^{T_p}(f^{T_p}(x)) J^{T_p}(x) = (J_p(x))^r \ , \quad (8.1)$$

emphasize - eigenvalues are NOT multiplicative unless on a periodic orbit

where $J_p(x) = J^{T_p}(x)$ is the fundamental matrix for a single traversal of the prime cycle p, $x \in p$ is any point on the cycle, and $f^{rT_p}(x) = x$ as $f^t(x)$ returns to x every multiple of the period T_p. Hence, it suffices to restrict our considerations to the stability of prime cycles.

⇓PRELIMINARY

8.1.1 Better nomenclature

do not like name "stability multipliers", as Λ's are not multiplicative

The linearized Poincaré $[(d-1) \times (d-1)]$ return map of a periodic orbit is called the *monodromy matrix*, and its eigenvalues are called the *Floquet multipliers*. (sometimes *characteristic multipliers* or *roots*).

Floquet theory describes the solutions of linear ordinary differential equations with periodic coefficients, such as the equation of variations (4.2) evaluated on a periodic orbit p,

$$\dot{\delta x} = A(t)\delta x \, , \qquad A(t) = A(x(t)) = A(t+T_p) \, . \quad (8.2)$$

The space of solutions is finite-dimensional and any solution

$$M_p \delta x(x(t)) = \delta x(x(t+T_p))$$

can be expanded in terms of periodic functions $u_j(t) = u_j(t+T_p)$ in the basis

$$u(x(t)) = \sum_j u_j(t) e^{T_p(\mu_{p,j}+i\mu_{p,j})} \mathbf{e}_j(x)$$

spaned by the eigenvectors of the monodromy matrix. If $\Lambda_{p,j}$ is an eigenvalue of M_p and $\mathbf{e}_j(x(t))$ is the corresponding eigenvector, then

$$M_p \mathbf{e}_j(x) = \Lambda_{p,j} \mathbf{e}_j(x) \, .$$

The number $\Lambda_{p,j}$ is called the Floquet multiplier of $\mathbf{e}_j$, and $\lambda_{p,j} + i\theta_{p,j}$ the Floquet or characteristic exponent, where $\Lambda_{p,j} = e^{T_p(\mu_{p,j}+i\nu_{p,j})}$.

⇑PRELIMINARY

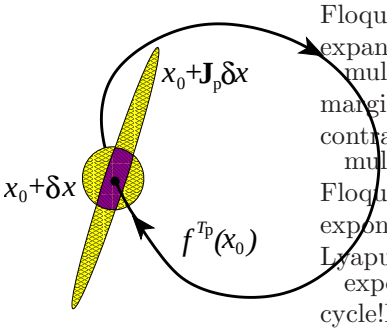

Figure 8.1: For a prime cycle p, fundamental matrix J_p returns an infinitesimal spherical neighborhood of $x_0 \in p$ stretched into an ellipsoid, with overlap ratio along the eigenvector $\mathbf{e}_i$ of $J_p(x)$ given by the eigenvalue $\Lambda_{p,i}$. These ratios are invariant under smooth nonlinear reparametrizations of state space coordinates, and are intrinsic property of cycle p.

Floquet!multipliers
expanding!Floquet multipliers
marginal!stability
contracting!Floquet multipliers
Floquet!exponents
exponent!Floquet
Lyapunov exponent!cycle
cycle!Lyapunov exponent

8.1.2 Fundamental matrix eigenvalues and exponents

We sort the *Floquet multipliers* $\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}$ of the $[d\times d]$ fundamental matrix J_p evaluated on the p-cycle into sets $\{e, m, c\}$

$$\begin{aligned} \text{expanding:} \quad & \{\Lambda_p\}_e &&= \{\Lambda_{p,j} : |\Lambda_{p,j}| > 1\} \\ \text{marginal:} \quad & \{\Lambda_p\}_m &&= \{\Lambda_{p,j} : |\Lambda_{p,j}| = 1\} \\ \text{contracting:} \quad & \{\Lambda_p\}_c &&= \{\Lambda_{p,j} : |\Lambda_{p,j}| < 1\}\,. \end{aligned} \tag{8.3}$$

and denote by Λ_p (no jth eigenvalue index) the product of *expanding* Floquet multipliers

$$\Lambda_p = \prod_e \Lambda_{p,e}\,. \tag{8.4}$$

As J_p is a real matrix, complex eigenvalues always come in complex conjugate pairs, $\Lambda_{p,i+1} = \Lambda^*_{p,i}$, so the product of expanding eigenvalues Λ_p is always real.

Cycle *Floquet exponents* are the stretching/contraction rates per unit time

$$\mu_{p,i} = \frac{1}{T_p} \ln |\Lambda_{p,i}|\ . \tag{8.5}$$

This definition is motivated by the form of the Floquet exponents for the linear dynamical systems, for example (4.15), as well as the fact that exponents so defined can be interpreted as Lyapunov exponents (10.32) evaluated on the prime cycle p. As in the three cases of (8.3), we sort the Floquet exponents $\lambda = \mu \pm \nu$ into three sets

☞ sect. 10.3

$$\begin{aligned} \text{expanding:} \quad & \{\lambda_p\}_e &&= \{\lambda_{p,i} :\ \mu_{p,i} > 0\} \\ \text{marginal:} \quad & \{\lambda_p\}_m &&= \{\lambda_{p,i} :\ \mu_{p,i} = 0\} \\ \text{contracting:} \quad & \{\lambda_p\}_c &&= \{\lambda_{p,i} :\ \mu_{p,i} < 0\}\,. \end{aligned} \tag{8.6}$$

A periodic orbit p of a d-dimensional flow or a map is *stable* if all its Floquet exponents (other than the vanishing longitudinal exponent, to be

stability!window
hyperbolic!flow
flow!hyperbolic
nonhyperbolic!flow
flow!nonhyperbolic
flow!elliptic
critical!point
cycle!superstable
superstable cycle
cycle!stable
stable!cycle
unstable!cycle
☞ sect. 5.3

explained in sect. 8.2.1 below) are strictly negative, $\mu_{p,i} < 0$. The region of system parameter values for which a periodic orbit p is stable is called the *stability window* of p. The set $\mathcal{M}_p$ of initial points that are asymptotically attracted to p as $t \to +\infty$ (for a fixed set of system parameter values) is called the *basin of attraction* of p.

If *all* Floquet exponents (other than the vanishing longitudinal exponent) of *all* periodic orbits of a flow are strictly bounded away from zero, $|\mu_i| \geq \mu_{min} > 0$, the flow is said to be *hyperbolic*. Otherwise the flow is said to be *nonhyperbolic*. In particular, if all $\mu_i = 0$, the orbit is said to be *elliptic*. Such orbits proliferate in Hamiltonian flows. [2]

We often do care about the sign of $\Lambda_{p,i}$ and, if $\Lambda_{p,i}$ is complex, its phase

$$\Lambda_{p,j} = \pm e^{\lambda_{p,j} T_p} = e^{(\mu_{p,j} \pm i\nu_{p,j})T_p} \,. \tag{8.7}$$

☞ sect. 5.2

Keeping track of this by case-by-case enumeration is a self-inflicted, unnecessary nuisance, followed in much of the literature. To avoid this, almost all of our formulas will be stated in terms of the Floquet multipliers Λ_j rather than in the terms of the overall signs, Floquet exponents λ_j and phases θ_j.

Example 8.1 Stability of 1-d map cycles: *The simplest example of cycle stability is afforded by 1-dimensional maps. The stability of a prime cycle p follows from the chain rule (4.39) for stability of the n_pth iterate of the map*

$$\Lambda_p = \frac{d}{dx_0} f^{n_p}(x_0) = \prod_{m=0}^{n_p-1} f'(x_m) \,, \quad x_m = f^m(x_0) \,. \tag{8.8}$$

Λ_p is a property of the cycle, not the initial point, as taking any periodic point in the p cycle as the initial point yields the same result.

A critical point *x_c is a value of x for which the mapping $f(x)$ has vanishing derivative, $f'(x_c) = 0$. For future reference we note that a periodic orbit of a 1-dimensional map is* stable *if*

$$|\Lambda_p| = \left|f'(x_{n_p}) f'(x_{n_p-1}) \cdots f'(x_2) f'(x_1)\right| < 1 \,,$$

and superstable *if the orbit includes a critical point, so that the above product vanishes. For a stable periodic orbit of period n the slope of the nth iterate $f^n(x)$ evaluated on a periodic point x (fixed point of the nth iterate) lies between -1 and 1. If $|\Lambda_p| > 1$, p-cycle is* unstable.

Example 8.2 Stability of cycles for maps: *No matter what method we had used to determine the unstable cycles, the theory to be developed here requires that their Floquet multipliers be evaluated as well. For maps a fundamental matrix is easily*

[2] Mason: for infinite dimensional flows one cannot have λ_{min}; in my thesis have a sequence $|\lambda_i| \to 0$, and all $|\lambda_i| \neq 0$.
PC: I think you are mixing up stability eigenvalues Λ with stability exponents λ in this and the next comment. My λ_{min} is a lower bound on $|\lambda_i|$, not λ_i, so I'm OK unless I have eigenvalues arbitrarily close to 0; but then the flow is not hypebolic

evaluated by picking any cycle point as a starting point, running once around a prime cycle, and multiplying the individual cycle point fundamental matrices according to (4.40). For example, the fundamental matrix M_p for a Hénon map (3.15) prime cycle p of length n_p is given by (4.41),

Henon@Hénon!map!stability
stability!billiards
billiards!stability

$$M_p(x_0) = \prod_{k=n_p}^{1} \begin{pmatrix} -2ax_k & b \\ 1 & 0 \end{pmatrix}, \qquad x_k \in p\,,$$

and the fundamental matrix M_p for a 2-dimensional billiard prime cycle p of length n_p

$$M_p = (-1)^{n_p} \prod_{k=n_p}^{1} \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_k & 1 \end{pmatrix}$$

follows from (6.11). We shall compute Floquet multipliers of Hénon map cycles once we learn how to find their periodic orbits, see exercise 17.11.

add exercise for by-hand fixed point, 2-cycle for Hénon

8.2 Cycle Floquet multipliers are cycle invariants

The 1-dimensional map cycle Floquet multiplier Λ_p is a product of derivatives over all points around the cycle, and is therefore independent of which periodic point is chosen as the initial one. In higher dimensions the form of the fundamental matrix $J_p(x_0)$ in (8.1) does depend on the choice of coordinates and the initial point $x_0 \in p$. Nevertheless, as we shall now show, the cycle *Floquet multipliers* are intrinsic property of a cycle also for multi-dimensional flows. Consider the ith eigenvalue, eigenvector pair $(\Lambda_{p,i}, \mathbf{e}_i)$ computed from J_p evaluated at a cycle point,

$$J_p(x)\mathbf{e}_i(x) = \Lambda_{p,i}\mathbf{e}_i(x)\,, \quad x \in p\,. \tag{8.9}$$

Consider another point on the cycle at time t later, $x' = f^t(x)$ whose fundamental matrix is $J_p(x')$. By the group property (4.33), $J^{T_p+t} = J^{t+T_p}$, and the fundamental matrix at x' can be written either as

$$J^{T_p+t}(x) = J^{T_p}(x')J^t(x) = J_p(x')J^t(x)\,, \quad \text{or} \quad J_p(x')J^t(x) = J^t(x)J_p(x)\,.$$

Multiplying (8.9) by $J^t(x)$, we find that the fundamental matrix evaluated at x' has the same eigenvalue,

$$J_p(x')\mathbf{e}_i(x') = \Lambda_{p,i}\mathbf{e}_i(x')\,, \quad \mathbf{e}_i(x') = J^t(x)\mathbf{e}_i(x)\,, \tag{8.10}$$

but with the eigenvector $\mathbf{e}_i$ transported along the flow $x \to x'$ to $\mathbf{e}_i(x') = J^t(x)\mathbf{e}_i(x)$. Hence, J_p evaluated anywhere along the cycle has the same set of Floquet multipliers $\{\Lambda_{p,1}, \Lambda_{p,2}, \cdots \Lambda_{p,d-1}, 1\}$. As quantities such as $\operatorname{tr} J_p(x)$, $\det J_p(x)$ depend only on the eigenvalues of $J_p(x)$ and not on

invariance!of flows
nonhyperbolic!flow
cycle!marginal stability
marginal stability!cycle
Poincar"'e!return map!cycle
stability!Poincar"'e map cycle

the starting point x, in expressions such as $\det\left(\mathbf{1}-M_p^r(x)\right)$ we may omit reference to any particular cycle point x:

$$\det\left(\mathbf{1}-M_p^r\right)=\det\left(\mathbf{1}-M_p^r(x)\right)\quad \text{for any } x\in p\,. \tag{8.11}$$

We postpone the proof that the cycle Floquet multipliers are smooth conjugacy invariants of the flow to sect. 8.5.

8.2.1 Marginal eigenvalues

The presence of marginal eigenvalues signals either a continuous symmetry of the flow (which one should immediately exploit to simplify the problem), or a non-hyperbolicity of a flow (a source of much pain, hard to avoid).

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8.4
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Example 8.3 A periodic orbit of a flow has a marginal eigenvalue: *As $J^t(x)$ transports the velocity field $v(x)$ by (4.7), after a complete period*

$$J_p(x)v(x)=v(x)\,, \tag{8.12}$$

so a periodic orbit of a flow *always has an eigenvector $\mathbf{e}_d(x)=v(x)$ parallel to the local velocity field with the unit eigenvalue*

$$\Lambda_{p,d}=1\,. \tag{8.13}$$

The continuous invariance that gives rise to this marginal eigenvalues is the invariance of a cycle under a translation of its points along the cycle: two points on the cycle (see figure 4.2) initially distance δx apart, $x'(0)-x(0)=\delta x(0)$, are separated by the exactly same δx after a full period T_p. As we shall see in sect. 8.3, this marginal stability direction can be eliminated by cutting the cycle by a Poincaré section and eliminating the continuous flow fundamental matrix in favor of the fundamental matrix of the Poincaré return map.

7.2
page 118
Ole: clarify this — draw a picture.

explain "monodromy"?

If the flow is governed by a time-independent Hamiltonian, the energy is conserved, and that leads to an additional marginal eigenvalue (remember, by symplectic invariance (5.17) real eigenvalues come in pairs). For the collinear helium of sect. 7.3 this marginal eigenvalue is made explicit in appendix D.2.

⇓PRELIMINARY

⇑PRELIMINARY

8.3 Stability of Poincaré map cycles

(R. Paškauskas and P. Cvitanović)

If a continuous flow periodic orbit p pierces the Poincaré section $\mathcal{P}$ once, the section point is a fixed point of the Poincaré return map P with stability (4.45)

$$\hat{J}_{ij}=\left(\delta_{ik}-\frac{v_i\,U_k}{(v\cdot U)}\right)J_{kj}\,, \tag{8.14}$$

with all primes dropped, as the initial and the final points coincide, $x' = f^{T_p}(x) = x$. If the periodic orbit p pierces the set of Poincaré sections $\mathcal{P}$ n times, the same observation applies to the nth iterate of P.

We have already established in (4.46) that the velocity $v(x)$ is a zero-eigenvector of the Poincaré section fundamental matrix, $\hat{J}v = 0$. Consider next $(\Lambda_{p,\alpha}, \mathbf{e}_\alpha)$, the full state space αth (eigenvalue, eigenvector) pair (8.9), evaluated at a cycle point on a Poincaré section,

$$J(x)\mathbf{e}_\alpha(x) = \Lambda_\alpha \mathbf{e}_\alpha(x)\,, \quad x \in \mathcal{P}\,. \tag{8.15}$$

Multiplying (8.14) by $\mathbf{e}_\alpha$ and inserting (8.15), we find that the full state space fundamental matrix and the Poincaré section fundamental matrix $\hat{J}$ has the same eigenvalue

$$\hat{J}(x)\hat{\mathbf{e}}_\alpha(x) = \Lambda_\alpha \hat{\mathbf{e}}_\alpha(x)\,, \quad x \in \mathcal{P}\,, \tag{8.16}$$

where $\hat{\mathbf{e}}_\alpha$ is a projection of the full state space eigenvector onto the Poincaré section:

note: normalization not preserved

$$(\hat{\mathbf{e}}_\alpha)_i = \left(\delta_{ik} - \frac{v_i\, U_k}{(v\cdot U)}\right)(\mathbf{e}_\alpha)_k\,. \tag{8.17}$$

draw figure

Hence, $\hat{J}_p$ evaluated on any Poincaré section point along the cycle p has the same set of Floquet multipliers $\{\Lambda_{p,1}, \Lambda_{p,2}, \cdots \Lambda_{p,d}\}$ as the full state space fundamental matrix J_p.

8.4 Rectification of a 1-dimensional periodic orbit

In sect. 7.4.1 we have constructed the conjugation function for a fixed point. Here we turn to the problem of constructing it for periodic orbits. Each point around the cycle has a differently distorted neighborhood, with differing second and higher order derivatives, so we need to compute a different conjugation function h_a at each cycle point x_a. We expand the map f around each cycle point along the cycle,

$$y_a(\phi) = f_a(\phi) - x_{a+1} = \phi f_{a,1} + \phi^2 f_{a,2} + \dots \tag{8.18}$$

where x_a is a point on the cycle, $f_a(\phi) = f(x_a+\phi)$ is centered on the periodic orbit, and the index k in $f_{a,k}$ refers to the kth order in the expansion (7.20).

For a periodic orbit the conjugation formula (7.19) generalizes to

$$f_a(\phi) = h_{a+1}^{-1}(f_a'(0)h_a(\phi))\,, \qquad a = 1, 2, \cdots, n\,,$$

Floquet!multiplier, metric invariants
metric!invariant!Floquet multiplier

point by point. The conjugationg functions h_a are obtained in the same way as before, by equating coefficients of the expansion (7.20), and assuming that the cycle Floquet multiplier $\Lambda = \prod_{a=0}^{n-1} f'(x_a)$ is not marginal, $|\Lambda| \neq 1$. The explicit expressions for h_a in terms of f are obtained by iterating around the whole cycle,

compute only onece, then propagate around the cycle by iteration?

$$f^n(x_a + \phi) = h_a^{-1}(\Lambda h_a(\phi)) + x_a \,. \tag{8.19}$$

☞ remark 7.2

evaluated at each cycle point a. Again we have the freedom to set $h'_a(0) = 1$ for all a.

8.4.1 Repeats of cycles

We have traded in our initial nonlinear map f for a (locally) linear map Λy and an equally complicated conjugation function h. What is gained by rewriting the map f in terms of the conjugacy function h? Once the neighborhood of a fixed point is linearized, the repeats of it are trivialized; from the conjugation formula (7.20) one can compute the derivatives of a function composed with itself r times:

$$f^r(x) = h^{-1}(\Lambda^r h(x)) \,.$$

One can already discern the form of the expansion for arbitrary repeats; the answer will depend on the conjugacy function $h(x)$ computed for a *single* repeat, and all the dependence on the repeat number will be carried by factors polynomial in Λ^r, a considerable simplification. The beauty of the idea is difficult to gauge at this stage - an appreciation only sets in when one starts computing perturbative corrections, be it in celestial mechanics (where the method was born), be it the quantum or stochastic corrections to "semiclassical" approximations.

8.5 Smooth conjugacies and cycle Floquet multipliers

In sect. 8.2 we have established that for a given flow the cycle Floquet multipliers are intrinsic to a given cycle, independent of the staring point along the cycle. Now we can prove a much stronger statement; cycle Floquet multipliers are *metric invariants* of the flow, the same in *any* representation of the dynamical system.

That the cycle Floquet multipliers are an invariant property of the given dynamical system follows from elementary considerations of sect. 7.1: If the same dynamics is given by a map f in x coordinates, and a map g in the $y = h(x)$ coordinates, then f and g (or any other good representation) are related by (7.2), a reparametrization and a coordinate transformation

$g = h \circ f \circ h^{-1}$. As both f and g are arbitrary representations of the dynamical system, the explicit form of the conjugacy h is of no interest, only the properties invariant under any transformation h are of general import. Furthermore, a good representation should not mutilate the data; h must be a *smooth conjugacy* which maps nearby cycle points of f into nearby cycle points of g. This smoothness guarantees that the cycles are not only topological invariants, but that their linearized neighborhoods are also metrically invariant. For a fixed point $f(x) = x$ of a 1-dimensional map this follows from the chain rule for derivatives,

smooth!conjugacy
conjugacy!smooth
metric!invariant
invariant!metric
conjugacy!invariant
neighborhood

$$\begin{aligned} g'(y) &= h'(f \circ h^{-1}(y)) f'(h^{-1}(y)) \frac{1}{h'(x)} \\ &= h'(x) f'(x) \frac{1}{h'(x)} = f'(x)\,, \end{aligned} \tag{8.20}$$

and the generalization to the Floquet multipliers of periodic orbits of d-dimensional flows is immediate.

Is it? add d-dimensional case

As stability of a flow can always be rewritten as stability of a Poincaré section return map, we find that the Floquet multipliers of any cycle, for a flow or a map in arbitrary dimension, is a metric invariant of the dynamical system.

argue $det \to tr$ can be waste of time
add d-dimensional case

in depth:
appendix D.1, p. 903

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8.6 There goes the neighborhood

In what follows, our task will be to determine the size of a *neighborhood* of $x(t)$, and that is why we care about the Floquet multipliers, and especially the unstable (expanding) ones. Nearby points aligned along the stable (contracting) directions remain in the neighborhood of the trajectory $x(t) = f^t(x_0)$; the ones to keep an eye on are the points which leave the neighborhood along the unstable directions. The sub-volume $|\mathcal{M}_i| = \prod_i^e \Delta x_i$ of the set of points which get no further away from $f^t(x_0)$ than L, the typical size of the system, is fixed by the condition that $\Delta x_i \Lambda_i = O(L)$ in each expanding direction i. Hence the neighborhood size scales as $\propto 1/|\Lambda_p|$ where Λ_p is the product of expanding eigenvalues (8.4) only; contracting ones play a secondary role. So secondary that even infinitely many of them (for example, the infinity of contracting eigendirections of the spatiotemporal dynamics of sect. 25.1) will not matter.

add a figure of hyperbolic flow
⇓PRELIMINARY
⇑PRELIMINARY
fix

So the physically important information is carried by the expanding sub-volume, not the total volume computed so easily in (4.37). That is also the reason why the dissipative and the Hamiltonian chaotic flows are much more alike than one would have naively expected for "compressible" *vs.* "incompressible" flows. In hyperbolic systems what matters are the

expanding directions. Whether the contracting eigenvalues are inverses of the expanding ones or not is of secondary importance. As long as the number of unstable directions is finite, the same theory applies both to the finite-dimensional ODEs and infinite-dimensional PDEs.

⇓PRELIMINARY

Commentary

Remark 8.1 A resonance condition. In the hyperbolic case there is a resonance condition that must be satisfied: none of the Floquet exponents may be related by ratios of integers. That is, if $\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}$ are the Floquet multipliers of the fundamental matrix, then they are in resonance if there exist integers $n_1, \ldots, n_d$ such that

$$(\Lambda_{p,1})^{n_1}(\Lambda_{p,2})^{n_2}\cdots(\Lambda_{p,d})^{n_d} = 1\,.$$

If there is resonance, then one may get corrections to the basic conjugation formulas in the form of monomials in the variables of the map.

give reference

(R. Mainieri)

⇑PRELIMINARY

Résumé

Periodic orbits play a central role in any invariant characterization of the dynamics, because (a) their existence and inter-relations are a *topological*, coordinate-independent property of the dynamics, and (b) their Floquet multipliers form an infinite set of *metric invariants*: The Floquet multipliers of a periodic orbit remain invariant under any smooth nonlinear change of coordinates $f \to h \circ f \circ h^{-1}$.

We shall show in chapter 11 that extending their local stability eigendirections into stable and unstable manifolds yields important global information about the topological organization of state space.

In hyperbolic systems what matters are the expanding directions. The physically important information is carried by the unstable manifold, and the expanding sub-volume characterized by the product of expanding eigenvalues of J_p. As long as the number of unstable directions is finite, the theory can be applied to flows of arbitrarily high dimension.

References

[8.1] J. Moehlis and K. Josić, "Periodic Orbit," www.scholarpedia.org/article/Periodic_Orbit.

[8.2] G. Floquet, "Sur les equations differentielles lineaires a coefficients periodique," *Ann. Ecole Norm. Ser. 2,* **12**, 47 (1883).

Exercises

three-disk@3-disk!cycle!analytically
stability!exact

Exercise 8.1 **Fundamental domain fixed points.** *Use the formula (6.11) for billiard fundamental matrix to compute the periods T_p and the expanding eigenvalues Λ_p of the fundamental domain $\overline{0}$ (the 2-cycle of the complete 3-disk space) and $\overline{1}$ (the 3-cycle of the complete 3-disk space) fixed points:*

	T_p	Λ_p
$\overline{0}$:	$R-2$	$R-1+R\sqrt{1-2/R}$
$\overline{1}$:	$R-\sqrt{3}$	$-\frac{2R}{\sqrt{3}}+1-\frac{2R}{\sqrt{3}}\sqrt{1-\sqrt{3}/R}$

(8.21)

We have set the disk radius to $a=1$.

Exercise 8.2 **Fundamental domain 2-cycle.** *Verify that for the $\overline{10}$-cycle the cycle length and the trace of the fundamental matrix are given by*

$$\begin{aligned} L_{10} &= 2\sqrt{R^2-\sqrt{3}R+1}-2, \\ \operatorname{tr}\mathbf{J}_{10} &= 2L_{10}+2+\frac{1}{2}\,\frac{L_{10}(L_{10}+2)^2}{\sqrt{3}R/2-1}\ . \end{aligned} \tag{8.22}$$

The $\overline{10}$-cycle is drawn in figure 11.6. The unstable eigenvalue Λ_{10} follows from (4.19).

recheck - fixed up by Per

⇓PRIVATE

Exercise 8.3 **Driven damped harmonic oscillator limit cycle.** *Driven damped harmonic oscillator equations of motion are:*

⇑PRIVATE

Exercise 8.4 **A limit cycle with analytic stability exponent.** *There are only two examples of nonlinear flows for which the stability eigenvalues can be evaluated analytically. Both are cheats. One example is the 2-d flow*

$$\dot{q}=\ p+q(1-q^2-p^2)\,, \qquad \dot{p}=-q+p(1-q^2-p^2)\,. \tag{8.23}$$

Determine all periodic solutions of this flow, and determine analytically their stability exponents. Hint: go to polar coordinates $(q,p)=(r\cos\theta, r\sin\theta)$.

G. Bard Ermentrout

Exercise 8.5 **The other example of a limit cycle with analytic stability exponent.** *What is the other example of a nonlinear flow for which the stability eigenvalues can be evaluated analytically? Hint: email G.B. Ermentrout.*

Exercise 8.6 **Yet another example of a limit cycle with analytic stability exponent.** *Prove G.B. Ermentrout wrong by solving a third example (or more) of a nonlinear flow for which the stability eigenvalues can be evaluated analytically.*

confession!St. Augustine
St. Augustine

Chapter 9

Transporting densities

Paulina: I'll draw the curtain:
My lord's almost so far transported that
He'll think anon it lives.
W. Shakespeare: *The Winter's Tale*

what does "anon it lives" refer to?

(P. Cvitanović, R. Artuso, L. Rondoni, and E.A. Spiegel)

In chapters 2, 3, 5 and 6 we learned how to track an individual trajectory, and saw that such a trajectory can be very complicated. In chapter 4 we studied a small neighborhood of a trajectory and learned that such neighborhood can grow exponentially with time, making the concept of tracking an individual trajectory for long times a purely mathematical idealization.

While the trajectory of an individual representative point may be highly convoluted, the density of these points might evolve in a manner that is relatively smooth. The evolution of the density of representative points is for this reason (and other that will emerge in due course) of great interest. So are the behaviors of other properties carried by the evolving swarm of representative points.

We shall now show that the global evolution of the density of representative points is conveniently formulated in terms of evolution operators.

9.1 Measures

Do I then measure, O my God, and know not what I measure?
St. Augustine, *The confessions of Saint Augustine*

A fundamental concept in the description of dynamics of a chaotic system is that of *measure*, which we denote by $d\mu(x) = \rho(x)dx$. An intuitive way to define and construct a physically meaningful measure is by a process

coarse-graining
partition!state space
state space!partition
characteristic!function
measure
density

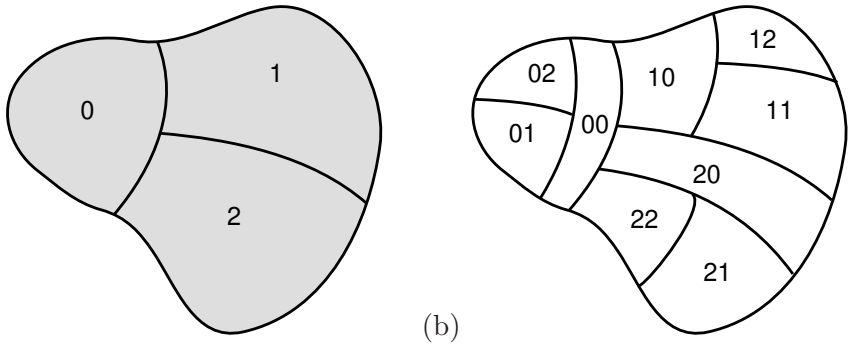

Figure 9.1: (a) First level of partitioning: A coarse partition of $\mathcal{M}$ into regions $\mathcal{M}_0$, $\mathcal{M}_1$, and $\mathcal{M}_2$. (b) $n = 2$ level of partitioning: A refinement of the above partition, with each region $\mathcal{M}_i$ subdivided into $\mathcal{M}_{i0}$, $\mathcal{M}_{i1}$, and $\mathcal{M}_{i2}$.

of *coarse-graining*. Consider a sequence 1, 2, ..., n, ... of increasingly refined partitions of state space, figure 9.1, into regions $\mathcal{M}_i$ defined by the characteristic function

$$\chi_i(x) = \begin{cases} 1 & \text{if } x \in \mathcal{M}_i\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{9.1}$$

A coarse-grained measure is obtained by assigning the "mass", or the fraction of trajectories contained in the ith region $\mathcal{M}_i \subset \mathcal{M}$ at the nth level of partitioning of the state space:

$$\Delta\mu_i = \int_{\mathcal{M}} d\mu(x)\chi_i(x) = \int_{\mathcal{M}_i} d\mu(x) = \int_{\mathcal{M}_i} dx\,\rho(x)\,. \tag{9.2}$$

The function $\rho(x) = \rho(x,t)$ denotes the *density* of representative points in state space at time t. This density can be (and in chaotic dynamics, often is) an arbitrarily ugly function, and it may display remarkable singularities; for instance, there may exist directions along which the measure is singular with respect to the Lebesgue measure. As our intent is to sprinkle state space with a finite number of initial points (repeat an experiment a finite number of times), we shall assume that the measure can be normalized

AH, MAP: define Lebesgue measure

$$\sum_i^{(n)} \Delta\mu_i = 1\,, \tag{9.3}$$

where the sum is over subregions i at the nth level of partitioning. The infinitesimal measure $dx\rho(x)$ can be thought of as an infinitely refined partition limit of $\Delta\mu_i = |\mathcal{M}_i|\rho(x_i)\,,\ x_i \in \mathcal{M}_i$, with normalization

$$\int_{\mathcal{M}} dx\ \rho(x) = 1\,. \tag{9.4}$$

So far, any arbitrary sequence of partitions will do. What are intelligent ways of partitioning state space? We postpone the answer to chapter 11, after we have developed some intuition about how the dynamics transports densities.

Jacobian
Dirac delta function

☞ chapter 11

9.2 Perron-Frobenius operator

Given a density, the question arises as to what it might evolve into with time. Consider a swarm of representative points making up the measure contained in a region $\mathcal{M}_i$ at time $t = 0$. As the flow evolves, this region is carried into $f^t(\mathcal{M}_i)$, as in figure 2.1 (b). No trajectory is created or destroyed, so the conservation of representative points requires that

$$\int_{f^t(\mathcal{M}_i)} dx\, \rho(x,t) = \int_{\mathcal{M}_i} dx_0\, \rho(x_0,0)\,.$$

If the flow is invertible and the transformation $x_0 = f^{-t}(x)$ is single-valued, we can transform the integration variable in the expression on the left to

$$\int_{\mathcal{M}_i} dx_0\, \rho(f^t(x_0),t)\left|\det M^t(x_0)\right|\,.$$

We conclude that the density changes with time as the inverse of the Jacobian (4.36)

$$\rho(x,t) = \frac{\rho(x_0,0)}{|\det M^t(x_0)|}\,, \qquad x = f^t(x_0)\,, \tag{9.5}$$

which makes sense: the density varies inversely to the infinitesimal volume occupied by the trajectories of the flow.

The manner in which a flow transports densities may be recast into the language of operators, by writing

$$\rho(x,t) = \mathcal{L}^t\rho(x) = \int_{\mathcal{M}} dx_0\, \delta\big(x - f^t(x_0)\big)\, \rho(x_0,0)\,. \tag{9.6}$$

Let us check this formula. Integrating Dirac delta functions is easy: $\int_{\mathcal{M}} dx\, \delta(x) = 1$ if $0 \in \mathcal{M}$, zero otherwise. The integral over a one-dimensional Dirac delta function picks up the Jacobian of its argument evaluated at all of its zeros:

$$\int dx\, \delta(h(x)) = \sum_{\{x:h(x)=0\}} \frac{1}{|h'(x)|}\,, \tag{9.7}$$

Dirac delta function
delta function!Dirac
Perron-Frobenius!operator
operator!Perron-Frobenius

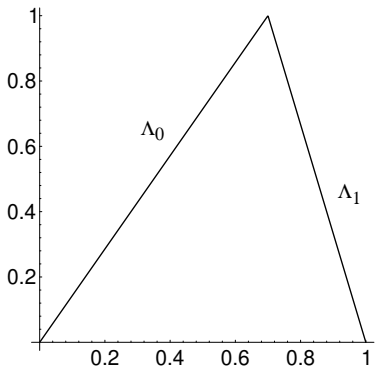

Figure 9.2: A piecewise-linear skew "Ulam tent" map (9.11) ($\Lambda_0 = 4/3$, $\Lambda_1 = -4$).

and in d dimensions the denominator is replaced by

9.1 page 146

$$\int dx\, \delta(h(x)) = \sum_{\{x:h(x)=0\}} \frac{1}{\left|\det \frac{\partial h(x)}{\partial x}\right|} \,. \tag{9.8}$$

use my transparencies here.

9.2 page 146

Now you can check that (9.6) is just a rewrite of (9.5):

$$\begin{aligned} \mathcal{L}^t \rho(x) &= \sum_{x_0=f^{-t}(x)} \frac{\rho(x_0)}{|f^{t\prime}(x_0)|} && \text{(1-dimensional)} \\ &= \sum_{x_0=f^{-t}(x)} \frac{\rho(x_0)}{|\det M^t(x_0)|} && (d\text{-dimensional})\,. \end{aligned} \tag{9.9}$$

For a deterministic, invertible flow x has only one preimage x_0; allowing for multiple preimages also takes account of noninvertible mappings such as the "stretch & fold" maps of the interval, to be discussed briefly in the next example, and in more detail in sect. 11.3.1.

We shall refer to the kernel of (9.6) as the *Perron-Frobenius operator*:

9.3 page 146

$$\mathcal{L}^t(x,y) = \delta\big(x - f^t(y)\big) \,. \tag{9.10}$$

remember to link with (10.24)

example 16.7

remark 15.4

If you do not like the word "kernel" you might prefer to think of $\mathcal{L}^t(x,y)$ as a matrix with indices x, y. The Perron-Frobenius operator assembles the density $\rho(x,t)$ at time t by going back in time to the density $\rho(x_0,0)$ at time $t=0$.

move this remark to ?: (for nomenclature, see remark 15.4).

in depth: appendix E, p. 907

Example 9.1 Perron-Frobenius operator for a piecewise-linear map: *Assume the expanding 1-d map $f(x)$ of figure 9.2, a piecewise-linear 2–branch map with slopes $\Lambda_0 > 1$ and $\Lambda_1 = -\Lambda_0/(\Lambda_0 - 1) < -1$:*

9.7 page 148

$$f(x) = \begin{cases} f_0(x) = \Lambda_0 x\,, & x \in \mathcal{M}_0 = [0, 1/\Lambda_0) \\ f_1(x) = \frac{\Lambda_0}{\Lambda_0 - 1}(1 - x)\,, & x \in \mathcal{M}_1 = (1/\Lambda_0, 1]\,. \end{cases} \tag{9.11}$$

transfer!matrix
stationary!state
measure!invariant
invariant!measure

Both $f(\mathcal{M}_0)$ and $f(\mathcal{M}_1)$ map onto the entire unit interval $\mathcal{M} = [0, 1]$. Assume a piecewise constant density

$$\rho(x) = \begin{cases} \rho_0 & \text{if } x \in \mathcal{M}_0 \\ \rho_1 & \text{if } x \in \mathcal{M}_1 \end{cases}\,. \tag{9.12}$$

As can be easily checked using (9.9), the Perron-Frobenius operator acts on this piecewise constant function as a [2×2] "transfer" matrix with matrix elements

9.1 page 146

$$\begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} \to \mathcal{L}\rho = \begin{pmatrix} \frac{1}{|\Lambda_0|} & \frac{1}{|\Lambda_1|} \\ \frac{1}{|\Lambda_0|} & \frac{1}{|\Lambda_1|} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix}\,, \tag{9.13}$$

9.5 page 147

introduce, explain "transfer" matrix

stretching both ρ_0 and ρ_1 over the whole unit interval Λ. In this example the density is constant after one iteration, so $\mathcal{L}$ has only a unit eigenvalue $e^{s_0} = 1/|\Lambda_0| + 1/|\Lambda_1| = 1$, with constant density eigenvector $\rho_0 = \rho_1$. The quantities $1/|\Lambda_0|$, $1/|\Lambda_1|$ are, respectively, the fractions of state space taken up by the $|\mathcal{M}_0|$, $|\mathcal{M}_1|$ intervals. This simple explicit matrix representation of the Perron-Frobenius operator is a consequence of the piecewise linearity of f, and the restriction of the densities ρ to the space of piecewise constant functions. The example gives a flavor of the enterprise upon which we are about to embark in this book, but the full story is much subtler: in general, there will exist no such finite-dimensional representation for the Perron-Frobenius operator. (Continued in example 10.1.)

To a student with a practical bent the example suggests a strategy for constructing evolution operators for smooth maps, as limits of partitions of state space into regions $\mathcal{M}_i$, with a piecewise-linear approximations f_i to the dynamics in each region, but that would be too naive; much of the physically interesting spectrum would be missed. As we shall see, the choice of function space for ρ is crucial, and the physically motivated choice is a space of smooth functions, rather than the space of piecewise constant functions.

chapter 16

9.3 Invariant measures

A *stationary* or *invariant density* is a density left unchanged by the flow

rethink - where did the Jacobian in (9.5) go?

$$\rho(x, t) = \rho(x, 0) = \rho(x)\,. \tag{9.14}$$

Conversely, if such a density exists, the transformation $f^t(x)$ is said to be *measure-preserving*. As we are given deterministic dynamics and our goal is the computation of asymptotic averages of observables, our task is to identify interesting invariant measures for a given $f^t(x)$. Invariant measures remain unaffected by dynamics, so they are fixed points (in the infinite-dimensional function space of ρ densities) of the Perron-Frobenius operator (9.10), with the unit eigenvalue:

repeller measures?

9.3 page 146

equilibrium!point
confession!C.N. Yang
Yang, C.N.
natural measure
measure!natural

$$\mathcal{L}^t\rho(x) = \int_{\mathcal{M}} dy\, \delta(x - f^t(y))\rho(y) = \rho(x). \tag{9.15}$$

resurect after replacing with the no-escape example the above.

⇑PRELIMINARY

⇓PRELIMINARY

For the piecewise linear map example worked out above, in example 10.1, we have already constructed explicitly such eigenfunction, $\rho(y) = \text{const.}$, with eigenvalue 1. In general, depending on the choice of $f^t(x)$ and the function space for $\rho(x)$, there may be no, one, or many solutions of the eigenfunction condition (9.15). For instance, a singular measure $d\mu(x) = \delta(x - x_q)dx$ concentrated on an equilibrium point $x_q = f^t(x_q)$, or any linear combination of such measures, each concentrated on a different equilibrium point, is stationary. There are thus infinitely many stationary measures that can be constructed. Almost all of them are unnatural in the sense that the slightest perturbation will destroy them.

From a physical point of view, there is no way to prepare initial densities which are singular, so it makes sense to concentrate on measures which are limits of transformations experienced by an initial smooth distribution $\rho(x)$ under the action of f, rather than as a limit computed from a single trajectory,

$$\rho_0(x) = \lim_{t\to\infty} \int_{\mathcal{M}} dy\, \delta(x - f^t(y))\rho(y,0)\,, \qquad \int_{\mathcal{M}} dy\, \rho(y,0) = 1\,. \tag{9.16}$$

Intuitively, the "natural" measure (or measures) should be the least sensitive to facts of life, such as noise (no matter how weak).

connect to chapter 35: make the robustness under external noise more convincing here

9.3.1 Natural measure

add to refsMeasure.tex

> **Huang:** Chen-Ning, do you think ergodic theory gives us useful insight into the foundation of statistical mechanics?
> **Yang:** I don't think so.
> Kerson Huang, *C.N. Yang interview*

The *natural* or *equilibrium measure* can be defined as the limit

$$\overline{\rho}_{x_0}(y) \quad = \quad \begin{cases} \lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau\, \delta(y - f^\tau(x_0)) & \text{flows} \\ \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta\big(y - f^k(x_0)\big) & \text{maps}\,, \end{cases} \tag{9.17}$$

9.8 page 148
9.9 page 148

where x_0 is a generic inital point. [1] Staring at an average over infinitely many Dirac deltas is not a prospect we cherish. Generated by the action of f, the natural measure satisfies the stationarity condition (9.15) and is thus invariant by construction. From a computational point of view, the natural

[1] Driebe: emphasize generic, not periodic: use normal (Borel definition of irrationals) vs. non-normal. Explained in the green book edited by Series

measure is the visitation frequency defined by coarse-graining, integrating (9.17) over the $\mathcal{M}_i$ region

visitation frequency
observable
average!space
space!average
functional
time!average
average!time
ergodic!theory
ergodic!average
Birkhoff!ergodic theorem

$$\Delta\overline{\mu}_i = \lim_{t\to\infty} \frac{t_i}{t}\,, \tag{9.18}$$

where t_i is the accumulated time that a trajectory of total duration t spends in the $\mathcal{M}_i$ region, with the initial point x_0 picked from some smooth density $\rho(x)$.

Let $a = a(x)$ be any *observable*. In the mathematical literature $a(x)$ is a function belonging to some function space, for instance the space of integrable functions L^1, that associates to each point in state space a number or a set of numbers. In physical applications the observable $a(x)$ is necessarily a smooth function. The observable reports on some property of the dynamical system. Several examples will be given in sect. 10.1.

put pointer, refer to the example there

The *space average* of the observable a with respect to a measure ρ is given by the d-dimensional integral over the state space $\mathcal{M}$:

$$\begin{aligned} \langle a\rangle_\rho &= \frac{1}{|\rho_{\mathcal{M}}|}\int_{\mathcal{M}} dx\,\rho(x)a(x) \\ |\rho_{\mathcal{M}}| &= \int_{\mathcal{M}} dx\,\rho(x) = \text{mass in } \mathcal{M}\,. \end{aligned} \tag{9.19}$$

For now we assume that the state space $\mathcal{M}$ has a finite dimension and a finite volume. By definition, $\langle a\rangle_\rho$ is a function(al) of ρ.

Inserting the right-hand-side of (9.17) into (9.19), we see that the natural measure corresponds to a *time average* of the observable a along a trajectory of the initial point x_0,

$$\overline{a_{x_0}} = \lim_{t\to\infty} \frac{1}{t}\int_0^t d\tau\, a(f^\tau(x_0))\,. \tag{9.20}$$

☞ appendix A

Analysis of the above asymptotic time limit is the central problem of ergodic theory. The Birkhoff ergodic theorem asserts that if a natural measure ρ exists, the limit $\overline{a(x_0)}$ for the time average (9.20) exists for all initial x_0. As we shall not rely on this result in what follows we forgo a proof here. Furthermore, if the dynamical system is *ergodic*, the time average over almost any trajectory tends to the space average

definition of ergodic (Webster?)

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t d\tau\, a(f^\tau(x_0)) = \langle a\rangle \tag{9.21}$$

for "almost all" initial x_0. By "almost all" we mean that the time average is independent of the initial point apart from a set of ρ-measure zero.

explain we drop suffix ρ if ρ is the natural measure

mixing
Henon@Hénon!attractor
deterministic flow
flow!deterministic
stochastic flow
flow!stochastic
evolution!kernel!probabilistic

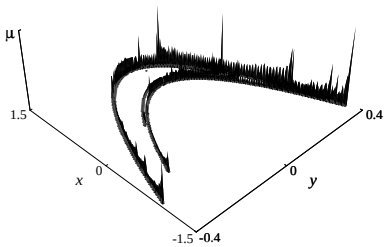

Figure 9.3: Natural measure (9.18) for the Hénon map (3.15) strange attractor at parameter values $(a, b) = (1.4, 0.3)$. See figure 3.5 for a sketch of the attractor without the natural measure binning. (Courtesy of J.-P. Eckmann)

For future reference, we note a further property that is stronger than ergodicity: if the space average of a product of any two variables decorrelates with time, explain *mixing*

$$\lim_{t\to\infty} \langle a(x)b(f^t(x))\rangle = \langle a\rangle \langle b\rangle \ , \tag{9.22}$$

☞ sect. 19.4

the dynamical system is said to be *mixing*.

Example 9.2 The Hénon attractor natural measure: *A numerical calculation of the natural measure (9.18) for the Hénon attractor (3.15) is given by the histogram in figure 9.3. The state space is partitioned into many equal-size areas $\mathcal{M}_i$, and the coarse grained measure (9.18) is computed by a long-time iteration of the Hénon map, and represented by the height of the column over area $\mathcal{M}_i$. What we see is a typical invariant measure - a complicated, singular function concentrated on a fractal set.*

If an invariant measure is quite singular (for instance a Dirac δ concentrated on a fixed point or a cycle), its existence is most likely of limited physical import. (See however discussion of orbital measures, sect. **??**). No smooth inital density will converge to this measure if the dynamics is unstable. In practice the average (9.17) is problematic and often hard to control, as generic dynamical systems are neither uniformly hyperbolic nor structurally stable: it is not known whether even the simplest model of a strange attractor, the Hénon attractor, is a strange attractor or merely a long stable cycle.

⇓PRELIMINARY
⇑PRELIMINARY

10.1
page 169

Genuflect here to refs. [12.34, 12.35, 12.36]

9.3.2 Determinism vs. stochasticity

While dynamics can lead to very singular ρ's, in any physical setting we cannot do better than to measure it averaged over some region $\mathcal{M}_i$; the coarse-graining is not an approximation but a physical necessity. One is free to think of a measure as a probability density, as long as one keeps in mind the distinction between deterministic and stochastic flows.
In deterministic evolution the evolution kernels are not probabilistic; the density of trajectories is transported *deterministically*. What this distinction means will became apparent later: for deterministic flows our trace and determinant formulas will be *exact*, while for quantum and stochastic flows they will only be the leading saddlepoint (stationary phase, steepest descent) approximations.

☞ chapter 15

refer to noise.tex as well

☞ chapter 34

Clearly, while deceptively easy to define, measures spell trouble. The good news is that if you hang on, you will *never need to compute them*, at least not in this book. How so? The evolution operators to which we next turn, and the trace and determinant formulas to which they will lead us, will assign the correct weights to desired averages without recourse to any explicit computation of the coarse-grained measure $\Delta\rho_i$.

continuity equation
semigroup
group!semi-

9.4 Density evolution for infinitesimal times

Consider the evolution of a smooth density $\rho(x) = \rho(x,0)$ under an infinitesimal step $\delta\tau$, by expanding the action of $\mathcal{L}^{\delta\tau}$ to linear order in $\delta\tau$:

$$\begin{aligned}
\mathcal{L}^{\delta\tau}\rho(y) &= \int_{\mathcal{M}} dx\, \delta\Big(y - f^{\delta\tau}(x)\Big)\, \rho(x) \\
&= \int_{\mathcal{M}} dx\, \delta(y - x - \delta\tau v(x))\, \rho(x) \\
&= \frac{\rho(y - \delta\tau v(y))}{\left|\det\left(1 + \delta\tau \frac{\partial v(y)}{\partial x}\right)\right|} = \frac{\rho(y) - \delta\tau \sum_{i=1}^{d} v_i(y)\partial_i\rho(y)}{1 + \delta\tau \sum_{i=1}^{d} \partial_i v_i(y)} \\
\rho(x,\delta\tau) &= \rho(x,0) - \delta\tau \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(v_i(x)\rho(x,0))\,. \qquad (9.23)
\end{aligned}$$

Here we have used the infinitesimal form of the flow (2.5), the Dirac delta Jacobian (9.9), and the $ln\, det = tr\, ln$ relation. Moving $\rho(y,0)$ to the left hand side and dividing by $\delta\tau$, we discover that the rate of the deformation of ρ under the infinitesimal action of the Perron-Frobenius operator is nothing but the *continuity equation* for the density:

4.1
page 78

$$\partial_t\rho + \partial\cdot(\rho v) = 0\,. \qquad (9.24)$$

The family of Perron-Frobenius operators operators $\left\{\mathcal{L}^t\right\}_{t\in\mathbb{R}_+}$ forms a semigroup parametrized by time

(a) $\mathcal{L}^0 = I$

(b) $\mathcal{L}^t\mathcal{L}^{t'} = \mathcal{L}^{t+t'} \qquad t,t' \geq 0 \qquad$ (semigroup property) .

From (9.23), time evolution by an infinitesimal step $\delta\tau$ is generated by

$$\mathcal{A}\rho(x) = +\lim_{\delta\tau\to 0^+} \frac{1}{\delta\tau}\left(\mathcal{L}^{\delta\tau} - I\right)\rho(x) = -\partial_i(v_i(x)\rho(x))\,. \qquad (9.25)$$

We shall refer to

$$\mathcal{A} = -\partial\cdot v + \sum_{i}^{d} v_i(x)\partial_i \qquad (9.26)$$

flow!generator of
looks like Hamiltonian
generator of flow but it is not
operator!semigroup!bounded
semigroup!operator
Laplace!transform
resolvent!operator
operator!resolvent

as the time evolution (semigroup) *generator*. If the flow is finite-dimensional and invertible, $\mathcal{A}$ is a generator of a full-fledged group. The left hand side of (9.25) is the definition of time derivative, so the evolution equation for $\rho(x)$ is

$$\left(\frac{\partial}{\partial t} - \mathcal{A}\right)\rho(x) \;=\; 0\,. \tag{9.27}$$

☞ appendix E.2

The finite time Perron-Frobenius operator (9.10) can be formally expressed by exponentiating the time evolution generator $\mathcal{A}$ as

$$\mathcal{L}^t = e^{t\mathcal{A}}\,. \tag{9.28}$$

⇓PRELIMINARY
?? page ??
⇑PRELIMINARY
9.10 page 148

The generator $\mathcal{A}$ is reminiscent of the generator of translations. Indeed, for a constant velocity field dynamical evolution is nothing but a translation by (time $\times$ velocity):

$$e^{-tv\frac{\partial}{\partial x}}a(x) = a(x - tv)\,. \tag{9.29}$$

As we will not need to implement a computational formula for general $e^{t\mathcal{A}}$ in what follows, we relegate making sense of such operators to appendix E.2.

☞ appendix E.2

9.4.1 Resolvent of $\mathcal{L}$

Here we limit ourselves to a brief remark about the notion of the "spectrum" of a linear operator.

The Perron-Frobenius operator $\mathcal{L}$ acts multiplicatively in time, so it is reasonable to suppose that there exist constants $M > 0$, $\beta \geq 0$ such that $||\mathcal{L}^t|| \leq Me^{t\beta}$ for all $t \geq 0$. What does that mean? The operator norm is defined in the same spirit in which one defines matrix norms (see appendix M.2): We are assuming that no value of $\mathcal{L}^t\rho(x)$ grows faster than exponentially for any choice of function $\rho(x)$, so that the fastest possible growth can be bounded by $e^{t\beta}$, a reasonable expectation in the light of the simplest example studied so far, the exact escape rate (10.20). If that is so, multiplying $\mathcal{L}^t$ by $e^{-t\beta}$ we construct a new operator $e^{-t\beta}\mathcal{L}^t = e^{t(\mathcal{A}-\beta)}$ which decays exponentially for large t, $||e^{t(\mathcal{A}-\beta)}|| \leq M$. We say that $e^{-t\beta}\mathcal{L}^t$ is an element of a *bounded* semigroup with generator $\mathcal{A} - \beta I$. Given this bound, it follows by the Laplace transform

☞ appendix M.2

EAS: need to define the norm of this operator

$$\int_0^\infty dt\, e^{-st}\mathcal{L}^t = \frac{1}{s - \mathcal{A}}\,, \qquad \mathrm{Re}\, s > \beta\,, \tag{9.30}$$

$\mathcal{A}$ sign wrong here

that the *resolvent* operator $(s - \mathcal{A})^{-1}$ is bounded ("resolvent" = able to

☞ sect. M.2

cause separation into constituents)

Poisson!bracket

be also as Fourier sform of (9.27), a en's" function

$$\left\| \frac{1}{s-\mathcal{A}} \right\| \leq \int_0^\infty dt\, e^{-st} M e^{t\beta} = \frac{M}{s-\beta}\,.$$

If one is interested in the spectrum of $\mathcal{L}$, as we will be, the resolvent operator is a natural object to study. The main lesson of this brief aside is that for continuous time flows, the Laplace transform is the tool that brings down the generator in (9.28) into the resolvent form (9.30) and enables us to study its spectrum.

in depth:
appendix E.2, p. 909

9.5 Liouville operator

A case of special interest is the Hamiltonian or symplectic flow defined by the Hamilton's equations of motion (5.1). A reader versed in quantum mechanics will have observed by now that with replacement $\mathcal{A} \to -\frac{i}{\hbar}\hat{H}$, where $\hat{H}$ is the quantum Hamiltonian operator, (9.27) looks rather like the time dependent Schrödinger equation, so this is probably the right moment to figure out what all this means in the case of Hamiltonian flows.

The Hamilton's evolution equations (5.1) for any time-independent quantity $Q = Q(q,p)$ are given by

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial Q}{\partial p_i}\frac{dp_i}{dt} = \frac{\partial H}{\partial p_i}\frac{\partial Q}{\partial q_i} - \frac{\partial Q}{\partial p_i}\frac{\partial H}{\partial q_i}\,. \tag{9.31}$$

As equations with this structure arise frequently for symplectic flows, it is convenient to introduce a notation for them, the *Poisson bracket*

$$\{A,B\} = \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}\,. \tag{9.32}$$

Gaspard has $\{A,B\}$ defined the other way around; use $\{,\}$

In terms of Poisson brackets the time evolution equation (9.31) takes the compact form

$$\frac{dQ}{dt} = \{H,Q\}\,. \tag{9.33}$$

The full state space flow velocity is $\dot{x} = (\dot{q},\dot{p})$, where the dot signifies time derivative for fixed initial point.

flow!incompressible
conservation!phase space volume
Liouville!theorem
density!phase space
phase space!density
continuity!equation
Liouville!equation
Poisson!bracket
symplectic!transformation

The discussion of sect. 9.4 applies to any deterministic flow. If the density itself is a material invariant, combining (??) and (9.24) we conclude that $\partial_i v_i = 0$ and $\det M^t(x_0) = 1$. An example of such incompressible flow is the Hamiltonian flow of sect. 5.2. For incompressible flows the continuity equation (9.24) becomes a statement of conservation of the state space volume (see sect. 5.2), or the *Liouville theorem*

⇓PRIVATE
⇑PRIVATE

$$\partial_t \rho + v_i \partial_i \rho = 0\,. \tag{9.34}$$

Hamilton's equations (5.1) imply that the flow is incompressible, $\partial_i v_i = 0$, so for Hamiltonian flows the equation for ρ reduces to the *continuity equation* for the phase space density:

☞ appendix E

confused - conflicts with (9.24) and (9.34)

$$\partial_t \rho + \partial_i(\rho v_i) = 0\,. \tag{9.35}$$

reference is Goldstein

Consider the evolution of the phase space density ρ of an ensemble of noninteracting particles; the particles are conserved, so

$$\frac{d}{dt}\rho(q,p,t) = \left(\frac{\partial}{\partial t} + \dot{q}_i\frac{\partial}{\partial q_i} + \dot{p}_i\frac{\partial}{\partial p_i}\right)\rho(q,p,t) = 0\,.$$

recheck sign!

Inserting Hamilton's equations (5.1) we obtain the *Liouville equation*, a special case of (9.27):

$$\frac{\partial}{\partial t}\rho(q,p,t) = -\mathcal{A}\rho(q,p,t) = \{H, \rho(q,p,t)\}\,, \tag{9.36}$$

rescue the Poisson brackets from appendix D

where $\{\ ,\ \}$ is the Poisson bracket (9.32). The generator of the flow (9.26) is now the generator of infinitesimal symplectic transformations,

$$\mathcal{A} = \dot{q}_i\frac{\partial}{\partial q_i} + \dot{p}_i\frac{\partial}{\partial p_i} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\,. \tag{9.37}$$

photoshop

For separable Hamiltonians of form $H = p^2/2m + V(q)$, the equations of motion are

$$\dot{q}_i = \frac{p_i}{m}\,, \qquad \dot{p}_i = -\frac{\partial V(q)}{\partial q_i}\,. \tag{9.38}$$

move to appendix F

and the action of the generator

$$\mathcal{A} = -\frac{p_i}{m}\frac{\partial}{\partial q_i} + \partial_i V(q)\frac{\partial}{\partial p_i}\,. \tag{9.39}$$

9.11
page 149

man notation is $-\mathcal{A}$

can be interpreted as a translation (9.29) in configuration space, followed by acceleration by force $\partial V(q)$ in the momentum space.

Liouville!operator
operator!Liouville
Poisson!bracket

This special case of the time evolution generator (9.26) for the case of symplectic flows is called the *Liouville operator*. You might have encountered it in statistical mechanics, while discussing what ergodicity means for 10^{23} hard balls. Here its action will be very tangible; we shall apply the evolution operator to systems as small as 1 or 2 hard balls and to our surprise learn that that suffices to alredy get a bit of a grip on foundations of the classical nonequilibrium statistical mechanics.

in depth:
sect. E.2, p. 909

Commentary

Remark 9.1 Ergodic theory: An overview of ergodic theory is outside the scope of this book: the interested reader may find it useful to consult ref. [9.1]. The existence of time average (9.20) is the basic result of ergodic theory, known as the Birkhoff theorem, see for example refs. [9.1, 1.20], or the statement of theorem 7.3.1 in ref. [9.8]. The natural measure (9.18) (more carefully defined than in the above sketch) is often referred to as the SRB or Sinai-Ruelle-Bowen measure [1.24, 1.22, 1.26].

Remark 9.2 Time evolution as a Lie group: Time evolution of sect. 9.4 is an example of a 1-parameter Lie group. Consult, for example, chapter 2. of ref. [9.9] for a clear and pedagogical introduction to Lie groups of transformations. For a discussion of the bounded semigroups of page 140 see, for example, Marsden and Hughes [9.2].

Remark 9.3 The sign convention of the Poisson bracket: The Poisson bracket is antisymmetric in its arguments and there is a freedom to define it with either sign convention. When such freedom exists, it is certain that both conventions are in use and this is no exception. In some texts [1.8, 9.3] you will see the right hand side of (9.32) defined as $\{B, A\}$ so that (9.33) is $\frac{dQ}{dt} = \{Q, H\}$. Other equally reputable texts [31.2] employ the convention used here. Landau and Lifshitz [9.4] denote a Poisson bracket by $[A, B]$, notation that we reserve here for the quantum-mechanical commutator. As long as one is consistent, there should be no problem.

Résumé

In physically realistic settings the initial state of a system can be specified only to a finite precision. If the dynamics is chaotic, it is not possible

to calculate accurately the long time trajectory of a given initial point. Depending on the desired precision, and given a deterministic law of evolution, the state of the system can then be tracked for a finite time.

The study of long-time dynamics thus requires trading in the evolution of a single state space point for the evolution of a *measure*, or the *density* of representative points in state space, acted upon by an *evolution operator*. Essentially this means trading in nonlinear dynamical equations on finite dimensional spaces $x = (x_1, x_2 \cdots x_d)$ for linear equations on infinite dimensional vector spaces of density functions $\rho(x)$. The most physical of stationary measures is the natural measure, a measure robust under perturbations by weak noise.

Reformulated this way, classical dynamics takes on a distinctly quantum-mechanical flavor. If the Lyapunov time (1.1), the time after which the notion of an individual deterministic trajectory loses meaning, is much shorter than the observation time, the "sharp" observables are those dual to time, the eigenvalues of evolution operators. This is very much the same situation as in quantum mechanics; as atomic time scales are so short, what is measured is the energy, the quantum-mechanical observable dual to the time. For long times the dynamics is described in terms of stationary measures, i.e., fixed points of certain evolution operators. Both in classical and quantum mechanics one has a choice of implementing dynamical evolution on densities ("Schrödinger picture", sect. 9.4) or on observables ("Heisenberg picture", sect. 10.2 and chapter 14). By "Schrödinger picture" we mean computer eigenvalues...., and then expectation values by extreme sandwiching the observables between the eigen.....

⇓PRELIMINARY

⇑PRELIMINARY

In what follows we shall find the second formulation more convenient, but the alternative is worth keeping in mind when posing and solving invariant density problems. However, as classical evolution operators are not unitary, their eigenstates can be quite singular and difficult to work with. In what follows we shall learn how to avoid dealing with these eigenstates altogether. As a matter of fact, what follows will be a labor of radical deconstruction; after having argued so strenuously here that only smooth measures are "natural", we shall merrily proceed to erect the whole edifice of our theory on periodic orbits, i.e., objects that are δ-functions in state space. The trick is that each comes with an interval, its neighborhood – cycle points only serve to pin these intervals, just as the millimeter marks on a measuring rod partition continuum into intervals.

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need more realistic set of refs on ergodic theory

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[9.2] J.E. Marsden and T.J.R. Hughes, *Mathematical Foundations of Elasticity* (Prentice-Hall, Englewood Cliffs, New Jersey 1983)

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[9.3] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, 1980).

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[9.5] P. Cvitanović, C.P. Dettmann, R. Mainieri and G. Vattay, *Trace formulas for stochastic evolution operators: Weak noise perturbation theory*, *J. Stat. Phys.* **93**, 981 (1998); `chao-dyn/9807034`.

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[9.9] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations* (Springer, New York 1989).

Dirac delta function
Gaussian!integral
Dirac delta derivatives

Exercises

Exercise 9.1 **Integrating over Dirac delta functions.** *Let us verify a few of the properties of the delta function and check (9.9), as well as the formulas (9.7) and (9.8) to be used later.*

(a) *If $f : \mathbb{R}^d \to \mathbb{R}^d$, show that*

$$\int_{\mathbb{R}^d} dx\, \delta\left(f(x)\right) = \sum_{x\in f^{-1}(0)} \frac{1}{|\det \partial_x f|}\,.$$

(b) *The delta function can be approximated by a sequence of Gaussians*

$$\int dx\, \delta(x) f(x) = \lim_{\sigma\to 0} \int dx\, \frac{e^{-\frac{x^2}{2\sigma}}}{\sqrt{2\pi\sigma}} f(x)\,.$$

Use this approximation to see whether the formal expression

$$\int_{\mathbb{R}} dx\, \delta(x^2)$$

makes sense.

Exercise 9.2 **Derivatives of Dirac delta functions.** *Consider $\delta^{(k)}(x) = \frac{\partial^k}{\partial x^k}\delta(x)$.*

(a) *Using integration by parts, determine the value of*

$$\int_{\mathbb{R}} dx\, \delta'(y)\,.$$

where $y = f(x) - x$.

(b)
$$\int dx\, \delta^{(2)}\left(y\right) = \sum_{\{x:y(x)=0\}} \frac{1}{|y'|}\left\{3\frac{(y'')^2}{(y')^4} - \frac{y'''}{(y')^3}\right\}\,. \tag{9.40}$$

(c)
$$\int dx\, b(x)\delta^{(2)}(y) = \sum_{\{x:y(x)=0\}} \frac{1}{|y'|}\left\{\frac{b''}{(y')^2} - \frac{b'y''}{(y')^3} + b\left(3\frac{(y'')^2}{(y')^4} - \frac{y'''}{(y')^3}\right)\right\}\,. \tag{9.41}$$

These formulas are useful for computing effects of weak noise on deterministic dynamics [9.5].

Exercise 9.3 **$\mathcal{L}^t$ generates a semigroup.** *Check that the Perron-Frobenius operator has the semigroup property,*

$$\int_M dz \mathcal{L}^{t_2}(y,z)\, \mathcal{L}^{t_1}(z,x) = \mathcal{L}^{t_2+t_1}(y,x)\,, \qquad t_1, t_2 \geq 0\,. \tag{9.42}$$

As the flows in which we tend to be interested are invertible, the $\mathcal{L}$'s that we will use often do form a group, with $t_1, t_2 \in \mathbb{R}$.

Exercise 9.4 Escape rate of the tent map.

(a) *Calculate by numerical experimentation the log of the fraction of trajectories remaining trapped in the interval* $[0,1]$ *for the tent map*

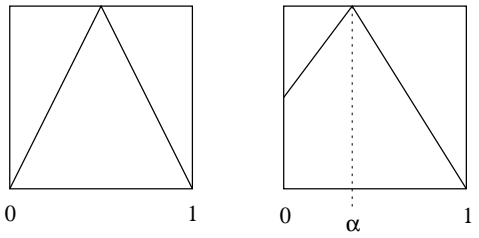
$$f(x) = a(1-2|x-0.5|)$$

for several values of a. [3]

(b) *Determine analytically the* a *dependence of the escape rate* $\gamma(a)$.

(c) *Compare your results for (a) and (b).*

Exercise 9.5 Invariant measure. *We will compute the invariant measure for two different piecewise linear maps.*



(a) *Verify the matrix* $\mathcal{L}$ *representation (10.19).*

(b) *The maximum value of the first map is 1. Compute an invariant measure for this map.*

(c) *Compute the leading eigenvalue of* $\mathcal{L}$ *for this map.*

(d) *For this map there is an infinite number of invariant measures, but only one of them will be found when one carries out a numerical simulation. Determine that measure, and explain why your choice is the natural measure for this map.*

(e) *In the second map the maximum occurs at* $\alpha = (3-\sqrt{5})/2$ *and the slopes are* $\pm(\sqrt{5}+1)/2$. *Find the natural measure for this map. Show that it is piecewise linear and that the ratio of its two values is* $(\sqrt{5}+1)/2$.

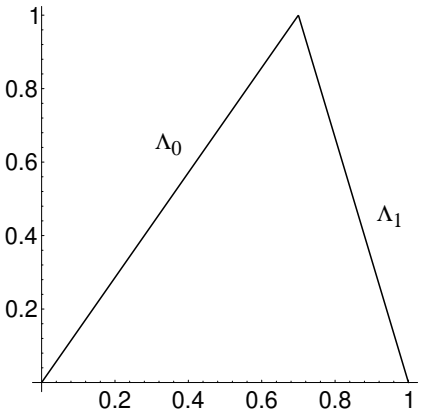
(medium difficulty)

Exercise 9.6 Escape rate for a flow conserving map. *Adjust* Λ_0, Λ_1 *in (10.17) so that the gap between the intervals* $\mathcal{M}_0$, $\mathcal{M}_1$ *vanishes. Show that the escape rate equals zero in this situation.*

Exercise 9.7 Eigenvalues of the Perron-Frobenius operator for the skew Ulam tent map. *Show that for the skew Ulam tent map*

[3] Mason: suggest specific values

Gauss map
invariant measure!Gauss map



$$f(x) = \begin{cases} f_0(x) = \Lambda_0 x\,, & x \in \mathcal{M}_0 = [0, 1/\Lambda_0) \\ f_1(x) = \frac{\Lambda_0}{\Lambda_0 - 1}(1-x)\,, & x \in \mathcal{M}_1 = (1/\Lambda_0, 1]\,. \end{cases} \tag{9.43}$$

the eigenvalues are available analytically, compute the first few.

Exercise 9.8 "Kissing disks"* *(continuation of exercises 6.1 and 6.2). Close off the escape by setting $R = 2$, and look in real time at the density of the Poincaré section iterates for a trajectory with a randomly chosen initial condition. Does it look uniform? Should it be uniform? (Hint - phase space volumes are preserved for Hamiltonian flows by the Liouville theorem). Do you notice the trajectories that loiter near special regions of phase space for long times? These exemplify "intermittency", a bit of unpleasantness to which we shall return in chapter 21.*

Exercise 9.9 Invariant measure for the Gauss map. *Consider the Gauss map (we shall need this map in chapter 28):*

$$f(x) = \begin{cases} \frac{1}{x} - \left[\frac{1}{x}\right] & x \neq 0 \\ 0 & x = 0 \end{cases}$$

where [ ] denotes the integer part.

(a) *Verify that the density*

$$\rho(x) \;=\; \frac{1}{\log 2}\frac{1}{1+x}$$

is an invariant measure for the map.

(b) *Is it the natural measure?*

Exercise 9.10 $\mathcal{A}$ as a generator of translations. *Verify that for a constant velocity field the evolution generator $\mathcal{A}$ in (9.29) is the generator of translations,*

$$e^{tv\frac{\partial}{\partial x}} a(x) = a(x+tv)\,.$$

4

[4] Mason: discuss the Lie group perspective in an appendix

Exercise 9.11 Incompressible flows. *Show that (9.9) implies that $\rho_0(x) = 1$ is an eigenfunction of a volume-preserving flow with eigenvalue $s_0 = 0$. In particular, this implies that the natural measure of hyperbolic and mixing Hamiltonian flows is uniform. Compare this results with the numerical experiment of exercise 9.8.*

Chapter 10

Averaging

> For it, the mystic evolution;
> Not the right only justified
> – what we call evil also justified.
> Walt Whitman,
> *Leaves of Grass: Song of the Universal*

We start by discussing the necessity of studying the averages of observables in chaotic dynamics, and then cast the formulas for averages in a multiplicative form that motivates the introduction of evolution operators and further formal developments to come. The main result is that any *dynamical* average measurable in a chaotic system can be extracted from the spectrum of an appropriately constructed evolution operator. In order to keep our toes closer to the ground, in sect. 10.3 we try out the formalism on the first quantitative diagnosis that a system's got chaos, Lyapunov exponents.

10.1 Dynamical averaging

In chaotic dynamics detailed prediction is impossible, as any finitely specified initial condition, no matter how precise, will fill out the entire accessible state space. Hence for chaotic dynamics one cannot follow individual trajectories for a long time; what is attainable is a description of the geometry of the set of possible outcomes, and evaluation of long time averages. Examples of such averages are transport coefficients for chaotic dynamical flows, such as escape rate, mean drift and diffusion rate; power spectra; and a host of mathematical constructs such as generalized dimensions, entropies and Lyapunov exponents. Here we outline how such averages are evaluated within the evolution operator framework. The key idea is to replace the expectation values of observables by the expectation values of generating functionals. This associates an evolution operator with a given observable, and relates the expectation value of the observable to the leading eigenvalue of the evolution operator.

observable
integrated observable
time!averaging
averaging!time

10.1.1 Time averages

Let $a = a(x)$ be any *observable*, a function that associates to each point in state space a number, a vector, or a tensor. The observable reports on a property of the dynamical system. It is a device, such as a thermometer or laser Doppler velocitometer. The device itself does not change during the measurement. The velocity field $a_i(x) = v_i(x)$ is an example of a vector observable; the length of this vector, or perhaps a temperature measured in an experiment at instant τ are examples of scalar observables. We define the *integrated observable* A^t as the time integral of the observable a evaluated along the trajectory of the initial point x_0,

$$A^t(x_0) = \int_0^t d\tau\, a(f^\tau(x_0))\,. \tag{10.1}$$

If the dynamics is given by an iterated mapping and the time is discrete, $t \to n$, the integrated observable is given by

$$A^n(x_0) = \sum_{k=0}^{n-1} a(f^k(x_0)) \tag{10.2}$$

(we suppress possible vectorial indices for the time being). For example, if the observable is the velocity, $a_i(x) = v_i(x)$, its time integral $A_i^t(x_0)$ is the trajectory $A_i^t(x_0) = x_i(t)$. Another familiar example, for Hamiltonian flows, is the action associated with a trajectory $x(t) = [q(t), p(t)]$ passing through a phase space point $x_0 = [q(0), p(0)]$ (this function will be the key to the semiclassical quantization of chapter 36):

$$A^t(x_0) = \int_0^t d\tau\, \dot{\mathbf{q}}(\tau) \cdot \mathbf{p}(\tau)\,. \tag{10.3}$$

The *time average* of the observable along a trajectory is defined by

$$\overline{a(x_0)} = \lim_{t\to\infty} \frac{1}{t} A^t(x_0)\,. \tag{10.4}$$

If a does not behave too wildly as a function of time – for example, if $a_i(x)$ is the Chicago temperature, bounded between -80^oF and $+130^oF$ for all times – $A^t(x_0)$ is expected to grow not faster than t, and the limit (10.4) exists. For an example of a time average - the Lyapunov exponent - see sect. 10.3.

The time average depends on the trajectory, but not on the initial point on that trajectory: if we start at a later state space point $f^T(x_0)$ we get a

couple of extra finite contributions that vanish in the $t \to \infty$ limit:

measure!smooth
smooth!measure

$$\begin{aligned}\overline{a(f^T(x_0))} &= \lim_{t\to\infty}\frac{1}{t}\int_T^{t+T} d\tau\, a(f^\tau(x_0)) \\ &= \overline{a(x_0)} - \lim_{t\to\infty}\frac{1}{t}\left(\int_0^T d\tau\, a(f^\tau(x_0)) - \int_t^{t+T} d\tau\, a(f^\tau(x_0))\right) \\ &= \overline{a(x_0)}\,.\end{aligned}$$

The integrated observable $A^t(x_0)$ and the time average $\overline{a(x_0)}$ take a particularly simple form when evaluated on a periodic orbit. Define

4.5 page 79

$$\begin{aligned}\text{flows:}\quad A_p &= a_p T_p = \int_0^{T_p} a\left(f^\tau(x_0)\right) d\tau\,, \qquad x_0 \in p \\ \text{maps:}\quad &= a_p n_p = \sum_{i=0}^{n_p-1} a\left(f^i(x_0)\right)\,, \end{aligned} \tag{10.5}$$

where p is a prime cycle, T_p is its period, and n_p is its discrete time period in the case of iterated map dynamics. A_p is a loop integral of the observable along a single parcourse of a prime cycle p, so it is an intrinsic property of the cycle, independent of the starting point $x_0 \in p$. (If the observable a is not a scalar but a vector or matrix we might have to be more careful in defining an average which is independent of the starting point on the cycle). If the trajectory retraces itself r times, we just obtain A_p repeated r times. Evaluation of the asymptotic time average (10.4) requires therefore only a single traversal of the cycle:

$$a_p = A_p/T_p\,. \tag{10.6}$$

However, $\overline{a(x_0)}$ is in general a wild function of x_0; for a hyperbolic system ergodic with respect to a smooth measure, it takes the same value $\langle a\rangle$ for almost all initial x_0, but a different value (10.6) on any periodic orbit, i.e., on a dense set of points (figure 10.1 (b)). For example, for an open system such as the Sinai gas of sect. 24.1 (an infinite 2-dimensional periodic array of scattering disks) the phase space is dense with initial points that correspond to periodic runaway trajectories. The mean distance squared traversed by any such trajectory grows as $x(t)^2 \sim t^2$, and its contribution to the diffusion rate $D \approx x(t)^2/t$, (10.4) evaluated with $a(x) = x(t)^2$, diverges. Seemingly there is a paradox; even though intuition says the typical motion should be diffusive, we have an infinity of ballistic trajectories.

draw (b) in figure 10.1
⇓PRELIMINARY
⇑PRELIMINARY
☞ chapter 24

For chaotic dynamical systems, this paradox is resolved by robust averaging, i.e., averaging also over the initial x, and worrying about the measure of the "pathological" trajectories.

averaging!space
space!averaging
natural measure
measure!natural

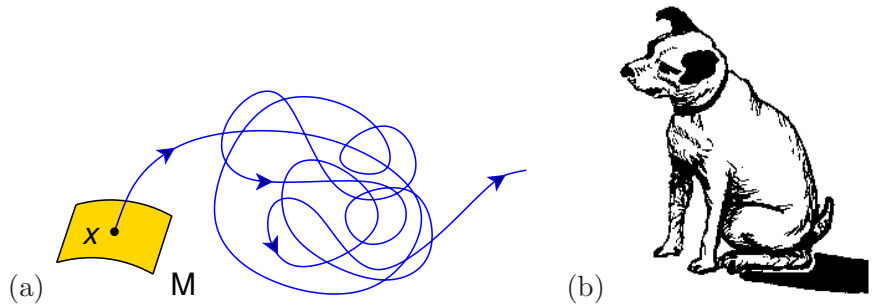


Figure 10.1: (a) A typical chaotic trajectory explores the phase space with the long time visitation frequency building up the natural measure $\rho_0(x)$. (b) time average evaluated along an atypical trajectory such as a periodic orbit fails to explore the entire accessible state space. (A. Johansen)

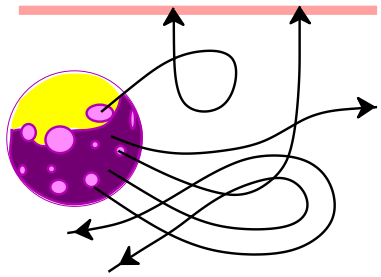


Figure 10.2: (A. Johansen)

10.1.2 Space averages

The *space average* of a quantity a that may depend on the point x of state space $\mathcal{M}$ and on the time t is given by the d-dimensional integral over the d coordinates of the dynamical system:

$$\begin{aligned} \langle a\rangle(t) &= \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} dx\; a(x(t)) \\ |\mathcal{M}| &= \int_{\mathcal{M}} dx \;= \text{volume of } \mathcal{M}\,. \end{aligned} \tag{10.7}$$

The space $\mathcal{M}$ is assumed to have finite dimension and volume (open systems like the 3-disk game of pinball are discussed in sect. 10.1.3).

What is it we *really* do in experiments? We cannot measure the time average (10.4), as there is no way to prepare a single initial condition with infinite precision. The best we can do is to prepare some initial density $\rho(x)$ perhaps concentrated on some small (but always finite) neighborhood $\rho(x) = \rho(x,0)$, so one should abandon the uniform space average (10.7), and consider instead

$$\langle a\rangle_\rho(t) = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} dx\; \rho(x)a(x(t))\,. \tag{10.8}$$

We do not bother to lug the initial $\rho(x)$ around, as for the ergodic and mixing systems that we shall consider here *any* smooth initial density will

expectation value

tend to the asymptotic natural measure $t \to \infty$ limit $\rho(x,t) \to \rho_0(x)$, so we can just as well take the initial $\rho(x) = \text{const.}$. The worst we can do is to start out with $\rho(x) = \text{const.}$, as in (10.7); so let us take this case and define the *expectation value* $\langle a \rangle$ of an observable a to be the asymptotic time and space average over the state space $\mathcal{M}$

$$\langle a \rangle = \lim_{t\to\infty} \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} dx \, \frac{1}{t} \int_0^t d\tau \, a(f^\tau(x)) \,. \tag{10.9}$$

We use the same $\langle \cdots \rangle$ notation as for the space average (10.7), and distinguish the two by the presence of the time variable in the argument: if the quantity $\langle a \rangle(t)$ being averaged depends on time, then it is a space average, if it does not, it is the expectation value $\langle a \rangle$.

The expectation value is a space average of time averages, with every $x \in \mathcal{M}$ used as a starting point of a time average. The advantage of averaging over space is that it smears over the starting points which were problematic for the time average (like the periodic points). While easy to define, the expectation value $\langle a \rangle$ turns out not to be particularly tractable in practice. Here comes a simple idea that is the basis of all that follows: Such averages are more conveniently studied by investigating instead of $\langle a \rangle$ the space averages of form

$$\left\langle e^{\beta \cdot A^t} \right\rangle = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} dx \, e^{\beta \cdot A^t(x)}. \tag{10.10}$$

In the present context β is an auxiliary variable of no particular physical significance. In most applications β is a scalar, but if the observable is a d-dimensional vector $a_i(x) \in \mathbb{R}^d$, then β is a conjugate vector $\beta \in \mathbb{R}^d$; if the observable is a $d \times d$ tensor, β is also a rank-2 tensor, and so on. Here we will mostly limit the considerations to scalar values of β.

If the limit $\overline{a(x_0)}$ for the time average (10.4) exists for "almost all" initial x_0 and the system is ergodic and mixing (in the sense of sect. 1.3.1), we expect the time average along almost all trajectories to tend to the same value $\overline{a}$, and the integrated observable A^t to tend to $t\overline{a}$. The space average (10.10) is an integral over exponentials, and such integral also grows exponentially with time. So as $t \to \infty$ we would expect the space average of $\left\langle \exp(\beta \cdot A^t) \right\rangle$ itself to grow exponentially with time

$$\left\langle e^{\beta \cdot A^t} \right\rangle \propto e^{ts(\beta)} \,,$$

and its rate of growth to go to a limit

$$s(\beta) = \lim_{t\to\infty} \frac{1}{t} \ln \left\langle e^{\beta \cdot A^t} \right\rangle \,. \tag{10.11}$$

diffusion!constant
open systems
systems!open

Now we understand one reason for why it is smarter to compute $\langle \exp(\beta \cdot A^t) \rangle$ rather than $\langle a \rangle$: the expectation value of the observable (10.9) and the moments of the integrated observable (10.1) can be computed by evaluating the derivatives of $s(\beta)$

$$\begin{aligned} \left.\frac{\partial s}{\partial \beta}\right|_{\beta=0} &= \lim_{t\to\infty} \frac{1}{t} \langle A^t \rangle = \langle a \rangle \ , \\ \left.\frac{\partial^2 s}{\partial \beta^2}\right|_{\beta=0} &= \lim_{t\to\infty} \frac{1}{t} \left(\langle A^t A^t \rangle - \langle A^t \rangle \langle A^t \rangle \right) \\ &= \lim_{t\to\infty} \frac{1}{t} \langle (A^t - t\langle a \rangle)^2 \rangle \ , \end{aligned} \tag{10.12}$$

and so forth. We have written out the formulas for a scalar observable; the vector case is worked out in the exercise 10.2. If we can compute the function $s(\beta)$, we have the desired expectation value without having to estimate any infinite time limits from finite time data.

10.2
page 169

Suppose we could evaluate $s(\beta)$ and its derivatives. What are such formulas good for? A typical application is to the problem of describing a particle scattering elastically off a 2-dimensional triangular array of disks. If the disks are sufficiently large to block any infinite length free flights, the particle will diffuse chaotically, and the transport coefficient of interest is the diffusion constant given by $\langle x(t)^2 \rangle \approx 4Dt$. In contrast to D estimated numerically from trajectories $x(t)$ for finite but large t, the above formulas yield the asymptotic D without any extrapolations to the $t \to \infty$ limit. For example, for $a_i = v_i$ and zero mean drift $\langle v_i \rangle = 0$, the diffusion constant is given by the curvature of $s(\beta)$ at $\beta = 0$,

$$D = \lim_{t\to\infty} \frac{1}{2dt} \langle x(t)^2 \rangle = \frac{1}{2d} \sum_{i=1}^{d} \left.\frac{\partial^2 s}{\partial \beta_i^2}\right|_{\beta=0} \ , \tag{10.13}$$

sect. 24.1

so if we can evaluate derivatives of $s(\beta)$, we can compute transport coefficients that characterize deterministic diffusion. As we shall see in chapter 24, periodic orbit theory yields an explicit closed form expression for D.

fast track:
sect. 10.2, p. 159

10.1.3 Averaging in open systems

If the $\mathcal{M}$ is a compact region or set of regions to which the dynamics is confined for all times, (10.9) is a sensible definition of the expectation value. However, if the trajectories can exit $\mathcal{M}$ without ever returning,

$$\int_{\mathcal{M}} dy\, \delta(y - f^t(x_0)) = 0 \qquad \text{for } t > t_{exit} \,, \qquad x_0 \in \mathcal{M} \,,$$

we might be in trouble. In particular, for a repeller the trajectory $f^t(x_0)$ will eventually leave the region $\mathcal{M}$, unless the initial point x_0 is on the repeller, so the identity

$$\int_{\mathcal{M}} dy\, \delta(y-f^t(x_0)) = 1\,, \quad t>0\,, \qquad \text{iff } x_0 \in \text{ non–wandering set} \tag{10.14}$$

might apply only to a fractal subset of initial points a set of zero Lebesgue measure. Clearly, for open systems we need to modify the definition of the expectation value to restrict it to the dynamics on the non–wandering set, the set of trajectories which are confined for all times.

repeller
M@$“pS$ state space volume
state space!volume $“pS$
escape!rate
piecewise linear map!repeller
repeller!piecewise-linear

Note by $\mathcal{M}$ a state space region that encloses all interesting initial points, say the 3-disk Poincaré section constructed from the disk boundaries and all possible incidence angles, and denote by $|\mathcal{M}|$ the volume of $\mathcal{M}$. The volume of the state space containing all trajectories which start out within the state space region $\mathcal{M}$ and recur within that region at the time t

$$|\mathcal{M}(t)| = \int_{\mathcal{M}} dxdy\, \delta\big(y - f^t(x)\big) \;\sim\; |\mathcal{M}|e^{-\gamma t} \tag{10.15}$$

is expected to decrease exponentially, with the escape rate γ. The integral over x takes care of all possible initial points; the integral over y checks whether their trajectories are still within $\mathcal{M}$ by the time t. For example, any trajectory that falls off the pinball table in figure 1.1 is gone for good.

☞ sect. 1.4.2

☞ sect. 19.1

The non–wandering set can be very difficult object to describe; but for any finite time we can construct a normalized measure from the finite-time covering volume (10.15), by redefining the space average (10.10) as

$$\left\langle e^{\beta\cdot A^t} \right\rangle = \int_{\mathcal{M}} dx\, \frac{1}{|\mathcal{M}(t)|} e^{\beta\cdot A^t(x)} \sim \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} dx\, e^{\beta\cdot A^t(x)+\gamma t}\,. \tag{10.16}$$

in order to compensate for the exponential decrease of the number of surviving trajectories in an open system with the exponentially growing factor $e^{\gamma t}$. What does this mean? Once we have computed γ we can replenish the density lost to escaping trajectories, by pumping in $e^{\gamma t}$ in such a way that the overall measure is correctly normalized at all times, $\langle 1\rangle = 1$.

credit Grassberger

Example 10.1 A piecewise-linear example: *(continuation of example 9.1)*
What is gained by reformulating the dynamics in terms of “operators”? We start by considering a simple example in which the operator is a $[2\times 2]$ *matrix. Assume the expanding 1-d map* $f(x)$ *of figure 10.3, a piecewise-linear 2–branch repeller with slopes* $\Lambda_0 > 1$ *and* $\Lambda_1 < -1$ *:*

label branches, change font size in figure 10.3

$$f(x) = \begin{cases} f_0 = \Lambda_0 x & \text{if} \quad x\in\mathcal{M}_0 = [0, 1/\Lambda_0] \\ f_1 = \Lambda_1(x-1) & \text{if} \quad x\in\mathcal{M}_1 = [1+1/\Lambda_1, 1] \end{cases}\,. \tag{10.17}$$

transfer!matrix
escape!rate

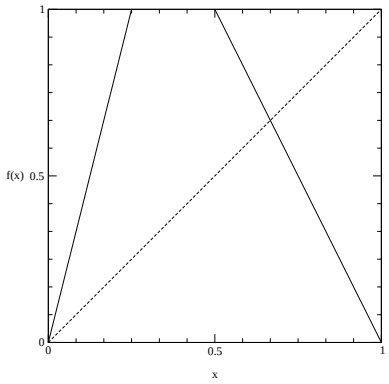


Figure 10.3: A piecewise-linear repeller (10.17): All trajectories that land in the gap between the f_0 and f_1 branches escape ($\Lambda_0 = 4$, $\Lambda_1 = -2$).

Both $f(\mathcal{M}_0)$ and $f(\mathcal{M}_1)$ map onto the entire unit interval $\mathcal{M} = [0,1]$. Assume a piecewise constant density

$$\rho(x) = \begin{cases} \rho_0 & \text{if } x \in \mathcal{M}_0 \\ \rho_1 & \text{if } x \in \mathcal{M}_1 \end{cases} . \tag{10.18}$$

There is no need to define $\rho(x)$ in the gap between $\mathcal{M}_0$ and $\mathcal{M}_1$, as any point that lands in the gap escapes.

The physical motivation for studying this kind of mapping is the pinball game: f is the simplest model for the pinball escape, figure 1.8, with f_0 and f_1 modelling its two strips of survivors.

As can be easily checked using (9.9), the Perron-Frobenius operator acts on this piecewise constant function as a [2×2] "transfer" matrix with matrix elements

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9.5 page 147

introduce, explain "transfer" matrix

$$\begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} \to \mathcal{L}\rho = \begin{pmatrix} \frac{1}{|\Lambda_0|} & \frac{1}{|\Lambda_1|} \\ \frac{1}{|\Lambda_0|} & \frac{1}{|\Lambda_1|} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} , \tag{10.19}$$

stretching both ρ_0 and ρ_1 over the whole unit interval Λ, and decreasing the density at every iteration. In this example the density is constant after one iteration, so $\mathcal{L}$ has only one non-zero eigenvalue $e^{s_0} = 1/|\Lambda_0| + 1/|\Lambda_1|$, with constant density eigenvector $\rho_0 = \rho_1$. The quantities $1/|\Lambda_0|$, $1/|\Lambda_1|$ are, respectively, the sizes of the $|\mathcal{M}_0|$, $|\mathcal{M}_1|$ intervals, so the exact *escape rate (1.3) – the log of the fraction of survivors at each iteration for this linear repeller – is given by the sole eigenvalue of $\mathcal{L}$:*

$$\gamma = -s_0 = -\ln(1/|\Lambda_0| + 1/|\Lambda_1|) . \tag{10.20}$$

Voila! Here is the rationale for introducing operators – in one time step we have solved the problem of evaluating escape rates at infinite time. This simple explicit matrix representation of the Perron-Frobenius operator is a consequence of the piecewise linearity of f, and the restriction of the densities ρ to the space of piecewise constant functions. The example gives a flavor of the enterprise upon which we are about to embark in this book, but the full story is much subtler: in general, there will exist no such finite-dimensional representation for the Perron-Frobenius operator.

We now turn to the problem of evaluating $\left\langle e^{\beta \cdot A^t} \right\rangle$.

10.2 Evolution operators

Dirac delta function
evolution!operator
operator!evolution

The above simple shift of focus, from studying $\langle a \rangle$ to studying $\langle \exp(\beta \cdot A^t) \rangle$ is the key to all that follows. Make the dependence on the flow explicit by rewriting this quantity as

$$\left\langle e^{\beta \cdot A^t} \right\rangle = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} dx \int_{\mathcal{M}} dy \, \delta\big(y - f^t(x)\big) \, e^{\beta \cdot A^t(x)} \,. \tag{10.21}$$

Here $\delta\big(y - f^t(x)\big)$ is the Dirac delta function: for a deterministic flow an initial point x maps into a unique point y at time t. Formally, all we have done above is to insert the identity

$$1 = \int_{\mathcal{M}} dy \, \delta\big(y - f^t(x)\big) \,, \tag{10.22}$$

into (10.10) to make explicit the fact that we are averaging only over the trajectories that remain in $\mathcal{M}$ for all times. However, having made this substitution we have replaced the study of individual trajectories $f^t(x)$ by the study of the evolution of density of *the totality* of initial conditions. Instead of trying to extract a temporal average from an arbitrarily long trajectory which explores the phase space ergodically, we can now probe the entire state space with short (and controllable) finite time pieces of trajectories originating from every point in $\mathcal{M}$.

As a matter of fact (and that is why we went to the trouble of defining the generator (9.26) of infinitesimal transformations of densities) *infinitesimally* short time evolution can suffice to determine the spectrum and eigenvalues of $\mathcal{L}^t$.

draw a field of grass

We shall refer to the kernel of $\mathcal{L}^t = e^{t\mathcal{A}}$ in the state space representation (10.21) as the *evolution operator*

$$\mathcal{L}^t(y,x) \;=\; \delta\big(y - f^t(x)\big) \; e^{\beta \cdot A^t(x)} \,. \tag{10.23}$$

recheck, probably $\delta\big(y - f^{-t}(x)\big)$

The simplest example is the Perron-Frobenius operator introduced in section 5.2. Another example - designed to deliver the Lyapunov exponent - will be the evolution operator (10.35). The evolution operator acts on scalar functions $\phi(x)$ as

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⇑PRELIMINARY

figure here

$$\mathcal{L}^t \phi(y) = \int_{\mathcal{M}} dx \, \delta\big(y - f^t(x)\big) \, e^{\beta \cdot A^t(x)} \phi(x) \,. \tag{10.24}$$

In terms of the evolution operator, the expectation value of the generating function (10.21) is given by

$$\left\langle e^{\beta \cdot A^t} \right\rangle = \langle \mathcal{L}^t \iota \rangle \,,$$

semigroup!evolution
evolution!semigroup
attractor!strange
strange attractor
limit!cycle
cycle!limit

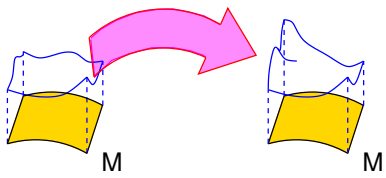


Figure 10.4: Space averaging pieces together the time average computed along the $t \to \infty$ trajectory of figure 10.1 by a space average over infinitely many short t trajectory segments starting at all initial points at once. (A. Johansen)

where the initial density $\iota(x)$ is the constant function that always returns 1.

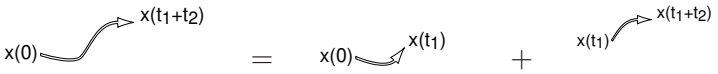
replace ι by something sensible

The evolution operator is different for different observables, as its definition depends on the choice of the integrated observable A^t in the exponential. Its job is deliver to us the expectation value of a, but before showing that it accomplishes that, we need to verify the semigroup property of evolution operators.

did not yet edit text from here to (14.3)

By its definition, the integral over the observable a is additive along the trajectory

redraw this



$$
\begin{aligned}
A^{t_1+t_2}(x_0) &= \int_0^{t_1} d\tau\, a(x(\tau)) + \int_{t_1}^{t_1+t_2} d\tau\, a(x(\tau)) \\
&= A^{t_1}(x_0) + A^{t_2}(f^{t_1}(x_0))\,.
\end{aligned}
$$

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If $A^t(x)$ is additive along the trajectory, the evolution operator generates a semigroup

sect. 9.4

$$
\mathcal{L}^{t_1+t_2}(y,x) = \int_{\mathcal{M}} dz\, \mathcal{L}^{t_2}(y,z)\mathcal{L}^{t_1}(z,x)\,, \tag{10.25}
$$

as is easily checked by substitution

$$
\mathcal{L}^{t_2}\mathcal{L}^{t_1}a(x) = \int_{\mathcal{M}} dy\, \delta(x - f^{t_2}(y)) e^{\beta\cdot A^{t_2}(y)} (\mathcal{L}^{t_1}a)(y) = \mathcal{L}^{t_1+t_2}a(x)\,.
$$

This semigroup property is the main reason why (10.21) is preferable to (10.9) as a starting point for evaluation of dynamical averages: it recasts averaging in form of operators multiplicative along the flow.

10.3 Lyapunov exponents

(J. Mathiesen and P. Cvitanović)

Let us apply the newly acquired tools to the fundamental diagnostics in this subject: Is a given system "chaotic"? And if so, how chaotic? If all points

sensitivity to initial conditions
Lyapunov exponent

sect. 1.3.1

in a neighborhood of a trajectory converge toward the same trajectory, the attractor is a fixed point or a limit cycle. However, if the attractor is strange, two trajectories

$$x(t) = f^t(x_0) \quad \text{and} \quad x(t) + \delta x(t) = f^t(x_0 + \delta x(0)) \tag{10.26}$$

that start out very close to each other separate exponentially with time, and in a finite time their separation attains the size of the accessible state space. This *sensitivity to initial conditions* can be quantified as

$$|\delta x(t)| \approx e^{\lambda t} |\delta x(0)| \tag{10.27}$$

where λ, the mean rate of separation of trajectories of the system, is called the *Lyapunov exponent*.

10.3.1 Lyapunov exponent as a time average

We can start out with a small δx and try to estimate λ from (10.27), but now that we have quantified the notion of linear stability in chapter 4 and defined the dynamical time averages in sect. 10.1.1, we can do better. The problem with measuring the growth rate of the distance between two points is that as the points separate, the measurement is less and less a local measurement. In study of experimental time series this might be the only option, but if we have the equations of motion, a better way is to measure the growth rate of vectors transverse to a given orbit.

The mean growth rate of the distance $|\delta x(t)|/|\delta x(0)|$ between neighboring trajectories (10.27) is given by the *Lyapunov exponent*

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln |\delta x(t)|/|\delta x(0)| \tag{10.28}$$

(For notational brevity we shall often suppress the dependence of quantities such as $\lambda = \lambda(x_0)$, $\delta x(t) = \delta x(x_0, t)$ on the initial point x_0 and the time t). For infinitesimal δx we know the $\delta x_i(t)/\delta x_j(0)$ ratio exactly, as this is by definition the fundamental matrix (4.32)

$$\lim_{\delta x\to 0} \frac{\delta x_i(t)}{\delta x_j(0)} = \frac{\partial x_i(t)}{\partial x_j(0)} = M^t_{ij}(x_0)\,,$$

so the leading Lyapunov exponent can be computed from the linear approximation (4.25)

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\left|M^t(x_0)\delta x(0)\right|}{|\delta x(0)|} = \lim_{t\to\infty} \frac{1}{2t} \ln \left|\hat{n}^T (M^t)^T M^t \hat{n}\right| . \tag{10.29}$$

time!average
Lyapunov exponent!numerically
Rossler@R“”ossler system

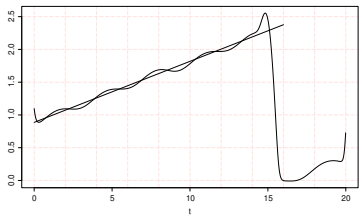

Figure 10.5: A numerical estimate of the leading Lyapunov exponent for the Rössler system (2.14) from the dominant expanding eigenvalue formula (10.29). The leading Lyapunov exponent $\lambda \approx 0.09$ is positive, so numerics supports the hypothesis that the Rössler attractor is strange. (J. Mathiesen)

In this formula the scale of the initial separation drops out, only its orientation given by the unit vector $\hat{n} = \delta x/|\delta x|$ matters. The eigenvalues of M are either real or come in complex conjugate pairs. As M is in general not symmetric and not diagonalizable, it is more convenient to work with the symmetric and diagonalizable matrix $\mathbf{M} = (M^t)^T M^t$, with real positive eigenvalues $\{|\Lambda_1|^2 \geq \ldots \geq |\Lambda_d|^2\}$, and a complete orthonormal set of eigenvectors of $\{u_1, \ldots, u_d\}$. Expanding the initial orientation $\hat{n} = \sum(\hat{n} \cdot u_i)u_i$ in the $\mathbf{M}u_i = |\Lambda_i|u_i$ eigenbasis, we have

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remark 4.2
⇑PRELIMINARY

$$\hat{n}^T \mathbf{M} \hat{n} = \sum_{i=1}^{d} (\hat{n}\cdot u_i)^2 |\Lambda_i|^2 = (\hat{n}\cdot u_1)^2 e^{2\lambda_1 t} \left(1 + O(e^{-2(\lambda_1-\lambda_2)t})\right) , \quad (10.30)$$

where $t\lambda_i = \ln|\Lambda_i(x_0,t)|$, and we assume that $\lambda_1 > \lambda_2 \geq \lambda_3 \cdots$. For long times the largest Lyapunov exponent dominates exponentially (10.29), provided the orientation $\hat{n}$ of the initial separation was not chosen perpendicular to the dominant expanding eigendirection u_1. The Lyapunov exponent is the time average

$$\begin{aligned} \overline{\lambda(x_0)} &= \lim_{t\to\infty} \frac{1}{t} \left\{ \ln|\hat{n}\cdot u_1| + \ln|\Lambda_1(x_0,t)| + O(e^{-2(\lambda_1-\lambda_2)t}) \right\} \\ &= \lim_{t\to\infty} \frac{1}{t} \ln|\Lambda_1(x_0,t)| , \end{aligned} \quad (10.31)$$

where $\Lambda_1(x_0,t)$ is the leading eigenvalue of $M^t(x_0)$. By choosing the initial displacement such that $\hat{n}$ is normal to the first (i-1) eigendirections we can define not only the leading, but all Lyapunov exponents as well:

$$\overline{\lambda_i(x_0)} = \lim_{t\to\infty} \frac{1}{t} \ln|\Lambda_i(x_0,t)| , \qquad i = 1, 2, \cdots, d . \quad (10.32)$$

edit figure 10.5

make problem set, mention rescaling to avoid overflows

The leading Lyapunov exponent now follows from the fundamental matrix by numerical integration of (4.34). The equations can be integrated accurately for a finite time, hence the infinite time limit of (10.29) can be only estimated from plots of $\frac{1}{2}\ln|\hat{n}^T\mathbf{M}\hat{n}|$ as function of time, such as the figure 10.5 for the Rössler system (2.14). As the local expansion and contraction rates vary along the flow, the temporal dependence exhibits small and large humps. The sudden fall to a low level is caused by a close passage to a folding point of the attractor, an illustration of why numerical evaluation of the Lyapunov exponents, and proving the very existence of a

strange attractor is a very difficult problem. The approximately monotone part of the curve can be used (at your own peril) to estimate the leading Lyapunov exponent by a straight line fit.

local!divergence rate
divergence rate!local
natural measure
Lyapunov exponent!natural measure

As we can already see, we are courting difficulties if we try to calculate the Lyapunov exponent by using the definition (10.31) directly. First of all, the state space is dense with atypical trajectories; for example, if x_0 happened to lie on a periodic orbit p, $\overline{\lambda}$ would be simply $\ln|\Lambda_p|/T_p$, a local property of cycle p, not a global property of the dynamical system. Furthermore, even if x_0 happens to be a "generic" state space point, it is still not obvious that $\ln|\Lambda(x_0,t)|/t$ should be converging to anything in particular. In a Hamiltonian system with coexisting elliptic islands and chaotic regions, a chaotic trajectory gets every so often captured in the neighborhood of an elliptic island and can stay there for arbitrarily long time; as there the orbit is nearly stable, during such episode $\ln|\Lambda(x_0,t)|/t$ can dip arbitrarily close to 0^+. For state space volume non-preserving flows the trajectory can traverse locally contracting regions, and $\ln|\Lambda(x_0,t)|/t$ can occasionally go negative; even worse, one never knows whether the asymptotic attractor is periodic or "strange", so any finite estimate of $\overline{\lambda}$ might be dead wrong. [1]

⇓PRELIMINARY
☞ sect. 43.2
⇑PRELIMINARY
10.1
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10.3.2 Evolution operator evaluation of Lyapunov exponents

A cure to these problems was offered in sect. 10.2. We shall now replace time averaging along a single trajectory by action of a multiplicative evolution operator on the entire state space, and extract the Lyapunov exponent from its leading eigenvalue. If the chaotic motion fills the whole state space, we are indeed computing the asymptotic Lyapunov exponent. If the chaotic motion is transient, leading eventually to some long attractive cycle, our Lyapunov exponent, computed on nonwandering set, will characterize the chaotic transient; this is actually what any experiment would measure, as even very small amount of external noise will suffice to destabilize a long stable cycle with a minute immediate basin of attraction.

Due to the chain rule (4.40) for the derivative of an iterated map, the stability of a 1-d mapping is multiplicative along the flow, so the integral (10.1) of the observable $a(x) = \ln|f'(x)|$, the local trajectory divergence rate, evaluated along the trajectory of x_0 is additive:

$$A^n(x_0) = \ln\left|f^{n\prime}(x_0)\right| = \sum_{k=0}^{n-1} \ln\left|f'(x_k)\right| \ . \tag{10.33}$$

The Lyapunov exponent is then the expectation value (10.9) given by a spatial integral (10.8) weighted by the natural measure

continue exercise 9.5: Lyapunovs

[1] Mason: there is a good Matlab module that computes a Lyapunov exp - will provide a reference

$$\lambda = \left\langle \ln |f'(x)| \right\rangle = \int_{\mathcal{M}} dx\, \rho_0(x) \ln |f'(x)| \,. \tag{10.34}$$

The associated (discrete time) evolution operator (10.23) is

$$\mathcal{L}(y,x) \;=\; \delta(y - f(x))\; e^{\beta \ln |f'(x)|} \,. \tag{10.35}$$

☞ appendix I.1

Here we have restricted our considerations to $1-d$ maps, as for higher-dimensional flows only the fundamental matrices are multiplicative, not the individual eigenvalues. Construction of the evolution operator for evaluation of the Lyapunov spectra in the general case requires more cleverness than warranted at this stage in the narrative: an extension of the evolution equations to a flow in the tangent space.

in depth:
appendix I.1, p. 945

All that remains is to determine the value of the Lyapunov exponent

$$\lambda = \left\langle \ln |f'(x)| \right\rangle = \left. \frac{\partial s(\beta)}{\partial \beta} \right|_{\beta=1} = s'(1) \tag{10.36}$$

from (10.12), the derivative of the leading eigenvalue $s_0(\beta)$ of the evolution operator (10.35).

☞ example 18.1

The only question is: how?

10.4 Why not just run it on a computer?

(R. Artuso and P. Cvitanović)

All of the insight gained in this chapter and the preceding one was nothing but an elegant way of thinking of the evolution operator, $\mathcal{L}$, as a matrix (this point of view will be further elaborated in chapter 16). There are many textbook methods of approximating an operator $\mathcal{L}$ by sequences of finite matrix approximations $\mathcal{L}$, but in what follows the great achievement will be that we shall avoid constructing any matrix approximation to $\mathcal{L}$ altogether. Why a new method? Why not just run it on a computer, as many do with such relish in diagonalizing quantum Hamiltonians?

The simplest possible way of introducing a state space discretization, figure 10.6, is to partition the state space $\mathcal{M}$ with a non-overlapping collection of sets $\mathcal{M}_i$, $i = 1, \ldots, N$, and to consider piece-wise constant densities (9.2), constant on each $\mathcal{M}_i$:

$$\rho(x) \;=\; \sum_{i=1}^{N} \rho_i \frac{\chi_i(x)}{|\mathcal{M}_i|}$$

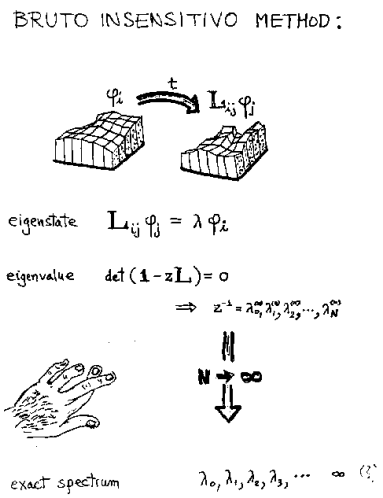

Figure 10.6: State space discretization approach to computing averages.

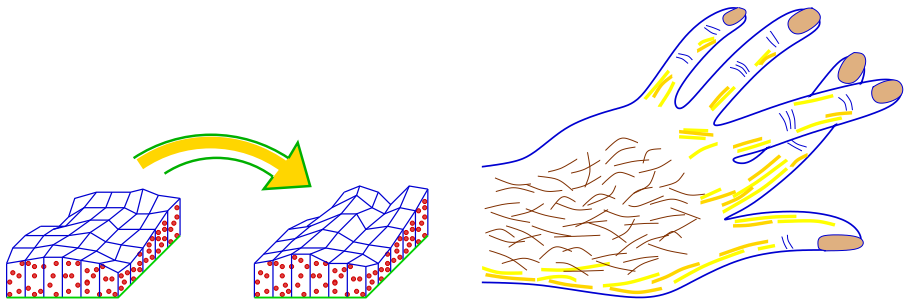

Figure 10.7: State space discretization approach to computing averages. (A. Johansen)

where $\chi_i(x)$ is the characteristic function (9.1) of the set $\mathcal{M}_i$. The density ρ_i at a given instant is related to the densities at the previous step in time by the action of the Perron-Frobenius operator, as in (9.6):

recheck, missing f' jacobian here?

$$\begin{aligned}\rho'_j &= \int_{\mathcal{M}} dy\, \chi_j(y)\rho'(y) = \int_{\mathcal{M}} dx\, dy\, \chi_j(y)\,\delta(y-f(x))\,\rho(x) \\ &= \sum_{i=1}^{N} \rho_i \frac{|\mathcal{M}_i \cap f^{-1}(\mathcal{M}_j)|}{|\mathcal{M}_i|}\,.\end{aligned}$$

In this way

$$\mathbf{L}_{ij} = \frac{|\mathcal{M}_i \cap f^{-1}(\mathcal{M}_j)|}{|\mathcal{M}_i|}\,, \quad \rho' = \rho\mathbf{L} \tag{10.37}$$

is a matrix approximation to the Perron-Frobenius operator, and its leading left eigenvector is a piecewise constant approximation to the invariant measure. It is an old idea of Ulam that such an approximation for the Perron-Frobenius operator is a meaningful one.

⇓PRELIMINARY

replace figure 10.6 by figure 10.7

⇑PRELIMINARY

The problem with such state space discretization approaches is that they are blind, the grid knows not what parts of the state space are more

pressure!thermodynamic
thermodynamical!pressure
partition function
pressure
convexity
Perron-Frobenius!operator
operator!Perron-Frobenius
Ruelle!-Pollicott resonances
resonances!Ruelle-Pollicott
Pollicott, M
Ruelle, D

☞ chapter 16

or less important. This observation motivates the next step in developing the theory of long-time dynamics of chaotic systems: in chapter 11 we shall exploit the intrinsic topology of the flow to give us both an invariant partition of the state space and a measure of the partition volumes, in the spirit of figure 1.10.

Furthermore, a piecewise constant ρ belongs to an unphysical function space, and with such approximations one is plagued by numerical artifacts such as spurious eigenvalues. In chapter 16 we shall employ a more refined approach to extracting spectra, by expanding the initial and final densities ρ, ρ' in some basis φ_0, φ_1, $\varphi_2, \cdots$ (orthogonal polynomials, let us say), and replacing $\mathcal{L}(y,x)$ by its φ_α basis representation $\mathbf{L}_{\alpha\beta} = \langle\varphi_\alpha|\mathcal{L}|\varphi_\beta\rangle$. The art is then the subtle art of finding a "good" basis for which finite truncations of $\mathbf{L}_{\alpha\beta}$ give accurate estimates of the eigenvalues of $\mathcal{L}$.

Regardless of how sophisticated the choice of basis might be, the basic problem cannot be avoided - as illustrated by the natural measure for the Hénon map (3.15) sketched in figure 9.3, eigenfunctions of $\mathcal{L}$ are complicated, singular functions concentrated on fractal sets, and in general cannot be represented by a nice basis set of smooth functions. We shall resort to matrix representations of $\mathcal{L}$ and the φ_α basis approach only insofar this helps us prove that the spectrum that we compute is indeed the correct one, and that finite periodic orbit truncations do converge.

in depth:
chapter 1, p. 1

Commentary

Remark 10.1 "Pressure". The quantity $\langle\exp(\beta\cdot A^t)\rangle$ is called a "partition function" by Ruelle [15.1]. Mathematicians decorate it with considerably more Greek and Gothic letters than is the case in this treatise. By a somewhat unfortuitous analogy either Ruelle [15.2] or Bowen [10.1] had given name "pressure" $P(a)$ to $s(\beta)$ (where a is the observable introduced here in sect. 10.1.1), defined by the "large system" limit (10.11). As we shall apply the theory also to computation of the physical gas pressure exerted on the walls of a container by a bouncing particle(in sections still unwritten), we prefer to refer to $s(\beta)$ as simply the leading eigenvalue of the evolution operator introduced in sect. 9.4. The "convexity" properties such as $P(a) \leq P(|a|)$ will be pretty obvious consequence of the definition (10.11). In the case that $\mathcal{L}$ is the Perron-Frobenius operator (9.10), the eigenvalues $\{s_0(\beta), s_1(\beta), \cdots\}$ are called the *Ruelle-Pollicott resonances* [10.2, 10.3, 10.4], with the leading one, $s(\beta) = s_0(\beta)$ being the one of main physical interest. In order to aid the reader in digesting the mathematics literature, we shall try to point out the notational correspondences whenever appropriate. The rigorous formalism is replete with lims, sups, infs, Ω-sets which are not really essential to understanding the physical applications of the theory, and are avoided in this presentation.

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add someplace $h(\sigma) + \sigma(a) \leq P(a)$

remark also that Kolimolgerov and typological "entropres" are not entrogies, but entropy production rates

Remark 10.2 Microcanonical ensemble. In statistical mechanics the space average (10.7) performed over the Hamiltonian system constant energy surface invariant measure $\rho(x)dx = dqdp\,\delta(H(q,p)-E)$ of volume $|\mathcal{M}| = \int_{\mathcal{M}} dqdp\,\delta(H(q,p)-E)$

$$\langle a(t)\rangle = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} dqdp\,\delta(H(q,p)-E)a(q,p,t) \tag{10.38}$$

is called the *microcanonical ensemble average*.

microcanonical ensemble
ensemble!microcanonical
multiplicative ergodic theorem
ergodic!theorem!multiplicative
Oseledec multiplicative ergodic theorem
Lyapunov exponent!numerical

Remark 10.3 Lyapunov exponents. The Multiplicative Ergodic Theorem of Oseledec [10.5, 10.6] states that the limit (10.32) exists for almost all points x_0 and all tangent vectors $\hat{n}$. There are at most d distinct values of λ as we let $\hat{n}$ range over the tangent space. These are the Lyapunov exponents [10.7] $\lambda_i(x_0)$.

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There is much literature on numerical computation of the Lyapunov exponents, see for example refs. [4.4, 10.13, 10.14, 10.15].

Résumé

The expectation value $\langle a\rangle$ of an observable $a(x)$ measured $A^t(x) = \int_0^t d\tau a(x(\tau))$ and averaged along the flow $x \to f^t(x)$ is given by the derivative $\partial s/\partial\beta$ of the leading eigenvalue $e^{ts(\beta)}$ of the corresponding evolution operator $\mathcal{L}^t$.

Using the Perron-Frobenius operator (9.10) whose leading eigenfunction, the natural measure, once computed, yields expectation value (9.19) of any observable $a(x)$ a separate evolution operator $\mathbf{L}$ has to be constructed for each and every observable. However, by the time the scaffolding is removed both $\mathcal{L}$'s and their eigenfunctions will be gone, and only the formulas for expectation value of observables will remain.

☞ chapter 18

The next question is: how do we evaluate the eigenvalues of $\mathcal{L}$? We saw in example 10.1, in the case of piecewise-linear dynamical systems, that these operators reduce to finite matrices, but for generic smooth flows, they are infinite-dimensional linear operators, and finding smart ways of computing their eigenvalues requires some thought. In chapter 11 we take the first step, and replace the *ad hoc* partitioning (10.37) by the intrinsic, topologically invariant partitioning. In chapter 13 we apply this information to our first application of the evolution operator formalism, evaluation of the topological entropy, the growth rate of the number of topologically distinct orbits. This small victory will then be refashioned in chapters 14 and 15 into a systematic method for computing eigenvalues of evolution operators in terms of periodic orbits.

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place sensible references here

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⇓PRELIMINARY

find Pollicot book ref - for Oseledec theory

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⇑PRELIMINARY

complete ref. [10.7]

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Exercises

Henon@Hénon!attractor!Lyapunov exponent
expectation value
kurtosis

Exercise 10.1 **How unstable is the Hénon attractor?**

(a) *Evaluate numerically the Lyapunov exponent λ by iterating the Hénon map*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 - ax^2 + y \\ bx \end{bmatrix}$$

for $a = 1.4$, $b = 0.3$.

(b) *Now check how robust is the Lyapunov exponent for the Hénon attractor? Evaluate numerically the Lyapunov exponent by iterating the Hénon map for $a = 1.39945219$, $b = 0.3$. How much do you trust now your result for the part (a) of this exercise?*

Exercise 10.2 **Expectation value of a vector observable and its moments.** *Check and extend the expectation value formulas (10.12) by evaluating the derivatives of $s(\beta)$ up to 4-th order for the space average $\langle \exp(\beta \cdot A^t) \rangle$ with a_i a vector quantity:*

(a)

$$\left.\frac{\partial s}{\partial \beta_i}\right|_{\beta=0} = \lim_{t\to\infty} \frac{1}{t} \langle A_i^t \rangle = \langle a_i \rangle \ , \tag{10.39}$$

(b)

$$\begin{aligned} \left.\frac{\partial^2 s}{\partial \beta_i \partial \beta_j}\right|_{\beta=0} &= \lim_{t\to\infty} \frac{1}{t} \left(\langle A_i^t A_j^t \rangle - \langle A_i^t \rangle \langle A_j^t \rangle \right) \\ &= \lim_{t\to\infty} \frac{1}{t} \left\langle (A_i^t - t\langle a_i \rangle)(A_j^t - t\langle a_j \rangle) \right\rangle \ . \end{aligned} \tag{10.40}$$

Note that the formalism is cmart: it automatically yields the variance *from the mean, rather than simply the 2nd moment $\langle a^2 \rangle$.*

(c) *compute the third derivative of $s(\beta)$.*

(d) *compute the fourth derivative assuming that the mean in (10.39) vanishes, $\langle a_i \rangle = 0$. The 4-th order moment formula*

$$K(t) = \frac{\langle x^4(t) \rangle}{\langle x^2(t) \rangle^2} - 3 \tag{10.41}$$

that you have derived is known as kurtosis*: it measures a deviation from what the 4-th order moment would be were the distribution a pure Gaussian (see (24.22) for a concrete example). If the observable is a vector, the kurtosis is given by*

$$K(t) = \frac{\sum_{ij} \left[\langle A_i A_i A_j A_j \rangle + 2 \left(\langle A_i A_j \rangle \langle A_j A_i \rangle - \langle A_i A_i \rangle \langle A_j A_j \rangle \right) \right]}{\left(\sum_i \langle A_i A_i \rangle \right)^2} \tag{10.42}$$

Rossler@Rösseler!attractor!Lyapunov exponent
three-disk@3-disk!escape rate

Exercise 10.3 Rösseler attractor Lyapunov exponent.

(a) *Evaluate numerically the Lyapunov exponent λ of the Rösseler attractor (??) for $a =?$, $b =?$.*

(b) *plot your own version of Fig 10.4.*

(c) *give your best estimate of λ*

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Exercise 10.4 Pinball escape rate from numerical simulation*. *Estimate the escape rate for $R : a = 6$ 3-disk pinball by shooting 100,000 randomly initiated pinballs into the 3-disk system and plotting the logarithm of the number of trapped orbits as function of time. For comparison, a numerical simulation of ref. [6.2] yields $\gamma = .410\ldots$.*

put here $\langle n \rangle$ numbers from Per.

list numerical results for different R.

Chapter 11

Qualitative dynamics, for pedestrians

> The classification of the constituents of a chaos, nothing less is here essayed.
> Herman Melville, *Moby Dick*, chapter 32

In this chapter we begin to learn how to use qualitative properties of a flow in order to *partition* the state space in a topologically invariant way, and *name* topologically distinct orbits. This will enable us – in chapter 13 – to *count* the distinct orbits, and in the process touch upon all the main themes of this book, going the whole distance from diagnosing chaotic dynamics to computing zeta functions.

We start by a simple physical example, symbolic dynamics of a 3-disk game of pinball, and then show that also for smooth flows the qualitative dynamics of stretching and folding flows enables us to partition the state space and assign symbolic dynamics itineraries to trajectories. Here we illustrate the method on a $1-d$ approximation to Rössler flow. In chapter 13 we turn this topological dynamics into a multiplicative operation on the state space partitions by means of transition matrices/Markov graphs, the simplest examples of evolution operators. Deceptively simple, this subject can get very difficult very quickly, so in this chapter we do the first pass, at a pedestrian level, postponing the discussion of higher-dimensional, cyclist level issues to chapter 12.

Even though by inclination you might only care about the serious stuff, like Rydberg atoms or mesoscopic devices, and resent wasting time on things formal, this chapter and chapter 13 are good for you. Read them.

11.1 Qualitative dynamics

(R. Mainieri and P. Cvitanović)

recurrence
symbolic dynamics
dynamics!symbolic
topological!dynamics
dynamics!topological
obscure!jargon
state
partition
alphabet
state!set
itinerary

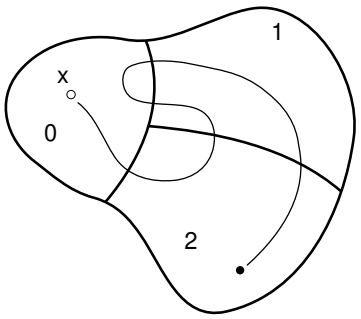

Figure 11.1: A trajectory with itinerary 021012.

What can a flow do to the state space points? This is a very difficult question to answer because we have assumed very little about the evolution function f^t; continuity, and differentiability a sufficient number of times. Trying to make sense of this question is one of the basic concerns in the study of dynamical systems. One of the first answers was inspired by the motion of the planets: they appear to repeat their motion through the firmament. Motivated by this observation, the first attempts to describe dynamical systems were to think of them as periodic.

However, periodicity is almost never quite exact. What one tends to observe is *recurrence*. A recurrence of a point x_0 of a dynamical system is a return of that point to a neighborhood of where it started. How close the point x_0 must return is up to us: we can choose a volume of any size and shape, and call it the neighborhood $\mathcal{M}_0$, as long as it encloses x_0. For chaotic dynamical systems, the evolution might bring the point back to the starting neighborhood infinitely often. That is, the set

made into a problem set

$$\{y \in \mathcal{M}_0 : \quad y = f^t(x_0), \quad t > t_0\} \tag{11.1}$$

will in general have an infinity of recurrent episodes.

To observe a recurrence we must look at neighborhoods of points. This suggests another way of describing how points move in state space, which turns out to be the important first step on the way to a theory of dynamical systems: qualitative, topological dynamics, or, as it is usually called, *symbolic dynamics*. As the subject can get quite technical, a summary of the basic notions and definitions of symbolic dynamics is relegated to sect. 11.6; check that section whenever you run into obscure symbolic dynamics jargon.

We start by cutting up the state space up into regions $\mathcal{M}_A, \mathcal{M}_B, \ldots, \mathcal{M}_Z$. This can be done in many ways, not all equally clever. Any such division of the state space into topologically distinct regions is a *partition*, and we associate with each region (sometimes referred to as a *state*) a symbol s from an N-letter *alphabet* or *state set* $\mathcal{A} = \{A, B, C, \cdots, Z\}$. As the dynamics moves the point through the state space, different regions will be visited. The visitation sequence - forthwith referred to as the *itinerary* - can be represented by the letters of the alphabet $\mathcal{A}$. If, as in the example sketched in figure 11.1, the state space is divided into three regions $\mathcal{M}_0$,

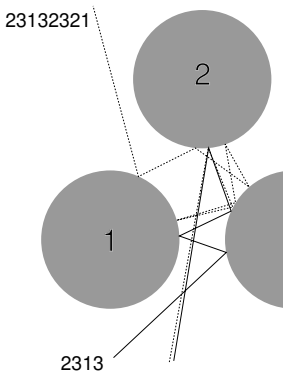

Figure 11.2: Two pinballs that start out very close to each other exhibit the same qualitative dynamics _2313_ for the first three bounces, but due to the exponentially growing separation of trajectories with time, follow different itineraries thereafter: one escapes after _2313_, the other one escapes after _23132321_

metric!indecomposability
indecomposability!metric
transient
transition matrix
complete!N-ary dynamics
bolic!dynamics!complete ary
-disk@3- isk!transition matrix
transition matrix!3-disk

$\mathcal{M}_1$, and $\mathcal{M}_2$, the "letters" are the integers $\{0,1,2\}$, and the itinerary for the trajectory sketched in the figure is $0 \mapsto 2 \mapsto 1 \mapsto 0 \mapsto 1 \mapsto 2 \mapsto \cdots$.

If there is no way to reach partition $\mathcal{M}_i$ from partition $\mathcal{M}_j$, and conversely, partition $\mathcal{M}_j$ from partition $\mathcal{M}_i$, the state space consists of at least two disconnected pieces, and we can analyze it piece by piece. An interesting partition should be dynamically connected, i.e., one should be able to go from any region $\mathcal{M}_i$ to any other region $\mathcal{M}_j$ in a finite number of steps. A dynamical system with such partition is said to be *metrically indecomposable*.

insert a figure of indecomposable, transient partitions

In general one also encounters transient regions - regions to which the dynamics does not return to once they are exited. Hence we have to distinguish between (for us uninteresting) wandering trajectories that never return to the initial neighborhood, and the non–wandering set (2.2) of the *recurrent* trajectories.

recover "generating"

The allowed transitions between the regions of a partition are encoded in the $[N\times N]$-dimensional *transition matrix* whose elements take values

$$T_{ij} \;=\; \begin{cases} 1 & \text{if a transition } \mathcal{M}_j \to \mathcal{M}_i \text{ is possible} \\ 0 & \text{otherwise}\,. \end{cases} \tag{11.2}$$

The transition matrix encodes the topological dynamics as an invariant law of motion, with the allowed transitions at any instant independent of the trajectory history, requiring no memory.

Example 11.1 Complete N-ary dynamics: *All transition matrix entries equal unity (one can reach any region from any other region in one step):*

$$T_c = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} . \tag{11.3}$$

Further examples of transition matrices, such as the 3-disk transition matrix (11.5) and the 1-step memory sparse matrix (11.16), are peppered throughout the text.

map!expanding
Markov!partition!finite
partition!Markov
three-disk@3-disk!symbolic dynamics
symbolic dynamics!3-disk
three-disk@3-disk!prime cycles
prime cycle!3-disk

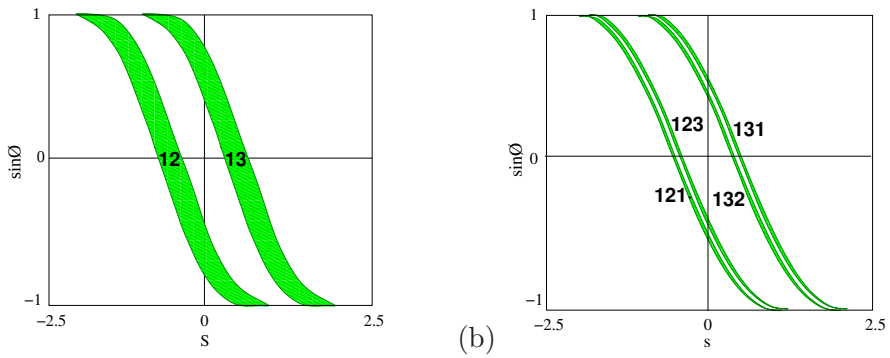

Figure 11.3: The 3-disk game of pinball Poincaré section, trajectories emanating from the disk 1 with x_0 = (arclength, parallel momentum) = (s_0, p_0) , disk radius : center separation ratio a:R = 1:2.5. (a) Strips of initial points $\mathcal{M}_{12}$, $\mathcal{M}_{13}$ which reach disks 2, 3 in one bounce, respectively. (b) Strips of initial points $\mathcal{M}_{121}$, $\mathcal{M}_{131}$ $\mathcal{M}_{132}$ and $\mathcal{M}_{123}$ which reach disks 1, 2, 3 in two bounces, respectively. (Y. Lan)

need a se definition transversality, hyperbolicity ins of this

However, knowing that a point from $\mathcal{M}_i$ reaches $\mathcal{M}_j$ in one step is not quite good enough. We would be happier if we knew that *any* point in $\mathcal{M}_i$ reaches $\mathcal{M}_j$; otherwise we have to subpartition $\mathcal{M}_i$ into the points which land in $\mathcal{M}_j$, and those which do not, and often we will find ourselves partitioning *ad infinitum*.

Such considerations motivate the notion of a *Markov partition*, a partition for which no memory of preceding steps is required to fix the transitions allowed in the next step. Dynamically, *finite Markov partitions* can be generated by *expanding* d-dimensional iterated mappings $f : \mathcal{M} \to \mathcal{M}$, if $\mathcal{M}$ can be divided into N regions $\{\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_{N-1}\}$ such that in one step points from an initial region $\mathcal{M}_i$ either fully cover a region $\mathcal{M}_j$, or miss it altogether,

$$\text{either} \quad \mathcal{M}_j \cap f(\mathcal{M}_i) = \emptyset \quad \text{or} \quad \mathcal{M}_j \subset f(\mathcal{M}_i)\,. \tag{11.4}$$

Let us illustrate what this means by our favorite example, the game of pinball.

Example 11.2 3-disk symbolic dynamics: *Consider the motion of a free point particle in a plane with* 3 *elastically reflecting convex disks. After a collision with a disk a particle either continues to another disk or escapes, and any trajectory can be labeled by the disk sequence. For example, if we label the three disks by* 1, 2 *and* 3, *the two trajectories in figure 11.2 have itineraries* _2313_, _23132321_ *respectively. The 3-disk prime cycles given in figures 1.6 and 11.6 are further examples of such itineraries.*

At each bounce a cone of initially nearby trajectories defocuses (see figure 1.8), and in order to attain a desired longer and longer itinerary of bounces the initial point $x_0 = (s_0, p_0)$ *has to be specified with a larger and larger precision, and lie within initial state space strips drawn in figure 11.3. Similarly, it is intuitively clear that as we go backward in time (in this case, simply reverse the velocity vector), we also*

1.1
page 32

fix figure 11.2 - itineraries do not match

recheck whether figure 11.3 was computed at a:R = 1:2.5

itinerary!bi-infinite
pruning!rules
subshift!finite type

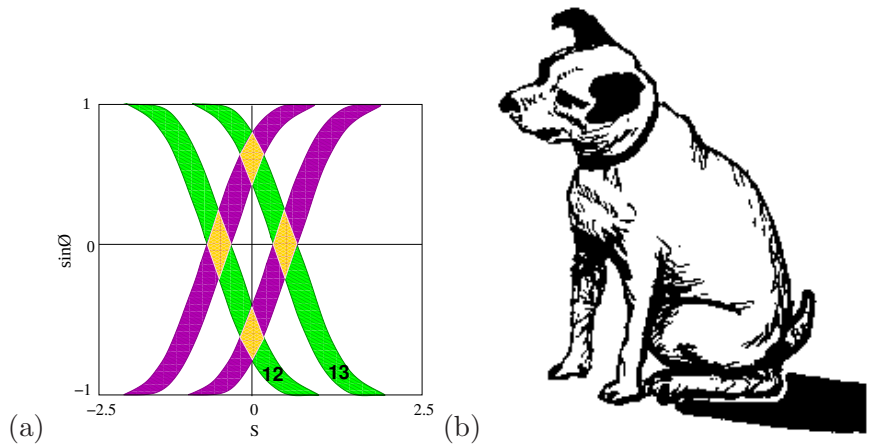

Figure 11.4: The Poincaré section of the state space for the binary labeled pinball. For definitiveness, this set is generated by starting from disk 1, preceded by disk 2. Indicated are the fixed points $\overline{0}$, $\overline{1}$ and the 2-cycle periodic points $\overline{01}$, $\overline{10}$, together with strips which survive 1, 2, ... bounces. Iteration corresponds to the decimal point shift; for example, all points in the rectangle $[01.01]$ map into the rectangle $[010.1]$ in one iteration. See also figure 11.6 (b).

need increasingly precise specification of $x_0 = (s_0, p_0)$ in order to follow a given past itinerary. Another way to look at the survivors after two bounces is to plot $\mathcal{M}_{s_1.s_2}$, the intersection of $\mathcal{M}_{.s_2}$ with the strips $\mathcal{M}_{s_1.}$ obtained by time reversal (the velocity changes sign $\sin\phi \to -\sin\phi$). $\mathcal{M}_{s_1.s_2}$, figure 11.4, is a "rectangle" of nearby trajectories which have arrived from the disk s_1 and are heading for the disk s_2.

We see that a finite length trajectory is not uniquely specified by its finite itinerary, but an isolated unstable cycle is: its itinerary is an infinitely repeating block of symbols. More generally, for hyperbolic flows the intersection of the future and past itineraries, the bi-infinite itinerary $S^-.S^+ = \cdots s_{-2}s_{-1}s_0.s_1s_2s_3\cdots$ specifies a unique trajectory. This is intuitively clear for our 3-disk game of pinball, and is stated more formally in the definition (11.4) of a Markov partition. The definition requires that the dynamics be expanding forward in time in order to ensure that the cone of trajectories with a given itinerary becomes sharper and sharper as the number of specified symbols is increased.

⇓PRIVATE
⇑PRIVATE
in figure 11.4 right, with a blowup and binary

Example 11.3 Pruning rules for a 3-disk alphabet: *As the disks are convex, there can be no two consecutive reflections off the same disk, hence the covering symbolic dynamics consists of all sequences which include no symbol repetitions 11, 22, 33. This is a finite set of finite length* pruning rules, *hence, the dynamics is a subshift of finite type (see (11.25) for definition), with the transition matrix (11.2) given by*

$$T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} . \qquad (11.5)$$

For convex disks the separation between nearby trajectories increases at every reflection, implying that the stability matrix has an expanding eigenvalue. By the Liouville phase space volume conservation (5.23), the other transverse eigenvalue is contracting. This example demonstrates that finite Markov partitions can be constructed for hyperbolic

Markov!partition!finite
dynamics!hyperbolic
symmetry!discrete
Markov!partition!not unique
symbolic dynamics!recoding
recoding

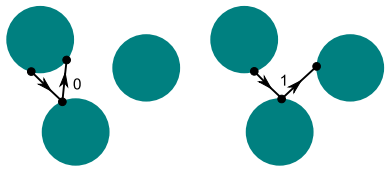

Figure 11.5: Binary labeling of the 3-disk pinball trajectories; a bounce in which the trajectory returns to the preceding disk is labeled 0, and a bounce which results in continuation to the third disk is labeled 1.

dynamical systems which are expanding in some directions, contracting in others. Further examples are the 1*-dimensional expanding mapping sketched in figure 11.8, and more examples are worked out in sect. 24.2.*

Determining whether the symbolic dynamics is complete (as is the case for sufficiently separated disks), pruned (for example, for touching or overlapping disks), or only a first coarse graining of the topology (as, for example, for smooth potentials with islands of stability) requires case-by-case investigation, a discussion we postpone to sect. 11.4 and chapter 12. For the time being we assume that the disks are sufficiently separated that there is no additional pruning beyond the prohibition of self-bounces.

fast track:
sect. 11.3, p. 178

11.2 A brief detour; recoding, symmetries, tilings

Though a useful tool, Markov partitioning is not without drawbacks. One glaring shortcoming is that Markov partitions are not unique: any of many different partitions might do the job. The 3-disk system offers a simple illustration of different Markov partitioning strategies for the same dynamical system.

The $\mathcal{A} = \{1, 2, 3\}$ symbolic dynamics for 3-disk system is neither unique, nor necessarily the smartest one - before proceeding it pays to exploit the symmetries of the pinball in order to obtain a more efficient description. In chapter 22 we shall be handsomely rewarded for our labors.

As the three disks are equidistantly spaced, our game of pinball has a sixfold symmetry. For instance, the cycles $\overline{12}$, $\overline{23}$, and $\overline{13}$ are related to each other by rotation by $\pm 2\pi/3$ or, equivalently, by a relabeling of the disks. Further examples of such symmetries are shown in figure 1.6. The disk labels are arbitrary; what is important is how a trajectory evolves as it hits subsequent disks, not what label the starting disk had. We exploit this symmetry by *recoding*, in this case replacing the absolute disk labels by relative symbols, indicating the type of the collision. For the 3-disk game of pinball there are two topologically distinct kinds of collisions, figure 11.5:

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11.2 page 195

$$s_i = \begin{cases} 0 & : \quad \text{pinball returns to the disk it came from} \\ 1 & : \quad \text{pinball continues to the third disk}\,. \end{cases} \tag{11.6}$$

n_p	p
1	0
	1
2	01
3	001
	011
4	0001
	0011
	0111
5	00001
	00011
	00101
	00111
	01011
	01111
6	000001
	000011
	000101
	000111
	001011
	001101
	001111
	010111
	011111
7	0000001
	0000011
	0000101

n_p	p
7	0001001
	0000111
	0001011
	0001101
	0010011
	0010101
	0001111
	0010111
	0011011
	0011101
	0101011
	0011111
	0101111
	0110111
	0111111
8	00000001
	00000011
	00000101
	00001001
	00000111
	00001011
	00001101
	00010011
	00010101
	00011001
	00100101

n_p	p
8	00001111
	00010111
	00011011
	00011101
	00100111
	00101011
	00101101
	00110101
	00011111
	00101111
	00110111
	00111011
	00111101
	01010111
	01011011
	00111111
	01011111
	01101111
	01111111
9	000000001
	000000011
	000000101
	000001001
	000010001
	000000111
	000001011

n_p	p
9	000001101
	000010011
	000010101
	000011001
	000100011
	000100101
	000101001
	000001111
	000010111
	000011011
	000011101
	000100111
	000101011
	000101101
	000110011
	000110101
	000111001
	001001011
	001001101
	001010011
	001010101
	000011111
	000101111
	000110111
	000111011
	000111101

n_p	p
9	001011111
	001101111
	001110111
	001111011
	001111101
	010101111
	010110111
	010111011
	001111111
	010111111
	011011111
	011101111
	011111111

prime cycle!binary
binary!prime cycles
prime cycle!ternary
ternary!prime cycles

Table 11.1: Prime cycles for the binary symbolic dynamics up to length 9.

This *binary* symbolic dynamics has two immediate advantages over the ternary one; the prohibition of self-bounces is automatic, and the coding utilizes the symmetry of the 3-disk pinball game in elegant manner. If the disks are sufficiently far apart there are no further restrictions on symbols, the symbolic dynamics is complete, and *all* binary sequences are admissible itineraries. As this type of symbolic dynamics pops up frequently, we list the shortest binary prime cycles in table 11.1.

11.3
page 195

put some text on the same page as table 11.1

Example 11.4 Recoding ternary symbolic dynamics in binary: *Given a ternary sequence and labels of 2 preceding disks, rule (11.6) fixes the subsequent binary symbols. Here we list an arbitrary ternanry itinerary, and the corresponding binary sequence:*

$$\begin{array}{rcccccccccccccccc} \textit{ternary} & : & 3 & 1 & 2 & 1 & 3 & 1 & 2 & 3 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 \\ \textit{binary} & : & \cdot & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & \end{array} \tag{11.7}$$

The first 2 disks initialize the trajectory and its direction; $3 \mapsto 1 \mapsto 2 \mapsto \cdots$. *Due to the 3-disk symmetry the six 3-disk sequences initialized by 12, 13, 21, 23, 31, 32 respectively have the same weights, the same size partitions, and are coded by a single binary sequence. For periodic orbits, the equivalent ternary cycles reduce to binary cycles of 1/3, 1/2 or the same length. How this works is best understood by inspection of table 11.2, figure 11.6 and figure 22.3.*

The 3-disk game of pinball is tiled by six copies of the *fundamental domain*, a one-sixth slice of the full 3-disk system, with the symmetry

fundamental!domain
three-disk@3-disk!symmetry
symmetry!3-disk

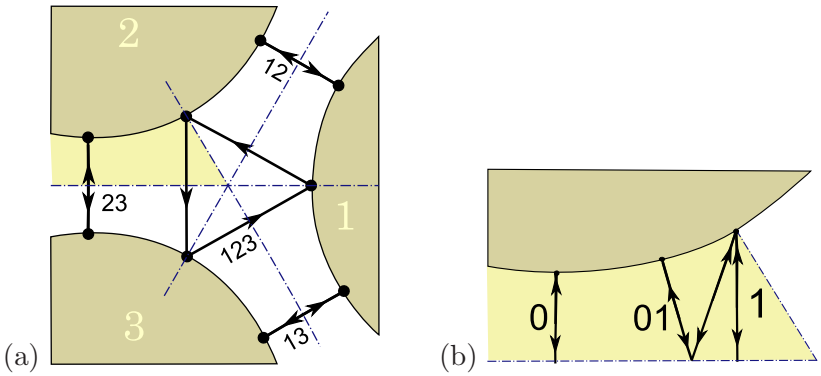

Figure 11.6: The 3-disk game of pinball with the disk radius : center separation ratio a:R = 1:2.5. (a) 2-cycles $\overline{12}$, $\overline{13}$, $\overline{23}$, and 3-cycles $\overline{123}$ and $\overline{132}$ (not drawn). (b) The fundamental domain, i.e., the small 1/6th wedge indicated in (a), consisting of a section of a disk, two segments of symmetry axes acting as straight mirror walls, and an escape gap. The above five cycles restricted to the fundamental domain are the two fixed points $\overline{0}$, $\overline{1}$. See figure 22.4 for cycle $\overline{10}$ and further examples.

axes acting as reflecting mirrors, see figure 11.6 (b). Every global 3-disk trajectory has a corresponding fundamental domain mirror trajectory obtained by replacing every crossing of a symmetry axis by a reflection.

Depending on the symmetry of the global trajectory, a repeating binary symbols block corresponds either to the full periodic orbit or to an irreducible segment (examples are shown in figure 11.6 and table 11.2). An irreducible segment corresponds to a periodic orbit in the fundamental domain. Table 11.2 lists some of the shortest binary periodic orbits, together with the corresponding full 3-disk symbol sequences and orbit symmetries. For a number of reasons that will be elucidated in chapter 22, life is much simpler in the fundamental domain than in the full system, so whenever possible our computations will be carried out in the fundamental domain.

recheck with Freddy!
11.5
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Inspecting the figure 11.3 we see that the relative ordering of regions with differing finite itineraries is a qualitative, topological property of the flow, so it makes sense to define a simple "canonical" representative partition which in a simple manner exhibits spatial ordering common to an entire class of topologically similar nonlinear flows.

in depth:
chapter 22, p. 427

11.3 Stretch and fold

rewrite this

Symbolic dynamics for N-disk game of pinball is so straightforward that one may altogether fail to see the connection between the topology of hyperbolic flows and their symbolic dynamics. This is brought out more clearly by the 1-dimensional visualization of "stretch & fold" flows to which we turn now.

$\tilde{p}$	p	$\mathbf{g}_{\tilde{p}}$
0	1 2	σ_{12}
1	1 2 3	C_3
01	12 13	σ_{23}
001	121 232 313	C_3
011	121 323	σ_{13}
0001	1212 1313	σ_{23}
0011	1212 3131 2323	C_3^2
0111	1213 2123	σ_{12}
00001	12121 23232 31313	C_3
00011	12121 32323	σ_{13}
00101	12123 21213	σ_{12}
00111	12123	e
01011	12131 23212 31323	C_3
01111	12132 13123	σ_{23}

$\tilde{p}$	p	$\mathbf{g}_{\tilde{p}}$
000001	121212 131313	σ_{23}
000011	121212 313131 232323	C_3^2
000101	121213	e
000111	121213 212123	σ_{12}
001011	121232 131323	σ_{23}
001101	121231 323213	σ_{13}
001111	121231 232312 313123	C_3
010111	121312 313231 232123	C_3^2
011111	121321 323123	σ_{13}
0000001	1212121 2323232 3131313	C_3
0000011	1212121 3232323	σ_{13}
0000101	1212123 2121213	σ_{12}
0000111	1212123	e
...	...	...

Table 11.2: C_{3v} correspondence between the binary labeled fundamental domain prime cycles $\tilde{p}$ and the full 3-disk ternary labeled cycles p, together with the C_{3v} transformation that maps the end point of the $\tilde{p}$ cycle into the irreducible segment of the p cycle, see sect. 22.2.2. Breaks in the ternary sequences mark repeats of the irreducible segment. The degeneracy of p cycle is $m_p = 6n_{\tilde{p}}/n_p$. The shortest pair of the fundamental domain cycles related by time reversal (but no spatial symmetry) are the 6-cycles $\overline{001011}$ and $\overline{001101}$.

Suppose concentrations of certain chemical reactants worry you, or the variations in the Chicago temperature, humidity, pressure and winds affect your mood. All such properties vary within some fixed range, and so do their rates of change. Even if we are studying an open system such as the 3-disk pinball game, we tend to be interested in a finite region around the disks and ignore the escapees. So a typical dynamical system that we care about is *bounded*. If the price for keeping going is high - for example, we try to stir up some tar, and observe it come to a dead stop the moment we cease our labors - the dynamics tends to settle into a simple limiting state. However, as the resistance to change decreases - the tar is heated up and we are more vigorous in our stirring - the dynamics becomes unstable.

If a flow is locally unstable but globally bounded, any open ball of initial points will be stretched out and then folded back.

At this juncture we show how this works on the simplest example: unimodal mappings of the interval. The erudite reader should skim through this chapter and then take a more demanding path, via the Smale horseshoes of chapter 12. Unimodal maps are easier, but physically less motivated. The Smale horseshoes are the high road, more complicated, but the right tool to generalize what we learned from the 3-disk dynamics, and begin analysis of general dynamical systems. It is up to you - unimodal maps suffice to get quickly to the heart of this treatise.

11.3.1 Temporal ordering: itineraries

In this section we learn how to *name* (and, in chapter 13, how to *count*) periodic orbits for the simplest, and nevertheless very instructive case, for

Rossler@R“”ossler system
critical!point
unimodal map
map!unimodal
map!tent
map!quadratic

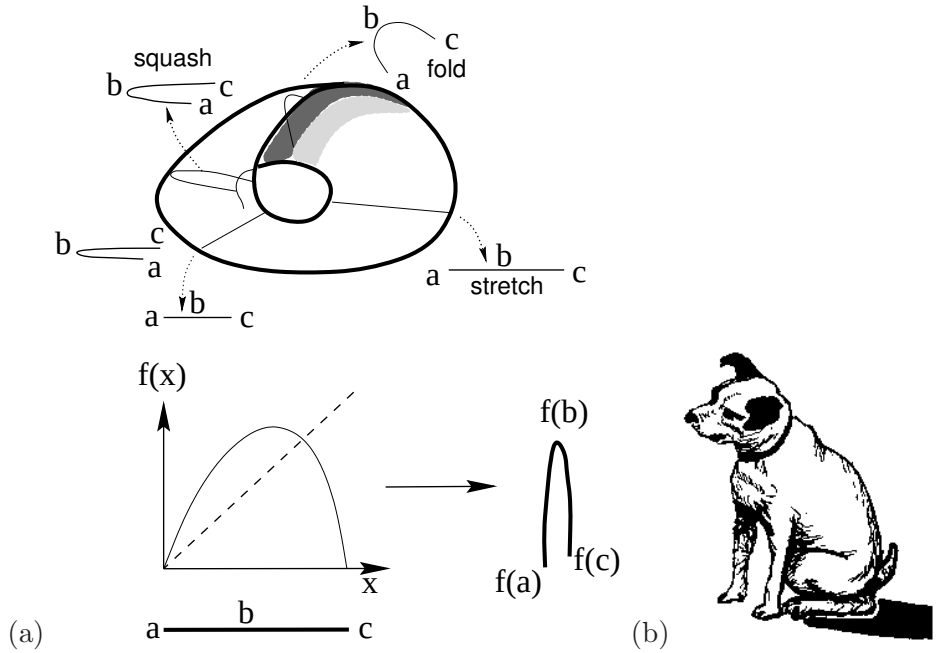

Figure 11.7: (a) A recurrent flow that stretches and folds. (b) The “stretch & fold” return map on the Poincaré section.

link to Rössler examples

1-dimensional maps of an interval.

Suppose that the compression of the folded interval in figure 11.7 is so fierce that we can neglect the thickness of the attractor. For example, the Rössler flow (2.14) is volume contracting, and an interval transverse to the attractor is stretched, folded and pressed back into a nearly 1-dimensional interval, typically compressed transversally by a factor of $\approx 10^{13}$ in one Poincaré section return. In such cases it makes sense to approximate the return map of a “stretch & fold” flow by a 1-dimensional map.

ask Freddy

The simplest mapping of this type is *unimodal;* interval is stretched and folded only once, with at most two points mapping into a point in the refolded interval. A unimodal map $f(x)$ is a 1-dimensional function $\mathbb{R} \to \mathbb{R}$ defined on an interval $\mathcal{M} \in \mathbb{R}$ with a monotonically increasing (or decreasing) branch, a *critical point* (or interval) x_c for which $f(x_c)$ attains the maximum (minimum) value, followed by a monotonically decreasing (increasing) branch. *Uni*-modal means that the map is a one-humped map with one critical point within interval $\mathcal{M}$. A *multi*-modal map has several critical points within interval $\mathcal{M}$.

draw figure 11.8a!

Example 11.5 Complete tent map, quadratic map: *The simplest examples of unimodal maps are the* complete tent map, *figure 11.8 (a),*

$$f(\gamma) = 1 - 2|\gamma - 1/2|\,, \tag{11.8}$$

and the quadratic map (sometimes also called the logistic map*)*

$$x_{t+1} = 1 - a x_t^2\,, \tag{11.9}$$

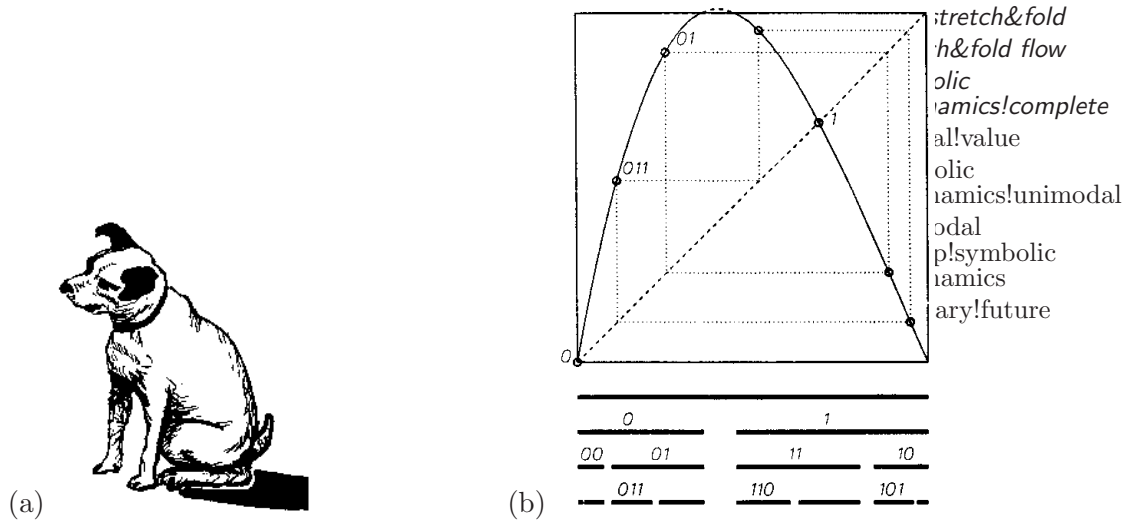

Figure 11.8: (a) The complete tent map together with intervals that follow the indicated itinerary for n steps. (b) A unimodal repeller with the remaining intervals after 1, 2 and 3 iterations. Intervals marked $s_1 s_2 \cdots s_n$ are unions of all points that do not escape in n iterations, and follow the itinerary $S^+ = s_1 s_2 \cdots s_n$. Note that the spatial ordering does not respect the binary ordering; for example $x_{00} < x_{01} < x_{11} < x_{10}$. Also indicated: the fixed points x_0, x_1, the 2-cycle $\overline{01}$, and the 3-cycle $\overline{011}$.

with the one critical point at $x_c = 0$. Further examples are the repelling unimodal map of figure 11.8 (b) and the piecewise linear map ??.

Such dynamical systems are irreversible (the inverse of f is double-valued), but, as we shall show in sect. 12.2, they may nevertheless serve as effective descriptions of invertible 2-dimensional hyperbolic flows.

need correct eq. ref. in figure 11.8

For the unimodal maps of figure 11.8 a Markov partition of the unit interval $\mathcal{M}$ is given by the two intervals $\{\mathcal{M}_0, \mathcal{M}_1\}$. We refer to (11.8) as the "complete" tent map because its symbolic dynamics is complete binary: as both $f(\mathcal{M}_0)$ and $f(\mathcal{M}_1)$ fully cover $\mathcal{M}_0$ and $\mathcal{M}_1$, the corresponding transition matrix is a [2×2] matrix with all entries equal to 1, as in (11.3). As binary symbolic dynamics pops up frequently in applications, we list the shortest binary prime cycles in table 11.1.

make table 11.1 columns narrower

11.3

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The *critical value* denotes either the maximum or the minimum value of $f(x)$ on the defining interval; we assume here that it is a maximum, $f(x_c) \geq f(x)$ for all $x \in \mathcal{M}$. The critical value $f(x_c)$ belongs neither to the left nor to the right partition $\mathcal{M}_i$, and is denoted by its own symbol $s = C$. As we shall see, its preimages serve as partition boundary points.

The trajectory $x_1, x_2, x_3, \ldots$ of the initial point x_0 is given by the iteration $x_{n+1} = f(x_n)$. Iterating f and checking whether the point lands to the left or to the right of x_c generates a *temporally* ordered topological itinerary (11.18) for a given trajectory,

bring (11.18) back into the text

$$s_n = \begin{cases} 1 & \text{if } x_n > x_c \\ 0 & \text{if } x_n < x_c \end{cases} . \tag{11.10}$$

We shall refer to $S^+(x_0) = .s_1 s_2 s_3 \cdots$ as the *future itinerary*. Our next task is to answer the reverse problem: given an itinerary, what is the corresponding *spatial* ordering of points that belong to a given trajectory?

Gatto
Nero!professor
Nero, G.
ordering!spatial
combinatorics!teaching
teaching!combinatorics

11.3.2 Spatial ordering, 1-d maps

> Tired of being harassed by your professors? Finish, get a job, do combinatorics your own way, while you still know everything.
>
> Professor Gatto Nero

Suppose you have succeeded in constructing a covering symbolic dynamics, such as for a well-separated 3-disk system. Now start moving the disks toward each other. At some critical separation a disk will start blocking families of trajectories traversing the other two disks. The order in which trajectories disappear is determined by their relative ordering in space; the ones closest to the intervening disk will be pruned first. Determining inadmissible itineraries requires that we relate the spatial ordering of trajectories to their time ordered itineraries.

12.9
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The easiest point of departure is to start out by working out this relation for the symbolic dynamics of 1-dimensional mappings. As it appears impossible to present this material without getting bogged down in a sea of 0's, 1's and subscripted subscripts, we announce the main result before embarking upon its derivation:

sect. 11.4

why doesn't fbox environment accept a line break?

The admissibility criterion eliminates *all* itineraries that cannot occur for a given unimodal map.

The tent map (11.8) consists of two straight segments joined at $x = 1/2$. The symbol s_n defined in (11.10) equals 0 if the function increases, and 1 if the function decreases. The piecewise linearity of the map makes it possible to analytically determine an initial point given its itinerary, a property that we now use to define a topological coordinatization common to all unimodal maps.

Here we have to face the fundamental problem of pedagogy: combinatorics cannot be taught. The best one can do is to state the answer, and then hope that you will figure it out by yourself. Once you figure it out, feel free to complain that the way the rule is stated here is incomprehensible, and shows us how you did it better.

Peles - this is confusing, motivate γ!

The tent map point $\gamma(S^+)$ with future itinerary S^+ is given by converting the sequence of s_n's into a binary number by the following algorithm:

$$
\begin{aligned}
w_{n+1} &= \begin{cases} w_n & \text{if } s_{n+1} = 0 \\ 1 - w_n & \text{if } s_{n+1} = 1 \end{cases}, \qquad w_1 = s_1 \\
\gamma(S^+) &= 0.w_1 w_2 w_3 \ldots = \sum_{n=1}^{\infty} w_n / 2^n . \qquad (11.11)
\end{aligned}
$$

This follows by inspection from the binary tree of figure 11.9.

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Example 11.6 Converting γ to S^+: *γ whose itinerary is $S^+ = 0110000\cdots$ is given by the binary number $\gamma = .010000\cdots$. Conversely, the itinerary of $\gamma = .01$ is $s_1 = 0$, $f(\gamma) = .1 \to s_2 = 1$, $f^2(\gamma) = f(.1) = 1 \to s_3 = 1$, etc..*

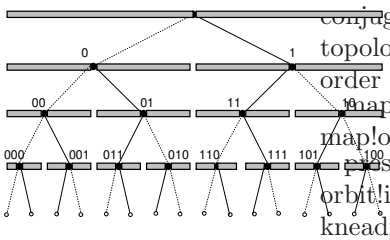


Figure 11.9: Alternating binary tree relates the itinerary labeling of the unimodal map figure 11.8 intervals to their spatial ordering. Dotted line stands for 0, full line for 1; the binary sub-tree whose root is a full line (symbol 1) reverses the orientation, due to the orientation reversing fold in figures 11.7 and 11.8.

topological!future coordinate
conjugacy!topological
topological!conjugacy
order preserving map
map!order preserving
orbit!inadmissible
kneading!theory
pruned!binary dynamics
symbolic!dynamics!pruned!binary
map!quadratic
show first Bernoulli map, show binary ordering - then show tent map and motivate alternating binary ordering

We shall refer to $\gamma(S^+)$ as the *(future) topological coordinate*. w_t's are the digits in the binary expansion of the starting point γ for the complete tent map (11.8). In the left half-interval the map $f(x)$ acts by multiplication by 2, while in the right half-interval the map acts as a flip as well as multiplication by 2, reversing the ordering, and generating in the process the sequence of s_n's from the binary digits w_n.

The mapping $x_0 \to S^+(x_0) \to \gamma_0 = \gamma(S^+)$ is a *topological conjugacy* which maps the trajectory of an initial point x_0 under iteration of a given unimodal map to that initial point γ for which the trajectory of the "canonical" unimodal map (11.8) has the same itinerary. The virtue of this conjugacy is that it *preserves the ordering* for any unimodal map in the sense that if $\overline{x} > x$, then $\overline{\gamma} > \gamma$.

11.4 Kneading theory

(K.T. Hansen and P. Cvitanović)

The main motivation for being mindful of spatial ordering of temporal itineraries is that this spatial ordering provides us with criteria that separate inadmissible orbits from those realizable by the dynamics. For 1-dimensional mappings the *kneading theory* provides such criterion of admissibility.

⇓PRIVATE

Example 11.7 Pruned binary dynamics: *Consider the 1-d quadratic map*

$$f(x) = Ax(1-x)\,, \qquad A = 3.8\,. \tag{11.12}$$

It is easy to check numerically that the kneading value (the itinerary of the critical point $x_c = 1/2$) is

$$K = 10110111101101111010111101111110\ldots,$$

see figure 11.10 (a). Corresponding itinerary for the dike map is illustrated by figure 11.10 (b).

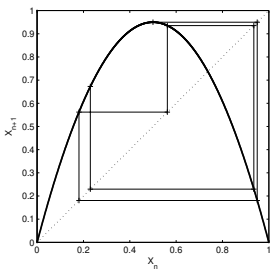
⇑PRIVATE

figure 11.10: Sharon's figure, and or Dane Freeman's dane.freeman[snail]asdl.gatech.edu version, especially dike map.

If the parameter in the quadratic map (11.9) is $a > 2$, then the iterates of the critical point x_c diverge for $n \to \infty$. As long as $a \geq 2$, any sequence S^+ composed of letters $s_i = \{0, 1\}$ is admissible, and any value of $0 \leq \gamma < 1$

kneading!sequence
kneading!value
dike map
map!dike

(a)



(b)

Figure 11.10: (a) The quadratic map $A = 3.8$ strange attractor partitioned by inverse iterates of the critical point, with the partition intervals after 1, 2 and 3 iterations indicated. Intervals marked $s_1 s_2 \cdots s_n$ are unions of all points that follow the itinerary $S^+ = s_1 s_2 \cdots s_n$. The forward itinerary of the critical point x_c, corresponding to *kneading sequence* $K = 1011011\ldots$, prunes blocks 100, 10?? not revisited by recurrent dynamics. Note that the spatial ordering does not respect the binary ordering; for example $x_{00} < x_{01} < x_{11} < x_{10}$. (b) The dike map together with intervals that follow the indicated itinerary for n steps. (continued in sect. 13.6)

corresponds to an admissible orbit in the non–wandering set of the map. The corresponding repeller is a complete binary labeled Cantor set, the $n \to \infty$ limit of the nth level covering intervals sketched in figure 11.8.

For $a < 2$ only a subset of the points in the interval $\gamma \in [0, 1]$ corresponds to admissible orbits. The forbidden symbolic values are determined by observing that the largest x_n value in an orbit $x_1 \to x_2 \to x_3 \to \ldots$ has to be smaller than or equal to the image of the critical point, *the critical value* $f(x_c)$. Let $K = S^+(x_c)$ be the itinerary of the critical point x_c, denoted the *kneading sequence* of the map. The corresponding topological coordinate is called the *kneading value*

$$\kappa = \gamma(K) = \gamma(S^+(x_c)). \tag{11.13}$$

A map with the same kneading sequence K as $f(x)$, such as the dike map figure 11.11, is obtained by slicing off all $\gamma\left(S^+(x_0)\right) > \kappa$,

$$f(\gamma) = \begin{cases} f_0(\gamma) = 2\gamma & \gamma \in I_0 = [0, \kappa/2) \\ f_c(\gamma) = \kappa & \gamma \in I_c = [\kappa/2, 1-\kappa/2] \\ f_1(\gamma) = 2(1-\gamma) & \gamma \in I_1 = [1-\kappa/2, 1] \end{cases} . \tag{11.14}$$

The dike map is the complete tent map figure 11.8 (a) with the top sliced off. It is convenient for coding the symbolic dynamics, as those γ values that survive the pruning are the same as for the complete tent map figure 11.8 (a), and are easily converted into admissible itineraries by (11.11).

If $\gamma(S^+) > \gamma(K)$, the point x whose itinerary is S^+ would exceed the critical value, $x > f(x_c)$, and hence cannot be an admissible orbit. Let

$$\hat{\gamma}(S^+) = \sup_m \gamma(\sigma^m(S^+)) \tag{11.15}$$

pruning!primary interval
topological!parameter

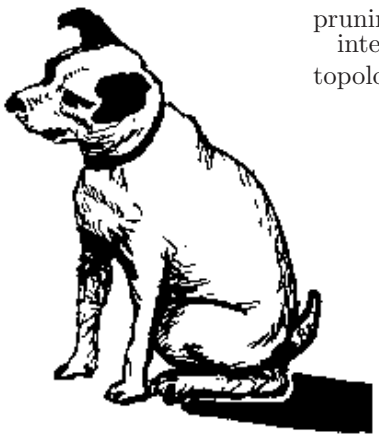


Figure 11.11: The "dike" map obtained by slicing of a top portion of the tent map figure 11.8 (a). Any orbit that visits the primary pruning interval $(\kappa, 1]$ is inadmissible. The admissible orbits form the Cantor set obtained by removing from the unit interval the primary pruning interval and all its iterates. Any admissible orbit has the same topological coordinate and itinerary as the corresponding tent map figure 11.8 (a) orbit.

be the *maximal value*, the highest topological coordinate reached by the orbit $x_1 \to x_2 \to x_3 \to \ldots$. We shall call the interval $(\kappa, 1]$ the *primary pruned interval*. The orbit S^+ is inadmissible if γ of any shifted sequence of S^+ falls into this interval.

Criterion of admissibility: *Let κ be the kneading value of the critical point, and $\hat{\gamma}(S^+)$ be the maximal value of the orbit S^+. Then the orbit S^+ is admissible if and only if $\hat{\gamma}(S^+) \leq \kappa$.*

While a unimodal map may depend on many arbitrarily chosen parameters, its dynamics determines the unique kneading value κ. We shall call κ the *topological parameter* of the map. Unlike the parameters of the original dynamical system, the topological parameter has no reason to be either smooth or continuous. The jumps in κ as a function of the map parameter such as a in (11.9) correspond to inadmissible values of the topological parameter. Each jump in κ corresponds to a stability window associated with a stable cycle of a smooth unimodal map. For the quadratic map (11.9) κ increases monotonically with the parameter a, but for a general unimodal map such monotonicity need not hold.

For further details of unimodal dynamics, the reader is referred to appendix F.1. As we shall see in sect. 12.4, for higher dimensional maps and flows there is no single parameter that orders dynamics monotonically; as a matter of fact, there is an infinity of parameters that need adjustment for a given symbolic dynamics. This difficult subject to which we shall return in sect. 12.5 is beyond our current ambition horizon.

⇓PRELIMINARY
⇑PRELIMINARY

11.5 Markov graphs

⇓PRELIMINARY

check this section against ref. [12.5]

So far we have established and related the *temporally* and *spatially* ordered topological dynamics for a class of "stretch & fold" dynamical systems, and given several examples of pruning of inadmissible trajectories. Now we use these results to generate the totality of admissible itineraries. This task will be relatively easy for repellers with complete Smale horseshoes and for subshifts of finite type.

combine spatial and temporal - how regions map in time

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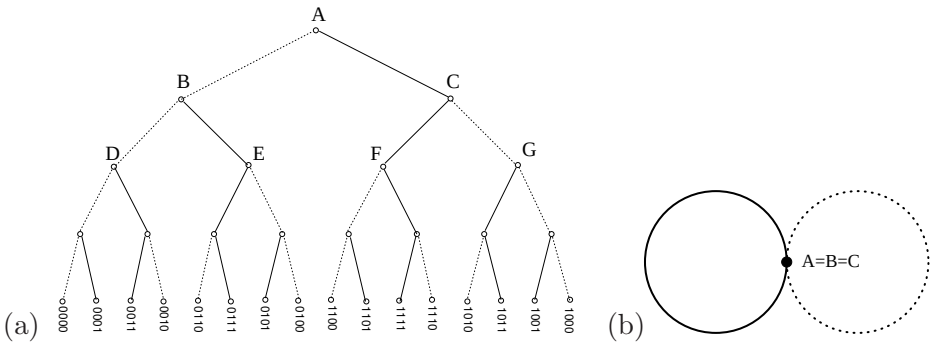

Figure 11.12: (a) The self-similarity of the complete binary symbolic dynamics represented by a binary tree (b) identification of nodes $B = A$, $C = A$ leads to the finite 1-node, 2-links Markov graph. All admissible itineraries are generated as walks on this finite Markov graph.

11.5.1 Finite memory

In the complete N-ary symbolic dynamics case (see example (11.3)) the choice of the next symbol requires no memory of the previous ones. However, any further refinement of the partition requires finite memory.

For example, for the binary labeled repeller with complete binary symbolic dynamics, we might chose to partition the state space into four regions $\{\mathcal{M}_{00}, \mathcal{M}_{01}, \mathcal{M}_{10}, \mathcal{M}_{11}\}$, a 1-step refinement of the initial partition $\{\mathcal{M}_0, \mathcal{M}_1\}$. Such partitions are drawn in figures 11.4 and 37.2, as well as figure 1.9. Topologically f acts as a left shift (12.7), and its action on the rectangle [.01] is to move the decimal point to the right, to [0.1], forget the past, [.1], and land in either of the two rectangles $\{[.10], [.11]\}$. Filling in the matrix elements for the other three initial states we obtain the 1-step memory transition matrix acting on the 4-state vector

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⇑PRIVATE

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$$\phi' = T\phi \;=\; \begin{pmatrix} T_{00,00} & 0 & T_{00,10} & 0 \\ T_{01,00} & 0 & T_{01,10} & 0 \\ 0 & T_{10,01} & 0 & T_{10,11} \\ 0 & T_{11,01} & 0 & T_{11,11} \end{pmatrix} \begin{pmatrix} \phi_{00} \\ \phi_{01} \\ \phi_{10} \\ \phi_{11} \end{pmatrix} . \qquad (11.16)$$

By the same token, for M-step memory the only nonvanishing matrix elements are of the form $T_{s_1 s_2 \ldots s_{M+1}, s_0 s_1 \ldots s_M}$, $s_{M+1} \in \{0, 1\}$. This is a sparse matrix, as the only non vanishing entries in the $m = s_0 s_1 \ldots s_M$ column of T_{dm} are in the rows $d = s_1 \ldots s_M 0$ and $d = s_1 \ldots s_M 1$. If we increase the number of steps remembered, the transition matrix grows big quickly, as the N-ary dynamics with M-step memory requires an $[N^{M+1} \times N^{M+1}]$ matrix. Since the matrix is very sparse, it pays to find a compact representation for T. Such representation is afforded by Markov graphs, which are not only compact, but also give us an intuitive picture of the topological dynamics.

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loops need arrows in figure 11.12

Construction of a good Markov graph is, like combinatorics, unexplainable. The only way to learn is by some diagrammatic gymnastics, so we work our way through a sequence of exercises in lieu of plethora of baffling definitions.

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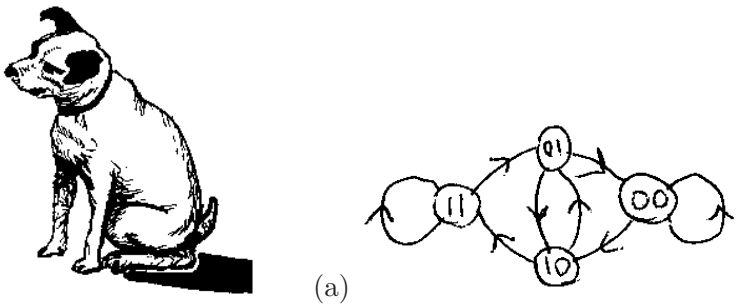
memory!finite



(b) (a)

Figure 11.13: (a) The 2-step memory Markov graph, links version obtained by identifying nodes $A = D = E = F = G$ in figure 11.12 (a). Links of this graph correspond to the matrix entries in the transition matrix (11.16). (b) the 2-step memory Markov graph, node version.

To start with, what do finite graphs have to do with infinitely long trajectories? To understand the main idea, let us construct a graph that enumerates all possible itineraries for the case of complete binary symbolic dynamics.

Mark a dot "·" on a piece of paper. Draw two short lines out of the dot, end each with a dot. The full line will signify that the first symbol in an itinerary is "1", and the dotted line will signifying "0". Repeat the procedure for each of the two new dots, and then for the four dots, and so on. The result is the binary tree of figure 11.12 (a). Starting at the top node, the tree enumerates exhaustively all distinct finite itineraries

$$\{0,1\},\{00,01,10,11\},\{000,001,010,\cdots\},\cdots .$$

The $M = 4$ nodes in figure 11.12 (a) correspond to the 16 distinct binary strings of length 4, and so on. By habit we have drawn the tree as the alternating binary tree of figure 11.9, but that has no significance as far as enumeration of itineraries is concerned - an ordinary binary tree would serve just as well.

The trouble with an infinite tree is that it does not fit on a piece of paper. On the other hand, we are not doing much - at each node we are turning either left or right. Hence all nodes are equivalent, and can be identified. To say it in other words, the tree is self-similar; the trees originating in nodes B and C are themselves copies of the entire tree. The result of identifying $B = A$, $C = A$ is a single node, 2-link Markov graph of figure 11.12 (b): any itinerary generated by the binary tree figure 11.12 (a), no matter how long, corresponds to a walk on this graph.

This is the most compact encoding of the complete binary symbolic dynamics. Any number of more complicated Markov graphs can do the job as well, and might be sometimes preferable. For example, identifying the trees originating in D, E, F and G with the entire tree leads to the 2-step memory Markov graph of figure 11.13a. The corresponding transition matrix is given by (11.16).

draw Markov graph figure 11.13a.

missing letter B in figure 11.14; dotted line in the loop in wrong place

symbolic dynamics
dynamics!symbolic
topological!dynamics
dynamics!topological
itinerary!future
itinerary!past
bi-infinite itinerary
itinerary!bi-infinite

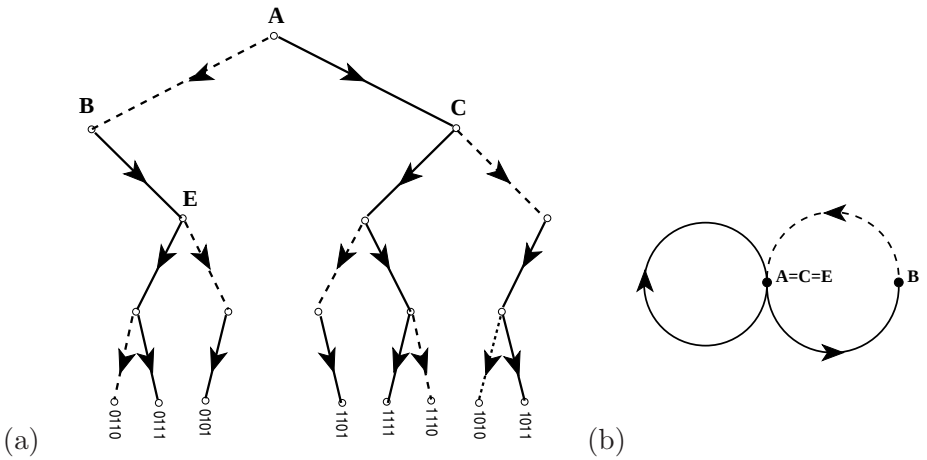


Figure 11.14: (a) The self-similarity of the _00_ pruned binary tree: trees originating from nodes C and E are the same as the entire tree. (b) Identification of nodes $A = C = E$ leads to the finite 2-node, 3-links Markov graph; as 0 is always followed by 1, the walks on this graph generate only the admissible itineraries.

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11.6 Symbolic dynamics, basic notions

In this section we collect the basic notions and definitions of symbolic dynamics. The reader might prefer to skim through this material on first reading, return to it later as the need arises.

Shifts. We associate with every initial point $x_0 \in \mathcal{M}$ the *future itinerary*, a sequence of symbols $S^+(x_0) = s_1 s_2 s_3 \cdots$ which indicates the order in which the regions are visited. If the trajectory $x_1, x_2, x_3, \ldots$ of the initial point x_0 is generated by

$$x_{n+1} = f(x_n)\,, \tag{11.17}$$

then the itinerary is given by the symbol sequence

$$s_n = s \qquad \text{if} \qquad x_n \in \mathcal{M}_s\,. \tag{11.18}$$

Similarly, the *past itinerary* $S^-(x_0) = \cdots s_{-2} s_{-1} s_0$ describes the history of x_0, the order in which the regions were visited before arriving to the point x_0. To each point x_0 in the dynamical space we thus associate a bi-infinite itinerary

Kitchen's convention is $\cdots s_{-2}s_{-1}.s_0 s_1 s_2 s_3 \cdots$

$$S(x_0) = (s_k)_{k\in\mathbb{Z}} = S^-.S^+ = \cdots s_{-2} s_{-1} s_0.s_1 s_2 s_3 \cdots\,. \tag{11.19}$$

The itinerary will be finite for a scattering trajectory, entering and then escaping $\mathcal{M}$ after a finite time, infinite for a trapped trajectory, and infinitely repeating for a periodic trajectory.

The set of all bi-infinite itineraries that can be formed from the letters of the alphabet $\mathcal{A}$ is called the *full shift*

$$\mathcal{A}^{\mathbb{Z}} = \{(s_k)_{k\in\mathbb{Z}} : s_k \in \mathcal{A} \text{ for all k } \in \mathbb{Z}\}\,. \tag{11.20}$$

The jargon is not thrilling, but this is how professional dynamicists talk to each other. We will stick to plain English to the extent possible.

We refer to this set of all conceivable itineraries as the *covering* symbolic dynamics. The name *shift* is descriptive of the way the dynamics acts on these sequences. As is clear from the definition (11.18), a forward iteration $x \to x' = f(x)$ shifts the entire itinerary to the left through the "decimal point". This operation, denoted by the shift operator σ,

$$\sigma(\cdots s_{-2}s_{-1}s_0.s_1s_2s_3\cdots) = \cdots s_{-2}s_{-1}s_0s_1.s_2s_3\cdots\,, \tag{11.21}$$

demoting the current partition label s_1 from the future S^+ to the "has been" itinerary S^-. The inverse shift σ^{-1} shifts the entire itinerary one step to the right.

A finite sequence $b = s_k s_{k+1}\cdots s_{k+n_b-1}$ of symbols from $\mathcal{A}$ is called a *block* of length n_b. A state space trajectory is *periodic* if it returns to its initial point after a finite time; in the shift space the trajectory is periodic if its itinerary is an infinitely repeating block p^∞. We shall refer to the set of periodic points that belong to a given periodic orbit as a *cycle*

$$p = \overline{s_1 s_2\cdots s_{n_p}} = \{x_{s_1 s_2\cdots s_{n_p}}, x_{s_2\cdots s_{n_p}s_1}, \cdots, x_{s_{n_p}s_1\cdots s_{n_p-1}}\}\,. \tag{11.22}$$

By its definition, a cycle is invariant under cyclic permutations of the symbols in the repeating block. A bar over a finite block of symbols denotes a periodic itinerary with infinitely repeating basic block; we shall omit the bar whenever it is clear from the context that the trajectory is periodic. Each *cycle point* is labeled by the first n_p steps of its future itinerary. For example, the 2nd cycle point is labeled by

$$x_{s_2\cdots s_{n_p}s_1} = x_{\overline{s_2\cdots s_{n_p}s_1}.\overline{s_2\cdots s_{n_p}s_1}}\,.$$

A *prime* cycle p of length n_p is a single traversal of the orbit; its label is a block of n_p symbols that cannot be written as a repeat of a shorter block (in literature such cycle is sometimes called *primitive*; we shall refer to it as "prime" throughout this text).

shift!full
plain English
English!plain
symbolic!dynamics!covering
covering!symbolic dynamics
shift
shift!operator
operator!shift
block!finite sequence
point!periodic
periodic!point
periodic!orbit
orbit!periodic
cycle!point
prime cycle
cycle!prime

partition!generating
generating partition
partition
topological!dynamics
dynamics!topological
subshift
shift!sub-
inadmissible symbol sequence
symbol!sequence!inadmissible
pruning!symbolic dynamics
symbolic dynamics!pruned
grammar!symbolic dynamics
symbolic dynamics!grammar
symbolic dynamics!coding
block, pruning
pruning!block
shift!finite type
symbolic dynamics!recoding
recoding

Partitions. A partition is called *generating* if every infinite symbol sequence corresponds to a distinct point in the state space. Finite Markov partition (11.4) is an example. Constructing a generating partition for a given system is a difficult problem. In examples to follow we shall concentrate on cases which allow finite partitions, but in practice almost any generating partition of interest is infinite.

A mapping $f : \mathcal{M} \to \mathcal{M}$ together with a partition $\mathcal{A}$ induces *topological dynamics* (Σ, σ), where the *subshift*

$$\Sigma = \{(s_k)_{k\in\mathbb{Z}}\}\,, \tag{11.23}$$

is the set of all *admissible* infinite itineraries, and $\sigma : \Sigma \to \Sigma$ is the shift operator (11.21). The designation "subshift" comes form the fact that $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ is the subset of the full shift (11.20). One of our principal tasks in developing symbolic dynamics of dynamical systems that occur in nature will be to determine Σ, the set of all bi-infinite itineraries S that are actually realized by the given dynamical system.

A partition too coarse, coarser than, for example, a Markov partition, would assign the same symbol sequence to distinct dynamical trajectories. To avoid that, we often find it convenient to work with partitions finer than strictly necessary. Ideally the dynamics in the refined partition assigns a unique infinite itinerary $\cdots s_{-2}s_{-1}s_0.s_1s_2s_3\cdots$ to each distinct trajectory, but there might exist full shift symbol sequences (11.20) which are not realized as trajectories; such sequences are called *inadmissible*, and we say that the symbolic dynamics is *pruned*. The word is suggested by "pruning" of branches corresponding to forbidden sequences for symbolic dynamics organized hierarchically into a tree structure, as explained in sect. 11.5.

Pruning. If the dynamics is pruned, the alphabet must be supplemented by a *grammar*, a set of pruning rules. After the inadmissible sequences have been pruned, it is often convenient to parse the symbolic strings into words of variable length - this is called *coding*. Suppose that the grammar can be stated as a finite number of pruning rules, each forbidding a block of finite length,

$$\mathcal{G} = \{b_1, b_2, \cdots b_k\}\ , \tag{11.24}$$

where a *pruning block* b is a sequence of symbols $b = s_1s_2\cdots s_{n_b}$, $s \in \mathcal{A}$, of finite length n_b. In this case we can always construct a finite Markov partition (11.4) by replacing finite length words of the original partition by letters of a new alphabet. In particular, if the longest forbidden block is of length $M + 1$, we say that the symbolic dynamics is a shift of finite type with M-step memory. In that case we can *recode* the symbolic dynamics in terms of a new alphabet, with each new letter given by an admissible block of at most length M. In the new alphabet the grammar rules are implemented by setting $T_{ij} = 0$ in (11.3) for forbidden transitions.

topological!dynamics
subshift!finite type
Markov!graph
graph!Markov
self-similar!fractal
partition
vertex
state
link
edge
arc
irreducible!graph
strongly connected graph
graph!irreducible

(a) $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ (b)

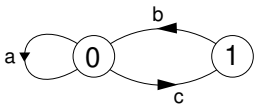

Figure 11.15: (a) The transition matrix for a simple subshift on two-state partition $\mathcal{A} = \{0, 1\}$, with grammar $\mathcal{G}$ given by a single pruning block $b = 11$ (consecutive repeat of symbol 1 is inadmissible): the state $\mathcal{M}_0$ maps both onto $\mathcal{M}_0$ and $\mathcal{M}_1$, but the state $\mathcal{M}_1$ maps only onto $\mathcal{M}_0$. (b) The corresponding finite 2-node, 3-link Markov graph, with nodes coding the symbols. All admissible itineraries are generated as walks on this finite Markov graph.

A topological dynamical system (Σ, σ) for which all admissible itineraries are generated by a finite transition matrix

$$\Sigma = \left\{ (s_k)_{k\in\mathbb{Z}} \,:\, T_{s_k s_{k+1}} = 1 \quad \text{for all } k \right\} \tag{11.25}$$

is called a subshift of *finite type*. Such systems are particularly easy to handle; the topology can be converted into symbolic dynamics by representing the transition matrix by a finite directed *Markov graph*, a convenient visualization of topological dynamics.

Markov graphs. A Markov graph describes compactly the ways in which the state space regions map into each other, accounts for finite memory effects in dynamics, and generates the totality of admissible trajectories as the set of all possible walks along its links. A Markov graph is also the precise statement of what is meant topologically by a "self-similar" fractal; supplemented by scaling information, the Markov graph generates a self-similar fractal, see chapter 20.

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need to resurrect parts of sect s-ThermDynFract

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A Markov graph consists of a set of *nodes* (or *vertices*, or *states*), one for each state in the alphabet $\mathcal{A} = \{A, B, C, \cdots, Z\}$, connected by a set of directed *links* (*edges*, *arcs*). Node i is connected by a directed link to node j whenever the transition matrix element (11.2) takes value $T_{ij} = 1$. There might be a set of links connecting two nodes, or links that originate and terminate on the same node. Two graphs are isomorphic if one can be obtained from the other by relabeling links and nodes; for us they are one and the same graph. As we are interested in recurrent dynamics, we restrict our attention to *irreducible* or *strongly connected* graphs, i.e., graphs for which there is a path from any node to any other node.

The simplest example is given in figure 11.15.

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alternating!binary ordering
binary ordering!alternating
Gray codes

Commentary

Remark 11.1 Symbolic dynamics, history and good taste. For a brief history of symbolic dynamics, from J. Hadamard in 1898 onward, see Notes to chapter 1 of Kitchens monograph [11.1], a very clear and enjoyable mathematical introduction to topics discussed here. Finite Markov graphs or finite automata are discussed in refs. [11.2, 11.3, 11.4, 11.5]. They belong to the category of regular languages. A good hands-on introduction to symbolic dynamics is given in ref. [11.6].

The binary labeling of the once-folding map periodic points was introduced by Myrberg [11.7] for one-dimensional maps, and its utility to two-dimensional maps has been emphasized in refs. [3.7, 3.11]. For one-dimensional maps it is now customary to use the R-L notation of Metropolis, Stein and Stein [11.8, 11.9], indicating that the point x_n lies either to the left or to the right of the critical point in figure 11.8. The symbolic dynamics of such mappings has been extensively studied by means of the Smale horseshoes, see for example ref. [11.10]. Using letters rather than numerals in symbol dynamics alphabets probably reflects good taste. We prefer numerals for their computational convenience, as they speed up the implementation of conversions into the topological coordinates (δ, γ) introduced in sect. 12.3.1. The alternating binary ordering of figure 11.9 is related to the Gray codes of computer science [2.8].

read up on Gray codes - are they well-ordered sequences?

Remark 11.2 Inflating Markov graphs. In the above examples the symbolic dynamics has been encoded by labeling links in the Markov graph. Alternatively one can encode the dynamics by labeling the nodes, as in figure 11.13, where the 4 nodes refer to 4 Markov partition regions $\{\mathcal{M}_{00}, \mathcal{M}_{01}, \mathcal{M}_{10}, \mathcal{M}_{11}\}$, and the 8 links to the 8 non-zero entries in the 2-step memory transition matrix (11.16).

Résumé

In chapters 14 and 15 we will establish that spectra of evolution operators can be extracted from periodic orbit sums:

$$\sum (\text{spectral eigenvalues}) \;=\; \sum (\text{periodic orbits}) \;.$$

In order to implement this theory we need to know what periodic orbits can exist, and the symbolic dynamics developed above and in chapter 12 is an invaluable tool toward this end.

fast track:
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⇑PRELIMINARY

Exercises

three-disk@3-disk!symbolic dynamics
symbolic dynamics!3-disk
map!contracting
map!dissipative
contracting!map
dissipative!map

Exercise 11.1 **Binary symbolic dynamics.** *Verify that the shortest prime binary cycles of the unimodal repeller of figure 11.8 are* $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$. *Compare with table 11.1. Try to sketch them in the graph of the unimodal function* $f(x)$*; compare ordering of the periodic points with figure 11.9. The point is that while overlayed on each other the longer cycles look like a hopeless jumble, the cycle points are clearly and logically ordered by the alternating binary tree.*

Exercise 11.2 **3-disk fundamental domain symbolic dynamics.** *Try to sketch* $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$. *in the fundamental domain, figure 11.6, and interpret the symbols* $\{0,1\}$ *by relating them to topologically distinct types of collisions. Compare with table 11.2. Then try to sketch the location of periodic points in the Poincaré section of the billiard flow. The point of this exercise is that while in the configuration space longer cycles look like a hopeless jumble, in the Poincaré section they are clearly and logically ordered. The Poincaré section is always to be preferred to projections of a flow onto the configuration space coordinates, or any other subset of state space coordinates which does not respect the topological organization of the flow.*

Exercise 11.3 **Generating prime cycles.** *Write a program that generates all binary prime cycles up to given finite length.*

Exercise 11.4 **A contracting baker's map.** *Consider a contracting (or "dissipative") baker's defined in exercise 4.5.*

The symbolic dynamics encoding of trajectories is realized via symbols 0 $(y \leq 1/2)$ *and* 1 $(y > 1/2)$. *Consider the observable* $a(x,y) = x$. *Verify that for any periodic orbit* p $(\epsilon_1 \ldots \epsilon_{n_p})$, $\epsilon_i \in \{0,1\}$

$$A_p = \frac{3}{4} \sum_{j=1}^{n_p} \delta_{j,1} \, .$$

Exercise 11.5 **Reduction of 3-disk symbolic dynamics to binary.**

(a) *Verify that the 3-disk cycles*
$\{\overline{1\,2}, \overline{1\,3}, \overline{2\,3}\}$, $\{\overline{1\,2\,3}, \overline{1\,3\,2}\}$, $\{\overline{12\,13}$ *+ 2 perms.*$\}$,
$\{\overline{121\,232\,313}$ *+ 5 perms.*$\}$, $\{\overline{121\,323}$*+ 2 perms.*$\}$, $\cdots$,
correspond to the fundamental domain cycles $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$ *respectively.*

(b) *Check the reduction for short cycles in table 11.2 by drawing them both in the full 3-disk system and in the fundamental domain, as in figure 11.6.*

(c) *Optional: Can you see how the group elements listed in table 11.2 relate irreducible segments to the fundamental domain periodic orbits?*

kneading value!unimodal
unimodal!kneading value
well ordered symbols!unimodal
unimodal!well ordered symbols
partition!infinite
kneading!sequence
kneading!value
dike map
map!dike

Exercise 11.6 Unimodal map symbolic dynamics. *Show that the tent map point $\gamma(S^+)$ with future itinerary S^+ is given by converting the sequence of s_n's into a binary number by the algorithm (11.11). This follows by inspection from the binary tree of figure 11.9.*

Exercise 11.7 Unimodal map kneading value. *Consider the 1-d quadratic map*

$$f(x) = Ax(1-x)\,, \qquad A = 3.8\,. \tag{11.26}$$

(a) *(easy) Plot (11.26), and the first 4-8 (whatever looks better) iterates of the critical point $x_c = 1/2$.*

(b) *(hard) Draw corresponding intervals of the partition of the unit interval as levels of a Cantor set, as in the symbolic dynamics partition of figure 11.8 (b). Note, however, that some of the intervals of figure 11.8 (b) do not appear in this case - they are* pruned*.*

(c) *(medium) Produce ChaosBook.org quality figure 11.8 (a).*

(d) *(easy) Check numerically that $K = S^+(x_c)$, the itinerary or the "kneading sequence" of the critical point is*

$$K = 10110111101101111010111101111110\ldots$$

The tent map point $\gamma(S^+)$ with future itinerary S^+ is given by converting the sequence of s_n's into a binary number by the algorithm (11.11),

$$\begin{aligned} w_{n+1} &= \begin{cases} w_n & \text{if } s_{n+1}=0 \\ 1-w_n & \text{if } s_{n+1}=1 \end{cases}\,, \qquad w_1 = s_1 \\ \gamma(S^+) &= 0.w_1w_2w_3\ldots = \sum_{n=1}^{\infty} w_n/2^n\,. \end{aligned}$$

(e) *(medium) List the the corresponding kneading value (11.13) sequence $\kappa = \gamma(K)$ to the same number of digits as K.*

(f) *(hard) Plot the missing dike map, figure 11.11, in ChaosBook.org quality, with the same kneading sequence K as $f(x)$. The dike map is obtained by slicing off all $\gamma\left(S^+(x_0)\right) > \kappa$, from the complete tent map figure 11.8 (a), see (11.14).*

How this kneading sequence is converted into a series of pruning rules is a dark art, relegated to sect. 13.6 and appendix F.1.

⇓PRELIMINARY
this might be an obsolete version; if not, use Malloppo text
copy Freddy p.18, 3rd paragraph

Exercise 11.8 One-dimensional repellers. *The simplest example of symbolic dynamics is afforded by 1-d unimodal maps,* ie. *maps with one raising and one decreasing branch. The alphabet consists of two letters, describing the branch on which the iterate is located.*

the quadratic polynomial

$$f(x) = x^2 + c\,, \tag{11.27}$$

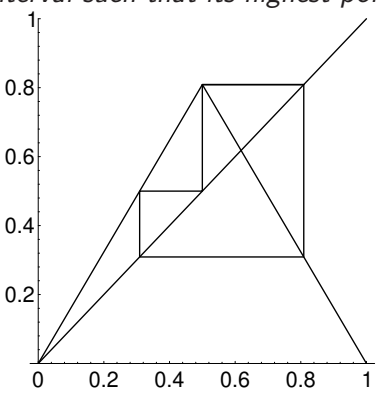
Per's p.29 *the cubic polynomial*

$$f(x) = (1+\epsilon)\mu_c x(1-x^2)\,, \tag{11.28}$$

For $\epsilon > 0$ the map repels, meaning that almost any initial x will iterate out of the unit interval and escape, see figure 11.8.

⇑PRELIMINARY

Exercise 11.9 **"Golden mean" pruned map.** *Consider a symmetrical tent map on the unit interval such that its highest point belongs to a 3-cycle:*



(a) *Find the absolute value Λ for the slope (the two different slopes $\pm\Lambda$ just differ by a sign) where the maximum at $1/2$ is part of a period three orbit, as in the figure.*

(b) *Show that no orbit of this map can visit the region $x > (1+\sqrt{5})/4$ more than once. Verify that once an orbit exceeds $x > (\sqrt{5}-1)/4$, it does not reenter the region $x < (\sqrt{5}-1)/4$.*

(c) *If an orbit is in the interval $(\sqrt{5}-1)/4 < x < 1/2$, where will it be on the next iteration?*

(d) *If the symbolic dynamics is such that for $x < 1/2$ we use the symbol 0 and for $x > 1/2$ we use the symbol 1, show that no periodic orbit will have the substring $_00_$ in it.*

(e) *On the second thought, is there a periodic orbit that violates the above $_00_$ pruning rule?*

For continuation, see exercise 13.6 and exercise 13.8. See also exercise 13.7 and exercise 13.9.

Exercise 11.10 **Binary 3-step transition matrix.** *Construct [8×8] binary 3-step transition matrix analogous to the 2-step transition matrix (11.16). Convince yourself that the number of terms of contributing to $\mathrm{tr}\,T^n$ is independent of the memory length, and that this [$2^m \times 2^m$] trace is well defined in the infinite memory limit $m \to \infty$.*

add somewhere this statement: "Critical points are special - they define the boundary between intervals, ie. interval is split into 0 [left part], x_c [critical point] and 1 [right part]. For the dike map x_c is the whole interval of points along the flat top of the map, but usually it is a point."

⇓PRELIMINARY

Exercise 11.11 Heavy pruning. *Implement the prunning grammar (12.11), with the pruned blocks*

$$10010, 101, 01001, 01101, 111, 10110,$$

by a method of your own devising, or following the the last example of sect. 11.5 illustrated in figure 13.3. For continuation, see exercise 13.10.

⇑PRELIMINARY

Smale, S.

Chapter 12

Qualitative dynamics, for cyclists

> **I.1. Introduction to conjugacy problems for diffeomorphisms.** This is a survey article on the area of global analysis defined by differentiable dynamical systems or equivalently the action (differentiable) of a Lie group G on a manifold M. Here Diff(M) is the group of all diffeomorphisms of M and a diffeomorphism is a differentiable map with a differentiable inverse. (...) Our problem is to study the global structure, i.e., all of the orbits of M.
>
> Stephen Smale, *Differentiable Dynamical Systems*

In sects. 9.1 and 11.1 we introduced the concept of partitioning the state space, in any way you please. In chapter 8 we established that stability eigenvalues of periodic orbits are invariants of a given flow. The invariance of stabilities of a periodic orbit is a local property of the flow.

For the Rössler flow of example 3.3, we have learned that the attractor is very thin, but otherwise the return maps that we found were disquieting – figure 3.3 did not appear to be a one-to-one map. This apparent loss of invertibility is an artifact of projection of higher-dimensional return maps onto lower-dimensional subspaces. As the choice of lower-dimensional subspace is arbitrary, the resulting snapshots of return maps look rather arbitrary, too. Other projections might look even less suggestive.

needs expanding!

Such observations beg a question: Does there exist a "natural", intrinsically optimal coordinate system in which we should plot of a return map?

explain return map here

As we shall now argue (see also sect. 17.1), the answer is yes: The intrinsic coordinates are given by the stable/unstable manifolds, and a return map should be plotted as a map from the unstable manifold back onto the immediate neighborhood of the unstable manifold.

In this chapter we show that every equilibrium point and every periodic orbit carries with it stable and unstable manifolds which provide a topologically

stable!manifold
unstable!manifold
manifold!stable
stable!manifold!flow
unstable!manifold, flow
stable!manifold!map
unstable!manifold!map

invariant *global* foliation of the state space. This qualitative dynamics of stretching and mixing enables us to partition the state space and assign symbolic dynamics itineraries to trajectories.

Given an itinerary, the topology of stretching and folding fixes the relative spatial ordering of trajectories, and separates the admissible and inadmissible itineraries. The level is distinctly cyclist, in distincition to the pedestrian tempo of the preceeding chapter. Skip this chapter unless you really need to get into nitty-gritty details of symbolic dynamics.

fast track:
chapter 13, p. 225

12.1 Going global: Stable/unstable manifolds

A neighborhood of a trajectory deforms as it is transported by the flow. In the linear approximation, the stability matrix A describes this shearing of an infinitesimal neighborhood in an infinitesimal time step. The shearing after finite time is described by the fundamental matrix M^t. Its eigenvalues and eigendirections describe deformation of an initial infinitesimal sphere of neighboring trajectories into an ellipsoid time t later. Nearby trajectories separate exponentially along the unstable directions, approach each other along the stable directions, and maintain their distance along the marginal directions.

The fixed or periodic point x^* stability matrix $M_p(x^*)$ eigenvectors (8.10) form a rectilinear coordinate frame in which the flow into, out of, or encircling the fixed point is linear in the sense of sect. 4.2. These eigendirections are numerically continued into global curvilinear invariant manifolds as follows.

The global continuations of the local stable, unstable eigendirections are called the *stable*, respectively *unstable manifolds*. They consist of all points which march into the fixed point forward, respectively backward in time

$$\begin{aligned} W^s &= \left\{x \in \mathcal{M} : f^t(x) - x^* \to 0 \text{ as } t \to \infty\right\} \\ W^u &= \left\{x \in \mathcal{M} : f^{-t}(x) - x^* \to 0 \text{ as } t \to \infty\right\} . \end{aligned} \tag{12.1}$$

The stable/unstable manifolds of a flow are rather hard to visualize, so as long as we are not worried about a global property such as the number of times they wind around a periodic trajectory before completing a parcourse, we might just as well look at their Poincaré section return maps. Stable, unstable manifolds for maps are defined by

define y^*; note 2-dimensional only example

$$\begin{aligned} W^s &= \{x \in \mathcal{P} : f^n(x) - x^* \to 0 \text{ as } n \to \infty\} \\ W^u &= \left\{x \in \mathcal{P} : f^{-n}(x) - x^* \to 0 \text{ as } n \to \infty\right\} . \end{aligned} \tag{12.2}$$

For $n \to \infty$ any finite segment of W^s, respectively W^u converges to the linearized map eigenvector $\mathbf{e}^{(e)}$, respectively $\mathbf{e}^{(c)}$. In this sense each eigenvector defines a (curvilinear) axis of the stable, respectively unstable manifold.

Conversely, we can use an arbitrarily small segment of a fixed point eigenvector to construct a finite segment of the associated manifold. Precise construction depends on the type of the eigenvalue(s).

Expanding real and positive eigendirection. Consider ith expanding eigenvalue, eigenvector pair $(\Lambda_i, \mathbf{e}_i)$ computed from J evaluated at a cycle point,

$$J(x)\mathbf{e}_i(x) = \Lambda_i \mathbf{e}_i(x)\,, \quad x \in p\,, \quad \Lambda_i > 1\,. \tag{12.3}$$

Take an infinitesimal eigenvector $\epsilon\,\mathbf{e}_i(x)$, $\epsilon \ll 1$, and its image $J_p(x)\epsilon\,\mathbf{e}_i(x) = \Lambda_i \epsilon\,\mathbf{e}_i(x)$. Sprinkle the interval $|\Lambda_i - 1|\epsilon$ with a large number of points x_m, equidistantly spaced on logarithmic scale $\ln|\Lambda_i - 1| + \ln\epsilon$. The successive images of these points $f(x_j)$, $f^2(x_j)$, $\cdots$, $f^m(x_j)$ trace out the curvilinear unstable manifold in direction $\mathbf{e}_i$. Repeat for $-\epsilon\,\mathbf{e}_i(x)$.

Contractiong real, positive eigendirection. Reverse the action of the map backwards in time. This turns a contracting direction into an expanding one, tracing out the curvilinear stable manifold in continuation of $\epsilon\,\mathbf{e}_j$.

Expanding/contracting real negative eigendirection. As above, but every even iterate $f^2(x_j)$, $f^4(x_j)$, $f^6(x_j)$ continues in the direction $\mathbf{e}_i$, every odd one in the direction $-\mathbf{e}_i$.

Complex eigenvalue pair. Construct an orthonormal pair of eigenvectors spanning the plane $\{\epsilon\,\mathbf{e}_j, \epsilon\,\mathbf{e}_{j+1}\}$. Iteration of the annulus between an infinitesimal circle and its image by J spans the spiralling/circle unstable manifold of the complex eigenvalue pair $\{\Lambda_i, \Lambda_{i+1} = \Lambda_i^*\}$.

⇓PRELIMINARY

Alternatively, one can write down a parametric equation for the stable (unstable) manifold and generate the manifold by integrating the equation.

explain, draw figure

⇑PRELIMINARY

12.2 Horseshoes

If a flow is locally unstable but globally bounded, any open ball of initial points will be stretched out and then folded back. An example is a 3-dimensional invertible flow sketched in figure 11.7 which returns an area of a Poincaré section of the flow stretched and folded into a "horseshoe", such that the initial area is intersected at most twice(see figure 12.4). Run backwards, the flow generates the backward horseshoe which intersects the forward horseshoe at most 4 times, and so forth. Such flows exist, and

⇓PRIVATE

⇑PRIVATE

motivate 1-d maps with this, replace figure 11.7 by hand-drawn version

Rossler@R"ossler system
horseshoe
map!once-folding
Poincar"e!section
return map
map!return
Henon@H"enon!map
Henon@H"enon!map!fixed points
Add Predrag's pictures
Henon@H"enon!map!inverse
stable!manifold
unstable!manifold
make example for negative b?

are easily constructed - an example is the Rössler system , discussed in example 3.3.

Now we shall construct an example of a locally unstable but globally bounded mapping which returns an initial area stretched and folded into a "horseshoe", such that the initial area is intersected at most twice. We shall refer to such mappings with at most 2^n transverse self-intersections at the nth iteration as the *once-folding* maps.

12.1 page 220

As an example is afforded by the 2-dimensional *Hénon map*

$$\begin{aligned} x_{n+1} &= 1 - ax_n^2 + by_n \\ y_{n+1} &= x_n \,. \end{aligned} \tag{12.4}$$

The Hénon map models qualitatively the Poincaré section return map of figure 11.7. For $b = 0$ the Hénon map reduces to the parabola (11.9), and, as shown in sects. 3.4 and 37.1, for $b \neq 0$ it is kind of a fattened parabola; by construction, it takes a rectangular initial area and returns it bent as a horseshoe.

For definitiveness, fix the parameter values to $a = 6$, $b = 0.9$. The map is quadratic, so it has 2 fixed points $x_0 = f(x_0)$, $x_1 = f(x_1)$ indicated in figure 12.1 (a). For the parameter values at hand, they are both unstable.

add exercise

If you start with a small ball of initial points centered around x_1, and iterate the map, the ball will be stretched and squashed along the line W_1^u.

Similarly, a small ball of initial points centered around the other fixed point x_0 iterated backward in time,

$$\begin{aligned} x_{n-1} &= y_n \\ y_{n-1} &= -\frac{1}{b}(1 - ay_n^2 - x_n) \,, \end{aligned} \tag{12.5}$$

sect?

traces out the line W_0^s. W_0^s is the stable manifold of x_0 fixed point, and W_1^u is the unstable manifold of x_1 fixed point, defined in sect. 12.1. Their intersections enclose the crosshatched region $\mathcal{M}_{\cdot}$. Any point outside W_1^u border of $\mathcal{M}_{\cdot}$ escapes to infinity forward in time, while any point outside W_0^s border escapes to infinity backwards in time. In this way the unstable - stable manifolds define topologically, invariant and optimal $\mathcal{M}_{\cdot}$ initial region; all orbits that stay confined for all times are confined to $\mathcal{M}_{\cdot}$.

Iterated one step forward, the region $\mathcal{M}_{\cdot}$ is stretched and folded into a *smale* horseshoe drawn in figure 12.1 (b). The horseshoe fattened parabolla shape is the consequence og the quadratic form x^2 in (12.4). Parameter a controls the amount of stretching, while the parameter b controls the amount of compression of the folded horseshoe. The case $a = 6$, $b = 0.9$ considered here corresponds to strong stretching and weak compression. Label the two forward intersections $f(\mathcal{M}_{\cdot}) \cap \mathcal{M}_{\cdot}$ by $\mathcal{M}_{s\cdot}$, with $s \in \{0, 1\}$, figure 12.1 (b). The horseshoe consists of the two strips $\mathcal{M}_{0\cdot}, \mathcal{M}_{1\cdot}$, and the

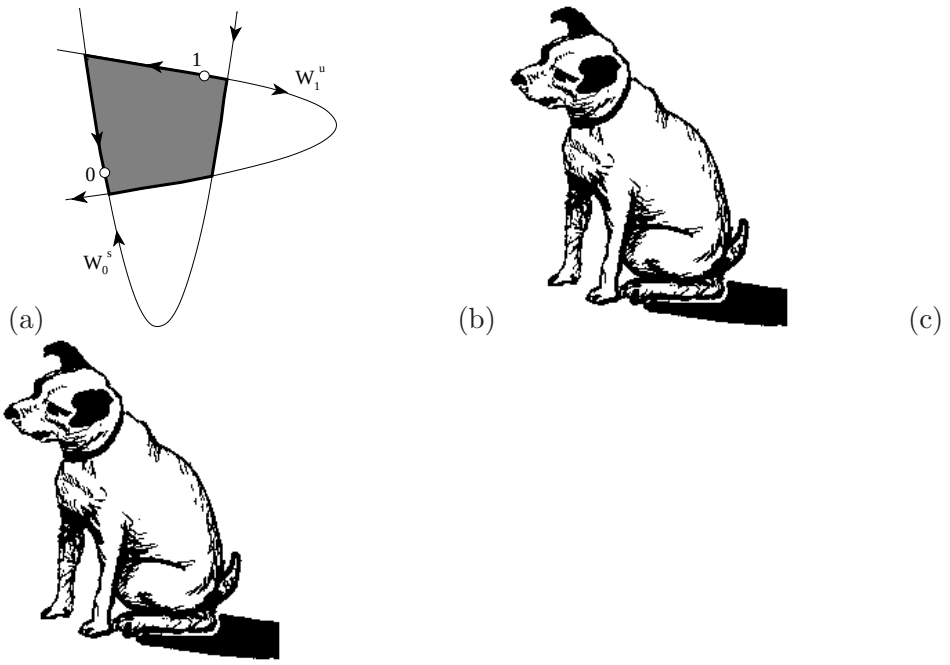


Figure 12.1: The Hénon map for $a = 6$, $b = .9$. (a) The fixed points $\overline{0}$, $\overline{1}$, and the segments of the W_0^s stable manifold, W_1^u unstable manifold that enclose the initial (crosshatched) region $\mathcal{M}_{.}$. (b) The forward horseshoe $f(\mathcal{M}_{.})$. (c) The backward horseshoe $f^{-1}(\mathcal{M}_{.})$. Iteration yields a complete Smale horseshoe, with every forward fold intersecting every backward fold.

bent segment that lies entirely outside the W_1^u line. As all points in this segment escape to infinity under forward iteration, this region can safely be cut out and thrown away.

discuss forward contraction, foliation, backward ditto

Iterated one step backwards, the region $\mathcal{M}_{.}$ is again stretched and folded into a horseshoe, figure 12.1 (c). As stability and instability are interchanged under time reversal, this horseshoe is transverse to the forward one. Again the points in the horseshoe bend wonder off to infinity as $n \to -\infty$, and we are left with the two (backward) strips $\mathcal{M}_{.0}, \mathcal{M}_{.1}$. Iterating two steps forward we obtain the four strips $\mathcal{M}_{11.}, \mathcal{M}_{01.}, \mathcal{M}_{00.}, \mathcal{M}_{10.}$, and iterating backwards we obtain the four strips $\mathcal{M}_{.00}, \mathcal{M}_{.01}, \mathcal{M}_{.11}, \mathcal{M}_{.10}$ transverse to the forward ones just as for 3-disk pinball game figure 11.3. Iterating three steps forward we get an 8 strips, and so on *ad infinitum*.

What is the significance of the subscript $_{.011}$ which labels the $\mathcal{M}_{.011}$ backward strip? The two strips $\mathcal{M}_{.0}, \mathcal{M}_{.1}$ partition the state space into two regions labeled by the two-letter alphabet $\mathcal{A} = \{0, 1\}$. $S^+ = .011$ is the *future itinerary* for all $x \in \mathcal{M}_{.011}$. Likewise, for the forward strips all $x \in \mathcal{M}_{s_{-m}\cdots s_{-1}s_0.}$ have the *past itinerary* $S^- = s_{-m}\cdots s_{-1}s_0$. Which partition we use to present pictorially the regions that do not escape in m iterations is a matter of taste, as the backward strips are the preimages of the forward ones

$$\mathcal{M}_{0.} = f(\mathcal{M}_{.0})\,, \qquad \mathcal{M}_{1.} = f(\mathcal{M}_{.1})\,.$$

non-wandering set
set!non-wandering
symbolic dynamics!complete
structural stability
stability!structural
horseshoe!complete
structural stability
stability!structural
ordering!spatial

Ω, the non–wandering set (2.2) of $\mathcal{M}_{.}$, is the union of all points whose forward and backward trajectories remain trapped for all time. given by the intersections of all images and preimages of $\mathcal{M}$:

$$\Omega = \left\{ x : x \in \lim_{m,n\to\infty} f^m(\mathcal{M}_{.}) \bigcap f^{-n}(\mathcal{M}_{.}) \right\} . \tag{12.6}$$

Two important properties of the Smale horseshoe are that it has a *complete binary symbolic dynamics* and that it is *structurally stable*.

For a *complete* Smale horseshoe every forward fold $f^n(\mathcal{M})$ intersects transversally every backward fold $f^{-m}(\mathcal{M})$, so a unique bi-infinite binary sequence can be associated to every element of the non–wandering set. A point $x \in \Omega$ is labeled by the intersection of its past and future itineraries $S(x) = \cdots s_{-2}s_{-1}s_0.s_1s_2\cdots$, where $s_n = s$ if $f^n(x) \in \mathcal{M}_{.s}$, $s \in \{0,1\}$ and $n \in \mathbb{Z}$. For sufficiently separated disks, the 3-disk game of pinball figure 11.3, is another example of a complete Smale horseshoe; in this case the "folding" region of the horseshoe is cut out of the picture by allowing the pinballs that fly between the disks to fall off the table and escape.

The system is said to be *structurally stable* if all intersections of forward and backward iterates of $\mathcal{M}$ remain transverse for sufficiently small perturbations $f \to f + \delta$ of the flow, for example, for slight displacements of the disks, or sufficiently small variations of the Hénon map parameters a, b while structural stability is exceedingly desirable, it is also exceedingly rare. About this, more later.

⇓PRELIMINARY

Inspecting the figure 12.1 (c) we see that the relative ordering of regions with differing finite itineraries is a qualitative, topological property of the flow, so it makes sense to define a simple "canonical" representative partition for the entire class of topologically similar flows.

⇑PRELIMINARY

12.3 Spatial ordering

Consider a system for which you have succeeded in constructing a covering symbolic dynamics, such as a well-separated 3-disk system. Now start moving the disks toward each other. At some critical separation a disk will start blocking families of trajectories traversing the other two disks. The order in which trajectories disappear is determined by their relative ordering in space; the ones closest to the intervening disk will be pruned first. Determining inadmissible itineraries requires that we relate the spatial ordering of trajectories to their time ordered itineraries.

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page 223

So far we have rules that, given a state space partition, generate a *temporally* ordered itinerary for a given trajectory. Our next task is the reverse: given a set of itineraries, what is the *spatial* ordering of corresponding points along the trajectories? In answering this question we will be aided by Smale's visualization of the relation between the topology of a flow and its symbolic dynamics by means of "horseshoes".

symbol plane@"topp
symbol!square
rectangle
symbol plane@"topp

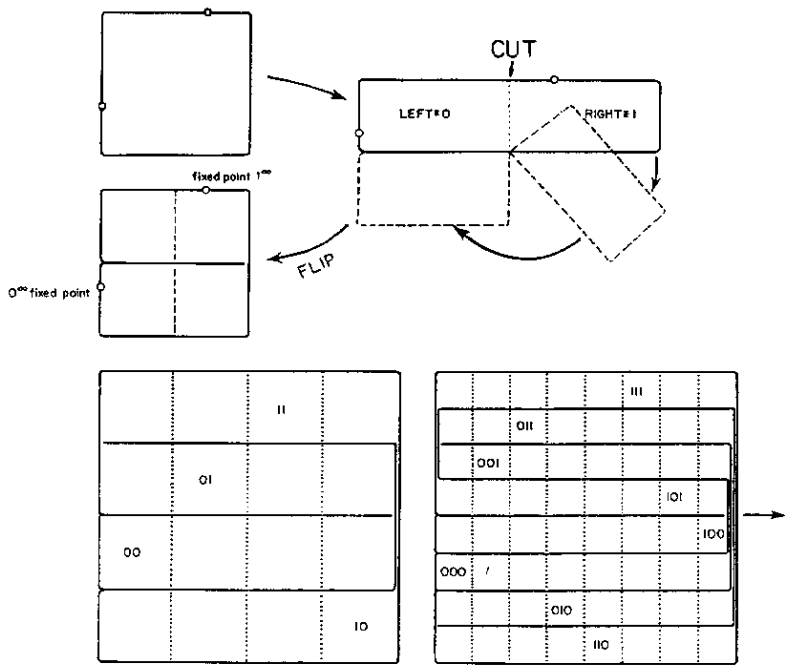

FIG. 4. Iterative construction of the symbol plane.

Figure 12.2: Kneading Danish Pastry: symbol square representation of an orientation reversing once-folding map obtained by fattening the Smale horseshoe intersections of figure 12.1 into a unit square. In the symbol square the dynamics maps rectangles into rectangles by a decimal point shift.

12.3.1 Symbol square

complement figure 12.2 with a figure of rectangles mapping into rectangles

For a better visualization of 2-dimensional non–wandering sets, fatten the intersection regions until they completely cover a unit square, as in figure 12.2.

say symbol square is a product of two tent maps

We shall refer to such a "map" of the topology of a given "stretch & fold" dynamical system as the *symbol square*. The symbol square is a topologically accurate representation of the non–wandering set and serves as a street map for labeling its pieces. Finite memory of m steps and finite foresight of n steps partitions the symbol square into *rectangles* $[s_{-m+1}\cdots s_0.s_1s_2\cdots s_n]$.

rescan the knife into figure 12.2

In the binary dynamics symbol square the size of such rectangle is $2^{-m}\times 2^{-n}$; it corresponds to a region of the dynamical state space which contains all points that share common n future and m past symbols. This region maps in a nontrivial way in the state space, but in the symbol square its dynamics is exceedingly simple; all of its points are mapped by the decimal point shift (11.21)

12.2
page 220

$$\sigma(\cdots s_{-2}s_{-1}s_0.s_1s_2s_3\cdots) = \cdots s_{-2}s_{-1}s_0s_1.s_2s_3\cdots\,, \qquad (12.7)$$

For example, the square [01.01] gets mapped into the rectangle $\sigma[01.01] = [010.1]$, see figure 12.4 (b).

⇓PRELIMINARY

⇑PRELIMINARY

As the horseshoe mapping is a simple repetitive operation, we expect a simple relation between the symbolic dynamics labeling of the horseshoe

draw a better figure!

12.3
page 220

past "topc

strips, and their relative placement. The symbol square points $\gamma(S^+)$ with future itinerary S^+ are constructed by converting the sequence of s_n's into a binary number by the algorithm (11.11). This follows by inspection from figure 12.2. In order to understand this relation between the topology of horseshoes and their symbolic dynamics, it might be helpful to backtrace to sect. 11.3.2 and work through and understand first the symbolic dynamics of one-dimensional unimodal mappings.

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into exercises - orientation preserving case

Under backward iteration the roles of 0 and 1 symbols are interchanged; $\mathcal{M}_0^{-1}$ has the same orientation as $\mathcal{M}$, while $\mathcal{M}_1^{-1}$ has the opposite orientation. We assign to an *orientation preserving* once-folding map the *past topological coordinate* $\delta = \delta(S^-)$ by the algorithm:

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recheck this

$$
\begin{aligned}
w_{n-1} &= \begin{cases} w_n & \text{if } s_n = 0 \\ 1 - w_n & \text{if } s_n = 1 \end{cases}, \qquad w_0 = s_0 \\
\delta(S^-) &= 0.w_0 w_{-1} w_{-2} \ldots = \sum_{n=1}^{\infty} w_{1-n}/2^n \,. \qquad (12.8)
\end{aligned}
$$

Such formulas are best derived by quiet contemplation of the action of a folding map, in the same way we derived the future topological coordinate (11.11).

The coordinate pair (δ, γ) maps a point (x, y) in the state space Cantor set of figure 12.1 into a point in the symbol square of figure 12.2, preserving the topological ordering; (δ, γ) serves as a topologically faithful representation of the non–wandering set of any once-folding map, and aids us in partitioning the set and ordering the partitions for any flow of this type.

Include results of refs. [12.66, 12.67]

add chapter on templates, braids?

12.4 Pruning

> The complexity of this figure will be striking, and I shall not even try to draw it.
> H. Poincaré, on his discovery of homoclinic tangles, *Les méthodes nouvelles de la méchanique céleste*

In general, not all possible itineraries are realized as physical trajectories. Trying to get from "here" to "there" we might find that a short path is excluded by some obstacle, such as a disk that blocks the path, or a potential ridge. To count correctly, we need to *prune* the inadmissible trajectories, i.e., specify the grammar of the admissible itineraries.

While the complete Smale horseshoe dynamics discussed so far is rather straightforward, we had to get through it in order to be able to approach a situation that resembles more the real life: adjust the parameters of a once-folding map so that the intersection of the backward and forward folds is still transverse, but no longer complete, as in figure 13.2 (a). The utility of

replace by the Biham Hénon example

Henon exercise
d on AACI,
CII

the symbol square lies in the fact that the surviving, admissible itineraries still maintain the same relative spatial ordering as for the complete case.

pruning!front
subshift!finite type
structural stability
stability!structural
Anosov flows

In the example of figure 13.2 (a) the rectangles [10.1], [11.1] have been pruned, and consequently *any* trajectory containing blocks $b_1 = 101$, $b_2 = 111$ is pruned. We refer to the border of this primary pruned region as the *pruning front*; another example of a pruning front is drawn in figure 13.2 (d). We call it a "front" as it can be visualized as a border between admissible and inadmissible; any trajectory whose periodic point would fall to the right of the front in figure 13.2 is inadmissible, i.e., pruned. The pruning front is a complete description of the symbolic dynamics of once-folding maps; we shall discuss this in more depth in sect. 12.5. For now we need this only as a concrete illustration of how pruning rules arise.

⇓PRELIMINARY
⇑PRELIMINARY

In the example at hand there are total of two forbidden blocks 101, 111, so the symbol dynamics is a subshift of finite type (11.25). For now we concentrate on this kind of pruning because it is particularly clean and simple. Unfortunately, for a generic dynamical system a subshift of finite type is the exception rather than the rule. Only some repelling sets (like our game of pinball) and a few purely mathematical constructs (called Anosov flows) are structurally stable - for most systems of interest an infinitesimal perturbation of the flow destroys and/or creates an infinity of trajectories, and specification of the grammar requires determination of pruning blocks of arbitrary length. The repercussions are dramatic and counterintuitive; for example, due to the lack of structural stability the transport coefficients such as the deterministic diffusion constant of sect. 24.2 are emphatically *not* smooth functions of the system parameters. This generic lack of structural stability is what makes nonlinear dynamics so hard; we shall return to the problem of approximating generic infinite Markov partitions in sect. 12.5.

⇓PRELIMINARY
⇑PRELIMINARY

The conceptually simpler finite subshift Smale horseshoes suffice to motivate most of the key concepts that we shall need for time being.

⇓PRELIMINARY

Example 12.1 A Hénon map pruning front:
figure 12.3

$$\det(1-zT) = 1 - t_0 - t_1 - t_{01} + t_0t_1 + t_{01}t_1 + t_0t_{01} - t_{0111} - t_0t_{01}t_1 - t_{00111} + t_0t_{0111} \tag{12.9}$$

$$1/\zeta_{top} = 1 - 2z + 2z^3 - 2z^4$$

Numbers of periodic points (13.25):

$$\begin{aligned}\sum_{n=1} N_n z^n &= \frac{2z - 6z^3 + 8z^4}{1 - 2z + 2z^3 - 2z^4} \\ &= 2z + 4z^2 + 2z^3 + 8z^4 + 12z^5 + 28z^6 + 44z^7 + 80z^8 \\ &\quad +128z^9 + 224z^{10} + 376z^{11} + 656z^{12} + 1120z^{13} + \\ &\quad 1936z^{14} + 3312z^{15} + 5696z^{16} + \dots \end{aligned} \tag{12.10}$$

⇑PRELIMINARY

subshift!finite type

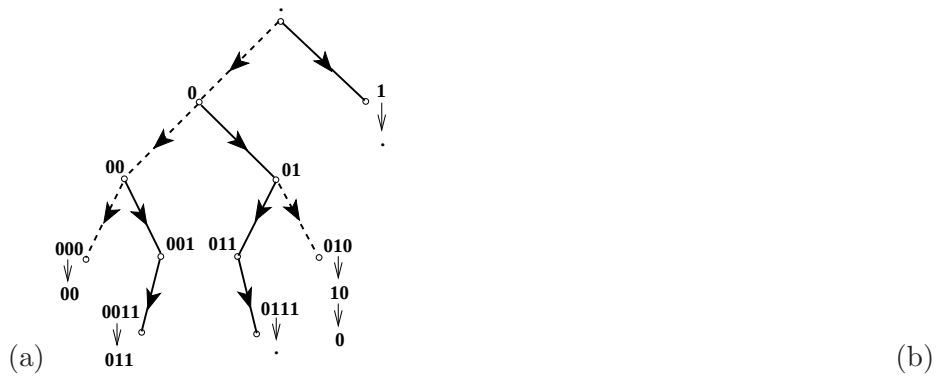

Figure 12.3: Conversion of the pruning front of figure 13.2 (d) into a finite Markov graph. (a) Starting with the start node ".", delineate all pruning blocks on the binary tree. Full line stands for "1" and the dashed line for "0". Label all internal nodes by reading the bits connecting ".", the base of the tree, to the node. The blocks 0010, 0110 are pruned by removing the dashed lines from the 001, 011 nodes. (b) Drop recursively the leading bits in the admissible blocks; if the truncated string corresponds to an internal node in (a), identical trees originate in both, so we connect them. All admissible sequences are generated as walks on this finite Markov graph. (d) Identify all distinct loops and construct the determinant (13.17).

12.4.1 Converting pruning blocks into Markov graphs

The complete binary symbolic dynamics is too simple to be illuminating, so we turn next to the simplest example of pruned symbolic dynamics, the finite subshift obtained by prohibition of repeats of one of the symbols, let us say _00_. This situation arises, for example, in studies of the circle maps, where this kind of symbolic dynamics describes "golden mean" rotations (we shall return to this example in chapter 28). Now the admissible itineraries are enumerated by the pruned binary tree of figure 11.14 (a), or the corresponding Markov graph figure 11.14 (b). We recognize this as the Markov graph example of figure 11.15.

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So we can already see the main ingradients of a general algorithm: (1) Markov graph encodes self-similarities of the tree of all itineraries, and (2) if we have a pruning block of length M, we need to descend M levels before we can start identifying the self-similar sub-trees.

Suppose now that, by hook or crook, you have been so lucky fishing for pruning rules that you now know the grammar (11.24) in terms of a finite set of pruning blocks $\mathcal{G} = \{b_1, b_2, \cdots b_k\}$, of lengths $n_{b_m} \leq M$. Our task is to generate all admissible itineraries. What to do?

A Markov graph algorithm.

1. Starting with the root of the tree, delineate all branches that correspond to all pruning blocks; implement the pruning by removing the last node in each pruning block.

2. Label all nodes internal to pruning blocks by the itinerary connecting

the root point to the internal node. Why? So far we have pruned forbidden branches by looking n_b steps into future for all pruning blocks. into future for pruning block $b = 10010$. However, the blocks with a right combination of past and future [1.0110], [10.110], [101.10] and [1011.0] are also pruned. In other words, any node whose near past coincides with the beginning of a pruning block is potentially dangerous - a branch further down the tree might get pruned.

3. Add to each internal node all remaining branches allowed by the alphabet, and label them. Why? Each one of them is the beginning point of an infinite tree, a tree that should be similar to another one originating closer to the root of the whole tree.

Peles: this one is tricky

4. Pick one of the free external nodes closest to the root of the entire tree, forget the most distant symbol in its past. Does the truncated itinerary correspond to an internal node? If yes, identify the two nodes. If not, forget the next symbol in the past, repeat. If no such truncated past corresponds to any internal node, identify with the root of the tree.

   This is a little bit abstract, so let's say the free external node in question is [1010.]. Three time steps back the past is [010.]. That is not dangerous, as no pruning block in this example starts with 0. Now forget the third step in the past: [10.] is dangerous, as that is the start of the pruning block [10.110]. Hence the free external node [1010.] should be identified with the internal node [10.].

5. Repeat until all free nodes have been tied back into the internal nodes.

6. Clean up: check whether every node can be reached from every other node. Remove the transient nodes, i.e., the nodes to which dynamics never returns.

7. The result is a Markov diagram. There is no guarantee that this is the smartest, most compact Markov diagram possible for given pruning (if you have a better algorithm, teach us), but walks around it do generate all admissible itineraries, and nothing else.

Heavy pruning.

describe transients in figure 13.3

We complete this training by examples by implementing the pruning of figure 13.2 (d). The pruning blocks are

$$[100.10], [10.1], [010.01], [011.01], [11.1], [101.10]. \tag{12.11}$$

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⇑PRELIMINARY

Blocks 01101, 10110 contain the forbidden block 101, so they are redundant as pruning rules. Draw the *pruning tree* as a section of a binary tree with 0 and 1 branches and label each internal node by the sequence of 0's and 1's connecting it to the root of the tree (figure 13.3 (a). These nodes are the potentially dangerous nodes - beginnings of blocks that might end

transient

up pruned. Add the side branches to those nodes (figure 13.3 (b). As we continue down such branches we have to check whether the pruning imposes constraints on the sequences so generated: we do this by knocking off the leading bits and checking whether the shortened strings coincide with any of the internal pruning tree nodes: $00 \to 0$; $110 \to 10$; $011 \to 11$; $0101 \to 101$ (pruned); $1000 \to 00 \to 00 \to 0$; $10011 \to 0011 \to 011 \to 11$; $01000 \to 0$.

As in the previous two examples, the trees originating in identified nodes are identical, so the tree is "self-similar". Now connect the side branches to the corresponding nodes, figure 13.3 (d). Nodes "." and 1 are transient nodes; no sequence returns to them, and as you are interested here only in infinitely recurrent sequences, delete them. The result is the finite Markov graph of figure 13.3 (d); the admissible bi-infinite symbol sequences are generated as all possible walks along this graph.

refer to K.T. Hanson's implementation, Politi?

add Per chapt. 6.1

⇓PRELIMINARY

Include sections of the paper with Kai.

RM: Finite and infinite subshifts. How the above stuff gets complicated. Serious pruning here.

12.5 Pruning dead itineraries

The splitting of cycles into the fundamental cycles and the curvature corrections depends on balancing long cycles t_{ab} against their pseudo-trajectory shadows $t_a t_b$. If the $\overline{ab}$ cycle or either of the shadows $\overline{a}$, $\overline{b}$ do not to exist, such curvature cancellation is unbalanced. In a generic dynamical system not every symbol sequence is realized as a physical trajectory; as one looks further and further, one discovers more and more rules which prohibit families of cycles, with unbalanced curvatures of any length. In the above we have used examples for which all possible orbits can be labeled by all possible binary sequences. In this chapter we discuss the cycle expansions for systems with more complicated symbolic dynamics.

Use 1-d inverse iterates to illustrate that pruning implies existence of complex orbits.

Symbolic dynamics of a generic dynamic system is arbitrarily complex; even for the logistic map the grammar is finite only for special parameter values. Our strategy is akin to bounding a real number by a sequence of rational approximants; we converge toward the non–wandering set under investigation by a sequence of self-similar Cantor sets. A "self-similar" Cantor set (in the sense in which we use the word here) is a Cantor set equipped with a *subshift of finite type* symbol dynamics, i.e., the corresponding grammar can be stated as a finite number of pruning rules, each forbidding a finite subsequence $_\epsilon_1\epsilon_2\ldots\epsilon_n_$. Here the notation $_\epsilon_1\epsilon_2\ldots\epsilon_n_$ stands for n consecutive symbols ϵ_1, ϵ_2, ..., ϵ_n, preceded and followed by arbitrary symbol strings.

The most important lesson of the pruning of the cycle expansions is that prohibition of a finite subsequence imbalances the head of a cycle expansion and increases the number of the fundamental cycles in (18.5). Hence the pruned expansions are expected to start converging only *after* all fundamental cycles have been incorporated - in the last example, the cycles $\overline{1}$, $\overline{10}$, $\overline{10100}$, $\overline{1011100}$. Without cycle expansions, no such crisp and clear cut definition of the fundamental set of scales is available.

Lozi map!bifurcation
bifurcation!Lozi map

A typical Hénon map is not complete. The problem of determining which symbol sequences are absent, or the problem orbit pruning, viewed in the x-y coordinates, appears formidable. It is equivalent to the problem of determining the location of all homoclinic tangencies, or all turnbacks of the Hénon attractor. They are dense on the attractor, and show no obvious self-similar structure in the x-y plane.

However, in the two dimensional Danish Pastry representation of the attractor, the pruning problem is easily solved; any periodic orbit which contains certain sub-sequence of symbols is forbidden. The forbidden sub-sequences are computed from the map, and are a function of its parameters.

We use the Lozi map to illustrate the orbit pruning problem. The Lozi map has the unstable-pruned orbit structure and non-trivial scaling structure of Hénon type, without the subtleties introduced by the coexistence of stable periodic orbits, or worries about the uniqueness of cycle points corresponding to a given symbol sequence. Furthermore, as the Lozi map is linear, it is computationally fast, and the attractor can be covered by hundreds of thousands of periodic points without straining patience of a desktop workstation.

For the Lozi map (3.17) the stable (unstable) manifolds W^s (W^u) are piecewise linear and easily plotted: figure **??** shows them for parameter values $(a, b) = (1.7, 0.5)$ and $(1.53, 0.5)$. Analysis of the Lozi map is much simpler than that of the Hénon map, as all primary turning points of W^u lie on the line $x = 0$, and the symbol corresponding to a point (x_n, x_{n+1}) is given by the sign of x_{t+1}. An example of a bifurcation of a primary turning point of the Lozi map is sketched in figure **??**.

missing figure **??**

KTH: The three turning points bifurcate together, and a jump of the primary turning point is 0.?

Beyond the orbit pruning and the scaling laws for the allowed unstable orbits, the Hénon attractors also own an infinity of attractive orbits coexisting with the strange attractor. We offer heuristic arguments and solid numerical evidence that the coexistence of attractive orbits does not destroy the strange attractor, which is also in this general case described by the two-dimensional Danish Pastry plot.

12.6 Pruning front

A generic once-folding map does not yield a complete horseshoe; some of the horseshoe pieces might be *pruned*, i.e., not realized for particular parameter values of the mapping. In 1 dimension, the criterion for whether a given symbolic sequence is realized by a given unimodal map is easily formulated; any orbit that strays to the right of the value computable from the *kneading sequence* (the orbit of the critical point) is pruned. This is a topological statement, independent of a particular unimodal map. Our objective is to generalize this notion to 2-d once-folding maps.

Our strategy is the following: we first construct a 2-d representation of the covering symbolic dynamics of the map. This is a "road map" in which

the various sheets of the stable and unstable manifolds are represented by straight sections, and the topology is preserved: the nearby periodic points in the symbol square represent nearby periodic points in the state space. Next we separate the admissible and the forbidden motions by means of a "pruning front", a boundary between the two kinds of orbits.

We make following assumptions:

(i) *The partition conjecture (Grassberger-Kantz) [37.3]:* the non–wandering set of a once-folding map can be described by a subset of a complete Smale horseshoe, partitioned by the set of primary turning points.

(ii) *The pruning-front conjecture (Cvitanović-Gunaratne-Procaccia) [12.22]:* kneading values of the set of all primary turning points separate the admissible from the forbidden orbits, and there are no other pruning rules.

(iii) *Multimodal map approximation [1.7]:* A 2-dimensional once-folding map can be systematically approximated by a sequence of 1-dimensional n-folding maps.

The intuition behind these conjectures is that the folding induced by a single iteration is the primary folding, and all other folds (turnbacks, homoclinic tangencies) are images or preimages of the primary ones. The asymptotic object is a collection of infinitely many 1-d sheets, and the pruning front is the set of the corresponding kneading sequences, one for each 1-d sheet.

Conjectures (i) and (ii) work for all examples that we have studied, and we are not aware of existence of any once-folding maps that contradict them.

Generic symbolic dynamics is not simple - as a matter of fact, already in one dimension its grammar can be arbitrarily complicated. However, the grammar is finite if the critical point is in the basin of attraction of an attractive periodic point; for example, if the critical point is attracted to a stable 3–cycle, the repeller consists of an isolated $\bar{0}$ fixed point and all orbits built from the two letters $\{1, 10\}$, see exercise ?!.

explain golden mean counting for unimodal maps somewhere

The analogous situation can be attained for a 2–dimensional once–folding map if the parameters of the map adjusted so that the intersection of the backward and forward folds is still transverse, but no longer complete, as in figure 13.2 (a). In this particular example the intersections _10.1_, _11.1_ have been lost, and consequently any trajectory containing substrings _101_, _111_ is pruned. We refer to the left border of this primary pruned region as the *pruning front*; an example of a pruning front is drawn in figure 13.2 (d). The pruning front is a 2–dimensional generalization of the 1–dimensional kneading sequence; the location of each vertical step in the pruning front is the kneading sequence of the corresponding primary turnback of the unstable manifolds. When correct, the pruning front conjecture is a complete description of the symbolic dynamics of once-folding maps in the same sense in which the kneading sequence defines the symbolic dynamics of a one–dimensional unimodal map. The intuition behind this conjecture is that the folding induced by a single iteration is the primary folding, and all

other folds (turnbacks, homoclinic tangencies) are images or preimages of the primary ones. The topology puts two obvious constraints on the form of a pruning front:

three-disk@3-disk!pruning front
pruning front!3-disk
Smale, S.

1. The pruning front is symmetric across the horizontal 1/2 line.

2. The pruning front is *monotone* across either half of the symbol square.

This is a consequence of the deterministic foliation; inner folds cannot pierce through the outer folds, and therefore have fewer transverse sections than the outer ones.

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12.7 Hénon map pruning front

Insert table like table 17.2 here for the pruned Hénon map values.

Hénon [3.1] has conjectured that for $(a, b) = (1.4, 0.3)$ the limit cycle of the Hénon map is a *strange attractor*. Its existence has never been proven. While for all practical purposes this is a strange attractor, pruning front techniques that we describe below enable us to find [18.1] stable attractors arbitrarily close by in the parameter space, such as the 13-cycle attractor at $(a, b) = (1.39945219, 0.3)$.

Benedicks and Carleson [12.35] have proven that in the vicinity of $(a, b) = (2, 0)$ strange attractors exist for a Lebesgue measure in the parameter space. PC: recheck the phrasing!

The Danish Pastry transformation, the simplest Baker's transformation appropriate to Hénon type mappings, yields a binary coordinatization of all possible cycle points.

explain Danish Pastry here

The symbolic dynamics of once-folding map is given by the Danish Pastry transformation. This generates both the longitudinal and transverse alternating binary tree. The longitudinal coordinate is given by the head of a symbolic sequence; the transverse coordinate is given by the tail of the symbolic sequence. The dynamics on this space is given by symbol shift permutations; volume preserving, with 2 expansion and 1/2 contraction.

12.8 3-disk pruning front

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Commentary

⇑PRELIMINARY

Remark 12.1 Smale horseshoe. S. Smale understood clearly that the crucial ingredient in the description of a chaotic flow is the topology of its non–wandering set, and he provided us with the simplest visualization of such sets as intersections of Smale horseshoes. In retrospect, much of the material covered here can already be found in Smale's fundamental paper [1.21], but a physicist who has run into a chaotic time series in his laboratory might not know that he is investigating the action (differentiable) of a Lie group G on a manifold M, and that the Lefschetz trace formula is the way to go. If you find yourself mystified by Smale's article abstract about "the action (differentiable) of a Lie group G on a manifold M",

refer also to his biographical notes

symplectic!transformation
transformation!canonical
transformation!symplectic
kneading!determinant
Markov!partition!infinite

quoted on page 201, rereading chapter 9 might help; for example, the Liouville operators form a Lie group (of symplectic, or canonical transformations) acting on the manifold (p, q).

Remark 12.2 Kneading theory. The admissible itineraries are studied in refs. [12.15, 11.8, 11.10, 11.11], as well as many others. We follow here the Milnor-Thurston exposition [12.16]. They study the topological zeta function for piecewise monotone maps of the interval, and show that for the finite subshift case it can be expressed in terms of a finite dimensional *kneading determinant*. As the kneading determinant is essentially the topological zeta function that we introduce in (13.4), we shall not discuss it here. Baladi and Ruelle have reworked this theory in a series of papers [12.18, 12.19, 12.20] and in ref. [12.21] replaced it by a power series manipulation. The kneading theory is covered here in P. Dahlqvist's appendix F.1.

Remark 12.3 Pruning fronts. The notion of a pruning front was introduced in ref. [12.22], and developed by K.T. Hansen for a number of dynamical systems in his Ph.D. thesis [1.7] and a series of papers [12.28]-[12.32]. Detailed studies of pruning fronts are carried out in refs. [12.23, 12.24, 12.53]; ref. [37.5] is the most detailed study carried out so far. The rigorous theory of pruning fronts has been developed by Y. Ishii [12.25, 12.26] for the Lozi map, and A. de Carvalho [12.27] in a very general setting.

Is ref. [37.5] the right one?

refine references, emhasize Carvalho Nonlinearity review

Remark 12.4 The unbearable growth of Markov graphs. A construction of finite Markov partitions is described in refs. [12.57, 12.58], as well as in the innumerably many other references.

If two regions in a Markov partition are not disjoint but share a boundary, the boundary trajectories require special treatment in order to avoid overcounting, see remark F.4 and sect. 22.3.1. If the image of a trial partition region cuts across only a part of another trial region and thus violates the Markov partition condition (11.4), a further refinement of the partition is needed to distinguish distinct trajectories - figure 13.2 is an example of such refinements.

⇓PRELIMINARY

⇑PRELIMINARY

use Mikkelsen termpaper to discuss border points subtleties

The finite Markov graph construction sketched above is not necessarily the minimal one; for example, the Markov graph of figure 13.3 does not generate only the "fundamental" cycles (see chapter 18), but shadowed cycles as well, such as t_{00011} in (13.17). For methods of reduction to a minimal graph, consult refs. [11.5, 12.52, 12.54]. Furthermore, when one implements the time reversed dynamics by the same algorithm, one usually gets a graph of very different topology even though both graphs generate the same admissible sequences, and have the same determinant. The algorithm described here makes some sense for 1-d dynamics, but is unnatural for 2-d maps whose dynamics it treats as one-dimensional. In practice, generic pruning grows longer and longer, and more plentiful pruning rules. For generic flows the refinements might never stop, and almost always we might have to deal with infinite Markov partitions, such as those that will be discussed in sect. 13.6. Not only do the Markov graphs get more and more unwieldy, they have the unpleasant property that every time we add a new rule, the graph has to be constructed from scratch, and it might look very different form the previous

one, even though it leads to a minute modification of the topological entropy. The most determined effort to construct such graphs may be the one of ref. [12.23]. Still, this seems to be the best technology available, unless the reader alerts us to something superior.

Résumé

Given a partition $\mathcal{A}$ of the state space $\mathcal{M}$, a dynamical system $(\mathcal{M}, f)$ induces topological dynamics (Σ, σ) on the space Σ of all admissible bi–infinite itineraries. The itinerary describes the time evolution of an orbit, while (for 2-d hyperbolic maps) the symbol square describes the spatial ordering of points along the orbit. The rule that everything to one side of the pruning front is forbidden might (in hindsight) seem obvious, but if you have ever tried to work out symbolic dynamics of some "generic" dynamical system, you should be struck by its simplicity: instead of pruning a Cantor set embedded within some larger Cantor set, the pruning front cleanly cuts out a *compact* region in the symbol square and that is all - there are no additional pruning rules.

The symbol square is a useful tool in transforming topological pruning into pruning rules for inadmissible sequences; those are implemented by constructing transition matrices and/or Markov graphs. These matrices are the simplest examples of evolution operators prerequisite to developing a theory of averaging over chaotic flows.

Importance of symbolic dynamics is often grossly unappreciated; as we shall see in chapters 16 and 18, coupled with uniform hyperbolicity, the existence of a finite grammar is the crucial prerequisite for construction of zeta functions with nice analyticity properties.

extract parts of refsKnead.tex

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remember to put improvements back to refs.all

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⇓PRELIMINARY

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⇑PRELIMINARY

Exercises

Exercise 12.1 A Smale horseshoe. *The Hénon map*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 - ax^2 + y \\ bx \end{bmatrix} \tag{12.12}$$

maps the (x, y) plane into itself - it was constructed by Hénon [3.1] in order to mimic the Poincaré section of once-folding map induced by a flow like the one sketched in figure 11.7. For definitivness fix the parameters to $a = 6$, $b = -1$.

a) *Draw a rectangle in the (x, y) plane such that its nth iterate by the Hénon map intersects the rectangle 2^n times.*

b) *Construct the inverse of the (12.12).*

c) *Iterate the rectangle back in the time; how many intersections are there between the n forward and m backward iterates of the rectangle?*

d) *Use the above information about the intersections to guess the (x, y) coordinates for the two fixed points, a 2-cycle point, and points on the two distinct 3-cycles from table 11.1. The exact cycle points are computed in exercise 17.11.*

Exercise 12.2 Kneading Danish pastry. *Write down the $(x, y) \to (x, y)$ mapping that implements the baker's map of figure 12.2, together with the inverse mapping. Sketch a few rectangles in symbol square and their forward and backward images. (Hint: the mapping is very much like the tent map (11.8)).*

Exercise 12.3 Kneading Danish without flipping. *The baker's map of figure 12.2 includes a flip - a map of this type is called an orientation reversing once-folding map. Write down the $(x, y) \to (x, y)$ mapping that implements an orientation preserving baker's map (no flip; Jacobian determinant = 1). Sketch and label the first few foldings of the symbol square.*

Exercise 12.4 Fix this manuscript. *Check whether the layers of the baker's map of figure 12.2 are indeed ordered as the branches of the alternating binary tree of figure 11.9. (They might not be - we have not rechecked them). Draw the correct binary trees that order both the future and past itineraries.*

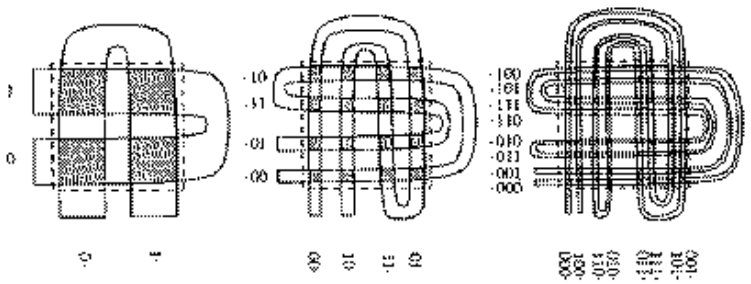
For once-folding maps there are four topologically distinct ways of laying out the stretched and folded image of the starting region,

(a) *orientation preserving: stretch, fold upward, as in figure 12.4*

(b) *orientation preserving: stretch, fold downward, as in figure 13.2*

(c) *orientation reversing: stretch, fold upward, flip, as in figure 12.5*

(d) *orientation reversing: stretch, fold downward, flip, as in figure 12.2,*



alternating binary tree
binary!tree, alternating

Figure 12.4: A complete Smale horseshoe iterated forwards and backwards, orientation preserving case: function f maps the dashed border square $\mathcal{M}$ into the vertical horseshoe, while the inverse map f^{-1} maps it into the horizontal horseshoe. a) One iteration, b) two iterations, c) three iterations. The non–wandering set is contained within the intersection of the forward and backward iterates (crosshatched). (from K.T. Hansen [1.7])

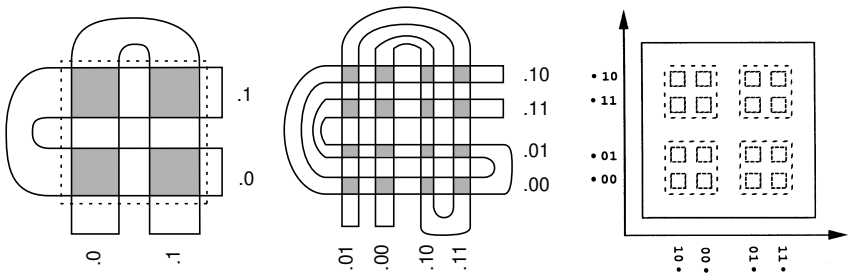


Figure 12.5: An orientation reversing Smale horseshoe map. Function $f = \{\text{stretch,fold,flip}\}$ maps the dashed border square $\mathcal{M}$ into the vertical horseshoe, while the inverse map f^{-1} maps it into the horizontal horseshoe. a) one iteration, b) two iterations, c) the non–wandering set cover by 16 rectangles, each labeled by the 2 past and the 2 future steps. (from K.T. Hansen [1.7])

with the corresponding four distinct binary-labeled symbol squares. For n-fold "stretch & fold" flows the labeling would be $n$ary. The intersection $\mathcal{M}_0$ for the orientation preserving Smale horseshoe, figure 12.4a, is oriented the same way as $\mathcal{M}$, while $\mathcal{M}_1$ is oriented opposite to $\mathcal{M}$. Brief contemplation of figure 12.2 indicates that the forward iteration strips are ordered relative to each other as the branches of the alternating binary tree in figure 11.9.

Check the labeling for all four cases.

lower labels in figure 12.5b wrong

Exercise 12.5 Orientation reversing once-folding map. *By adding a reflection around the vertical axis to the horseshoe map g we get the orientation reversing map $\tilde{g}$ shown in figure 12.5. $\tilde{Q}_0$ and $\tilde{Q}_1$ are oriented as Q_0 and Q_1, so the definition of the future topological coordinate γ is identical to the γ for the orientation preserving horseshoe. The inverse intersections $\tilde{Q}_0^{-1}$ and $\tilde{Q}_1^{-1}$ are oriented so that $\tilde{Q}_0^{-1}$ is opposite to Q, while $\tilde{Q}_1^{-1}$ has the same orientation as Q. Check that the past topological coordinate δ is given by*

$$
\begin{aligned}
w_{n-1} &= \begin{cases} 1-w_n & \text{if } s_n = 0 \\ w_n & \text{if } s_n = 1 \end{cases}, \qquad w_0 = s_0 \\
\delta(x) &= 0.w_0 w_{-1} w_{-2} \ldots = \sum_{n=1}^{\infty} w_{1-n}/2^n \,. \qquad (12.13)
\end{aligned}
$$

Exercise 12.6 Infinite symbolic dynamics. *Let σ be a function that returns zero or one for every infinite binary string: $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}$. Its value is represented by $\sigma(\epsilon_1, \epsilon_2, \ldots)$ where the ϵ_i are either 0 or 1. We will now define an operator $\mathcal{T}$ that acts on observables on the space of binary strings. A function a is an observable if it has bounded variation, that is, if*

$$\|a\| = \sup_{\{\epsilon_i\}} |a(\epsilon_1, \epsilon_2, \ldots)| < \infty \,.$$

For these functions

$$\mathcal{T} a(\epsilon_1, \epsilon_2, \ldots) = a(0, \epsilon_1, \epsilon_2, \ldots)\sigma(0, \epsilon_1, \epsilon_2, \ldots) + a(1, \epsilon_1, \epsilon_2, \ldots)\sigma(1, \epsilon_1, \epsilon_2, \ldots) \,.$$

(a) *(easy) Consider a finite version T_n of the operator $\mathcal{T}$:*

$$\begin{aligned} T_n a(\epsilon_1, \epsilon_2, \ldots, \epsilon_{1,n}) = \\ a(0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1})\sigma(0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}) + \\ a(1, \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1})\sigma(1, \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}) \,. \end{aligned}$$

Show that T_n is a $2^n \times 2^n$ matrix. Show that its trace is bounded by a number independent of n.

(b) *(medium) With the operator norm induced by the function norm, show that $\mathcal{T}$ is a bounded operator.*

(c) *(hard) Show that $\mathcal{T}$ is not trace class. (Hint: check if $\mathcal{T}$ is compact "trace class" is defined in appendix M.)*

Exercise 12.7 Time reversability.** *Hamiltonian flows are time reversible. Does that mean that their Markov graphs are symmetric in all node $\to$ node links, their transition matrices are adjacency matrices, symmetric and diagonalizable, and that they have only real eigenvalues?*

Exercise 12.8 Alphabet {0,1}, prune only the fixed point $\overline{0}$. *This is equivalent to the* infinite *alphabet {1, 2, 3, 4, ...} unrestricted symbolic dynamics. The prime cycles are labeled by all non-repeating sequences of integers, ordered lexically: $t_n, n > 0$; $t_{mn}, t_{mmn}, \ldots, n > m > 0$; $t_{mnr}, r > n > m > 0, \ldots$ (see sect. 21.3). Now the number of fundamental cycles is infinite as well:*

$$\begin{aligned} 1/\zeta \;=\; & 1 - \sum_{n>0} t_n - \sum_{n>m>0} (t_{mn} - t_n t_m) \\ & - \sum_{n>m>0} (t_{mmn} - t_m t_{mn}) - \sum_{n>m>0} (t_{mnn} - t_{mn} t_n) \qquad (12.14) \\ & - \sum_{r>n>m>0} (t_{mnr} + t_{mrn} - t_{mn} t_r - t_{mr} t_n - t_m t_{nr} + t_m t_n t_r) \qquad (12.15) \end{aligned}$$

⇓PRELIMINARY

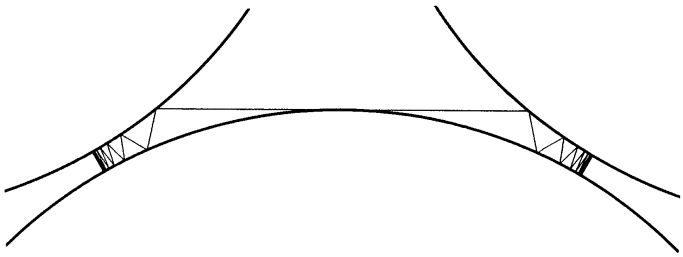
We have already encountered this sum in the Ulam map cycle expansion in sect. ?!.

⇑PRELIMINARY

As shown in sect. 21.3, this grammar plays an important role in description of fixed points of marginal stability.

Exercise 12.9 3-disk pruning *(Not easy) Show that for 3-disk game of pinball the pruning of orbits starts at* $R : a = 2.04821419\ldots$.

three-disk@3-disk!pruning front
pruning front!3-disk



(Kai T. Hansen)

Exercise 12.10 Alphabet {0,1}, prune $_1000_$, $_00100_$, $_01100_$. *This example is motivated by the pruning front description of the symbolic dynamics for the Hénon-type maps, sect. 12.7.*

⇓PRIVATE

⇑PRIVATE

step 1. $_1000_$ *prunes all cycles with a* $_000_$ *subsequence with the exception of the fixed point* $\overline{0}$*; hence we factor out* $(1-t_0)$ *explicitly, and prune* $_000_$ *from the rest. This means that* x_0 *is an isolated fixed point - no cycle stays in its vicinity for more than 2 iterations. In the notation of sect. 12.4.1, the alphabet is* $\{1, 2, 3; \overline{0}\}$*, and the remaining pruning rules have to be rewritten in terms of symbols 2=10, 3=100:*

step 2. *alphabet* $\{1, 2, 3; \overline{0}\}$*, prune* $_33_$, $_213_$,$_313_$. *This means that the 3-cycle* $\overline{3} = \overline{100}$ *is pruned and no long cycles stay close enough to it for a single* $_100_$ *repeat. As in example 1?!, prohibition of* $_33_$ *is implemented by dropping the symbol "3" and extending the alphabet by the allowed blocks 13, 23:*

restore right reference

step 3. *alphabet* $\{1, 2, \underline{13}, \underline{23}; \overline{0}\}$*, prune* $_\underline{213}_$, $_\underline{23}\,\underline{13}_$, $_\underline{13}\,\underline{13}_$*, where* $\underline{13} = 13$, $\underline{23} = 23$ *are now used as single letters. Pruning of the repetitions* $_\underline{13}\,\underline{13}_$ *(the 4-cycle* $\overline{\underline{13}} = \overline{1100}$ *is pruned) yields the*

result: *alphabet* $\{1, 2, \underline{23}, \underline{113}; \overline{0}\}$*, unrestricted 4-ary dynamics. The other remaining possible blocks* $_\underline{213}_$, $_\underline{2313}_$ *are forbidden by the rules of step 3. The cycle expansion is given by*

$$1/\zeta = (1-t_0)(1-t_1-t_2-t_{23}-t_{113}) \tag{12.16}$$

for unrestricted 4-letter alphabet $\{1, 2, \underline{23}, \underline{113}\}$.

add Markov graph exercises.

Redo exercises by Markov graphs.

admissible!trajectories, number of

Chapter 13

Counting

> That which is crooked cannot be made straight: and that which is wanting cannot be numbered.
>
> Ecclestiastes 1.15

We are now in a position to develop our first prototypical application of periodic orbit theory: cycle counting. This is the simplest illustration of the raison d'etre of periodic orbit theory; we shall develop a duality transformation that relates *local* information - in this case the next admissible symbol in a symbol sequence - to *global* averages, in this case the mean rate of growth of the number of admissible itineraries with increasing itinerary length. We shall transform the topological dynamics of chapter 11 into a multiplicative operation by means of transition matrices/Markov graphs, and show that the nth power of a transition matrix counts all itineraries of length n. The asymptotic growth rate of the number of admissible itineraries is therefore given by the leading eigenvalue of the transition matrix; the leading eigenvalue is in turn given by the leading zero of the characteristic determinant of the transition matrix, which is - in this context - called the *topological zeta function*. For flows with finite Markov graphs this determinant is a finite polynomial which can be read off the Markov graph.

The method goes well beyond the problem at hand, and forms the core of the entire treatise, making tangible a rather abstract notion of "spectral determinants" yet to come.

13.1 Counting itineraries

In the 3-disk system the number of admissible trajectories doubles with every iterate: there are $K_n = 3 \cdot 2^n$ distinct itineraries of length n. If disks are too close and some part of trajectories is pruned, this is only an upper bound and explicit formulas might be hard to discover, but we still might be able to establish a lower exponential bound of the form $K_n \geq Ce^{n\hat{h}}$. make up an exercise

topological!entropy
entropy!topological
transition matrix
Perron-Frobenius!matrix

Bounded exponentially by $3e^{n\ln 2} \geq K_n \geq Ce^{n\hat{h}}$ the number of trajectories must grow exponentially as a function of the itinerary length, with rate given by the *topological entropy*:

$$h = \lim_{n\to\infty} \frac{1}{n} \ln K_n \ . \tag{13.1}$$

We shall now relate this quantity to the leading, with rateriven eigenvalue of the transition matrix.

The transition matrix element $T_{ij} \in \{0, 1\}$ in (11.2) indicates whether the transition from the starting partition j into partition i in one step is allowed or not, and the (i, j) element of the transition matrix iterated n times is

13.1
page 246
add figure

$$(T^n)_{ij} = \sum_{k_1,k_2,\ldots,k_{n-1}} T_{ik_1} T_{k_1k_2} \ldots T_{k_{n-1}j}$$

receives a contribution 1 from every admissible sequence of transitions, so $(T^n)_{ij}$ is the number of admissible n symbol itineraries starting with j and ending with i.

Example 13.1 3-disk itinerary counting.

The $(T^2)_{13} = 1$ *element of* T^2 *for the 3-disk transition matrix (11.5)*

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} . \tag{13.2}$$

corresponds to $3 \to 2 \to 1$*, the only 2-step path from 3 to 1, while* $(T^2)_{33} = 2$ *counts the two itineraries 313 and 323.*

The total number of admissible itineraries of n symbols is

$$K_n = \sum_{ij} (T^n)_{ij} = (\,1, 1, \ldots, 1\,)\, T^n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} . \tag{13.3}$$

We can also count the number of prime cycles and pruned periodic points, but in order not to break up the flow of the main argument, we relegate these pretty results to sects. 13.5.2 and 13.7. Recommended reading if you ever have to compute lots of cycles.

The matrix T has non-negative integer entries. A matrix M is said to be *Perron-Frobenius* if some power k of M has strictly positive entries, $(M^k)_{rs} > 0$. In the case of the transition matrix T this means that

every partition eventually reaches all of the partitions, i.e., the partition is dynamically transitive or indecomposable, as assumed in (2.2). The notion of *transitivity* is crucial in ergodic theory: a mapping is transitive if it has a dense orbit. This notion is inherited by the shift operation once we introduce a symbolic dynamics. If that is not the case, state space decomposes into disconnected pieces, each of which can be analyzed separately by a separate indecomposable Markov graph. Hence it suffices to restrict our considerations to transition matrices of Perron-Frobenius type.

topological!transitivity
dynamical!transitivity
indecomposability

A finite $[N \times N]$ matrix T has eigenvalues $T\varphi_\alpha = \lambda_\alpha \varphi_\alpha$ and (right) eigenvectors $\{\varphi_0, \varphi_1, \cdots, \varphi_{M-1}\}$. Expressing the initial vector in (13.3) in this basis (which might be incomplete, $M \leq N$),

$$T^n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = T^n \sum_{\alpha=0}^{N-1} b_\alpha \varphi_\alpha = \sum_{\alpha=0}^{N-1} b_\alpha \lambda_\alpha^n \varphi_\alpha \,,$$

and contracting with $(\,1, 1, \ldots, 1\,)$, we obtain

rethink λ, later using e^s

$$K_n = \sum_{\alpha=0}^{N-1} c_\alpha \lambda_\alpha^n \,.$$

13.2
page 246

The constants c_α depend on the choice of initial and final partitions: In this example we are sandwiching T^n between the vector $(\,1, 1, \ldots, 1\,)$ and its transpose, but any other pair of vectors would do, as long as they are not orthogonal to the leading eigenvector φ_0. In a experiment the vector $(\,1, 1, \ldots, 1\,)$ would be replaced by a description of the initial state,and the right vector would describe the measure time n later.

Perron theorem states that a Perron-Frobenius matrix has a nondegenerate positive real eigenvalue $\lambda_0 > 1$ (with a positive eigenvector) which exceeds the moduli of all other eigenvalues. Therefore as n increases, the sum is dominated by the leading eigenvalue of the transition matrix, $\lambda_0 > |\mathrm{Re}\, \lambda_\alpha|$, $\alpha = 1, 2, \cdots, N-1$, and the topological entropy (13.1) is given by

$$\begin{aligned} h &= \lim_{n\to\infty} \frac{1}{n} \ln c_0 \lambda_0^n \left[1 + \frac{c_1}{c_0} \left(\frac{\lambda_1}{\lambda_0} \right)^n + \cdots \right] \\ &= \ln \lambda_0 + \lim_{n\to\infty} \left[\frac{\ln c_0}{n} + \frac{1}{n} \frac{c_1}{c_0} \left(\frac{\lambda_1}{\lambda_0} \right)^n + \cdots \right] \\ &= \ln \lambda_0 \,. \end{aligned} \tag{13.4}$$

What have we learned? The transition matrix T is a one-step *short time* operator, advancing the trajectory from a partition to the next admissible partition. Its eigenvalues describe the rate of growth of the total number of trajectories at the *asymptotic times*. Instead of painstakingly counting

emphasize positive definiteness

trace!local
transition matrix
cyclic!symmetry
symmetry!cyclic
admissible!periodic points
periodic!point!admissible

$K_1, K_2, K_3, \ldots$ and estimating (13.1) from a slope of a log-linear plot, we have the *exact* topological entropy if we can compute the leading eigenvalue of the transition matrix T. This is reminiscent of the way the free energy is computed from transfer matrix for one-dimensional lattice models with finite range interactions. Historically, it is analogy with statistical mechanics that led to introduction of evolution operator methods into the theory of chaotic systems, theory that will be developed further in chapter 20.

13.2 Topological trace formula

There are two standard ways of getting at eigenvalues of a matrix - by evaluating the trace $\mathrm{tr}\, T^n = \sum \lambda_\alpha^n$, or by evaluating the determinant $\det(1 - zT)$. We start by evaluating the trace of transition matrices.

Consider an M-step memory transition matrix, like the 1-step memory example (11.16). The trace of the transition matrix counts the number of partitions that map into themselves. In the binary case the trace picks up only two contributions on the diagonal, $T_{0\cdots0,0\cdots0} + T_{1\cdots1,1\cdots1}$, no matter how much memory we assume. We can even take infinite memory $M \to \infty$, in which case the contributing partitions are shrunk to the fixed points, $\mathrm{tr}\, T = T_{\overline{0},\overline{0}} + T_{\overline{1},\overline{1}}$.

11.10 page 197

More generally, each closed walk through n concatenated entries of T contributes to $\mathrm{tr}\, T^n$ a product of the matrix entries along the walk. Each step in such a walk shifts the symbolic string by one symbol; the trace ensures that the walk closes on a periodic string c. Define t_c to be the *local trace*, the product of matrix elements along a cycle c, each term being multiplied by a book keeping variable z. $z^n \mathrm{tr}\, T^n$ is then the sum of t_c for all cycles of length n. For example, for an $[8\times8]$ transition matrix $T_{s_1s_2s_3,s_0s_1s_2}$ version of (11.16), or any refined partition $[2^n\times2^n]$ transition matrix, n arbitrarily large, the periodic point $\overline{100}$ contributes $t_{100} = z^3 T_{\overline{100},\overline{010}} T_{\overline{010},\overline{001}} T_{\overline{001},\overline{100}}$ to $z^3 \mathrm{tr}\, T^3$. This product is manifestly cyclically symmetric, $t_{100} = t_{010} = t_{001}$, and so a prime cycle p of length n_p contributes n_p times, once for each periodic point along its orbit. For the binary labeled non–wandering set the first few traces are given by (consult tables 11.1 and 13.1)

11.10 page 197

$$\begin{aligned} z\,\mathrm{tr}\, T &= t_0 + t_1, \\ z^2 \mathrm{tr}\, T^2 &= t_0^2 + t_1^2 + 2t_{10}, \\ z^3 \mathrm{tr}\, T^3 &= t_0^3 + t_1^3 + 3t_{100} + 3t_{101}, \\ z^4 \mathrm{tr}\, T^4 &= t_0^4 + t_1^4 + 2t_{10}^2 + 4t_{1000} + 4t_{1001} + 4t_{1011}. \end{aligned} \tag{13.5}$$

For complete binary symbolic dynamics $t_p = z^{n_p}$ for every binary prime cycle p; if there is pruning $t_p = z^{n_p}$ if p is admissible cycle and $t_p = 0$ otherwise. Hence $\mathrm{tr}\, T^n$ counts the number of *admissible periodic points* of period n. In general, the nth order trace (13.5) picks up contributions

n	N_n	# of prime cycles of length n_p									
		1	2	3	4	5	6	7	8	9	10
1	2	2									
2	4	2	1								
3	8	2		2							
4	16	2	1		3						
5	32	2				6					
6	64	2	1	2			9				
7	128	2						18			
8	256	2	1		3				30		
9	512	2		2						56	
10	1024	2	1			6					99

Table 13.1: The total numbers of periodic points N_n of period n for binary symbolic dynamics. The numbers of prime cycles contributing illustrates the preponderance of long prime cycles of length n over the repeats of shorter cycles of lengths n_p, $n = rn_p$. Further listings of binary prime cycles are given in tables 11.1 and 13.2. (L. Rondoni)

trace!formula!topological
topological!trace formula
Laplace!transform, discrete

from all repeats of prime cycles, with each cycle contributing n_p periodic points, so the total number of periodic points of period n is given by

$$z^n N_n = z^n \mathrm{tr}\, T^n = \sum_{n_p|n} n_p t_p^{n/n_p} = \sum_p n_p \sum_{r=1}^{\infty} \delta_{n,n_p r} t_p^r \,. \tag{13.6}$$

Here $m|n$ means that m is a divisor of n, and (taking $z = 1$) $t_p = 1$ if the cycle is admissible, and $t_p = 0$ otherwise.

In order to get rid of the awkward divisibility constraint $n = n_p r$ in the above sum, we introduce the generating function for numbers of periodic points

credit Rondoni for figure 13.1

$$\sum_{n=1}^{\infty} z^n N_n = \mathrm{tr}\, \frac{zT}{1-zT} \,. \tag{13.7}$$

Substituting (13.6) into the left hand side, and replacing the right hand side by the eigenvalue sum $\mathrm{tr}\, T^n = \sum \lambda_\alpha^n$, we obtain our first example of a trace formula, the *topological trace formula*

$$\sum_{\alpha=0} \frac{z\lambda_\alpha}{1-z\lambda_\alpha} = \sum_p \frac{n_p t_p}{1-t_p} \,. \tag{13.8}$$

A trace formula relates the spectrum of eigenvalues of an operator - in this case the transition matrix - to the spectrum of periodic orbits of the dynamical system. The z^n sum in (13.7) is a discrete version of the Laplace transform (see chapter 14), and the resolvent on the left hand side is the antecedent of the more sophisticated trace formulas (14.9), (14.21) and (36.3). We shall now use this result to compute the spectral determinant of the transition matrix.

more specific

determinant!spectral
spectral!determinant
determinant!trace relation
cumulant!expansion

13.3 Determinant of a graph

Our next task is to determine the zeros of the *spectral determinant* of an $[M \times M]$ transition matrix

$$\det(1 - zT) = \prod_{\alpha=0}^{M-1} (1 - z\lambda_\alpha) \ . \tag{13.9}$$

We could now proceed to diagonalize T on a computer, and get this over with. It pays, however, to dissect $\det(1-zT)$ with some care; understanding this computation in detail will be the key to understanding the cycle expansion computations of chapter 18 for arbitrary dynamical averages. For T a finite matrix, (13.9) is just the characteristic equation for T. However, we shall be able to compute this object even when the dimension of T and other such operators goes to ∞, and for that reason we prefer to refer to (13.9) loosely as the "spectral determinant".

There are various definitions of the determinant of a matrix; they mostly reduce to the statement that the determinant is a certain sum over all possible permutation cycles composed of the traces $\operatorname{tr} T^k$, in the spirit of the determinant–trace relation (1.15):

4.1 page 78

$$\begin{aligned} \det(1 - zT) &= \exp(\operatorname{tr} \ln(1 - zT)) = \exp\left(-\sum_{n=1} \frac{z^n}{n} \operatorname{tr} T^n\right) \\ &= 1 - z \operatorname{tr} T - \frac{z^2}{2}\left((\operatorname{tr}\ T)^2 - \operatorname{tr}(T^2)\right) - \dots \end{aligned} \tag{13.10}$$

This is sometimes called a cumulant expansion. Formally, the right hand is an infinite sum over powers of z^n. If T is an $[M\times M]$ finite matrix, then the characteristic polynomial is at most of order M. In that case the coefficients of z^n, $n > M$ must vanish *exactly*.

We now proceed to relate the determinant in (13.10) to the corresponding Markov graph of chapter 11: to this end we start by the usual algebra textbook expression for a determinant as the sum of products of all permutations

$$\det(1-zT) = \sum_{\{\pi\}} (-1)^\pi (1-zT)_{1,\pi_1}(1-zT)_{2,\pi_2} \cdots (1-zT)_{M,\pi_M} \tag{13.11}$$

where T is a $[M\times M]$ matrix, $\{\pi\}$ denotes the set of permutations of M symbols, π_k is what k is permuted into by the permutation π, and $(-1)^\pi = \pm 1$ is the parity of permutation π. The right hand side of (13.11) yields a polynomial of order M in z: a contribution of order n in z picks up $M-n$ unit factors along the diagonal, the remaining matrix elements yielding

$$(-z)^n (-1)^{\tilde{\pi}} T_{\eta_1,\tilde{\pi}_{\eta_1}} \cdots T_{\eta_n,\tilde{\pi}_{\eta_n}} \tag{13.12}$$

where $\tilde{\pi}$ is the permutation of the subset of n distinct symbols $\eta_1 \dots \eta_n$ indexing T matrix elements. As in (13.5), we refer to any combination $t_c = T_{\eta_1\eta_2} T_{\eta_2\eta_3} \cdots T_{\eta_k\eta_1}$, for a given itinerary $c = \eta_1\eta_2 \cdots, \eta_k$, as the *local trace* associated with a closed loop c on the Markov graph. Each term of form (13.12) may be factored in terms of local traces $t_{c_1} t_{c_2} \cdots t_{c_k}$, that is loops on the Markov graph. These loops are non-intersecting, as each node may only be reached by *one* link, and they are indeed loops, as if a node is reached by a link, it has to be the starting point of another *single* link, as each η_j must appear exactly *once* as a row and column index.

characteristic!polynomial
polynomial!characteristic
loop!intersecting
fundamental!cycle
cycle!fundamental

GT: make this digestible

So the general structure is clear, a little more thinking is only required to get the sign of a generic contribution. We consider only the case of loops of length 1 and 2, and leave to the reader the task of generalizing the result by induction. Consider first a term in which only loops of unit length appear on (13.12), that is, only the diagonal elements of T are picked up. We have $k = n$ loops and an even permutation $\tilde{\pi}$ so the sign is given by $(-1)^k$, k being the number of loops. Now take the case in which we have i single loops and j loops of length $n = 2j+i$. The parity of the permutation gives $(-1)^j$ and the first factor in (13.12) gives $(-1)^n = (-1)^{2j+i}$. So once again these terms combine into $(-1)^k$, where $k = i + j$ is the number of loops. We may summarize our findings as follows:

lots of pictures needed

13.3
page 246

> The characteristic polynomial of a transition matrix/Markov graph is given by the sum of all possible partitions π of the graph into products of non-intersecting loops, with each loop trace t_p carrying a minus sign:

GT: (re)define k, f

$$\det(1 - zT) = \sum_{k=0}^{f} {\sum_{\pi}}' (-1)^k t_{p_1} \cdots t_{p_k} \qquad (13.13)$$

Any self-intersecting loop is *shadowed* by a product of two loops that share the intersection point. As both the long loop t_{ab} and its shadow $t_a t_b$ in the case at hand carry the same weight $z^{n_a+n_b}$, the cancellation is exact, and the loop expansion (13.13) is finite, with f the maximal number of non-intersecting loops.

GT: discuss non-intersecting, self-intersecting

We refer to the set of all non-self-intersecting loops $\{t_{p_1}, t_{p_2}, \cdots t_{p_f}\}$ as the *fundamental cycles*. This is not a very good definition, as the Markov graphs are not unique – the most we know is that for a given finite-grammar language, there exist Markov graph(s) with the minimal number of loops. Regardless of how cleverly a Markov graph is constructed, it is always true that for any finite Markov graph the number of fundamental cycles f is finite. If you know a better way to define the "fundamental cycles", let us know.

fast track:
sect. 13.4, p. 233

golden mean!pruning
pruning!golden mean

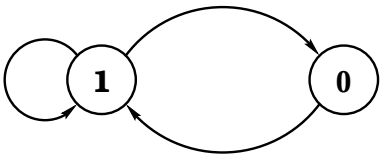


Figure 13.1: The golden mean pruning rule Markov graph, see also figure 11.14.

13.3.1 Topological polynomials: learning by examples

The above definition of the determinant in terms of traces is most easily grasped by working through a few examples. The complete binary dynamics Markov graph of figure 11.12 (b) is a little bit too simple, but let us start humbly.

Example 13.2 Topological polynomial for complete binary dynamics: *There are only two non-intersecting loops, yielding*

$$\det(1-zT) = 1-t_0-t_1 = 1-2z\,. \tag{13.14}$$

The leading (and only) zero of this characteristic polynomial yields the topological entropy $e^h = 2$. As we know that there are $K_n = 2^n$ binary strings of length N, we are not surprised by this result.

Similarly, for complete symbolic dynamics of N symbols the Markov graph has one node and N links, yielding

$$\det(1-zT) = 1-Nz\,, \tag{13.15}$$

whence the topological entropy $h = \ln N$.

Example 13.3 Golden mean pruning: *A more interesting example is the "golden mean" pruning of figure 13.1. There is only one grammar rule, that a repeat of symbol 0 is forbidden. The non-intersecting loops are of length 1 and 2, so the topological polynomial is given by*

13.4
page 246

$$\det(1-zT) = 1-t_1-t_{01} = 1-z-z^2\,. \tag{13.16}$$

The leading root of this polynomial is the golden mean, so the entropy (13.4) is the logarithm of the golden mean, $h = \ln\frac{1+\sqrt{5}}{2}$.

recheck appropriateness/correctness of figure 13.2.

replace this by Biham

Example 13.4 Nontrivial pruning: *The non-self-intersecting loops of the Markov graph of figure 13.3 (d) are indicated in figure 13.3 (e). The determinant can be written down by inspection, as the sum of all possible partitions of the graph into products of non-intersecting loops, with each loop carrying a minus sign:*

⇓PRELIMINARY

13.10
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⇑PRELIMINARY

$$\begin{aligned}\det(1-zT) &= 1-t_0-t_{0011}-t_{0001}-t_{00011}\\ &\quad +t_0t_{0011}+t_{0011}t_{0001}\,.\end{aligned} \tag{13.17}$$

With $t_p = z^{n_p}$, where n_p is the length of the p-cycle, the smallest root of

13.11
page 248

ov!graph!infinite
lant!expansion

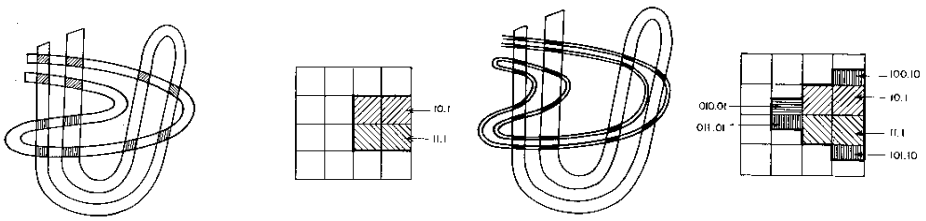

Figure 13.2: (a) An incomplete Smale horseshoe: the inner forward fold does not intersect the two rightmost backward folds. (b) The primary pruned region in the symbol square and the corresponding forbidden binary blocks. (c) An incomplete Smale horseshoe which illustrates (d) the monotonicity of the pruning front: the thick line which delineates the left border of the primary pruned region is monotone on each half of the symbol square. The backward folding in figures (a) and (c) is only schematic - in invertible mappings there are further missing intersections, all obtained by the forward and backward iterations of the primary pruned region.

$$0 = 1 - z - 2z^4 + z^8 \tag{13.18}$$

yields the topological entropy $h = -\ln z$, $z = 0.658779\ldots$, $h = 0.417367\ldots$, *significantly smaller than the entropy of the covering symbolic dynamics, the complete binary shift* $h = \ln 2 \;=\; 0.693\ldots$

in depth:
sect. S.1, p. 1175

13.4 Topological zeta function

What happens if there is no finite-memory transition matrix, if the Markov graph is infinite? If we are never sure that looking further into future will reveal no further forbidden blocks? There is still a way to define the determinant, and this idea is central to the whole treatise: the determinant is then defined by its *cumulant* expansion (13.10)

GT: put Newton formula in here? the take $N \to \infty$

4.1
page 78

$$\det(1 - zT) = 1 - \sum_{n=1}^{\infty} \hat{c}_n z^n \,. \tag{13.19}$$

For finite dimensional matrices the expansion is a finite polynomial, and (13.19) is an identity; however, for infinite dimensional operators the cumulant expansion coefficients $\hat{c}_n$ *define* the determinant.

Let us now evaluate the determinant in terms of traces for an arbitrary transition matrix. In order to obtain an expression for the spectral determinant (13.9) in terms of cycles, substitute (13.6) into (13.19) and sum over the repeats of prime cycles using $\ln(1-x) = \sum_r x^r/r$,

$$\det(1 - zT) = \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{t_p^r}{r}\right) = \prod_p (1 - t_p)\,, \tag{13.20}$$

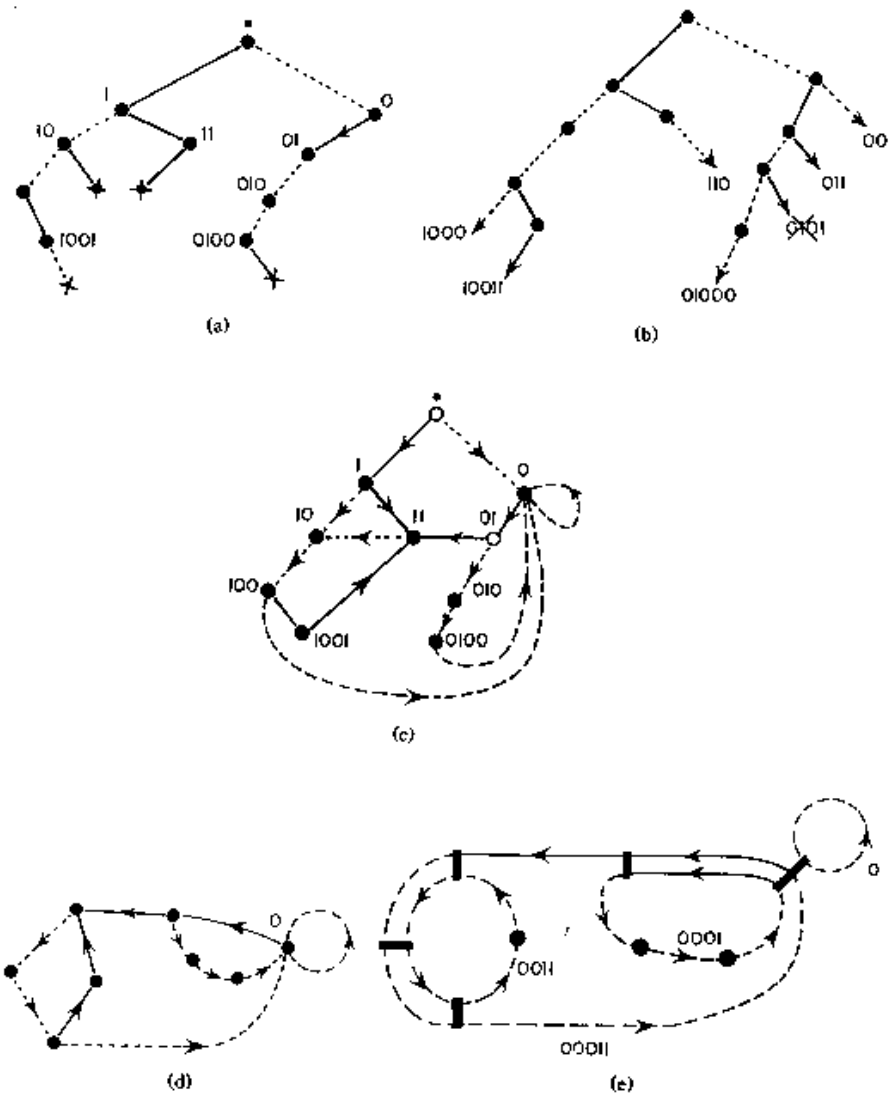

Figure 13.3: Conversion of the pruning front of figure 13.2 (d) into a finite Markov graph. (a) Starting with the start node ".", delineate all pruning blocks on the binary tree. A solid line stands for "1" and a dashed line for "0". Ends of forbidden strings are marked with $\times$. Label all internal nodes by reading the bits connecting ".", the base of the tree, to the node. (b) Indicate all admissible starting blocks by arrows. (c) Drop recursively the leading bits in the admissible blocks; if the truncated string corresponds to an internal node in (a), connect them. (d) Delete the transient, non-circulating nodes; all admissible sequences are generated as walks on this finite Markov graph. (e) Identify all distinct loops and construct the determinant (13.17).

where for the topological entropy the weight assigned to a prime cycle p of length n_p is $t_p = z^{n_p}$ if the cycle is admissible, or $t_p = 0$ if it is pruned. This determinant is called the *topological* or the *Artin-Mazur* zeta function, conventionally denoted by

$$1/\zeta_{\rm top} = \prod_p (1 - z^{n_p}) = 1 - \sum_{n=1} \hat{c}_n z^n \ . \tag{13.21}$$

Counting cycles amounts to giving each admissible prime cycle p weight $t_p = z^{n_p}$ and expanding the Euler product (13.21) as a power series in z. As the precise expression for coefficients $\hat{c}_n$ in terms of local traces t_p is more general than the current application to counting, we shall postpone its derivation to chapter 18.

The topological entropy h can now be determined from the leading zero $z = e^{-h}$ of the topological zeta function. For a finite $[M\times M]$ transition matrix, the number of terms in the characteristic equation (13.13) is finite, and we refer to this expansion as the *topological polynomial* of order $\leq M$. The power of defining a determinant by the cumulant expansion is that it works even when the partition is infinite, $M \to \infty$; an example is given in sect. 13.6, and many more later on.

fast track:
sect. 13.6, p. 240

13.4.1 Topological zeta function for flows

We now apply the method that we shall use in deriving (14.21) to the problem of deriving the topological zeta functions for flows. The time-weighted density of prime cycles of period t is

$$\Gamma(t) = \sum_p \sum_{r=1} T_p \, \delta(t - rT_p) \, . \tag{13.22}$$

As in (14.20), a Laplace transform smooths the sum over Dirac delta spikes and yields the *topological trace formula*

$$\sum_p \sum_{r=1} T_p \int_{0_+}^{\infty} dt \, e^{-st} \, \delta(t - rT_p) = \sum_p T_p \sum_{r=1}^{\infty} e^{-sT_p r} \tag{13.23}$$

and the *topological zeta function* for flows:

$$1/\zeta_{\rm top}(s) \ = \ \prod_p \left(1 - e^{-sT_p}\right) \ , \tag{13.24}$$

topological!zeta function
zeta function!topological
Artin-Mazur zeta function
zeta function!Artin-Mazur
topological!polynomial
polynomial!topological
Laplace!transform
Dirac delta function
trace!formula!topological
topological!zeta function
zeta function!topological

periodic!point!count

related to the trace formula by

$$\sum_p T_p \sum_{r=1}^{\infty} e^{-sT_p r} \;=\; -\frac{\partial}{\partial s} \ln 1/\zeta_{\rm top}(s)\,.$$

This is the continuous time version of the discrete time topological zeta function (13.21) for maps; its leading zero $s = -h$ yields the topological entropy for a flow.

recheck the sign of h

13.5 Counting cycles

In what follows we shall occasionally need to compute all cycles up to topological length n, so it is handy to know their exact number.

13.5.1 Counting periodic points

N_n, the number of periodic points of period n can be computed from (13.19) and (13.7) as a logarithmic derivative of the topological zeta function

$$\begin{aligned}\sum_{n=1} N_n z^n &= \mathrm{tr}\left(-z\frac{d}{dz}\ln(1-zT)\right) = -z\frac{d}{dz}\ln\det\,(1-zT) \\ &= \frac{-z\frac{d}{dz}1/\zeta_{\rm top}}{1/\zeta_{\rm top}}\,. \qquad (13.25)\end{aligned}$$

We see that the trace formula (13.8) diverges at $z \to e^{-h}$, as the denominator has a simple zero there.

Example 13.5 Complete N-ary dynamics: *As a check of formula (13.19) in the finite grammar context, consider the complete N-ary dynamics (11.3) for which the number of periodic points of period n is simply $\mathrm{tr}\, T_c^n = N^n$. Substituting*

$$\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}\, T_c^n = \sum_{n=1}^{\infty} \frac{(zN)^n}{n} = \ln(1-zN)\,,$$

into (13.19) we verify (13.15). The logarithmic derivative formula (13.25) in this case does not buy us much either, we recover

$$\sum_{n=1} N_n z^n = \frac{Nz}{1-Nz}\,.$$

replace by Biham

Example 13.6 Nontrivial pruned dynamics: *Consider the pruning of figure 13.3 (e). Substituting (13.18) we obtain*

prime cycle!count
Moebius inversion

$$\sum_{n=1} N_n z^n = \frac{z + 8z^4 - 8z^8}{1 - z - 2z^4 + z^8} \,. \tag{13.26}$$

Now the topological zeta function is not merely a tool for extracting the asymptotic growth of N_n; it actually yields the exact and not entirely trivial recursion relation for the numbers of periodic points: $N_1 = N_2 = N_3 = 1$, $N_n = 2n + 1$ for $n = 4, 5, 6, 7, 8$, and $N_n = N_{n-1} + 2N_{n-4} - N_{n-8}$ for $n > 8$.

13.5.2 Counting prime cycles

Having calculated the number of periodic points, our next objective is to evaluate the number of *prime* cycles M_n for a dynamical system whose symbolic dynamics is built from N symbols. The problem of finding M_n is classical in combinatorics (counting necklaces made out of n beads out of N different kinds) and is easily solved. There are N^n possible distinct strings of length n composed of N letters. These N^n strings include all M_d prime d-cycles whose period d equals or divides n. A prime cycle is a non-repeating symbol string: for example, $p = \overline{011} = \overline{101} = \overline{110} = \ldots 011011 \ldots$ is prime, but $\overline{0101} = 010101 \ldots = \overline{01}$ is not. A prime d-cycle contributes d strings to the sum of all possible strings, one for each cyclic permutation. The total number of possible periodic symbol sequences of length n is therefore related to the number of prime cycles by

$$N_n \;=\; \sum_{d|n} d M_d \,, \tag{13.27}$$

where N_n equals $\mathrm{tr}\, T^n$. The number of prime cycles can be computed recursively

$$M_n \;=\; \frac{1}{n}\left(N_n - \sum_{d|n}^{d<n} d M_d \right) \,,$$

or by the *Möbius inversion formula*

13.12
page 248

$$M_n \;=\; n^{-1} \sum_{d|n} \mu\left(\frac{n}{d}\right) N_d \,. \tag{13.28}$$

where the Möbius function $\mu(1) = 1$, $\mu(n) = 0$ if n has a squared factor, and $\mu(p_1 p_2 \ldots p_k) = (-1)^k$ if all prime factors are different.

all tables need centering, fine tuning

We list the number of prime cycles up to length 10 for 2-, 3- and 4-letter complete symbolic dynamics in table 13.2. The number of *prime* cycles follows by Möbius inversion (13.28).

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move the chilli to the left of the whole example

pruning!individual cycles
cycle!pruning

n	$M_n(N)$	$M_n(2)$	$M_n(3)$	$M_n(4)$
1	N	2	3	4
2	$N(N-1)/2$	1	3	6
3	$N(N^2-1)/3$	2	8	20
4	$N^2(N^2-1)/4$	3	18	60
5	$(N^5-N)/5$	6	48	204
6	$(N^6-N^3-N^2+N)/6$	9	116	670
7	$(N^7-N)/7$	18	312	2340
8	$N^4(N^4-1)/8$	30	810	8160
9	$N^3(N^6-1)/9$	56	2184	29120
10	$(N^{10}-N^5-N^2+N)/10$	99	5880	104754

Table 13.2: Number of prime cycles for various alphabets and grammars up to length 10. The first column gives the cycle length, the second the formula (13.28) for the number of prime cycles for complete N-symbol dynamics, columns three through five give the numbers for $N = 2, 3$ and 4.

Example 13.7 Counting N-disk periodic points: *A simple example of pruning is the exclusion of "self-bounces" in the N-disk game of pinball. The number of points that are mapped back onto themselves after n iterations is given by $N_n = \mathrm{tr}\, T^n$. The pruning of self-bounces eliminates the diagonal entries, $T_{N-disk} = T_c - \mathbf{1}$, so the number of the N-disk periodic points is*

$$N_n = \mathrm{tr}\, T^n_{N-disk} = (N-1)^n + (-1)^n(N-1) \tag{13.29}$$

(here T_c is the complete symbolic dynamics transition matrix (11.3)). For the N-disk pruned case (13.29) Möbius inversion (13.28) yields

$$\begin{aligned} M_n^{N-disk} &= \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)(N-1)^d + \frac{N-1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)(-1)^d \\ &= M_n^{(N-1)} \quad \text{for} \quad n>2\ . \end{aligned} \tag{13.30}$$

There are no fixed points, $M_1^{N-disk} = 0$. The number of periodic points of period 2 is $N^2 - N$, hence there are $M_2^{N-disk} = N(N-1)/2$ prime cycles of length 2; for lengths $n > 2$, the number of prime cycles is the same as for the complete $(N-1)$-ary dynamics of table 13.2.

Example 13.8 Pruning individual cycles: *Consider the 3-disk game of pinball. The prohibition of repeating a symbol affects counting only for the fixed points and the 2-cycles. Everything else is the same as counting for a complete binary dynamics (eq (13.30)). To obtain the topological zeta function, just divide out the binary 1- and 2-cycles $(1-zt_0)(1-zt_1)(1-z^2t_{01})$ and multiply with the correct 3-disk 2-cycles $(1-z^2t_{12})(1-z^2t_{13})(1-z^2t_{23})$:*

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13.17 page 250

$$\begin{aligned} 1/\zeta_{3-disk} &= (1-2z)\frac{(1-z^2)^3}{(1-z)^2(1-z^2)} \\ &= (1-2z)(1+z)^2 = 1-3z^2-2z^3\ . \end{aligned} \tag{13.31}$$

The factorization reflects the underlying 3-disk symmetry; we shall rederive it in (22.25). As we shall see in chapter 22, symmetries lead to factorizations of topological polynomials and topological zeta functions.

n	M_n	N_n	S_n	$m_p \cdot \hat{p}$
1	0	0	0	
2	3	6=3·2	1	3·12
3	2	6=2·3	1	2·123
4	3	18=3·2+3·4	1	3·1213
5	6	30=6·5	1	6·12123
6	9	66=3·2+2·3+9·6	2	6·121213 + 3·121323
7	18	126=18·7	3	6·1212123 + 6·1212313 + 6·1213123
8	30	258=3·2+3·4+30·8	6	6·12121213 + 3·12121313 + 6·12121323 + 6·12123123 + 6·12123213 + 3·12132123
9	56	510=2·3+56·9	10	6·121212123 + 6·(121212313 + 121212323) + 6·(121213123 + 121213213) + 6·121231323 + 6·(121231213 + 121232123) + 2·121232313 + 6·121321323
10	99	1022	18	

Table 13.3: List of the 3-disk prime cycles up to length 10. Here n is the cycle length, M_n the number of prime cycles, N_n the number of periodic points and S_n the number of distinct prime cycles under the C_{3v} symmetry (see chapter 22 for further details). Column 3 also indicates the splitting of N_n into contributions from orbits of lengths that divide n. The prefactors in the fifth column indicate the degeneracy m_p of the cycle; for example, 3·12 stands for the three prime cycles $\overline{12}$, $\overline{13}$ and $\overline{23}$ related by $2\pi/3$ rotations. Among symmetry related cycles, a representative $\hat{p}$ which is lexically lowest was chosen. The cycles of length 9 grouped by parenthesis are related by time reversal symmetry, but not by any other C_{3v} transformation.

n	M_n	N_n	S_n	$m_p \cdot \hat{p}$
1	0	0	0	
2	6	12=6·2	2	4·12 + 2·13
3	8	24=8·3	1	8·123
4	18	84=6·2+18·4	4	8·1213 + 4·1214 + 2·1234 + 4·1243
5	48	240=48·5	6	8·(12123 + 12124) + 8·12313 + 8·(12134 + 12143) + 8·12413
6	116	732=6·2+8·3+116·6	17	8·121213 + 8·121214 + 8·121234 + 8·121243 + 8·121313 + 8·121314 + 4·121323 + 8·(121324 + 121423) + 4·121343 + 8·121424 + 4·121434 + 8·123124 + 8·123134 + 4·123143 + 4·124213 + 8·124243
7	312	2184	39	
8	810	6564	108	

Table 13.4: List of the 4-disk prime cycles up to length 8. The meaning of the symbols is the same as in table 13.3. Orbits related by time reversal symmetry (but no other symmetry) already appear at cycle length 5. List of the cycles of length 7 and 8 has been omitted.

topological!entropy
entropy!topological
partition!infinite

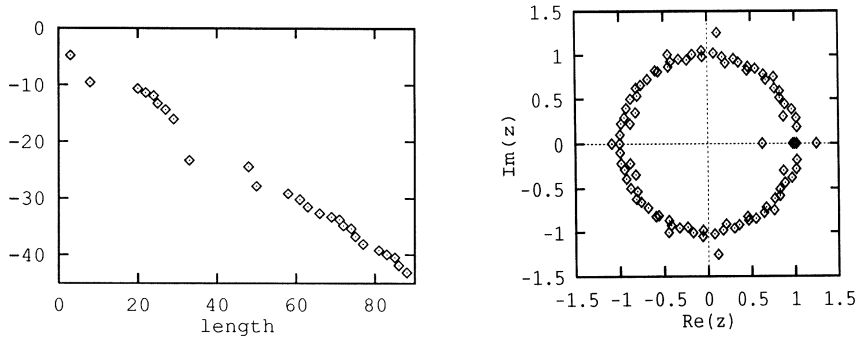

Figure 13.4: (a) The logarithm of the difference between the leading zero of the finite polynomial approximations to topological zeta function and our best estimate, as a function of the length for the quadratic map $A = 3.8$. (b) The 90 zeroes of the characteristic polynomial for the quadratic map $A = 3.8$ approximated by symbolic strings up to length 90. (from ref. [1.7])

draw Ulam tent map, prune 0 fixed point; show this leads to infinite alphabet

13.18
page 250

Example 13.9 Alphabet $\{a, cb^k;\ \overline{b}\}$: *(continuation of exercise 13.18) In the cycle counting case, the dynamics in terms of $a \to z$, $cb^k \to \frac{z}{1-z}$ is a complete binary dynamics with the explicit fixed point factor $(1 - t_b) = (1 - z)$:*

$$1/\zeta_{top} = (1-z)\left(1 - z - \frac{z}{1-z}\right) = 1 - 3z + z^2\,.$$

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page 251

recheck PC's NB 6/4-88 for the Artin-Mazur, Smale references...

13.6 Topological zeta function for an infinite partition

(K.T. Hansen and P. Cvitanović)

Now consider an example of a dynamical system which (as far as we know - there is no proof) has an infinite partition, or an infinity of longer and longer pruning rules. Take the 1-d quadratic map

$$f(x) = Ax(1-x)$$

with $A = 3.8$. It is easy to check numerically that the itinerary or the "kneading sequence" (see figure 11.10) of the critical point $x = 1/2$ is

$$K = 10110111101101111010111101111110\ldots$$

where the symbolic dynamics is defined by the partition of figure 11.8. How this kneading sequence is converted into a series of pruning rules is

ELIMINARY
ELIMINARY

shadowing

a dark art, relegated to appendix F.1 which also might get explained in chapter 12. For the moment it suffices to state the result, to give you a feeling for what a "typical" infinite partition topological zeta function looks like. Approximating the dynamics by a Markov graph corresponding to a repeller of the period 29 attractive cycle close to the $A = 3.8$ strange attractor (or, much easier, following the algorithm of appendix F.1) yields a Markov graph with 29 nodes and the characteristic polynomial [1]

$$\begin{aligned} 1/\zeta_{\rm top}^{(29)} &= 1 - z^1 - z^2 + z^3 - z^4 - z^5 + z^6 - z^7 + z^8 - z^9 - z^{10} \\ &\quad + z^{11} - z^{12} - z^{13} + z^{14} - z^{15} + z^{16} - z^{17} - z^{18} + z^{19} + z^{20} \\ &\quad - z^{21} + z^{22} - z^{23} + z^{24} + z^{25} - z^{26} + z^{27} - z^{28} . \end{aligned} \tag{13.32}$$

The smallest real root of this approximate topological zeta function is

$$z = 0.62616120\ldots \tag{13.33}$$

Constructing finite Markov graphs of increasing length corresponding to $A \to 3.8$ we find polynomials with better and better estimates for the topological entropy. For the closest stable period 90 orbit we obtain our best estimate of the topological entropy of the repeller:

$$h = -\ln 0.62616130424685\ldots = 0.46814726655867\ldots . \tag{13.34}$$

Figure 13.4 illustrates the convergence of the truncation approximations to the topological zeta function as a plot of the logarithm of the difference between the zero of a polynomial and our best estimate (13.34), plotted as a function of the length of the stable periodic orbit. The error of the estimate (13.33) is expected to be of order $z^{29} \approx e^{-14}$ because going from length 28 to a longer truncation yields typically combinations of loops with 29 and more nodes giving terms $\pm z^{29}$ and of higher order in the polynomial. Hence the convergence is exponential, with exponent of $-0.47 = -h$, the topological entropy itself.

recheck, perhaps the shadowing improves the estimate to e^{-2h}?

In figure 13.4 (b) we plot the zeroes of the polynomial approximation to the topological zeta function obtained by accounting for all forbidden strings of length 90 or less. The leading zero giving the topological entropy is the point closest to the origin. Most of the other zeroes are close to the unit circle; we conclude that for infinite Markov partitions the topological zeta function has a unit circle as the radius of convergence. The convergence is controlled by the ratio of the leading to the next-to-leading eigenvalues, which is in this case indeed $\lambda_1/\lambda_0 = 1/e^h = e^{-h}$.

prove elsewhere that $\lambda_1 = 1$

cite Grassberger, Politi

13.7 Shadowing

[1] Dahlqvist: divide by $(1 - z)$ once!

dynamical system!smooth
smooth!dynamics
structural stability
stability!structural

The topological zeta function is a pretty function, but the infinite product (13.20) should make you pause. For finite transfer matrices the left hand side is a determinant of a finite matrix, therefore a finite polynomial; so why is the right hand side an infinite product over the infinitely many prime periodic orbits of all periods?

The way in which this infinite product rearranges itself into a finite polynomial is instructive, and crucial for all that follows. You can already take a peek at the full cycle expansion (18.5) of chapter 18; all cycles beyond the fundamental t_0 and t_1 appear in the shadowing combinations such as

$$t_{s_1 s_2 \cdots s_n} - t_{s_1 s_2 \cdots s_m} t_{s_{m+1} \cdots s_n} \, .$$

For subshifts of finite type such shadowing combinations cancel *exactly*, if we are counting cycles as we do here, or if the dynamics is piecewise linear, as in exercise 15.2. As we have already argued in sect. 1.5.4 and appendix K.1.2, for nice hyperbolic flows whose symbolic dynamics is a subshift of finite type, the shadowing combinations *almost* cancel, and the spectral determinant is dominated by the fundamental cycles from (13.13), with longer cycles contributing only small "curvature" corrections.

Vattay has removed sect s-TracInterp from getused.tex - need other piecewise linear section examples

These exact or nearly exact cancellations depend on the flow being smooth and the symbolic dynamics being a subshift of finite type. If the dynamics requires infinite Markov partition with pruning rules for longer and longer blocks, most of the shadowing combinations still cancel, but the few corresponding to the forbidden blocks do not, leading to a finite radius of convergence for the spectral determinant as in figure 13.4 (b).

recheck sign

One striking aspect of the pruned cycle expansion (13.32) compared to the trace formulas such as (13.7) is that coefficients are not growing exponentially - indeed they all remain of order 1, so instead having a radius of convergence e^{-h}, in the example at hand the topological zeta function has the unit circle as the radius of convergence. In other words, exponentiating the spectral problem from a trace formula to a spectral determinant as in (13.19) increases the *analyticity domain*: the pole in the trace (13.8) at $z = e^{-h}$ is promoted to a smooth zero of the spectral determinant with a larger radius of convergence.

A detailed discussion of the radius of convergence is given in appendix F.1.

The very sensitive dependence of spectral determinants on whether the symbolic dynamics is or is not a subshift of finite type is the bad news that we should announce already now. If the system is generic and not structurally stable (see sect. 12.2), a smooth parameter variation is in no sense a smooth variation of topological dynamics - infinities of periodic orbits are created or destroyed, Markov graphs go from being finite to infinite and back. That will imply that the global averages that we intend to compute are generically nowhere differentiable functions of the system parameters, and averaging over families of dynamical systems

entropy!topological
determinant!graph

can be a highly nontrivial enterprise; a simple illustration is the parameter dependence of the diffusion constant computed in a remark in chapter 24.

You might well ask: What is wrong with computing the entropy from (13.1)? Does all this theory buy us anything? An answer: If we count K_n level by level, we ignore the self-similarity of the pruned tree - examine for example figure 11.14, or the cycle expansion of (13.26) - and the finite estimates of $h_n = \ln K_n/n$ converge nonuniformly to h, and on top of that with a slow rate of convergence, $|h - h_n| \approx O(1/n)$ as in (13.4). The determinant (13.9) is much smarter, as by construction it encodes the self-similarity of the dynamics, and yields the asymptotic value of h with no need for any finite n extrapolations.

So, the main lesson of learning how to count well, a lesson that will be affirmed over and over, is that while the trace formulas are a conceptually essential step in deriving and understanding periodic orbit theory, the spectral determinant is the right object to use in actual computations. Instead of resumming all of the exponentially many periodic points required by trace formulas at each level of truncation, spectral determinants incorporate only the small incremental corrections to what is already known - and that makes them more convergent and economical to use.

Commentary

Remark 13.1 "Entropy". The ease with which the topological entropy can be motivated obscures the fact that our definition does not lead to an invariant characterization of the dynamics, as the choice of symbolic dynamics is largely arbitrary: the same caveat applies to other entropies to be discussed in chapter 20, and to get proper invariants one needs to evaluate a supremum over all possible partitions. The key mathematical point that eliminates the need of such search is the existence of *generators*, i.e., partitions that under dynamics are able to probe the whole phase space on arbitrarily small scales: more precisely a generator is a finite partition $\Omega = \omega_1 \dots \omega_N$, with the following property: take $\mathcal{M}$ the subalgebra of the state space generated by Ω, and consider the partition built upon all possible intersections of sets $\phi^k(\beta_i)$, where ϕ is dynamical evolution, β_i is an element of $\mathcal{M}$ and k takes all possible integer values (positive as well as negative), then the closure of such a partition coincides with the algebra of all measurable sets. For a thorough (and readable) discussion of generators and how they allow a computation of the Kolmogorov entropy, see ref. [13.1] and chapter 20.

Remark 13.2 Perron-Frobenius matrices. For a proof of Perron theorem on the leading eigenvalue see ref. [1.20]. Sect. A4.1 of ref. [13.2] offers a clear discussion of the spectrum of the transition matrix.

Remark 13.3 Determinant of a graph. Many textbooks offer derivations of the loop expansions of characteristic polynomials for transition matrices and their Markov graphs, see for example refs. [13.3, 13.4, 13.5].

partition!infinite
Smale, S.

Remark 13.4 T is not trace class. Note to the erudite reader: the transition matrix T (in the infinite partition limit (13.19)) is *not* trace class in the sense of appendix M. Still the trace is well defined in the $n \to \infty$ limit.

Remark 13.5 Artin-Mazur zeta functions. Motivated by A. Weil's zeta function for the Frobenius map [13.6], Artin and Mazur [15.13] introduced the zeta function (13.21) that counts periodic points for diffeomorphisms (see also ref. [13.7] for their evaluation for maps of the interval). Smale [13.8] conjectured rationality of the zeta functions for Axiom A diffeomorphisms, later proved by Guckenheimer [13.9] and Manning [13.10]. See remark 15.4 on page 282 for more zeta function history.

what is the difference between [**?**] and [12.14]?

Remark 13.6 Ordering periodic orbit expansions. In sect. 18.6 we will introduce an alternative way of hierarchically organizing cumulant expansions, in which the order is dictated by stability rather than cycle length: such a procedure may be better suited to perform computations when the symbolic dynamics is not well understood.

Résumé

What have we accomplished? We have related the number of topologically distinct paths from "this region" to "that region" in a chaotic system to the leading eigenvalue of the transition matrix T. The eigenspectrum of T is given by a certain sum over traces $\mathrm{tr}\, T^n$, and in this way the periodic orbit theory has entered the arena, already at the level of the topological dynamics, the crudest description of dynamics.

The main result of this chapter is the cycle expansion (13.21) of the topological zeta function (i.e., the spectral determinant of the transition matrix):

$$1/\zeta_{\rm top}(z) \;=\; 1 - \sum_{k=1} \hat{c}_k z^k \,.$$

For subshifts of finite type, the transition matrix is finite, and the topological zeta function is a finite polynomial evaluated by the loop expansion (13.13) of $\det(1 - zT)$. For infinite grammars the topological zeta function is defined by its cycle expansion. The topological entropy h is given by the smallest zero $z = e^{-h}$. This expression for the entropy is *exact*; in contrast to the definition (13.1), no $n \to \infty$ extrapolations of $\ln K_n/n$ are required.

Historically, these topological zeta functions were the inspiration for applying the transfer matrix methods of statistical mechanics to the problem of computation of dynamical averages for chaotic flows. The key result was the dynamical zeta function to be derived in chapter 14, a weighted generalization of the topological zeta function.

exponential proliferation
complexity!algorithmic

Contrary to claims one sometimes encounters in the literature, "exponential proliferation of trajectories" is not the problem; what limits the convergence of cycle expansions is the proliferation of the grammar rules, or the "algorithmic complexity", as illustrated by sect. 13.6, and figure 13.4 in particular.

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Exercises

Check the IOP exercises format...

Exercise 13.1 **A transition matrix for 3-disk pinball.**

a) Draw the Markov graph corresponding to the 3-disk ternary symbolic dynamics, and write down the corresponding transition matrix corresponding to the graph. Show that iteration of the transition matrix results in two coupled linear difference equations, - one for the diagonal and one for the off diagonal elements. (Hint: relate $\mathrm{tr}\, T^n$ *to* $\mathrm{tr}\, T^{n-1} + \ldots$.*)*

b) Solve the above difference equation and obtain the number of periodic orbits of length n*. Compare with table 13.3.*

c) Find the eigenvalues of the transition matrix $\mathbf{T}$ *for the 3-disk system with ternary symbolic dynamics and calculate the topological entropy. Compare this to the topological entropy obtained from the binary symbolic dynamics* $\{0, 1\}$*.*

Exercise 13.2 **Sum of A_{ij} is like a trace.** *Let* A *be a matrix with eigenvalues* λ_k*. Show that*

$$\Gamma_n = \sum_{i,j} [A^n]_{ij} = \sum_k c_k \lambda_k^n \,.$$

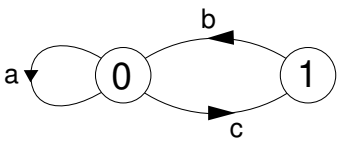
(a) *Use this to show that* $\ln |\mathrm{tr}\, A^n|$ *and* $\ln |\Gamma_n|$ *have the same asymptotic behavior as* $n \to \infty$*, i.e., their ratio converges to one.*

(b) *Do eigenvalues* λ_k *need to be distinct,* $\lambda_k \neq \lambda_l$ *for* $k \neq l$*?*

Exercise 13.3 **Loop expansions.** *Prove by induction the sign rule in the determinant expansion (13.13):*

$$\det (1 - z\mathbf{T}) = \sum_{k \geq 0} \sum_{p_1 + \cdots + p_k} (-1)^k t_{p_1} t_{p_2} \cdots t_{p_k} \,.$$

Exercise 13.4 **Transition matrix and cycle counting.** *Suppose you are given the Markov graph*



This diagram can be encoded by a matrix T*, where the entry* T_{ij} *means that there is a link connecting node* i *to node* j*. The value of the entry is the weight of the link.*

prime cycle
cycle!prime
three-disk@3-disk!prime cycles
prime cycle!3-disk

a) Walks on the graph are given the weight that is the product of the weights of all links crossed by the walk. Convince yourself that the transition matrix for this graph is:

$$T = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} .$$

b) Enumerate all the walks of length three on the Markov graph. Now compute T^3 and look at the entries. Is there any relation between the terms in T^3 and all the walks?

c) Show that T^n_{ij} is the number of walks from point i to point j in n steps. (Hint: one might use the method of induction.)

d) Try to estimate the number $N(n)$ of walks of length n for this simple Markov graph.

e) The topological entropy h measures the rate of exponential growth of the total number of walks $N(n)$ as a function of n. What is the topological entropy for this Markov graph?

Exercise 13.5 3-disk prime cycle counting. *A* prime *cycle p of length n_p is a single traversal of the orbit; its label is a non-repeating symbol string of n_p symbols. For example, $\overline{12}$ is prime, but $\overline{2121}$ is not, since it is $\overline{21} = \overline{12}$ repeated.*

Verify that a 3-disk pinball has 3, 2, 3, 6, 9, $\cdots$ prime cycles of length 2, 3, 4, 5, 6, $\cdots$.

Exercise 13.6 "Golden mean" pruned map. *Continuation of exercise 11.9: Show that the total number of periodic orbits of length n for the "golden mean" tent map is*

$$\frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n} .$$

For continuation, see exercise 13.8. See also exercise 13.9.

Exercise 13.7 Alphabet {0,1}, prune _00_ . *The Markov diagram figure 11.14 (b) implements this pruning rule. The pruning rule implies that "0" must always be bracketed by "1"s; in terms of a new symbol $2 = 10$, the dynamics becomes unrestricted symbolic dynamics with with binary alphabet {1,2}. The cycle expansion (13.13) becomes*

$$\begin{aligned} 1/\zeta &= (1-t_1)(1-t_2)(1-t_{12})(1-t_{112})\dots \\ &= 1 - t_1 - t_2 - (t_{12} - t_1 t_2) - (t_{112} - t_{12} t_1) - (t_{122} - t_{12} t_2) \dots \end{aligned} \quad (13.35)$$

In the original binary alphabet this corresponds to:

$$\begin{aligned} 1/\zeta &= 1 - t_1 - t_{10} - (t_{110} - t_1 t_{10}) \\ &\quad -(t_{1110} - t_{110} t_1) - (t_{11010} - t_{110} t_{10}) \dots \end{aligned} \quad (13.36)$$

This symbolic dynamics describes, for example, circle maps with the golden mean winding number, see chapter 28. For unimodal maps this symbolic dynamics is realized by the tent map of exercise 13.6.

add references here...

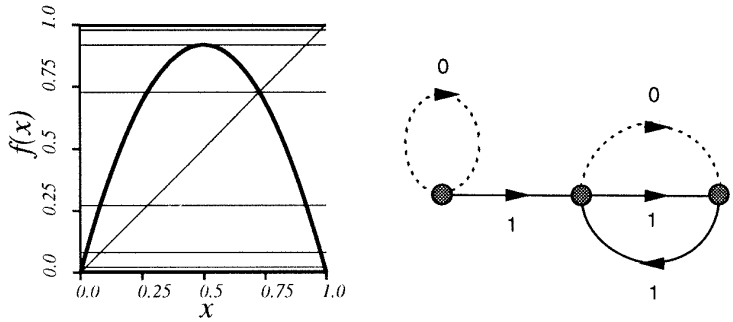


Figure 13.5: (a) A unimodal map for which the critical point maps into the right hand fixed point in three iterations, and (b) the corresponding Markov graph (Kai T. Hansen).

Exercise 13.8 **Spectrum of the "golden mean" pruned map.** *(medium - Exercise 13.6 continued)*

(a) *Determine an expression for* $\mathrm{tr}\,\mathcal{L}^n$*, the trace of powers of the Perron-Frobenius operator (9.10) for the tent map of exercise 13.6.*

(b) *Show that the spectral determinant for the Perron-Frobenius operator is*

$$\det(1-z\mathcal{L}) = \prod_{k\ \mathrm{even}}\left(1-\frac{z}{\Lambda^{k+1}}-\frac{z^2}{\Lambda^{2k+2}}\right)\prod_{k\ \mathrm{odd}}\left(1+\frac{z}{\Lambda^{k+1}}+\frac{z^2}{\Lambda^{2k+2}}\right). \quad (13.37)$$

Exercise 13.9 **A unimodal map example.** *Consider a unimodal map of figure 13.5 (a) for which the critical point maps into the right hand fixed point in three iterations,* $S^+ = 100\overline{1}$*. Show that the admissible itineraries are generated by the Markov graph figure 13.5 (b).*

(Kai T. Hansen)

⇓PRELIMINARY

Exercise 13.10 **Heavy pruning.** *(continuation of exercise 11.11.) Implement the grammar (12.11) by verifying all steps in the construction outlined in figure 13.3. Verify the entropy estimate (13.18). Perhaps count admissible trajectories up to some length of 5-10 symbols by your own method (generate all binary sequences, throw away the bad ones?), check whether this converges to the* h *value claimed in the text.*

⇑PRELIMINARY

Exercise 13.11 **Glitches in shadowing.**** *Note that the combination* t_{00011} *minus the "shadow"* $t_0 t_{0011}$ *in (13.17) cancels exactly, and does not contribute to the topological polynomial (13.18). Are you able to construct a smaller Markov graph than figure 13.3 (e)?*

Exercise 13.12 **Whence Möbius function?** *To understand where the Möbius function comes from consider the function*

$$f(n) = \sum_{d|n} g(d) \quad (13.38)$$

where $d|n$ stands for sum over all divisors d of n. Invert recursively this infinite tower of equations and derive the Möbius inversion formula

$$g(n) = \sum_{d|n} \mu(n/d) f(d) \tag{13.39}$$

Exercise 13.13 **Counting prime binary cycles.** *In order to get comfortable with Möbius inversion reproduce the results of the second column of table 13.2.*

Write a program that determines the number of prime cycles of length n. You might want to have this program later on to be sure that you have missed no 3-pinball prime cycles.

Exercise 13.14 **Counting subsets of cycles.** *The techniques developed above can be generalized to counting subsets of cycles. Consider the simplest example of a dynamical system with a complete binary tree, a repeller map (11.8) with two straight branches, which we label 0 and 1. Every cycle weight for such map factorizes, with a factor t_0 for each 0, and factor t_1 for each 1 in its symbol string. Prove that the transition matrix traces (13.5) collapse to $tr(T^k) = (t_0 + t_1)^k$, and $1/\zeta$ is simply*

$$\prod_p (1 - t_p) = 1 - t_0 - t_1 \tag{13.40}$$

Substituting (13.40) into the identity

$$\prod_p (1 + t_p) = \prod_p \frac{1 - {t_p}^2}{1 - t_p}$$

we obtain

$$\begin{aligned} \prod_p (1 + t_p) &= \frac{1 - t_0^2 - t_1^2}{1 - t_0 - t_1} = 1 + t_0 + t_1 + \frac{2t_0 t_1}{1 - t_0 - t_1} \\ &= 1 + t_0 + t_1 + \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} 2 \binom{n-2}{k-1} t_0^k t_1^{n-k}. \end{aligned} \tag{13.41}$$

Hence for $n \geq 2$ the number of terms in the cumulant expansion with k 0's and $n-k$ 1's in their symbol sequences is $2\binom{n-2}{k-1}$.

In order to count the number of prime cycles in each such subset we denote with $M_{n,k}$ $(n = 1, 2, \ldots ; k = \{0, 1\}$ for $n = 1$; $k = 1, \ldots, n-1$ for $n \geq 2)$ the number of prime n-cycles whose labels contain k zeros. Show that

$$\begin{aligned} M_{1,0} &= M_{1,1} = 1 \\ nM_{n,k} &= \sum_{m \left| \frac{n}{k} \right.} \mu(m) \binom{n/m}{k/m}, \quad n \geq 2\,, k = 1, \ldots, n-1 \end{aligned}$$

where the sum is over all m which divide both n and k.

Exercise 13.15 Logarithmic periodicity of $\ln N_n{}^*$. *Plot $\ln N_n - nh$ for a system with a nontrivial finite Markov graph. Do you see any periodicity? If yes, why?*

Exercise 13.16 4-disk pinball topological polynomial. *Show that the 4-disk pinball topological polynomial (the pruning affects only the fixed points and the 2-cycles) is given by*

$$\begin{aligned} 1/\zeta_{4-disk} &= (1-3z)\frac{(1-z^2)^6}{(1-z)^3(1-z^2)^3} \\ &= (1-3z)(1+z)^3 = 1-6z^2-8z^3-3z^4 \ . \end{aligned} \tag{13.42}$$

Exercise 13.17 N-disk pinball topological polynominal. *Show that for an N-disk pinball, the topological polynominal is given by*

$$\begin{aligned} 1/\zeta_{N-disk} &= (1-(N-1)z)\,\frac{(1-z^2)^{N(N-1)/2}}{(1-z)^{N-1}(1-z^2)^{(N-1)(N-2)/2}} \\ &= (1-(N-1)z)\,(1+z)^{N-1} \ . \end{aligned} \tag{13.43}$$

The topological polynomial has a root $z^{-1} = N-1$, as we already know it should from (13.29) or (13.15). We shall see in sect. 22.4 that the other roots reflect the symmetry factorizations of zeta functions.

Exercise 13.18 Alphabet $\{a,b,c\}$, prune $_ab_$. *The pruning rule implies that any string of "b"s must be preceeded by a "c"; so one possible alphabet is $\{a, cb^k;\ \overline{b}\}$, k=0,1,2.... As the rule does not prune the fixed point $\overline{b}$, it is explicitly included in the list. The cycle expansion (13.13) becomes*

$$\begin{aligned} 1/\zeta &= (1-t_a)(1-t_b)(1-t_c)(1-t_{cb})(1-t_{ac})(1-t_{cbb})\dots \\ &= 1-t_a-t_b-t_c+t_at_b-(t_{cb}-t_ct_b)-(t_{ac}-t_at_c)-(t_{cbb}-t_{cb}t_b)\dots \end{aligned}$$

The effect of the $_ab_$ pruning is essentially to unbalance the 2 cycle curvature $t_{ab}-t_at_b$; the remainder of the cycle expansion retains the curvature form.

Exercise 13.19 Alphabet $\{0,1\}$, prune n repeats. *of "0" $_000\dots00_$.*

This is equivalent to the n symbol alphabet $\{1, 2, \dots, n\}$ unrestricted symbolic dynamics, with symbols corresponding to the possible 10...00 block lengths: 2=10, 3=100, ..., n=100...00. The cycle expansion (13.13) becomes

$$1/\zeta = 1-t_1-t_2\dots-t_n-(t_{12}-t_1t_2)\dots-(t_{1n}-t_1t_n)\dots \ . \tag{13.44}$$

IVATE

IVATE

Exercise 13.20 **Alphabet {0,1}, prune _1000_, _00100_, _01100_.** *This example is motivated by the pruning front description of the symbolic dynamics for the Hénon-type maps, sect. 12.4.*

Show that the topological zeta function is given by

$$1/\zeta = (1-t_0)(1-t_1-t_2-t_{23}-t_{113}) \tag{13.45}$$

with the unrestricted 4-letter alphabet {1, 2, $\underline{23}$, $\underline{113}$}. Here 2, 3, refer to 10, 100 *respectively, as in exercise 13.19.*

Exercise 13.21 **Alphabet {0,1}, prune _1000_, _00100_, _01100_, _10011_.** *This example of pruning we shall use in sect. 12.7. The first three pruning rules were incorporated in the preceeding exercise.*

⇓PRIVATE

⇑PRIVATE

(a) Show that the last pruning rule _10011_ *leads (in a way similar to exercise 13.20) to the alphabet {$\underline{21^k}$, $\underline{23}$, $\underline{21^k113}$; $\overline{1}$, $\overline{0}$}, and the cycle expansion*

$$1/\zeta = (1-t_0)(1-t_1-t_2-t_{23}+t_1t_{23}-t_{2113}) \tag{13.46}$$

Note that this says that 1, 23, 2, 2113 are the fundamental cycles; not all cycles up to length 7 are needed, only 2113.

(b) Show that the topological polynomial is

$$1/\zeta_{top} = (1-z)(1-z-z^2-z^5+z^6-z^7) \tag{13.47}$$

and check that it yields the exact value of the entropy $h = 0.522737642\ldots$.

Redo exercises by Markov graphs.

Chapter 14

Trace formulas

> The trace formula is not a formula, it is an idea.
> Martin Gutzwiller

Dynamics is posed in terms of local equations, but the ergodic averages require global information. How can we use a local description of a flow to learn something about the global behavior? We have given a quick sketch of this program in sects. 1.5 and 1.6; now we redo the same material in greater depth. In chapter 10 we have related global averages to the eigenvalues of appropriate evolution operators. Traces of evolution operators can be evaluated as integrals over Dirac delta functions, and in this way the spectra of evolution operators become related to periodic orbits. If there is one idea that one should learn about chaotic dynamics, it happens in this chapter, and it is this: there is a fundamental local $\leftrightarrow$ global duality which says that

the spectrum of eigenvalues is dual to the spectrum of periodic orbits

For dynamics on the circle, this is called Fourier analysis; for dynamics on well-tiled manifolds, Selberg traces and zetas; and for generic nonlinear dynamical systems the duality is embodied in the trace formulas that we will now introduce. These objects are to dynamics what partition functions are to statistical mechanics.

14.1 Trace of an evolution operator

Our extraction of the spectrum of $\mathcal{L}$ commences with the evaluation of the trace. To compute an expectation value using (10.21) we have to integrate over all the values of the kernel $\mathcal{L}^t(x, y)$. If $\mathcal{L}^t$ were a matrix we would be computing a weighted sum of its eigenvalues which is dominated by the leading eigenvalue as $t \to \infty$. As the trace of $\mathcal{L}^t$ is also dominated by the leading eigenvalue as $t \to \infty$, we might just as well look at the trace

13.2
page 246

monodromy matrix
stability!eigenvalues
expanding!stability eigenvalues
marginal!stability eigenvalues
contracting!stability eigenvalues

$$\operatorname{tr}\mathcal{L}^t = \int dx\, \mathcal{L}^t(x,x) = \int dx\, \delta\big(x - f^t(x)\big)\, e^{\beta\cdot A^t(x)}\,. \tag{14.1}$$

Assume that $\mathcal{L}$ has a spectrum of discrete eigenvalues $s_0, s_1, s_2, \cdots$ ordered so that $\operatorname{Re} s_\alpha \geq \operatorname{Re} s_{\alpha+1}$. We ignore for the time being the question of what function space the eigenfunctions belong to, as we shall compute the eigenvalue spectrum without constructing any explicit eigenfunctions.

By definition, the trace is the sum over eigenvalues (for the time being we choose not to worry about convergence of such sums),

$$\operatorname{tr}\mathcal{L}^t = \sum_{\alpha=0}^{\infty} e^{s_\alpha t}\,. \tag{14.2}$$

On the other hand, we have learned in sect. 9.2 how to evaluate the delta-function integral (14.1).

As the case of discrete time mappings is somewhat simpler, we first derive the trace formula for maps, and then for flows. The final formula (14.21) covers both cases.

14.1.1 Hyperbolicity assumption

According to (9.8) the trace (14.1) picks up a contribution whenever $x - f^n(x) = 0$, i.e., whenever x belongs to a periodic orbit. For reasons which we will explain in sect. 14.3, it is wisest to start by focusing on discrete time systems. The contribution of an isolated prime cycle p of period n_p for a map f can be evaluated by restricting the integration to an infinitesimal open neighborhood $\mathcal{M}_p$ around the cycle,

$$\begin{aligned}\operatorname{tr}{}_p\mathcal{L}^{n_p} &= \int_{\mathcal{M}_p} dx\, \delta(x - f^{n_p}(x)) \\ &= \frac{n_p}{\left|\det\left(\mathbf{1} - M_p\right)\right|} = n_p \prod_{i=1}^{d} \frac{1}{|1-\Lambda_{p,i}|}\,.\end{aligned} \tag{14.3}$$

For the time being we set here and in (9.9) the observable $e^{\beta A_p} = 1$. Periodic orbit fundamental matrix M_p is also known as the *monodromy matrix* (from Greek *mono-* = alone, single, and *dromo* = run, racecourse), and its eigenvalues $\Lambda_{p,1}$, $\Lambda_{p,2}$, ..., $\Lambda_{p,d}$ as the Floquet multipliers.

We sort the eigenvalues $\Lambda_{p,1}$, $\Lambda_{p,2}$, ..., $\Lambda_{p,d}$ of the p-cycle [d×d] fundamental matrix M_p into expanding, marginal and contracting sets $\{e, m, c\}$, as in (8.3). As the integral (14.3) can be carried out only if M_p has no eigenvalue of unit magnitude, we assume that no eigenvalue is marginal (we shall show in sect. 14.3 that the longitudinal $\Lambda_{p,d+1} = 1$ eigenvalue for

flows can be eliminated by restricting the consideration to the transverse fundamental matrix M_p), and factorize the trace (14.3) into a product over the expanding and the contracting eigenvalues

hyperbolicity assumption
three-disk@3-disk!hyperbolicity
stability!elliptic

$$\left|\det\left(\mathbf{1}-M_p\right)\right|^{-1}=\frac{1}{|\Lambda_p|}\prod_e\frac{1}{1-1/\Lambda_{p,e}}\prod_c\frac{1}{1-\Lambda_{p,c}}\ , \tag{14.4}$$

where $\Lambda_p=\prod_e\Lambda_{p,e}$ is the product of expanding eigenvalues. Both $\Lambda_{p,c}$ and $1/\Lambda_{p,e}$ are smaller than 1 in absolute value, and as they are either real or come in complex conjugate pairs we are allowed to drop the absolute value brackets $|\cdots|$ in the above products.

The *hyperbolicity assumption* requires that the stabilities of all cycles included in the trace sums be exponentially bounded away from unity:

$$\begin{array}{rcll}|\Lambda_{p,e}| & > & e^{\lambda_e T_p} & \text{any } p\text{, any expanding } |\Lambda_{p,e}|>1\\ |\Lambda_{p,c}| & < & e^{-\lambda_c T_p} & \text{any } p\text{, any contracting } |\Lambda_{p,c}|<1\,,\end{array} \tag{14.5}$$

where $\lambda_e,\lambda_c>0$ are strictly positive bounds on the expanding, contracting cycle Lyapunov exponents. If a dynamical system satisfies the hyperbolicity assumption (for example, the well separated 3-disk system clearly does), the $\mathcal{L}^t$ spectrum will be relatively easy to control. If the expansion/contraction is slower than exponential, let us say $|\Lambda_{p,i}|\sim {T_p}^2$, the system may exhibit "phase transitions", and the analysis is much harder - we shall discuss this in chapter 21.

Elliptic stability, with a pair of purely imaginary exponents $\Lambda_m=e^{\pm i\theta}$ is excluded by the hyperbolicity assumption. While the contribution of a single repeat

$$\frac{1}{(1-e^{i\theta})(1-e^{-i\theta})}=\frac{1}{2(1-\cos\theta)}$$

does not make (9.9) diverge, for a generic θ repeats $\cos(r\theta)$ behave badly and by ergodicity $1-\cos(r\theta)$ is arbitrarily small, $1-\cos(r\theta)<\epsilon$, infinitely often. If $\Lambda_m=e^{i2\pi p/r}$ is rth root of unity, $1/\left|\det\left(\mathbf{1}-M_p^r\right)\right|$ diverges. Elliptic case will require a separate treatment.

It follows from (14.4) that for long times, $t=rT_p\to\infty$, only the product of expanding eigenvalues matters, $\left|\det\left(\mathbf{1}-M_p^r\right)\right|\to|\Lambda_p|^r$. We shall use this fact to motivate the construction of dynamical zeta functions in sect. 15.3. However, for evaluation of the full spectrum the exact cycle weight (14.3) has to be kept.

14.2 A trace formula for maps

If the evolution is given by a discrete time mapping, and all periodic points have stability eigenvalues $|\Lambda_{p,i}|\neq 1$ strictly bounded away from unity, the

Dirac delta function
prime cycle
cycle!prime
Laplace!transform, discrete
generating function
trace!formula!maps

trace $\mathcal{L}^n$ is given by the sum over all periodic points i of period n:

$$\mathrm{tr}\,\mathcal{L}^n = \int dx\, \mathcal{L}^n(x,x) = \sum_{x_i \in \mathrm{Fix} f^n} \frac{e^{\beta\cdot A_i}}{|\det\,(\mathbf{1} - M^n(x_i))|}\,. \tag{14.6}$$

Here Fix $f^n = \{x : f^n(x) = x\}$ is the set of all periodic points of period n, and A_i is the observable (10.5) evaluated over n discrete time steps along the cycle to which the periodic point x_i belongs. The weight follows from the properties of the Dirac delta function (9.8) by taking the determinant of $\partial_i(x_j - f^n(x)_j)$. If a trajectory retraces itself r times, its fundamental matrix is M_p^r, where M_p is the $[d\times d]$ fundamental matrix (4.6) evaluated along a single traversal of the prime cycle p. As we saw in (10.5), the integrated observable A^n is additive along the cycle: If a prime cycle p trajectory retraces itself r times, $n = rn_p$, we obtain A_p repeated r times, $A_i = A^n(x_i) = rA_p$, $x_i \in p$.

A prime cycle is a single traversal of the orbit, and its label is a non-repeating symbol string. There is only one prime cycle for each cyclic permutation class. For example, the four cycle points $\overline{0011} = \overline{1001} = \overline{1100} = \overline{0110}$ belong to the same prime cycle $p = 0011$ of length 4. As both the stability of a cycle and the weight A_p are the same everywhere along the orbit, each prime cycle of length n_p contributes n_p terms to the sum, one for each cycle point. Hence (14.6) can be rewritten as a sum over all prime cycles and their repeats

☞ chapter 11

$$\mathrm{tr}\,\mathcal{L}^n = \sum_p n_p \sum_{r=1}^{\infty} \frac{e^{r\beta\cdot A_p}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|}\delta_{n,n_p r}\,, \tag{14.7}$$

with the Kronecker delta $\delta_{n,n_p r}$ projecting out the periodic contributions of total period n. This constraint is awkward, and will be more awkward still for the continuous time flows, where it would yield a series of Dirac delta spikes. In both cases a Laplace transform rids us of the time periodicity constraint.

We *define* the trace formula for maps to be the sum

$$\sum_{n=1}^{\infty} z^n \mathrm{tr}\,\mathcal{L}^n = \mathrm{tr}\,\frac{z\mathcal{L}}{1 - z\mathcal{L}} = \sum_p n_p \sum_{r=1}^{\infty} \frac{z^{n_p r} e^{r\beta\cdot A_p}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|}\,. \tag{14.8}$$

Such discrete time Laplace transform of $\mathrm{tr}\,\mathcal{L}^n$ is usually referred to as a "generating function". Expressing the trace as in (14.2), in terms of the sum of the eigenvalues of $\mathcal{L}$, we obtain the *trace formula for maps*:

box this relation

$$\sum_{\alpha=0}^{\infty} \frac{z e^{s_\alpha}}{1 - z e^{s_\alpha}} = \sum_p n_p \sum_{r=1}^{\infty} \frac{z^{n_p r}\, e^{r\beta\cdot A_p}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|}\,. \tag{14.9}$$

This is our second example of the duality between the spectrum of eigenvalues and the spectrum of periodic orbits, announced in the introduction to this chapter.

space!density functions
piecewise constant function
⇓PRELIMINARY
function!space!piecewise constant

If the above derivation is too formal for your taste, you might want to read sect. ?? in parallel with this chapter.

Rondoni's section s-Still-another was killed by Vattay and needs resurrection as an appendix

⇑PRELIMINARY

fast track:
sect. 14.3, p. 258

Example 14.1 A trace formula for transfer operators: *For a piecewise-linear map (10.17), we can explicitly evaluate the trace formula. By the piecewise linearity and the chain rule* $\Lambda_p = \Lambda_0^{n_0}\Lambda_1^{n_1}$, *where the cycle* p *contains* n_0 *symbols* 0 *and* n_1 *symbols* 1, *the trace (14.6) reduces to*

$$\operatorname{tr}\mathcal{L}^n = \sum_{m=0}^{n}\binom{n}{m}\frac{1}{|1-\Lambda_0^m\Lambda_1^{n-m}|} = \sum_{k=0}^{\infty}\left(\frac{1}{|\Lambda_0|\Lambda_0^k}+\frac{1}{|\Lambda_1|\Lambda_1^k}\right)^n . \tag{14.10}$$

The eigenvalues are simply

$$e^{s_k} = \frac{1}{|\Lambda_0|\Lambda_0^k}+\frac{1}{|\Lambda_1|\Lambda_1^k} . \tag{14.11}$$

restore the smooth 2-branch case here.
⇓PRELIMINARY
move this to symbolic.tex

The simplest example of spectrum for such dynamical system is the spectrum for the symmetric piecewise-linear 2-branch repeller (10.17) for which $\Lambda = \Lambda_1 = -\Lambda_0$. *In this case all odd eigenvalues vanish, and the even eigenvalues are given by* $e^{s_k} = 2/\Lambda^k$, k *even.*

9.7
page 148

Asymptotically the spectrum (14.11) is dominated by the lesser of the two fixed point slopes $\Lambda = \Lambda_0$ *(if* $|\Lambda_0| < |\Lambda_1|$, *otherwise* $\Lambda = \Lambda_1$*), and the eigenvalues* e^{s_k} *fall off exponentially as* $1/\Lambda^k$, *just as in the single fixed-point examples.*

⇑PRELIMINARY

For $k = 0$ *this is in agreement with the explicit transfer matrix (10.19) eigenvalues (10.20). The alert reader should experience anxiety at this point. Is it not true that we have already written down explicitly the transfer operator in (10.19), and that it is clear by inspection that it has only* one *eigenvalue* $e^{s_0} = 1/|\Lambda_0| + 1/|\Lambda_1|$? *The example at hand is one of the simplest illustrations of necessity of defining the space that the operator acts on in order to define the spectrum. The transfer operator (10.19) is the correct operator on the space of functions piecewise constant on the two defining intervals* $\{\mathcal{M}_0, \mathcal{M}_1\}$; *on this space the operator indeed has only the eigenvalue* e^{s_0}. *As we shall see in example 16.1, the full spectrum (14.11) corresponds to the action of the transfer operator on the space of real analytic functions.*

The Perron-Frobenius operator trace formula for the piecewise-linear map (10.17) follows from (14.8)

$$\operatorname{tr}\frac{z\mathcal{L}}{1-z\mathcal{L}} = \frac{z\left(\frac{1}{|\Lambda_0-1|}+\frac{1}{|\Lambda_1-1|}\right)}{1-z\left(\frac{1}{|\Lambda_0-1|}+\frac{1}{|\Lambda_1-1|}\right)} , \tag{14.12}$$

verifying the trace formula (14.9).

Laplace!transform
trace!formula!flows

14.3 A trace formula for flows

> Amazing! I did not understand a single word.
> Fritz Haake
>
> (R. Artuso and P. Cvitanović)

For a continuous time flow, the Laplace transform of an evolution operator yields the resolvent (9.30). This is a delicate step, since the evolution operator becomes the identity in the $t \to 0^+$ limit. In order to make sense of the trace we regularize the Laplace transform by a lower cutoff ϵ smaller than the period of any periodic orbit, and write

$$\int_\epsilon^\infty dt\, e^{-st}\, \mathrm{tr}\, \mathcal{L}^t = \mathrm{tr}\, \frac{e^{-(s-\mathcal{A})\epsilon}}{s-\mathcal{A}} = \sum_{\alpha=0}^{\infty} \frac{e^{-(s-s_\alpha)\epsilon}}{s-s_\alpha}\,, \tag{14.13}$$

where $\mathcal{A}$ is the generator of the semigroup of dynamical evolution, see sect. 9.4. Our task is to evaluate $\mathrm{tr}\, \mathcal{L}^t$ from its explicit state space representation.

14.3.1 Integration along the flow

As any pair of nearby points on a cycle returns to itself exactly at each cycle period, the eigenvalue of the fundamental matrix corresponding to the eigenvector along the flow necessarily equals unity for all periodic orbits. Hence for flows the trace integral $\mathrm{tr}\, \mathcal{L}^t$ requires a separate treatment for the longitudinal direction. To evaluate the contribution of an isolated prime cycle p of period T_p, restrict the integration to an infinitesimally thin tube $\mathcal{M}_p$ enveloping the cycle (see figure 1.11), and consider a local coordinate system with a longitudinal coordinate $dx_\parallel$ along the direction of the flow, and $d-1$ transverse coordinates $x_\perp$,

☞ example 8.3

$$\mathrm{tr}_p\, \mathcal{L}^t = \int_{\mathcal{M}_p} dx_\perp dx_\parallel\, \delta\big(x_\perp - f^t_\perp(x)\big)\, \delta\big(x_\parallel - f^t(x_\parallel)\big)\ . \tag{14.14}$$

(we set $\beta = 0$ in the $\exp(\beta \cdot A^t)$ weight for the time being). Pick a point on the prime cycle p, and let

$$v(x_\parallel) = \left(\sum_{i=1}^{d} v_i(x)^2 \right)^{1/2} \tag{14.15}$$

be the magnitude of the tangential velocity at any point $x = (x_\parallel, 0, \cdots, 0)$ on the cycle p. The velocity $v(x)$ must be strictly positive, as otherwise the orbit would stagnate for infinite time at $v(x) = 0$ points, and that would get us nowhere.

Laplace!transform
relative!solutions

As $0 \le \tau < T_p$, the trajectory $x_\parallel(\tau) = f^\tau(x_p)$ sweeps out the entire cycle, and for larger times $x_\parallel$ is a cyclic variable of periodicity T_p,

$$x_\parallel(\tau) = x_\parallel(\tau + rT_p) \qquad r = 1, 2, \cdots \tag{14.16}$$

We parametrize both the longitudinal coordinate $x_\parallel(\tau)$ and the velocity $v(\tau) = v(x_\parallel(\tau))$ by the flight time τ, and rewrite the integral along the periodic orbit as

$$\oint_p dx_\parallel \, \delta\big(x_\parallel - f^t(x_\parallel)\big) \;=\; \oint_p d\tau \, v(\tau) \, \delta\big(x_\parallel(\tau) - x_\parallel(\tau + t)\big) \,. \tag{14.17}$$

By the periodicity condition (14.16) the Dirac δ function picks up contributions for $t = rT_p$, so the Laplace transform can be split as

$$\int_0^\infty dt \, e^{-st} \, \delta\big(x_\parallel(\tau) - x_\parallel(\tau + t)\big) = \sum_{r=1}^{\infty} e^{-sT_p r} \, I_r$$
$$I_r = \int_{-\epsilon}^{\epsilon} dt \, e^{-st} \, \delta\big(x_\parallel(\tau) - x_\parallel(\tau + rT_p + t)\big) \,.$$

Taylor expanding and applying the periodicity condition (14.16), we have $x_\parallel(\tau + rT_p + t) = x_\parallel(\tau) + v(\tau)t + \ldots$,

$$I_r = \int_{-\epsilon}^{\epsilon} dt \, e^{-st} \, \delta\big(x_\parallel(\tau) - x_\parallel(\tau + rT_p + t)\big) = \frac{1}{v(\tau)} \,,$$

so the remaing integral (14.17) over τ is simply the cycle period $\oint_p d\tau = T_p$. The contribution of the longitudinal integral to the Laplace transform is thus

$$\int_0^\infty dt \, e^{-st} \oint_p dx_\parallel \, \delta\big(x_\parallel - f^t(x_\parallel)\big) = T_p \sum_{r=1}^{\infty} e^{-sT_p r} \,. \tag{14.18}$$

This integration is a prototype of what needs to be done for each marginal direction, whenever existence of a conserved quantity (energy in Hamiltonian flows, angular momentum, translational invariance, etc.) implies exisistance of a smooth manifold of equivalent (equivariant) solutions of dynamical equations.

⇓PRIVATE

in depth:
sect. 23, p. 451

⇑PRIVATE

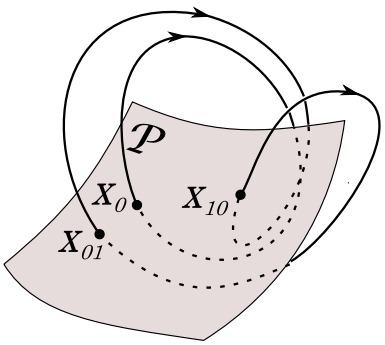


Figure 14.1: Continuation of figure 1.11 - RHS of the trace formula (14.21). Tubes still to be drawn

14.3.2 Stability in the transverse directions

Think of the $\tau = 0$ point in above integrals along the cycle p as a choice of a particular Poincaré section. As we have shown in sect. 8.3, the transverse stability eigenvalues do not depend on the choice of a Poincaré section, so ignoring the dependence on $x_\parallel(\tau)$ in evaluating the transverse integral in (14.14) is justified. For the transverse integration variables the fundamental matrix is defined in a reduced Poincaré surface of section $\mathcal{P}$ of fixed $x_\parallel$. Linearization of the periodic flow transverse to the orbit yields

$$\int_{\mathcal{P}} dx_\perp \delta\Big(x_\perp - f_\perp^{rT_p}(x)\Big) = \frac{1}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|}\ , \tag{14.19}$$

where M_p is the p-cycle $[d-1\times d-1]$ *transverse* fundamental matrix. As in (14.5) we have to assume hyperbolicity, i.e., that the magnitudes of all transverse eigenvalues are bounded away from unity.

Substitution (14.18), (14.19) in (14.14) leads to an expression for $\operatorname{tr}\mathcal{L}^t$ as a sum over all prime cycles p and their repetitions

$$\int_\epsilon^\infty dt\, e^{-st} \operatorname{tr}\mathcal{L}^t = \sum_p T_p \sum_{r=1}^\infty \frac{e^{r(\beta\cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|}\,. \tag{14.20}$$

The $\epsilon \to 0$ limit of the two expressions for the resolvent, (14.13) and (14.20), now yields the *classical trace formula for flows*

$$\sum_{\alpha=0}^\infty \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^\infty \frac{e^{r(\beta\cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|}\,. \tag{14.21}$$

14.1
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⇑PRELIMINARY

make a learned comment about how this is a generalization of the Poisson resummation technique

(If you are worried about the convergence of the resolvent sum, keep the ε regularization.)

This formula is still another example of the duality between the (local) cycles and (global) eigenvalues. If T_p takes only integer values, we can replace $e^{-s} \to z$ throughout, so the trace formula for maps (14.9) is a special

case of the trace formula for flows. The relation between the continuous and discrete time cases can be summarized as follows:

$$\begin{aligned} T_p &\leftrightarrow n_p \\ e^{-s} &\leftrightarrow z \\ e^{t\mathcal{A}} &\leftrightarrow \mathcal{L}^n \,. \end{aligned} \tag{14.22}$$

We could now proceed to estimate the location of the leading singularity of $\mathrm{tr}\,(s-\mathcal{A})^{-1}$ by extrapolating finite cycle length truncations of (14.21) by methods such as Padé approximants. However, it pays to first perform a simple resummation which converts this divergence of a trace into a *zero* of a spectral determinant. We shall do this in sect. 15.2, but first a brief refresher of how all this relates to the formula for escape rate (1.7) offered in the introduction might help digest the material.

fast track:
sect. 15, p. 267

14.4 An asymptotic trace formula

In order to illuminate the manipulations of sect. 14.2 and relate them to something we already possess intuition about, we now rederive the heuristic sum of sect. 1.5.1 from the exact trace formula (14.9). The Laplace transforms (14.9) or (14.21) are designed to capture the time $\to\infty$ asymptotic behavior of the trace sums. By the hyperbolicity assumption (14.5), for $t = T_p r$ large the cycle weight approaches

$$\left|\det\left(\mathbf{1}-M_p^r\right)\right| \to |\Lambda_p|^r \,, \tag{14.23}$$

where Λ_p is the product of the expanding eigenvalues of M_p. Denote the corresponding approximation to the nth trace (14.6) by

$$\Gamma_n = \sum_i^{(n)} \frac{1}{|\Lambda_i|} \,, \tag{14.24}$$

and denote the approximate trace formula obtained by replacing the cycle weights $\left|\det\left(\mathbf{1}-M_p^r\right)\right|$ by $|\Lambda_p|^r$ in (14.9) by $\Gamma(z)$. Equivalently, think of this as a replacement of the evolution operator (10.23) by a transfer operator (as in example 14.1). For concreteness consider a dynamical system whose symbolic dynamics is complete binary, for example the 3-disk system figure 1.5. In this case distinct periodic points that contribute

ceiling function

to the nth periodic points sum (14.7) are labeled by all admissible itineraries composed of sequences of letters $s_i \in \{0,1\}$:

$$\begin{aligned}\Gamma(z) &= \sum_{n=1}^{\infty} z^n \Gamma_n = \sum_{n=1}^{\infty} z^n \sum_{x_i \in \mathrm{Fix} f^n} \frac{e^{\beta\cdot A^n(x_i)}}{|\Lambda_i|} \\ &= z\left\{\frac{e^{\beta\cdot A_0}}{|\Lambda_0|} + \frac{e^{\beta\cdot A_1}}{|\Lambda_1|}\right\} + z^2\left\{\frac{e^{2\beta\cdot A_0}}{|\Lambda_0|^2} + \frac{e^{\beta\cdot A_{01}}}{|\Lambda_{01}|} + \frac{e^{\beta\cdot A_{10}}}{|\Lambda_{10}|} + \frac{e^{2\beta\cdot A_1}}{|\Lambda_1|^2}\right\} \\ &\quad + z^3\left\{\frac{e^{3\beta\cdot A_0}}{|\Lambda_0|^3} + \frac{e^{\beta\cdot A_{001}}}{|\Lambda_{001}|} + \frac{e^{\beta\cdot A_{010}}}{|\Lambda_{010}|} + \frac{e^{\beta\cdot A_{100}}}{|\Lambda_{100}|} + \ldots\right\} \qquad (14.25)\end{aligned}$$

Both the cycle averages A_i and the stabilities Λ_i are the same for all points $x_i \in p$ in a cycle p. Summing over repeats of all prime cycles we obtain

$$\Gamma(z) = \sum_p \frac{n_p t_p}{1-t_p}\,, \qquad t_p = z^{n_p} e^{\beta\cdot A_p}/|\Lambda_p|\,. \qquad (14.26)$$

This is precisely our initial heuristic estimate (1.8). Note that we could not perform such sum over r in the exact trace formula (14.9) as $\left|\det\left(\mathbf{1}-M_p^r\right)\right| \neq \left|\det\left(\mathbf{1}-M_p\right)\right|^r$; the correct way to resum the exact trace formulas is to first expand the factors $1/|1-\Lambda_{p,i}|$, as we shall do in (15.10).

☞ sect. 15.2

If the weights $e^{\beta A^n(x)}$ are multiplicative along the flow, and the flow is hyperbolic, for given β the magnitude of each $|e^{\beta A^n(x_i)}/\Lambda_i|$ term is bounded by some constant M^n. The total number of cycles grows as 2^n (or as e^{hn}, $h =$ topological entropy, in general), and the sum is convergent for z sufficiently small, $|z| < 1/2M$. For large n the nth level sum (14.6) tends to the leading $\mathcal{L}^n$ eigenvalue e^{ns_0}. Summing this asymptotic estimate level by level

$$\Gamma(z) \approx \sum_{n=1}^{\infty} (ze^{s_0})^n = \frac{ze^{s_0}}{1-ze^{s_0}} \qquad (14.27)$$

we see that we should be able to determine s_0 by determining the smallest value of $z = e^{-s_0}$ for which the cycle expansion (14.26) diverges.

If one is interested only in the leading eigenvalue of $\mathcal{L}$, it suffices to consider the approximate trace $\Gamma(z)$. We will use this fact in sect. 15.3 to motivate the introduction of dynamical zeta functions (15.15), and in sect. 15.5 we shall give the exact relation between the exact and the approximate trace formulas.

Commentary

Remark 14.1 Who's dunne it? Continuous time flow traces weighted by cycle periods were introduced by Bowen [14.1] who treated them as Poincaré section

suspensions weighted by the "time ceiling" function (3.5). They were used by Parry and Pollicott [14.2]. The derivation presented here [14.3] paralleles closely to the derivation of the Gutzwiller semiclassical trace formula, chapters 34 and 36.

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Remark 14.2 $t \to 0_+$ regularization. We remark again that in taking the Laplace transform that led to (14.21), we have ignored a possible $t \to 0_+$ diverging term, as we do not know how to profitably regularize the delta function kernel in this limit. In the quantum (or heat kernel) case such volume term gives rise to the Weyl mean density of states (see sect. 36.1.1). A more careful treatment would regularize the divergent sum in (14.21), and possibly assign to the volume term some interesting role in the theory of classical resonance spectra. [1] [2]

Tanner says he has fixed this.

⇑PRELIMINARY

Remark 14.3 Flat and sharp traces. In the above formal derivation of trace formulas we cared very little whether our sums were well posed. In the Fredholm theory traces like (14.1) require compact operators with continuous function kernels. This is not the case for our Dirac delta evolution operators: nevertheless, there is a large class of dynamical systems for which our results may be shown to be perfectly legal. In the mathematical literature expressions like (14.6) are called *flat* traces (see the review [14.4] and chapter 16). Other names for traces appear as well: for instance, in the context of 1-d mappings, *sharp* traces refer to generalizations of (14.6) where contributions of periodic points are weighted by the Lefschetz sign ± 1, reflecting whether the periodic point sits on a branch of nth iterate of the map which crosses the diagonal starting from below or starting from above [15.12]. Such traces are connected to the theory of kneading invariants (see ref. [14.4] and references therein). Traces weighted by ± 1 sign of the derivative of the fixed point have been used to study the period doubling repeller, leading to high precision estimates of the Feigenbaum constant δ, refs. [14.5, 18.6, 14.6].

Résumé

The description of a chaotic dynamical system in terms of cycles can be visualized as a tessellation of the dynamical system, figure 1.10, with a smooth flow approximated by its *periodic orbit skeleton*, each region $\mathcal{M}_i$ centered on a periodic point x_i of the topological length n, and the size of the region determined by the linearization of the flow around the periodic point. The integral over such topologically partitioned state space yields the *classical trace formula*

$$\sum_{\alpha=0}^{\infty} \frac{1}{s-s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta\cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|}\,.$$

[1] Artuso: the remark and the exercise almost coincide

[2] Baladi: (14.21) is nonzero only if $t > 0$ is an integer multiple of T. So the limit as $t \to 0_\pm$ is just zero, no? so what or why you should regularize? For $t = 0$ you get the identity operator whose trace is divergent so there should not be great hope to give a meaning to the l.h.s. of (14.21).

Now that we have a trace formula, we might ask for what is it good? As it stands, it is little more than a scary divergent formula which relates the unspeakable infinity of global eigenvalues to the unthinkable infinity of local unstable cycles. However, it is a good stepping stone on the way to construction of spectral determinants (to which we turn next), and a first hint that when the going is good, the theory might turn out to be convergent beyond our wildest dreams (chapter 16). In order to implement such formulas, we will have to determine "all" prime cycles. The first step is topological: enumeration of all admissible cycles undertaken in chapter 12. The more onerous enterprise of actually computing the cycles we first approach traditionally, as a numerical task in chapter 17, and then more boldly as a part and parcel of variational fundations of classical and quantum dynamics in chapter 37.

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replace by ref. [1.22]

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⇑PRELIMINARY

Exercises

Laplace!transform

Exercise 14.1 $t \to 0_+$ **regularization of eigenvalue sums******.** *In taking the Laplace transform (14.21) we have ignored the $t \to 0_+$ divergence, as we do not know how to regularize the delta function kernel in this limit. In the quantum (or heat kernel) case this limit gives rise to the Weyl or Thomas-Fermi mean eigenvalue spacing (see sect. 36.1.1). Regularize the divergent sum in (14.21) following (for example) the prescription of appendix M.6 and assign to such volume term some interesting role in the theory of classical resonance spectra. E-mail the solution to the authors.*

Exercise 14.2 **General weights.** *(easy) Let f^t be a flow and $\mathcal{L}^t$ the operator*

recheck whether this is backwards

$$\mathcal{L}^t g(x) = \int dy\, \delta(x - f^t(y)) w(t,y) g(y)$$

where w is a weight function. In this problem we will try and determine some of the properties w must satisfy.

(a) *Compute $\mathcal{L}^s \mathcal{L}^t g(x)$ to show that*

$$w(s, f^t(x)) w(t,x) = w(t+s, x)\,.$$

(b) *Restrict t and s to be integers and show that the most general form of w is*

$$w(n,x) = g(x) g(f(x)) g(f^2(x)) \cdots g(f^{n-1}(x))\,,$$

for some g that can be multiplied. Could g be a function from $\mathbb{R}^{n_1} \mapsto \mathbb{R}^{n_2}$? ($n_i \in \mathbb{N}$.)

⇓PRELIMINARY

(c) *(not yet tried out) Generalize the expression for w for continuous time. Use $g = e^p$ and convert the infinite product into an exponential of an integral.*

fix - done in the text?

⇑PRELIMINARY

⇓PRELIMINARY

Exercise 14.3 **Higher order traces.** *One way to define the determinant of a matrix is to consider in a d-dimensional vector space the multilinear function Δ (this will turn out to be the determinant function where the matrix is specified by the column vectors, or if you like wedge products $\Delta(e_1, \ldots, e_d = e_1 \wedge \cdots \wedge e_d$). It takes d vectors as arguments and is linear in each of them*

$$\Delta(e_1, \ldots, \alpha e_k + e'_k, \ldots, e_d) = \alpha \Delta(e_1, \ldots, e_k, \ldots, e_d) + \Delta(e_1, \ldots, e'_k, \ldots, e_d)\,.$$

This function is also anti-symmetric: if we swap the position of two of its arguments it changes sign

$$\Delta(e_1, \ldots, e_k, \ldots, e_j, \ldots, e_d) = -\Delta(e_1, \ldots, e_j, \ldots, e_k, \ldots, e_d)\,.$$

One can show that up to a multiplicative constant, this function is unique. Assume that it has been shown unique.

(a) *Show that* $\Delta(Ae_1,\ldots,Ae_d)$ *differs from* $\Delta(e_1,\ldots,e_d)$ *by a multiplicative constant (let us call it* $\det A$*), that is,*

$$\Delta(Ae_1,\ldots,Ae_d) = \det(A)\Delta(e_1,\ldots,e_d)$$

Take this as the definition of the determinant of a matrix.

(b) *Show that* $\det(AB) = \det(A)\det(B)$

(c) *Try and give a definition for trace of a matrix in terms of* Δ*. Think of* $\det(I+\epsilon A) = 1+\epsilon \mathrm{tr}\, A$ *as* $\epsilon \to 0$ *and show that*

$$\mathrm{tr}\, A = \sum_k \Delta(e_1,\ldots,Ae_k,\ldots,e_d)$$

(d) *For a* 3×3 *matrix, expand* $\det(I-A)$.

(e) *Try the general case.*

$$\det(I-A) = \sum_{k\geq 0}(-1)^k \mathrm{tr}^{\,k}(A)\,,$$

where $\mathrm{tr}^{\,k}(A)$ *means the summation*

$$\sum_{j_1,\ldots,j_k} \Delta(\underbrace{\ldots Ae_{j_1},\ldots,Ae_{j_k}}_{k\ A's})$$

(f) *If the* e_k *are a basis for the vector space, them*

$$\Delta(\underbrace{\ldots Ae_{j_1},\ldots,Ae_{j_k}}_{k\ A's})$$

is really the determinant of the matrix A *with only* k *of the columns, i.e., a* $k\times k$ *minor. So the expansion is really an expansion in minors.*

⇑PRELIMINARY

exponential!convergence
analyticity!domain
radius!of convergence
convergence!radius
Hadamard determinant
determinant!Hadamard
spectral!determinant
determinant!spectral

Chapter 15

Spectral determinants

> "It seems very pretty," she said when she had finished it, "but it's rather hard to understand!" (You see she didn't like to confess, even to herself, that she couldn't make it out at all.) "Somehow it seems to fill my head with ideas — only I don't exactly know what they are!"
>
> Lewis Carroll, *Through the Looking Glass*

The problem with the trace formulas (14.9), (14.21) and (14.26) is that they diverge at $z = e^{-s_0}$, respectively $s = s_0$, i.e., precisely where one would like to use them. While this does not prevent numerical estimation of some "thermodynamic" averages for iterated mappings, in the case of the Gutzwiller trace formula of chapter 36 this leads to a perplexing observation that crude estimates of the radius of convergence seem to put the entire physical spectrum out of reach. We shall now cure this problem by thinking, at no extra computational cost; while traces and determinats are formally equivalent, determinants are the tool of choice when it comes to computing spectra. The idea is illustrated by figure 1.13: Determinants tend to have larger analyticity domains because if $\mathrm{tr}\,\mathcal{L}/(1-z\mathcal{L}) = -\frac{d}{dz}\ln\det(1-z\mathcal{L})$ diverges at a particular value of z, then $\det(1-z\mathcal{L})$ might have an isolated zero there, and a zero of a function is easier to determine numerically than its poles.

chapter 36

chapter 16

15.1 Spectral determinants for maps

The eigenvalues z_k of a linear operator are given by the zeros of the determinant

$$\det(1-z\mathcal{L}) = \prod_k (1-z/z_k)\,. \tag{15.1}$$

recheck that eigenvalues are labeled either by α or k

For finite matrices this is the characteristic determinant; for operators this is the Hadamard representation of the *spectral determinant* (sparing the

transfer operator!Ruelle
operator!spectrum
spectral!determinant!entire
eigenvalue!exponential spacing
4.1
page 78

reader from pondering possible regularization factors). Consider first the case of maps, for which the evolution operator advances the densities by integer steps in time. In this case we can use the formal matrix identity

$$\ln\det(1-M) = \mathrm{tr}\,\ln(1-M) = -\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}\,M^n\,, \tag{15.2}$$

to relate the spectral determinant of an evolution operator for a map to its traces (14.7), and hence to periodic orbits:

$$\begin{aligned}\det(1-z\mathcal{L}) &= \exp\left(-\sum_{n}^{\infty}\frac{z^n}{n}\mathrm{tr}\,\mathcal{L}^n\right)\\ &= \exp\left(-\sum_{p}\sum_{r=1}^{\infty}\frac{1}{r}\frac{z^{n_p r}e^{r\beta\cdot A_p}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|}\right)\,.\end{aligned} \tag{15.3}$$

⇓PRELIMINARY

This seems out of place - Bernoulli shift spectrum?

Example 15.1 Bernoulli shift spectrum?: *... the determinant,*

$$det(1-z\mathcal{L}) = \exp\left(-\sum_{n=1}\frac{z^n}{n}\,\frac{2^n}{2^n-1}\right) = \prod_{k=0}\left(1-\frac{z}{z_k}\right), \tag{15.4}$$

which agrees with the fact that the eigenvalues z_k, $k=0,1,2,\ldots$.

⇑PRELIMINARY

Going the other way, the trace formula (14.9) can be recovered from the spectral determinant by taking a derivative

$$\mathrm{tr}\,\frac{z\mathcal{L}}{1-z\mathcal{L}} = -z\frac{d}{dz}\ln\det(1-z\mathcal{L})\,. \tag{15.5}$$

fast track:
sect. 15.2, p. 269

Example 15.2 Spectral determinants of transfer operators:

For a piecewise-linear map (10.17) with a finite Markov partition, an explicit formula for the spectral determinant follows by substituting the trace formula (14.10) into (15.3):

define transfer operator

$$\det(1-z\mathcal{L}) = \prod_{k=0}^{\infty}\left(1-\frac{t_0}{\Lambda_0^k}-\frac{t_1}{\Lambda_1^k}\right)\,, \tag{15.6}$$

where $t_s = z/|\Lambda_s|$. The eigenvalues are necessarily the same as in (14.11), which we already determined from the trace formula (14.9).

The exponential spacing of eigenvalues guarantees that the spectral determinant (15.6) is an entire function. It is this property that generalizes to piecewise smooth flows with finite Markov paritions, and singles out spectral determinants rather than the trace formulas or dynamical zeta functions as the tool of choice for evaluation of spectra.

refer to the convergence chapter

15.2 Spectral determinant for flows

Smale!wild idea
spectral!determinant!for flows
determinant!spectral!for flows
flow!spectral determinant

> ...an analogue of the [Artin-Mazur] zeta function for diffeomorphisms seems quite remote for flows. However we will mention a wild idea in this direction. $[\cdots]$ define $l(\gamma)$ to be the minimal period of γ $[\cdots]$ then define formally (another zeta function!) $Z(s)$ to be the infinite product
>
> $$Z(s) = \prod_{\gamma\in\Gamma}\prod_{k=0}^{\infty}\left(1-[\exp l(\gamma)]^{-s-k}\right).$$
>
> Stephen Smale, *Differentiable Dynamical Systems*

We write the formula for the spectral determinant for flows by analogy to (15.3)

might want to make the regularization prefactor explicit?

$$\det(s-\mathcal{A}) = \exp\left(-\sum_p\sum_{r=1}^{\infty}\frac{1}{r}\frac{e^{r(\beta\cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|}\right), \tag{15.7}$$

and then check that the trace formula (14.21) is the logarithmic derivative of the spectral determinant

Fried relates the zeros to correlation spectrum

$$\operatorname{tr}\frac{1}{s-\mathcal{A}} = \frac{d}{ds}\ln\det(s-\mathcal{A}). \tag{15.8}$$

⇓PRELIMINARY

To recover $\det(s-\mathcal{A})$ integrate both sides with respect to $s-\mathcal{A}$

$$\int_{s_0}^{s} ds\,\operatorname{tr}\frac{1}{s-\mathcal{A}} = \ln\frac{\det(s-\mathcal{A})}{\det(s_0-\mathcal{A})}.$$

Here we dance all to easily around the problem of possible zero-length orbits contribution.

In this form, the determinant is regularized, as the divergent xxx contribution cancels out

$$\begin{aligned}\frac{\det(s-\mathcal{A})}{\det(s_0-\mathcal{A})} &= \exp\left(\int_{s_0}^{s} ds\,\operatorname{tr}\frac{1}{s-\mathcal{A}}\right)\\ &= \mathbf{T}\exp\left(\int_0^{\infty} dt\int_{s_0}^{s} ds\,\operatorname{tr}\,e^{t(s-\mathcal{A})}\right)\\ &= \mathbf{T}\exp\left(\int_0^{\infty} dt\frac{1}{t}\operatorname{tr}\left(e^{t(s-\mathcal{A})}-e^{t(s_0-\mathcal{A})}\right)\right).\end{aligned}$$

⇑PRELIMINARY

point to Wirzba append.

With z set to $z=e^{-s}$ as in (14.22), the spectral determinant (15.7) has the same form for both maps and flows. We refer to (15.7) as *spectral determinant*, as the spectrum of the operator $\mathcal{A}$ is given by the zeros of

$$\det(s-\mathcal{A}) = 0. \tag{15.9}$$

spectral!determinant!infinite product rep.
tp@$t˙p$!cycle weight
cycle!weigth
spectral!determinant!1-d maps

We now note that the r sum in (15.7) is close in form to the expansion of a logarithm. This observation enables us to recast the spectral determinant into an infinite product over periodic orbits as follows:

Let M_p be the p-cycle $[d\times d]$ transverse fundamental matrix, with eigenvalues $\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}$. Expanding the expanding eigenvalue factors $1/(1-1/\Lambda_{p,e})$ and the contracting eigenvalue factors $1/(1-\Lambda_{p,c})$ in (14.4) as geometric series, substituting back into (15.7), and resumming the logarithms, we find that the spectral determinant is formally given by the infinite product

Insert Gregor's volume term.
probably better to write p product first — more correct as far as convergence is concerned.
Fried An Ec Norm Sup 1985, 1986 might contain this

$$\begin{aligned}
\det(s-\mathcal{A}) &= \prod_{k_1=0}^{\infty}\cdots\prod_{l_c=0}^{\infty}\frac{1}{\zeta_{k_1\cdots l_c}} \\
1/\zeta_{k_1\cdots l_c} &= \prod_p\left(1-t_p\frac{\Lambda_{p,e+1}^{l_1}\Lambda_{p,e+2}^{l_2}\cdots\Lambda_{p,d}^{l_c}}{\Lambda_{p,1}^{k_1}\Lambda_{p,2}^{k_2}\cdots\Lambda_{p,e}^{k_e}}\right) \qquad (15.10)\\
t_p &= t_p(z,s,\beta)=\frac{1}{|\Lambda_p|}e^{\beta\cdot A_p-sT_p}z^{n_p}\,. \qquad (15.11)
\end{aligned}$$

In such formulas t_p is a weight associated with the p cycle (letter t refers to the "local trace" evaluated along the p cycle trajectory), and the index p runs through all distinct prime cycles. When convenient, we inserts the z^{n_p} factor into cycle weights, as a formal parameter which keeps track of the topological cycle lengths. These factors will assists us in expanding zeta functions and determinants, eventually we shall set $z=1$. The subscripts e,c indicate that there are e expanding eigenvalues, and c contracting eigenvalues. The observable whose average we wish to compute contributes through the $A^t(x)$ term in the p cycle multiplicative weight $e^{\beta\cdot A_p}$. By its definition (10.1), the weight for maps is a product along the cycle points

☞ chapter 18

$$e^{A_p}=\prod_{j=0}^{n_p-1}e^{a(f^j(x_p))}\,,$$

and the weight for flows is an exponential of the integral (10.5) along the cycle

$$e^{A_p}=\exp\left(\int_0^{T_p}a(x(\tau))d\tau\right).$$

This formula is correct for scalar weighting functions; more general matrix valued weights require a time-ordering prescription as in the fundamental matrix of sect. 4.1 (we will discuss this in sect. 15.5.1).

⇓PRELIMINARY
⇑PRELIMINARY

Example 15.3 Expanding 1-d map: *For expanding 1-d mappings the spectral determinant (15.10) takes the form*

$$\det(1-z\mathcal{L})=\prod_p\prod_{k=0}^{\infty}\left(1-t_p/\Lambda_p^k\right)\,,\qquad t_p=\frac{e^{\beta A_p}}{|\Lambda_p|}z^{n_p}\,. \qquad (15.12)$$

Example 15.4 Two-degree of freedom Hamiltonian flows: *For a 2-degree of freedom Hamiltonian flows the energy conservation eliminates on phase space variable, and restriction to a Poincaré section eliminates the marginal longitudinal eigenvalue $\Lambda = 1$, so a periodic orbit of 2-degree of freedom hyperbolic Hamiltonian flow has one expanding transverse eigenvalue Λ, $|\Lambda| > 1$, and one contracting transverse eigenvalue $1/\Lambda$. The weight in (14.4) is expanded as follows:*

Hamiltonian!flow!spectral determinant
spectral!determinant!2-d hyperbolic Hamiltonian flow
dynamical!zeta function
zeta function!dynamical

$$\frac{1}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|} = \frac{1}{|\Lambda|^r(1-1/\Lambda_p^r)^2} = \frac{1}{|\Lambda|^r}\sum_{k=0}^{\infty}\frac{k+1}{\Lambda_p^{kr}} \ . \tag{15.13}$$

The spectral determinant exponent can be resummed,

$$-\sum_{r=1}^{\infty}\frac{1}{r}\frac{e^{(\beta A_p - sT_p)r}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|} = \sum_{k=0}^{\infty}(k+1)\log\left(1-\frac{e^{\beta A_p - sT_p}}{|\Lambda_p|\Lambda_p^k}\right) \ ,$$

and the spectral determinant for a 2-dimensional hyperbolic Hamiltonian flow rewritten as an infinite product over prime cycles

$$\det(s-\mathcal{A}) = \prod_p\prod_{k=0}^{\infty}\left(1-t_p/\Lambda_p^k\right)^{k+1} \ . \tag{15.14}$$

state here also the d-dimensional result ala Gaspard

16.4
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Now we are finally poised to deal with the problem posed at the beginning of chapter 14; how do we actually evaluate the averages introduced in sect. 10.1? The eigenvalues of the dynamical averaging evolution operator are given by the values of s for which the spectral determinant (15.7) of the evolution operator (10.23) vanishes. If we can compute the leading eigenvalue $s_0(\beta)$ and its derivatives, we are done. Unfortunately, the infinite product formula (15.10) is no more than a shorthand notation for the periodic orbit weights contributing to the spectral determinant; more work will be needed to bring such formulas into a tractable form. This shall be accomplished in chapter 18, but here it is natural to introduce still another variant of a determinant, the dynamical zeta function.

15.3 Dynamical zeta functions

It follows from sect. 14.1.1 that if one is interested only in the leading eigenvalue of $\mathcal{L}^t$, the size of the p cycle neighborhood can be approximated by $1/|\Lambda_p|^r$, the dominant term in the $rT_p = t \to \infty$ limit, where $\Lambda_p = \prod_e \Lambda_{p,e}$ is the product of the expanding eigenvalues of the fundamental matrix M_p. With this replacement the spectral determinant (15.7) is replaced by the *dynamical zeta function*

$$1/\zeta \ = \ \exp\left(-\sum_p\sum_{r=1}^{\infty}\frac{1}{r}t_p^r\right) \tag{15.15}$$

dynamical!zeta function!Euler product rep.
Euler!product rep.!dynamical zeta function

that we have already derived heuristically in sect. 1.5.2. Resumming the logarithms using $\sum_r t_p^r/r = -\ln(1-t_p)$ we obtain the *Euler product representation* of the dynamical zeta function:

$$1/\zeta = \prod_p (1 - t_p) \ . \tag{15.16}$$

In order to simplify the notation, we usually omit the explicit dependence of $1/\zeta$, t_p on z, s, β whenever the dependence is clear from the context.

The approximate trace formula (14.26) plays the same role *vis-à-vis* the dynamical zeta function (15.8)

$$\Gamma(s) = \frac{d}{ds} \ln \zeta^{-1} = \sum_p \frac{T_p t_p}{1 - t_p} \,, \tag{15.17}$$

as the exact trace formula (14.21) plays *vis-à-vis* the spectral determinant (15.7). The heuristically derived dynamical zeta function of sect. 1.5.2 now re-emerges as the $1/\zeta_{0\cdots0}(z)$ part of the *exact* spectral determinant; other factors in the infinite product (15.10) affect the non-leading eigenvalues of $\mathcal{L}$.

In summary, the dynamical zeta function (15.16) associated with the flow $f^t(x)$ is defined as the product over all prime cycles p. The quantities, T_p, n_p and Λ_p, denote the period, topological length and product of the expanding stability eigenvalues of prime cycle p, A_p is the integrated observable $a(x)$ evaluated on a single traversal of cycle p (see (10.5)), s is a variable dual to the time t, z is dual to the discrete "topological" time n, and $t_p(z, s, \beta)$ denotes the local trace over the cycle p. We have included the factor z^{n_p} in the definition of the cycle weight in order to keep track of the number of times a cycle traverses the surface of section. The dynamical zeta function is useful because the term

$$1/\zeta(s) = 0 \tag{15.18}$$

when $s = s_0$, Here s_0 is the leading eigenvalue of $\mathcal{L}^t = e^{t\mathcal{A}}$, which is often all that is necessary for application of this equation. The above argument completes our derivation of the trace and determinant formulas for classical chaotic flows. In chapters that follow we shall make these formulas tangible by working out a series of simple examples.

The remainder of this chapter offers examples of zeta functions.

fast track:
chapter 18, p. 337

15.3.1 A contour integral formulation

contour integral

The following observation is sometimes useful, in particular for zeta functions with richer analytic structure than just zeros and poles, as in the case of intermittency (chapter 21): Γ_n, the trace sum (14.24), can be expressed in terms of the dynamical zeta function (15.16)

$$1/\zeta(z) = \prod_p \left(1 - \frac{z^{n_p}}{|\Lambda_p|}\right) \quad . \tag{15.19}$$

as a contour integral

$$\Gamma_n = \frac{1}{2\pi i} \oint_{\gamma_r^-} z^{-n} \left(\frac{d}{dz} \log \zeta^{-1}(z)\right) dz \quad , \tag{15.20}$$

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where a small contour γ_r^- encircles the origin in negative (clockwise) direction. If the contour is small enough, i.e., it lies inside the unit circle $|z| = 1$, we may write the logarithmic derivative of $\zeta^{-1}(z)$ as a convergent sum over all periodic orbits. Integrals and sums can be interchanged, the integrals can be solved term by term, and the trace formula (14.24) is recovered. For hyperbolic maps, cycle expansions or other techniques provide an analytical continuation of the dynamical zeta function beyond the leading zero; we may therefore deform the original contour into a larger circle with radius R which encircles both poles and zeros of $\zeta^{-1}(z)$, as depicted in figure 15.1. Residue calculus turns this into a sum over the zeros z_α and poles z_β of the dynamical zeta function, that is

chapter 18

$$\Gamma_n = \sum_{|z_\alpha|<R}^{\text{zeros}} \frac{1}{z_\alpha^n} - \sum_{|z_\beta|<R}^{\text{poles}} \frac{1}{z_\beta^n} + \frac{1}{2\pi i} \oint_{\gamma_R^-} dz\, z^{-n} \frac{d}{dz} \log \zeta^{-1}, \tag{15.21}$$

where the last term gives a contribution from a large circle γ_R^-. It would be a miracle if you still remebered this, but in sect. 1.4.2 we interpreted Γ_n as fraction of survivors after n bounces, and defined the escape rate γ as the rate of the find exponential decay of Γ_n. We now see that this exponential decay is dominated by the leading zero or pole of $\zeta^{-1}(z)$.

15.3.2 Dynamical zeta functions for transfer operators

Ruelle's original dynamical zeta function was a generalization of the topological zeta function (13.21) to a function that assigns different weights to different cycles:

chapter 13

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \left(\sum_{x_i \in \text{Fix} f^n} \text{tr} \prod_{j=0}^{n-1} g(f^j(x_i)) \right) .$$

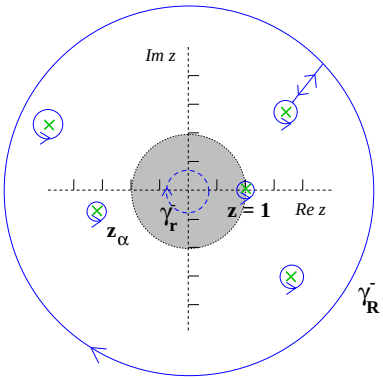

Figure 15.1: The survival probability Γ_n can be split into contributions from poles (x) and zeros (o) between the small and the large circle and a contribution from the large circle.

14.2 page 265

Here we sum over all periodic points x_i of period n, and $g(x)$ is any (matrix valued) weighting function, where the weight evaluated multiplicatively along the trajectory of x_i.

maybe incorporate parts of exercise 14.2 here

By the chain rule (4.39) the stability of any n-cycle of a 1-d map is given by $\Lambda_p = \prod_{j=1}^{n} f'(x_i)$, so the 1-$d$ map cycle stability is the simplest example of a multiplicative cycle weight $g(x_i) = 1/|f'(x_i)|$, and indeed - via the Perron-Frobenius evolution operator (9.9) - the historical motivation for Ruelle's more abstract construction.

In particular, for a piecewise-linear map with a finite Markov partition such as the map of example 9.1, the dynamical zeta function is given by a finite polynomial, (discussed in more detail in appendix B.5), a straightforward generalization of the topological transition matrix determinant (11.2). As explained in sect. 13.3, for a finite $[N\times N]$ dimensional matrix the determinant is given by

⇓PRELIMINARY
⇑PRELIMINARY

$$\prod_p (1 - t_p) = \sum_{n=1}^{N} z^n c_n \,,$$

where c_n is given by the sum over all non-self-intersecting closed paths of length n together with products of all non-intersecting closed paths of total length n.

Example 15.5 A piecewise linear repeller: *Due to piecewise linearity, the stability of any n-cycle of the piecewise linear repeller (10.17) factorizes as $\Lambda_{s_1 s_2 \ldots s_n} = \Lambda_0^m \Lambda_1^{n-m}$, where m is the total number of times the letter $s_j = 0$ appears in the p symbol sequence, so the traces in the sum (14.26) take the particularly simple form*

$$\operatorname{tr} T^n = \Gamma_n = \left(\frac{1}{|\Lambda_0|} + \frac{1}{|\Lambda_1|} \right)^n .$$

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The dynamical zeta function (15.15) evaluated by resumming the traces,

$$1/\zeta(z) = 1 - z/|\Lambda_0| - z/|\Lambda_1| \,, \tag{15.22}$$

is indeed the determinant $\det(1 - zT)$ *of the transfer operator (10.19), which is almost as simple as the topological zeta function (13.25).*

sect. 11.6

false zeros
zeros!false
Euler!product

More generally, piecewise-linear approximations to dynamical systems yield polynomial or rational polynomial cycle expansions, provided that the symbolic dynamics is a subshift of finite type.

Complete with Per's pp. 22-23

We see that the exponential proliferation of cycles so dreaded by quantum chaologians is a bogus anxiety; we are dealing with exponentially many cycles of increasing length and instability, but all that really matters in this example are the stabilities of the two fixed points. Clearly the information carried by the infinity of longer cycles is highly redundant; we shall learn in chapter 18 how to exploit this redundancy systematically.

Illustrate Fredholm with Per's p. 30

15.4 False zeros

Compare (15.22) with the Euler product (15.16). For simplicity consider two equal scales, $|\Lambda_0| = |\Lambda_1| = e^{\lambda}$. Our task is to determine the leading zero $z = e^{\gamma}$ of the Euler product. It is a novice error to assume that the infinite Euler product (15.16) vanishes whenever one of its factors vanishes. If that were true, each factor $(1 - z^{n_p}/|\Lambda_p|)$ would yield

add Rieman zeta exercise

$$0 = 1 - e^{n_p(\gamma - \lambda_p)}, \tag{15.23}$$

so the escape rate γ would equal the stability exponent of a repulsive cycle, one eigenvalue $\gamma = \gamma_p$ for each prime cycle p. This is false! The exponentially growing number of cycles with growing period conspires to shift the zeros of the infinite product. The correct formula follows from (15.22)

$$0 = 1 - e^{\gamma - \lambda + h}, \qquad h = \ln 2. \tag{15.24}$$

This particular formula for the escape rate is a special case of a general relation between escape rates, Lyapunov exponents and entropies that we shall derive in sect. **??**. Physically this means that the escape induced by the repulsion by each unstable fixed point is diminished by the rate of backscatter from other repelling regions, i.e., the entropy h; the positive entropy of orbits shifts the "false zeros" $z = e^{\lambda_p}$ of the Euler product (15.16) to the true zero $z = e^{\lambda - h}$.

⇓PRIVATE
rescue this from old thermodyn1.tex
??
page ??
⇑PRIVATE

15.5 Spectral determinants *vs.* dynamical zeta functions

In sect. 15.3 we derived the dynamical zeta function as an approximation to the spectral determinant. Here we relate dynamical zeta functions to

spectral!determinant!weighted

spectral determinants *exactly*, by showing that a dynamical zeta function can be expressed as a ratio of products of spectral determinants.

RM: Ref to right equation or problem - important to explain the wedge

The elementary identity for d-dimensional matrices

$$1 = \frac{1}{\det(1-M)} \sum_{k=0}^{d} (-1)^k \mathrm{tr}\left(\wedge^k M\right) , \tag{15.25}$$

define wedge

inserted into the exponential representation (15.15) of the dynamical zeta function, relates the dynamical zeta function to *weighted* spectral determinants.

Example 15.6 Dynamical zeta function in terms of determinants, 1-d maps: *For 1-d maps the identity*

$$1 = \frac{1}{(1-1/\Lambda)} - \frac{1}{\Lambda}\frac{1}{(1-1/\Lambda)}$$

substituted into (15.15) yields an expression for the dynamical zeta function for 1-d maps as a ratio of two spectral determinants

$$1/\zeta = \frac{\det(1-z\mathcal{L})}{\det(1-z\mathcal{L}_{(1)})} \tag{15.26}$$

make an exercise: cancelation for piecewise linear flows

define meromorphic

where the cycle weight in $\mathcal{L}_{(1)}$ is given by replacement $t_p \to t_p/\Lambda_p$. As we shall see in chapter 16, this establishes that for nice hyperbolic flows $1/\zeta$ is meromorphic, with poles given by the zeros of $\det(1-z\mathcal{L}_{(1)})$. The dynamical zeta function and the spectral determinant have the same zeros, although in exceptional circumstances some zeros of $\det(1-z\mathcal{L}_{(1)})$ might be cancelled by coincident zeros of $\det(1-z\mathcal{L}_{(1)})$. Hence even though we have derived the dynamical zeta function in sect. 15.3 as an "approximation" to the spectral determinant, the two contain the same spectral information.

define $\det(1-z\mathcal{L}_{(2)})$

Example 15.7 Dynamical zeta function in terms of determinants, 2-d Hamiltonian maps: *For 2-dimensional Hamiltonian flows the above identity yields*

$$\frac{1}{|\Lambda|} = \frac{1}{|\Lambda|(1-1/\Lambda)^2}(1-2/\Lambda+1/\Lambda^2) ,$$

so

$$1/\zeta = \frac{\det(1-z\mathcal{L})\det(1-z\mathcal{L}_{(2)})}{\det(1-z\mathcal{L}_{(1)})} . \tag{15.27}$$

compare with $1/\zeta = \frac{F^2}{F_1 F_{-1}}$ of the preceeding section

write out Ruelle's alternating product for any dimensions

This establishes that for nice 2-d hyperbolic flows the dynamical zeta function is meromorphic.

Example 15.8 Dynamical zeta functions for 2-d Hamiltonian flows: *The relation (15.27) is not particularly useful for our purposes. Instead we insert the identity*

$$1 = \frac{1}{(1-1/\Lambda)^2} - \frac{2}{\Lambda}\frac{1}{(1-1/\Lambda)^2} + \frac{1}{\Lambda^2}\frac{1}{(1-1/\Lambda)^2}$$

into the exponential representation (15.15) of $1/\zeta_k$*, and obtain*

$$1/\zeta_k = \frac{\det\left(1-z\mathcal{L}_{(k)}\right)\det\left(1-z\mathcal{L}_{(k+2)}\right)}{\det\left(1-z\mathcal{L}_{(k+1)}\right)^2} \ . \tag{15.28}$$

Even though we have no guarantee that $\det\left(1-z\mathcal{L}_{(k)}\right)$ *are entire, we do know (by arguments explained in sect. ?!) that the upper bound on the leading zeros of* $\det\left(1-z\mathcal{L}_{(k+1)}\right)$ *lies strictly below the leading zeros of* $\det\left(1-z\mathcal{L}_{(k)}\right)$*, and therefore we expect that for 2-dimensional Hamiltonian flows the dynamical zeta function* $1/\zeta_k$ *generically has a* double *leading pole coinciding with the leading zero of the* $\det\left(1-z\mathcal{L}_{(k+1)}\right)$ *spectral determinant. This might fail if the poles and leading eigenvalues come in wrong order, but we have not encountered such situations in our numerical investigations. This result can also be stated as follows: the theorem establishes that the spectral determinant (15.14) is entire, and also implies that the poles in* $1/\zeta_k$ *must have the right multiplicities to cancel in the* $\det\left(1-z\mathcal{L}\right) = \prod 1/\zeta_k^{k+1}$ *product.*

s the same as 7)?

eck, looks wrong

ELIMINARY

for the sect. referred to

↑PRELIMINARY

↓PRELIMINARY

15.5.1 Spectral determinants with matrix weights

We now generalize the results of chapter 14 to matrix-weighted evolution operators given by

$$\mathcal{L}^t_{\beta\alpha}(y,x) = g^t_{\beta\alpha}(x)\,\delta\big(y-f^t(x)\big) \ . \tag{15.29}$$

Here $g^t_{\beta\alpha}(x)$ is any weight multiplicative along the trajectory, with the precise functional form depending on the dynamical average under study. We have already introduced a class of such weights in (10.23). The indices α and β refer to possible extra matrix structure; for example, vector fields such as magnetic flux lines advected by the flow discussed in appendix I.2, or the group elements associated with discrete symmetries of chapter 22. The trace $\mathrm{tr}\, g^n = \sum_\alpha g^n_{\alpha\alpha}$ is taken over the possible matrix indices of the multiplicative weight in (15.29)

$$\sum_{\alpha=0}^{\infty}\frac{1}{s-s_\alpha} = \sum_p T_p \sum_r^{\infty}\frac{e^{-sT_p r}\mathrm{tr}\, g_p^r}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|} \ . \tag{15.30}$$

$$\det\left(s-\mathcal{A}\right) = \exp\left(-\sum_p\sum_{r\geq 1}\frac{1}{r}\frac{e^{-sT_p r}\mathrm{tr}\,\left(g_p^r\right)}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|}\right) \ , \tag{15.31}$$

eigenvalue
hyperbolic!flow
flow!hyperbolic

$$\det(s-\mathcal{A}) = \prod_{k_1=0}^{\infty}\cdots\prod_{l_c=0}^{\infty}\frac{1}{\zeta_{k_1\cdots l_c}(s)}$$

$$1/\zeta_{k_1\cdots l_c}(s) = \prod_p \det\left(1-\frac{e^{-sT_p}g_p}{|\Lambda_p|}\frac{\Lambda_{p,e+1}^{l_1}\Lambda_{p,e+2}^{l_2}\cdots\Lambda_{p,d}^{l_c}}{\Lambda_{p,1}^{k_1}\Lambda_{p,2}^{k_2}\cdots\Lambda_{p,e}^{k_e}}\right) \quad (15.32)$$

make up an exercise for this

Here "det" refers to the $g_{\beta\alpha}$ matrix indices, and g_p is the p cycle average of the multiplicative weight $g^t(x)$, which for maps is the product along the cycle points

$$g_p = \prod_{j=1}^{n_p} g(f^j(x_p))\,,$$

and an integral like (10.1) along the cycle for the flows

$$g_{p,\alpha\beta} = \left[\mathbf{T}e^{\int_0^{T_p}d\tau\mathbf{a}(x(\tau))}\right]_{\alpha\beta}\,, \quad (15.33)$$

where $\mathbf{T}$ stands for the time-ordered integration. As a matrix trace can be written as sum of its eigenvalues, $\mathrm{tr}\, g = \sum g_\alpha$, such determinants tend to factorize. We shall return to this in chapter 22 and appendix I.2.

⇑PRELIMINARY

15.6 All too many eigenvalues?

What does the 2-dimensional hyperbolic Hamiltonian flow spectral determinant (15.14) tell us? Consider one of the simplest conceivable hyperbolic flows: the game of pinball of figure 15.2 (a) consisting of two disks of equal size in a plane. There is only one periodic orbit, with the period T and expanding eigenvalue Λ given by elementary considerations (see exercise 8.1), and the resonances $\det(s_\alpha-\mathcal{A})=0$, $\alpha=\{k,n\}$ plotted in figure 15.2 (b):

$$s_\alpha = -(k+1)\lambda + n\frac{2\pi i}{T}\,, \quad n\in\mathbb{Z}\,,\ k\in\mathbb{Z}_+\,, \quad \text{multiplicity } k+1, \quad (15.34)$$

can be read off the spectral determinant (15.14) for a single unstable cycle:

$$\det(s-\mathcal{A}) = \prod_{k=0}^{\infty}\left(1-e^{-sT}/|\Lambda|\Lambda^k\right)^{k+1}\,. \quad (15.35)$$

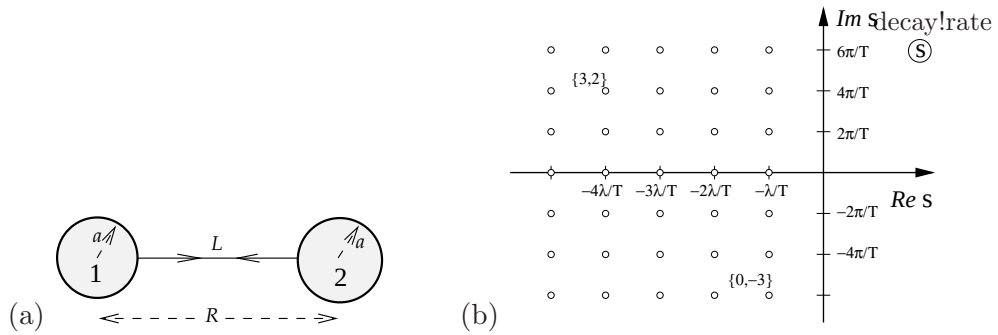

Figure 15.2: (a) A game of pinball consisting of two disks of equal size in a plane, with its only periodic orbit (A. Wirzba). (b) The classical resonances $\alpha = \{k, n\}$ for a 2-disk game of pinball, equation (15.34).

In the above $\lambda = \ln|\Lambda|/T$ is the cycle Lyapunov exponent. For an open system, the real part of the eigenvalue s_α gives the decay rate of αth eigenstate, and the imaginary part gives the "node number" of the eigenstate. The negative real part of s_α indicates that the resonance is unstable, and the decay rate in this simple case (zero entropy) equals the cycle Lyapunov exponent.

Rapidly decaying eigenstates with large negative $\mathrm{Re}\, s_\alpha$ are not a problem, but as there are eigenvalues arbitrarily far in the imaginary direction, this might seem like all too many eigenvalues. However, they are necessary - we can check this by explicit computation of the right hand side of (14.21), the trace formula for flows:

$$\begin{aligned}
\sum_{\alpha=0}^{\infty} e^{s_\alpha t} &= \sum_{k=0}^{\infty}\sum_{n=-\infty}^{\infty} (k+1) e^{(k+1)\lambda t + i2\pi n t/T} \\
&= \sum_{k=0}^{\infty} (k+1) \left(\frac{1}{|\Lambda|\Lambda^k}\right)^{t/T} \sum_{n=-\infty}^{\infty} e^{i2\pi n/T} \\
&= \sum_{k=0}^{\infty} \frac{k+1}{|\Lambda|^r \Lambda^{kr}} \sum_{r=-\infty}^{\infty} \delta(r - t/T) \\
&= T \sum_{r=-\infty}^{\infty} \frac{\delta(t - rT)}{|\Lambda|(1-1/\Lambda^r)^2}\,. \qquad (15.36)
\end{aligned}$$

Hence, the two sides of the trace formula (14.21) are verified. The formula is fine for $t > 0$; for $t \to 0_+$, however, sides are divergent and need regularization.

on dimensional grounds, need to either use $z = e^{-sT}$ everywhere, or set the time scale to $T = 1$

The reason why such sums do not occur for maps is that for discrete time we work with the variable $z = e^s$, so an infinite strip along $\mathrm{Im}\, s$ maps into an annulus in the complex z plane, and the Dirac delta sum in the above is replaced by the Kronecker delta sum in (14.7). In the case at hand there is only one time scale T, and we could just as well replace s by the variable $z = e^{-sT}$. In general, a continuous time flow has an infinity of irrationally related cycle periods, and the resonance arrays are more irregular, *cf.* figure 18.1.

add frequency beating section (PER PhD?), copy 3 disk resonances text; ask AW for figure

discuss trace class operators somewhere here

PRELIMINARY

equilibrium!point
Laplace!transform
spectral!determinant!equilibrium point
dynamical!zeta function!equilibrium point

15.7 Equilibrium points

this section looks all wrong

In our derivation of trace formulas for flows we have neglected the *equilibrium* (or *stagnation*) points x_*, i.e., the zero velocity points (2.7) that remain stationary under the flow

$$f^t(x_*) = x_* \qquad \text{for all } t\,.$$

Expanding around the equilibrium point x_*, using the fact that A, its matrix of its stability exponents (4.26), is constant, and assuming no marginal eigenvalues, we obtain the equilibrium point contribution to the trace sum

$$\operatorname{tr}\mathcal{L}^t\big|_* = \frac{e^{\beta\cdot a_* t}}{|\det\left(\mathbf{1}-e^{At}\right)|} = e^{(\beta\cdot a_*-\lambda_*)t}\prod_e \frac{1}{1-e^{-\lambda_{*,e}t}}\prod_c \frac{1}{1-e^{\lambda_{*,c}t}}\,.$$

Here the stability exponents (the eigenvalues of A) have been split into expanding ($\lambda_{*,e}>0$) and contracting ($\lambda_{*,c}<0$) exponents as in (14.4) and (15.10), $\lambda_* = \sum_e \lambda_{*,e}$ is the sum of the expanding exponents, and $a_* = a(x_*)$ is the value of the observable at the equilibrium point. Expanding each factor $1/(1-e^{-\lambda_{*,i}t})$ as a geometric series we obtain

$$\operatorname{tr}\mathcal{L}^t\big|_* = \sum_{k_1\cdots k_d}^{\infty} e^{(\beta\cdot a_*-\sum_e \lambda_{*,e}(1+k_e)+\sum_c \lambda_{*,c}k_c)t}\,. \tag{15.37}$$

Taking the Laplace transform as in (**??**) yields

$$\begin{aligned}\int_{0_+}^{\infty} dt\, e^{-st}\operatorname{tr}\mathcal{L}^t\big|_* &= \operatorname{tr}\frac{1}{s-\mathcal{A}}\Big|_* \qquad (15.38)\\ &= \sum_{k_1 k_2\cdots k_d=0}^{\infty}\frac{1}{s-\beta\cdot a_*+\sum_e \lambda_{*,e}(1+k_e)-\sum_c \lambda_{*,c}k_c}\,.\end{aligned}$$

A poor man's trace formula (14.26) contribution is obtained by setting all $k_i=0$. As in (15.8), the contribution to the determinant $\det(s-\mathcal{A})$ is obtained by integrating over $\int_{s_0}^{s} ds$. The spectral determinant (formally) has an extra factor associated with each equilibrium point x_*

$$\det(s-\mathcal{A})|_* = \prod_{k_1\cdots k_d=0}^{\infty}\left(s-\beta\cdot a_*+\sum_e \lambda_{s,e}(1+k_e)-\sum_c \lambda_{s,c}k_c\right), \tag{15.39}$$

and the dynamical zeta function (15.16) picks up an equilibrium point factor

$$1/\zeta = \prod_*(s-\beta\cdot a_*+\lambda_*)\prod_p\left(1-e^{-T_p(s-\beta\cdot a_p+\lambda_p)}\right)\,. \tag{15.40}$$

⇑PRELIMINARY

Commentary

transfer operator
Smale!wild idea
Smale, S.

Remark 15.1 Piecewise monotone maps. A partial list of cases for which the transfer operator is well defined: the expanding Hölder case, weighted subshifts of finite type, expanding differentiable case, see Bowen [1.22]: expanding holomorphic case, see Ruelle [16.9]; piecewise monotone maps of the interval, see Hofbauer and Keller [15.15] and Baladi and Keller [15.18].

Remark 15.2 Smale's wild idea. Smale's wild idea quoted on page 269 was technically wrong because 1) the Selberg zeta function yields the spectrum of a quantum mechanical Laplacian rather than the classical resonances, 2) the spectral determinant weights are different from what Smale conjectured, as the individual cycle weights also depend on the stability of the cycle, 3) the formula is not dimensionally correct, as k is an integer and s represents inverse time. Only for spaces of constant negative curvature do all cycles have the same Lyapunov exponent $\lambda = \ln|\Lambda_p|/T_p$. In this case, one can normalize time so that $\lambda = 1$, and the factors e^{-sT_p}/Λ_p^k in (15.10) simplify to $s^{-(s+k)T_p}$, as intuited in Smale's quote on page 269 (where $l(\gamma)$ is the cycle period denoted here by T_p). Nevertheless, Smale's intuition was remarkably on the target.

Remark 15.3 Is this a generalization of the Fourier analysis? Fourier analysis is a theory of the space $\leftrightarrow$ eigenfunction duality for dynamics on a circle. The way in which periodic orbit theory generalizes Fourier analysis to nonlinear flows is discussed in ref. [15.4], a very readable introduction to the Selberg Zeta function.

Remark 15.4 Zeta functions, antecedents. For a function to be deserving of the appellation "zeta function", one expects it to have an Euler product representation (15.16), and perhaps also satisfy a functional equation. Various kinds of zeta functions are reviewed in refs. [15.8, 15.9, 15.10]. Historical antecedents of the dynamical zeta function are the fixed-point counting functions introduced by Weil [15.11], Lefschetz [15.12] and Artin and Mazur [15.13], and the determinants of transfer operators of statistical mechanics [1.24].

In his review article Smale [1.21] already intuited, by analogy to the Selberg Zeta function, that the spectral determinant is the right generalization for continuous time flows. In dynamical systems theory, dynamical zeta functions arise naturally only for piecewise linear mappings; for smooth flows the natural object for the study of classical and quantal spectra are the spectral determinants. Ruelle derived the relation (15.3) between spectral determinants and dynamical zeta functions, but since he was motivated by the Artin-Mazur zeta function (13.21) and the statistical mechanics analogy, he did not consider the spectral determinant to be a more natural object than the dynamical zeta function. This has been put right in papers on "flat traces" [12.21, 16.22].

The nomenclature has not settled down yet; what we call evolution operators here is elsewhere called transfer operators [1.26], Perron-Frobenius operators [15.6] and/or Ruelle-Araki operators.

Here we refer to kernels such as (10.23) as evolution operators. We follow Ruelle in usage of the term "dynamical zeta function", but elsewhere in the

literature the function (15.16) is often called the Ruelle zeta function. Ruelle [1.27] points out that the corresponding transfer operator T was never considered by either Perron or Frobenius; a more appropriate designation would be the Ruelle-Araki operator. Determinants similar to or identical with our spectral determinants are sometimes called Selberg Zetas, Selberg-Smale zetas [1.8], functional determinants, Fredholm determinants, or even - to maximize confusion - dynamical zeta functions [15.14]. A Fredholm determinant is a notion that applies only to trace class operators - as we consider here a somewhat wider class of operators, we prefer to refer to their determinants loosely as "spectral determinants".

track down the reference to Araki.

add appropriate references

add link to Wirzba append

⇓PRELIMINARY

Remark 15.5 Discussion of a single equilibrium point. Gaspard [1.8] discusses at length the evolution operator spectrum for a system with a single equilibrium point. We will use this system in a similar way by focusing on a mapping with a single fixed point in example 16.1; this will develop our intuition about evolution operator spectra. A project length problem: We still need to show that the periodic orbits that come close to the equilibrium point shadow it.

make into a project

⇑PRELIMINARY

Résumé

The eigenvalues of evolution operators are given by the zeros of corresponding determinants, and one way to evaluate determinants is to expand them in terms of traces, using the matrix identity $\log\det = \mathrm{tr}\log$. Traces of evolution operators can be evaluated as integrals over Dirac delta functions, and in this way the spectra of evolution operators are related to periodic orbits. The spectral problem is now recast into a problem of determining zeros of either the *spectral determinant*

$$\det(s-\mathcal{A}) = \exp\left(-\sum_p\sum_{r=1}^{\infty}\frac{1}{r}\frac{e^{(\beta\cdot A_p-sT_p)r}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|}\right),$$

or the leading zeros of the *dynamical zeta function*

$$1/\zeta = \prod_p (1-t_p)\,, \qquad t_p = \frac{1}{|\Lambda_p|}e^{\beta\cdot A_p-sT_p}\,.$$

The spectral determinant is the tool of choice in actual calculations, as it has superior convergence properties (this will be discussed in chapter 16 and is illustrated, for example, by table 18.2). In practice both spectral determinants and dynamical zeta functions are preferable to trace formulas because they yield the eigenvalues more readily; the main difference is that while a trace diverges at an eigenvalue and requires extrapolation methods, determinants vanish at s corresponding to an eigenvalue s_α, and are analytic in s in an open neighborhood of s_α.

The critical step in the derivation of the periodic orbit formulas for spectral determinants and dynamical zeta functions is the hyperbolicity assumption (14.5) that no cycle stability eigenvalue is marginal, $|\Lambda_{p,i}| \neq 1$. By dropping the prefactors in (1.4), we have given up on any possibility of recovering the precise distribution of the initial x (return to the past is rendered moot by the chaotic mixing and the exponential growth of errors), but in exchange we gain an effective description of the asymptotic behavior of the system. The pleasant surprise (to be demonstrated in chapter 18) is that the infinite time behavior of an unstable system turns out to be as easy to determine as its short time behavior.

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Exercises

Exercise 15.1 **Escape rate for a 1-d repeller, numerically.** *Consider the quadratic map*

$$f(x) = Ax(1-x) \tag{15.41}$$

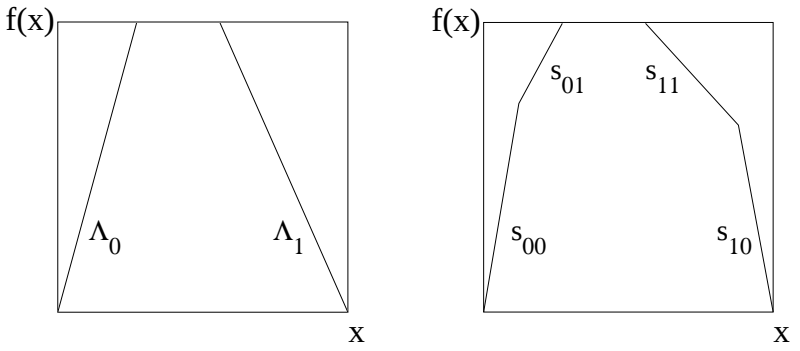
on the unit interval. The trajectory of a point starting in the unit interval either stays in the interval forever or after some iterate leaves the interval and diverges to minus infinity. Estimate numerically the escape rate (19.8), the rate of exponential decay of the measure of points remaining in the unit interval, for either $A = 9/2$ or $A = 6$. Remember to compare your numerical estimate with the solution of the continuation of this exercise, exercise 18.2.

Exercise 15.2 **Dynamical zeta functions.** *(easy)*

(a) *Evaluate in closed form the dynamical zeta function*

$$1/\zeta(z) = \prod_p \left(1 - \frac{z^{n_p}}{|\Lambda_p|}\right) ,$$

for the piecewise-linear map (10.17) with the left branch slope Λ_0, the right branch slope Λ_1.



(b) *What if there are four different slopes s_{00}, s_{01}, s_{10}, and s_{11} instead of just two, with the preimages of the gap adjusted so that junctions of branches s_{00}, s_{01} and s_{11}, s_{10} map in the gap in one iteration? What would the dynamical zeta function be?*

Exercise 15.3 **Dynamical zeta functions from Markov graphs.** *Extend sect. 13.3 to evaluation of dynamical zeta functions for piecewise linear maps with finite Markov graphs. This generalizes the results of exercise 15.2.*

Exercise 15.4 **Zeros of infinite products.** *Determination of the quantities of interest by periodic orbits involves working with infinite product formulas.*

(a) *Consider the infinite product*

$$F(z) = \prod_{k=0}^{\infty}(1 + f_k(z))$$

where the functions f_k are "sufficiently nice." This infinite product can be converted into an infinite sum by the use of a logarithm. Use the properties of infinite sums to develop a sensible definition of infinite products.

(b) *If z^* is a root of the function F, show that the infinite product diverges when evaluated at z^*.*

(c) *How does one compute a root of a function represented as an infinite product?*

(d) *Let p be all prime cycles of the binary alphabet $\{0, 1\}$. Apply your definition of $F(z)$ to the infinite product*

$$F(z) = \prod_{p}(1 - \frac{z^{n_p}}{\Lambda^{n_p}})$$

(e) *Are the roots of the factors in the above product the zeros of $F(z)$?*

(Per Rosenqvist)

Exercise 15.5 Dynamical zeta functions as ratios of spectral determinants. *(medium) Show that the zeta function*

$$1/\zeta(z) = \exp\left(-\sum_{p}\sum_{r=1}\frac{1}{r}\frac{z^{n_p}}{|\Lambda_p|^r}\right)$$

can be written as the ratio $1/\zeta(z) = \det(1 - z\mathcal{L}_{(0)})/\det(1 - z\mathcal{L}_{(1)})$, where $\det(1 - z\mathcal{L}_{(s)}) = \prod_p \prod_{k=0}^{\infty}(1 - z^{n_p}/|\Lambda_p|\Lambda_p^{k+s})$.

Exercise 15.6 Contour integral for survival probability. *Perform explicitly the contour integral appearing in (15.20).*

Exercise 15.7 Dynamical zeta function for maps. *In this problem we will compare the dynamical zeta function and the spectral determinant. Compute the exact dynamical zeta function for the skew Ulam tent map (9.43)*

$$1/\zeta(z) = \prod_{p\in P}\left(1 - \frac{z^{n_p}}{|\Lambda_p|}\right).$$

What are its roots? Do they agree with those computed in exercise 9.7?

Exercise 15.8 Dynamical zeta functions for Hamiltonian maps. *Starting from*

$$1/\zeta(s) = \exp\left(-\sum_{p}\sum_{r=1}^{\infty}\frac{1}{r}t_p^r\right)$$

for a 2-dimensional Hamiltonian map. Using the equality

$$1 = \frac{1}{(1-1/\Lambda)^2}(1-2/\Lambda+1/\Lambda^2)\,,$$

show that

$$1/\zeta = \det(1-\mathcal{L})\det(1-\mathcal{L}_{(2)})/\det(1-\mathcal{L}_{(1)})^2\,.$$

In this expression $\det(1-z\mathcal{L}_{(k)})$ *is the expansion one gets by replacing* $t_p \to t_p/\Lambda_p^k$ *in the spectral determinant.*

Exercise 15.9 Riemann ζ function. *The Riemann* ζ *function is defined as the sum*

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}\,, \qquad s\in\mathbb{C}\,.$$

(a) *Use factorization into primes to derive the Euler product representation*

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}\ .$$

The dynamical zeta function exercise 15.16 is called a "zeta" function because it shares the form of the Euler product representation with the Riemann zeta function.

(b) *(Not trivial:) For which complex values of* s *is the Riemann zeta sum convergent?*

(c) *Are the zeros of the terms in the product,* $s = -\ln p$*, also the zeros of the Riemann* ζ *function? If not, why not?*

Exercise 15.10 Finite truncations. *(easy) Suppose we have a one-dimensional system with complete binary dynamics, where the stability of each orbit is given by a simple multiplicative rule:*

$$\Lambda_p = \Lambda_0^{n_{p,0}}\Lambda_1^{n_{p,1}}\,, \qquad n_{p,0} = \#0 \text{ in } p\,,\ n_{p,1} = \#1 \text{ in } p\,,$$

so that, for example, $\Lambda_{00101} = \Lambda_0^3\Lambda_1^2$.

(a) *Compute the dynamical zeta function for this system; perhaps by creating a transfer matrix analogous to (10.19), with the right weights.*

(b) *Compute the finite* p *truncations of the cycle expansion, i.e. take the product only over the* p *up to given length with* $n_p \le N$*, and expand as a series in* z

$$\prod_p\left(1-\frac{z^{n_p}}{|\Lambda_p|}\right)\ .$$

Do they agree? If not, how does the disagreement depend on the truncation length N*?*

Chapter 16

Why does it work?

> Bloch: "Space is the field of linear operators." give ref to Bloch
> Heisenberg: "Nonsense, space is blue and birds fly through it."
> Felix Bloch, *Heisenberg and the early days of quantum mechanics*

[1]

(R. Artuso, H.H. Rugh and P. Cvitanović)

As we shall see, the trace formulas and spectral determinants work well, sometimes very well. The question is: Why? The heuristic manipulations of chapters 14 and 7 were naive and reckless, as we are facing infinite-dimensional vector spaces and singular integral kernels.

We now outline the key ingredients of proofs that put the trace and determinant formulas on solid footing. This requires taking a closer look at the evolution operators from a mathematical point of view, since up to now we have talked about eigenvalues without any reference to what kind of a function space the corresponding eigenfunctions belong to. We shall restrict our considerations to the spectral properties of the Perron-Frobenius operator for maps, as proofs for more general evolution operators follow along the same lines. What we refer to as a "the set of eigenvalues" acquires meaning only within a precisely specified functional setting: this sets the stage for a discussion of the analyticity properties of spectral determinants. In example 16.1 we compute explicitly the eigenspectrum for the three analytically tractable piecewise linear examples. In sect. 16.4 we review the basic facts of the classical Fredholm theory of integral equations. The program is sketched in sect. 16.5, motivated by an explicit study of eigenspectrum of the Bernoulli shift map, and in sect. 16.6 generalized to piecewise real-analytic hyperbolic maps acting on appropriate densities. We show on a very simple example that the spectrum is quite sensitive to the regularity properties of the functions considered.

[1] Artuso: exercises 7.9 and 7.19 have to be removed (7.9 migrated to chapter 8, while 7.19 is actually solved in the text ..).

entire function
single fixed point!repeller
repeller!single fixed point
dependent entire function
Perron-Frobenius!operator
eigenfunctions!Perron-Frobenius

For expanding and hyperbolic finite-subshift maps analyticity leads to a very strong result; not only do the determinants have better analyticity properties than the trace formulas, but the spectral determinants are singled out as entire functions in the complex s plane.

The goal of this chapter is not to provide an exhaustive review of the rigorous theory of the Perron-Frobenius operators and their spectral determinants, but rather to give you a feeling for how our heuristic considerations can be put on a firm basis. The mathematics underpinning the theory is both hard and profound.

If you are primarily interested in applications of the periodic orbit theory, you should skip this chapter on the first reading.

fast track:
chapter 17, p. 317

16.1 Linear maps: exact spectra

We start gently; in example 16.1 we work out the *exact* eigenvalues and eigenfunctions of the Perron-Frobenius operator for the simplest example of unstable, expanding dynamics, a linear 1-d map with one unstable fixed point. . Ref. [16.6] shows that this can be carried over to d-dimensions. Not only that, but in example 16.5 we compute the exact spectrum for the simplest example of a dynamical system with an *infinity* of unstable periodic orbits, the Bernoulli shift.

fix!

Example 16.1 The simplest eigenspectrum - a single fixed point: *In order to get some feeling for the determinants defined so formally in sect. 15.2, let us work out a trivial example: a repeller with only one expanding linear branch*

$$f(x) = \Lambda x\,, \qquad |\Lambda| > 1\,,$$

and only one fixed point $x^* = 0$. *The action of the Perron-Frobenius operator (9.10) is*

$$\mathcal{L}\phi(y) = \int dx\,\delta(y-\Lambda x)\,\phi(x) = \frac{1}{|\Lambda|}\phi(y/\Lambda)\,. \tag{16.1}$$

From this one immediately gets that the monomials y^k *are eigenfunctions:*

$$\mathcal{L}y^k = \frac{1}{|\Lambda|\Lambda^k}y^k\,, \qquad k = 0, 1, 2, \ldots \tag{16.2}$$

What are these eigenfunctions? Think of eigenfunctions of the Schrödinger equation: k labels the kth eigenfunction x^k in the same spirit in which the number of nodes labels the kth quantum-mechanical eigenfunction. A quantum-mechanical amplitude with more nodes has more variability,

hence a higher kinetic energy. Analogously, for a Perron-Frobenius operator, a higher k eigenvalue $1/|\Lambda|\Lambda^k$ is getting exponentially smaller because densities that vary more rapidly decay more rapidly under the expanding action of the map.

eigenfunction!"FPoper
eigenvalue!"FPoper
spectral!determinant!single fixed point
single fixed point!spectral determinant
trace formula!maps
spectral!radius

Example 16.2 The trace formula for a single fixed point: *The eigenvalues Λ^{-k-1} fall off exponentially with k, so the trace of $\mathcal{L}$ is a convergent sum*

$$\operatorname{tr}\mathcal{L} = \frac{1}{|\Lambda|}\sum_{k=0}^{\infty}\Lambda^{-k} = \frac{1}{|\Lambda|(1-\Lambda^{-1})} = \frac{1}{|f(0)'-1|}\,,$$

in agreement with (14.6). A similar result follows for powers of $\mathcal{L}$, yielding the single-fixed point version of the trace formula for maps (14.9):

$$\sum_{k=0}^{\infty}\frac{ze^{s_k}}{1-ze^{s_k}} = \sum_{r=1}^{\infty}\frac{z^r}{|1-\Lambda^r|}\,, \qquad e^{s_k} = \frac{1}{|\Lambda|\Lambda^k}\,. \tag{16.3}$$

The left hand side of (16.3) is a meromorphic function, with the leading zero at $z = |\Lambda|$. So what?

Example 16.3 Meromorphic functions and exponential convergence: *As an illustration of how exponential convergence of a truncated series is related to analytic properties of functions, consider, as the simplest possible example of a meromorphic function, the ratio*

$$h(z) \;=\; \frac{z-a}{z-b}$$

with a, b real and positive and $a < b$. Within the spectral radius *$|z| < b$ the function h can be represented by the power series*

$$h(z) \;=\sum_{k=0}^{\infty}\sigma_k z^k\,,$$

where $\sigma_0 = a/b$, and the higher order coefficients are given by $\sigma_j = (a-b)/b^{j+1}$. Consider now the truncation of order N of the power series

$$h_N(z) \;=\; \sum_{k=0}^{N}\sigma_k z^k \;=\; \frac{a}{b} \;+\; \frac{z(a-b)(1-z^N/b^N)}{b^2(1-z/b)}\,.$$

Let $\hat{z}_N$ be the solution of the truncated series $h_N(\hat{z}_N) = 0$. To estimate the distance between a and $\hat{z}_N$ it is sufficient to calculate $h_N(a)$. It is of order $(a/b)^{N+1}$, so finite order estimates converge exponentially to the asymptotic value.

This example shows that: (1) an estimate of the leading pole (the leading eigenvalue of $\mathcal{L}$) from a finite truncation of a trace formula converges exponentially, and (2) the non-leading eigenvalues of $\mathcal{L}$ lie outside of the radius of convergence of the trace formula and cannot be computed by means of such cycle expansion. However, as we shall now see, the whole spectrum is reachable at no extra effort, by computing it from a determinant rather than a trace.

Euler!formula
Bernoulli!shift
shift!Bernoulli
Bernoulli!polynomials

Example 16.4 The spectral determinant for a single fixed point: *The spectral determinant (15.3) follows from the trace formulas of example 16.2:*

$$\det(1-z\mathcal{L}) = \prod_{k=0}^{\infty}\left(1-\frac{z}{|\Lambda|\Lambda^k}\right) = \sum_{n=0}^{\infty}(-t)^n Q_n\,, \quad t=\frac{z}{|\Lambda|}\,, \tag{16.4}$$

where the cummulants Q_n are given explicitly by the Euler formula

16.3
page 316

$$Q_n = \frac{1}{1-\Lambda^{-1}}\,\frac{\Lambda^{-1}}{1-\Lambda^{-2}}\,\cdots\,\frac{\Lambda^{-n+1}}{1-\Lambda^{-n}}\quad . \tag{16.5}$$

(If you cannot figure out how to derive this formula, the solutions on p. 1130 offer several proofs.)

The main lesson to glean from this simple example is that the cummulants Q_n decay asymptotically *faster* than exponentially, as $\Lambda^{-n(n-1)/2}$. For example, if we approximate series such as (16.4) by the first 10 terms, the error in the estimate of the leading zero is $\approx 1/\Lambda^{50}$!

So far all is well for a rather boring example, a dynamical system with a single repelling fixed point. What about chaos? Systems where the number of unstable cycles increases exponentially with their length? We now turn to the simplest example of a dynamical system with an infinity of unstable periodic orbits.

copy Driebe p. 19, draw figure

Example 16.5 Bernoulli shift: *Consider next the Bernoulli shift map*

$$x \mapsto 2x \pmod 1\,, \qquad x\in[0,1]\,. \tag{16.6}$$

The associated Perron-Frobenius operator (9.9) assambles $\rho(y)$ from its two preimages

$$\mathcal{L}\rho(y) = \frac{1}{2}\rho\left(\frac{y}{2}\right)+\frac{1}{2}\rho\left(\frac{y+1}{2}\right)\,. \tag{16.7}$$

For this simple example the eigenfunctions can be written down explicitly: they coincide, up to constant prefactors, with the Bernoulli polynomials $B_n(x)$. These polynomials are generated by the Taylor expansion of the generating function

$$\mathcal{G}(x,t) = \frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!}\,, \quad B_0(x)=1\,,\ B_1(x)=x-\frac{1}{2}\,,\ldots$$

The Perron-Frobenius operator (16.7) acts on the generating function $\mathcal{G}$ as

$$\mathcal{L}\mathcal{G}(x,t) = \frac{1}{2}\left(\frac{te^{xt/2}}{e^t-1}+\frac{te^{t/2}e^{xt/2}}{e^t-1}\right) = \frac{t}{2}\frac{e^{xt/2}}{e^{t/2}-1} = \sum_{k=1}^{\infty}B_k(x)\frac{(t/2)^k}{k!}\,,$$

I do not get your drift here: The persistence of a finite essential spectral radius would suggest that traces and determinants do not exist in this case either. The pleasant surprise is that they do, see remark 16.3.

hence each $B_k(x)$ is an eigenfunction of $\mathcal{L}$ with eigenvalue $1/2^k$.

The full operator has two components corresponding to the two branches. For the n times iterated operator we have a full binary shift, and for each of the 2^n branches the above calculations carry over, yielding the same trace $(2^n-1)^{-1}$ for

every cycle on length n. Without further ado we substitute everything back and obtain the determinant,

escape!rate

$$\det(1-z\mathcal{L}) = \exp\left(-\sum_{n=1}\frac{z^n}{n}\,\frac{2^n}{2^n-1}\right) = \prod_{k=0}\left(1-\frac{z}{2^k}\right)\,, \tag{16.8}$$

verifying that the Bernoulli polynomials are eigenfunctions with eigenvalues $1,\ 1/2,\ \ldots,$ $1/2^n,\ \ldots$.

The Bernoulli map spectrum looks reminiscent of the single fixed-point spectrum (16.2), with the difference that the leading eigenvalue here is 1, rather than $1/|\Lambda|$. The difference is significant: the single fixed-point map is a repeller, with escape rate (1.6) given by the $\mathcal{L}$ leading eigenvalue $\gamma = \ln|\Lambda|$, while there is no escape in the case of the Bernoulli map. As already noted in discussion of the relation (15.24), for bound systems the local expansion rate (here $\ln|\Lambda| = \ln 2$) is balanced by the entropy (here $\ln 2$, the log of the number of preimages F_s), yielding zero escape rate.

link this to refsect??

So far we have demonstrated that our periodic orbit formulas are correct for two piecewise linear maps in 1 dimension, one with a single fixed point, and one with a full binary shift chaotic dynamics. For a single fixed point, eigenfunctions are monomials in x. For the chaotic example, they are orthogonal polynomials on the unit interval. What about higher dimensions? We check our formulas on a 2-d hyperbolic map next.

Example 16.6 The simplest of 2-d maps - a single hyperbolic fixed point: *We start by considering a very simple linear hyperbolic map with a single hyperbolic fixed point,*

$$f(x) = (f_1(x_1,x_2), f_2(x_1,x_2)) = (\Lambda_s x_1, \Lambda_u x_2)\,, \quad 0<|\Lambda_s|<1\,,\ \ |\Lambda_u|>1\,.$$

The Perron-Frobenius operator (9.10) acts on the 2-d density functions as

$$\mathcal{L}\rho(x_1,x_2) = \frac{1}{|\Lambda_s\Lambda_u|}\,\rho(x_1/\Lambda_s, x_2/\Lambda_u) \tag{16.9}$$

What are good eigenfunctions? Cribbing the 1-d eigenfunctions for the stable, contracting x_1 direction from example 16.1 is not a good idea, as under the iteration of $\mathcal{L}$ the high terms in a Taylor expansion of $\rho(x_1,x_2)$ in the x_1 variable would get multiplied by exponentially exploding eigenvalues $1/\Lambda_s^k$. This makes sense, as in the contracting directions hyperbolic dynamics crunches up initial densities, instead of smoothing them. So we guess instead that the eigenfunctions are of form

$$\varphi_{k_1k_2}(x_1,x_2) = x_2^{k_2}/x_1^{k_1+1}\,, \qquad k_1,\,k_2 = 0,1,2,\ldots\,, \tag{16.10}$$

a mixture of the Laurent series in the contraction x_1 direction, and the Taylor series in the expanding direction, the x_2 variable. The action of Perron-Frobenius operator on this set of basis functions

$$\mathcal{L}\varphi_{k_1k_2}(x_1,x_2) \;=\; \frac{\sigma}{|\Lambda_u|}\,\frac{\Lambda_s^{k_1}}{\Lambda_u^{k_2}}\,\varphi_{k_1k_2}(x_1,x_2)\,, \qquad \sigma = \Lambda_s/|\Lambda_s|$$

is smoothing, with the higher k_1, k_2 eigenvectors decaying exponentially faster, by $\Lambda_s^{k_1}/\Lambda_u^{k_2+1}$ factor in the eigenvalue. One verifies by an explicit calculation (undoing the geometric series expansions to lead to (15.10)) that the trace of $\mathcal{L}$ indeed equals $1/|\det(\mathbf{1}-M)| = 1/|(1-\Lambda_u)(1-\Lambda_s)|$, from which it follows that all our trace and spectral determinant formulas apply. The argument applies to any hyperbolic map linearized around the fixed point of form $f(x_1....,x_d) = (\Lambda_1 x_1, \Lambda_2 x_2, \ldots, \Lambda_d x_d)$.

recheck

So far we have checked the trace and spectral determinant formulas derived heuristically in chapters 14 and 15, but only for the case of 1- and 2-d linear maps. But for infinite-dimensional vector spaces this game is fraught with dangers, and we have already been mislead by piecewise linear examples into spectral confusions: contrast the spectra of example 9.1 and example 10.1 with the spectrum computed in example 14.1.

We show next that the above results do carry over to a sizable class of piecewise analytic expanding maps.

16.2 Evolution operator in a matrix representation

phase space → state space

The standard, and for numerical purposes sometimes very effective way to look at operators is through their matrix representations. Evolution operators are moving density functions defined over some state space, and as in general we can implement this only numerically, the temptation is to discretize the state space as in sect. 10.4. The problem with such state space discretization approaches that they sometimes yield plainly wrong spectra (compare example 10.1 with the result of example 14.1), so we have to think through carefully what is it that we *really* measure.

An expanding map $f(x)$ takes an initial smooth density $\phi_n(x)$, defined on a subinterval, stretches it out and overlays it over a larger interval, resulting in a new, smoother density $\phi_{n+1}(x)$. Repetition of this process smoothes the initial density, so it is natural to represent densities $\phi_n(x)$ by their Taylor series. Expanding

$$\phi_n(y) = \sum_{k=0}^{\infty} \phi_n^{(k)}(0)\frac{y^k}{k!}\,, \qquad \phi_{n+1}(y)_k = \sum_{\ell=0}^{\infty} \phi_{n+1}^{(\ell)}(0)\frac{y^\ell}{\ell!}\,,$$

$$\phi_{n+1}^{(\ell)}(0) = \int dx\ \delta^{(\ell)}(y-f(x))\phi_n(x)\Big|_{y=0}\,, \qquad x = f^{-1}(0)\,,$$

and substitute the two Taylor series into (9.6):

$$\phi_{n+1}(y) = (\mathcal{L}\phi_n)(y) = \int_{\mathcal{M}} dx\, \delta(y-f(x))\,\phi_n(x)\,.$$

The matrix elements follow by evaluating the integral

$$\mathbf{L}_{\ell k} = \frac{\partial^\ell}{\partial y^\ell} \int dx\, \mathcal{L}(y,x) \frac{x^k}{k!}\bigg|_{y=0} . \tag{16.11}$$

we obtain a matrix representation of the evolution operator

in a remark: credit Rugh [16.18] for this matrix rep

$$\int dx\, \mathcal{L}(y,x) \frac{x^k}{k!} = \sum_{k'} \frac{y^{k'}}{k'!} \mathbf{L}_{k'k} , \qquad k,k' = 0,1,2,\ldots$$

which maps the x^k component of the density of trajectories $\phi_n(x)$ into the $y^{k'}$ component of the density $\phi_{n+1}(y)$ one time step later, with $y = f(x)$.

add exercises

Rytis says drop $1 - x = f(x)$

We already have some practice with evaluating derivatives $\delta^{(\ell)}(y) = \frac{\partial^\ell}{\partial y^\ell}\delta(y)$ from sect. 9.2. This yields a representation of the evolution operator centered on the fixed point, evaluated recursively in terms of derivatives of the map f:

$$\begin{aligned} (\mathbf{L})_{\ell k} &= \int dx\, \delta^{(\ell)}(x - f(x)) \frac{x^k}{k!}\bigg|_{x=f(x)} \\ &= \frac{1}{|f'|} \left(\frac{d}{dx} \frac{1}{f'(x)} \right)^\ell \frac{x^k}{k!}\bigg|_{x=f(x)} . \end{aligned} \tag{16.12}$$

The matrix elements vanish for $\ell < k$, so $\mathbf{L}$ is a lower triangular matrix. The diagonal and the successive off-diagonal matrix elements are easily evaluated iteratively by computer algebra

$$\mathbf{L}_{kk} = \frac{1}{|\Lambda|\Lambda^k} , \quad \mathbf{L}_{k+1,k} = -\frac{(k+2)! f''}{2k!|\Lambda|\Lambda^{k+2}} , \quad \cdots .$$

For chaotic systems the map is expanding, $|\Lambda| > 1$. Hence the diagonal terms drop off exponentially, as $1/|\Lambda|^{k+1}$, the terms below the diagonal fall off even faster, and truncating $\mathbf{L}$ to a finite matrix introduces only exponentially small errors.

The trace formula (16.3) takes now a matrix form

$$\operatorname{tr} \frac{z\mathcal{L}}{1 - z\mathcal{L}} = \operatorname{tr} \frac{\mathbf{L}}{1 - z\mathbf{L}} . \tag{16.13}$$

In order to illustrate how this works, we work out a few examples.

In example 16.7 we show that these results carry over to any analytic single-branch 1-d repeller. Further examples motivate the steps that lead to a proof that spectral determinants for general analytic 1-dimensional expanding maps, and - in sect. 16.6, for 2-dimensional hyperbolic mappings - are also entire functions.

spectral!determinant!entire

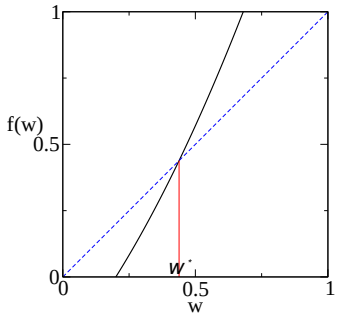


Figure 16.1: A nonlinear one-branch repeller with a single fixed point w^*.

Example 16.7 Perron-Frobenius operator in a matrix representation: *in example 16.1, we start with a map with a single fixed point, but this time with a nonlinear piecewise analytic map f with a nonlinear inverse $F = f^{-1}$, sign of the derivative $\sigma = \sigma(F') = F'/|F'|$, and the Perron-Frobenius operator acting on densities analytic in an open domain enclosing the fixed point $x = w^*$,*

compare to Dettmann? paper

rewrite in Cauchy int form

$$\mathcal{L}\phi(y) = \int dx\ \delta(y - f(x))\,\phi(x) = \sigma\ F'(y)\ \phi(F(y))\,.$$

Assume that F is a contraction of the unit disk in the complex plane, i.e.,

$$|F(z)| < \theta < 1 \quad \textit{and} \quad |F'(z)| < C < \infty \quad \textit{for} \quad |z| < 1\,, \tag{16.14}$$

and expand ϕ in a polynomial basis with the Cauchy integral formula

$$\phi(z) = \sum_{n=0}^{\infty} z^n \phi_n = \oint \frac{dw}{2\pi i}\ \frac{\phi(w)}{w - z}\,, \quad \phi_n = \oint \frac{dw}{2\pi i}\ \frac{\phi(w)}{w^{n+1}}$$

forward reference to (16.22)

Combining this with (16.22), we see that in this basis Perron-Frobenius operator $\mathcal{L}$ is represented by the matrix

$$\mathcal{L}\phi(w) = \sum_{m,n} w^m L_{mn} \phi_n\,, \quad L_{mn} = \oint \frac{dw}{2\pi i}\ \frac{\sigma\ F'(w)(F(w))^n}{w^{m+1}}\,. \tag{16.15}$$

Taking the trace and summing we get:

$$\text{tr}\ \mathcal{L} = \sum_{n\geq 0} L_{nn} = \oint \frac{dw}{2\pi i}\ \frac{\sigma\ F'(w)}{w - F(w)}\,.$$

do it for several poles

This integral has but one simple pole at the unique fixed point $w^* = F(w^*) = f(w^*)$. Hence

16.6 page 316

$$\text{tr}\ \mathcal{L} = \frac{\sigma\ F'(w^*)}{1 - F'(w^*)} = \frac{1}{|f'(w^*) - 1|}\,.$$

This super-exponential decay of cummulants Q_k ensures that for a repeller consisting of a single repelling point the spectral determinant (16.4) is *entire* in the complex z plane.

In retrospect, the matrix representation method for solving the density evolution problems is eminently sensible — after all, that is the way one

solves a close relative to classical density evolution equations, the Schrödinger equation. *When* available, matrix representations for $\mathcal{L}$ enable us to compute many more orders of cumulant expansions of spectral determinants and many more eigenvalues of evolution operators than the cycle expensions approach.

exponential!convergence
convergence!super-exponential
finite!matrix truncations
matrix!truncations

RM: The formula for product of zetas is missing

Now, if the spectral determinant is entire, formulas such as (15.26) imply that the dynamical zeta function is a meromorphic function. The practical import of this observation is that it guarantees that finite order estimates of zeroes of dynamical zeta functions and spectral determinants converge exponentially, or - in cases such as (16.4) - super-exponentially to the exact values, and so the cycle expansions to be discussed in chapter 18 represent a *true perturbative* approach to chaotic dynamics.

⇓PRELIMINARY

This result says that the single fixed-point piecewise-linear map example (16.2) applies to any analytic 1-d map with a single fixed point, with eigenfunctions $\phi_n = y^n$, and in this basis $\mathcal{L}$ diagonal, $L_{nm} = \delta_{nm}/|\Lambda|\Lambda^{-n}$. Thus we have rederived rigorously the trace formula (14.8) derived heuristically in sect. 14.2, so far only for the case of a 1-d expanding analytic map with a single fixed point. The requirement that the map be analytic is needed to substitute the bound (16.14) into the contour integral (16.15) in order to bound the $\mathcal{L}$ matrix elements

☞ appendix K.4

$$|L_{mn}| \leq \sup_{|w|\leq 1} |F'(w)|\, |F(w)|^n \leq C\theta^n$$

which shows that finite $[N\times N]$ matrix truncations approximate the operator within an error exponentially small in N. It also follows that eigenvalues decay as θ^n. In higher dimensions similar considerations show that the entries in the matrix decay as $1/\Lambda^{k^{1+1/d}}$ and eigenvalues decay as $1/\Lambda^{k^{1/d}}$.

The size of such matrices is infinite, but if the matrix entries decay exponentially fast with the index size, calculations based on finite matrix truncations have exponentially small errors.

Furthermore, from bounds on the elements L_{mn} one calculates bounds on the cummulants (??) $\mathrm{tr}\left(\wedge^k\mathcal{L}\right)$ and verifies that they fall off as $\Lambda^{-k^2/2}$, concluding that the $\mathcal{L}$ eigenvalues fall off exponentially for a general Axiom A 1-d map.

16.3 Eigenfunctions, an attractive fixed point

We have been hinting darkly all along that - in contrast to the more familiar quantum evolution operators acting on Hilbert spaces - evolution operators are in general *not* self-adjoint operators. For a finite-dimensional matrix lack of self-adjointness implies that the matrix cannot be brought to a diagonal form, and that its left and right eigenvectors are not (complex conjugate) transposes of each other.

Fredholm theory
Fredholm!integral operator

For the infinite-dimensional evolution operators the implications are stark. Consider a single fixed point as an example. For a repelling fixed point the action of the Perron-Frobenius operator is to expand and smooth out the initial distribution, and hence the natural basis for densities an expanding maps evolves are smooth functions, such as polynomials: the *right* eigenfunctions are smooth.

Consider now the action of the Perron-Frobenius operator for the same map backward in time. Now the fixed point is attractive, densities get more contracted and kinkier with time, concentrating asymptotical on the fixed point itself; the *left* eigenfunctions are highly singular.

⇑PRELIMINARY

Before turning to specifics we summarize a few facts about classical theory of integral equations, something you might prefer to skip on first reading. The purpose of this exercise is to understand that the Fredholm theory, a theory that works so well for the Hilbert spaces of quantum mechanics does not necessarily work for deterministic dynamics - the ergodic theory is much harder.

fast track:
sect. 16.5, p. 300

16.4 Classical Fredholm theory

He who would valiant be
'Gainst all disaster
Let him in constancy
Follow the Master.
John Bunyan, *Pilgrim's Progress*

The Perron-Frobenius operator

$$\mathcal{L}\phi(x) = \int dy\, \delta(x - f(y))\, \phi(y)$$

has the same appearance as a classical Fredholm integral operator

$$\mathcal{K}\varphi(x) \;=\; \int_{\mathcal{M}} dy\, \mathcal{K}(x,y)\varphi(y)\,, \tag{16.16}$$

and one is tempted to resort too classical Fredholm theory in order to establish analyticity properties of spectral determinants. This path to enlightenment is blocked by the singular nature of the kernel, which is a distribution, whereas the standard theory of integral equations usually concerns itself with regular kernels $\mathcal{K}(x,y) \in L^2(\mathcal{M}^2)$. Here we briefly

recall some steps of Fredholm theory, before working out the example of example 16.5.

Fredholm!integral equations
Hilbert-Schmidt! condition
resolvent!kernel
kernel!resolving

The general form of Fredholm integral equations of the second kind is

$$\varphi(x) \;=\; \int_{\mathcal{M}} dy\, \mathcal{K}(x,y)\varphi(y) \;+\; \xi(x) \tag{16.17}$$

where $\xi(x)$ is a given function in $L^2(\mathcal{M})$ and the kernel $\mathcal{K}(x,y) \in L^2(\mathcal{M}^2)$ (Hilbert-Schmidt condition). The natural object to study is then the linear integral operator (16.16), acting on the Hilbert space $L^2(\mathcal{M})$: the fundamental property that follows from the $L^2(Q)$ nature of the kernel is that such an operator is *compact*, that is close to a finite rank operator (see appendix M). A compact operator has the property that for every $\delta > 0$ only a *finite* number of linearly independent eigenvectors exist corresponding to eigenvalues whose absolute value exceeds δ, so we immediately realize (figure 16.4) that much work is needed to bring Perron-Frobenius operators into this picture.

We rewrite (16.17) in the form

$$\mathcal{T}\varphi \;=\; \xi\,, \qquad \mathcal{T} \;=\; \mathbb{1} - \mathcal{K}\,. \tag{16.18}$$

The Fredholm alternative is now applied to this situation as follows: the equation $\mathcal{T}\varphi = \xi$ has a unique solution for every $\xi \in L^2(\mathcal{M})$ **or** there exists a non-zero solution of $\mathcal{T}\varphi_0 = 0$, with an eigenvector of $\mathcal{K}$ corresponding to the eigenvalue 1. The theory remains the same if instead of $\mathcal{T}$ we consider the operator $\mathcal{T}_\lambda = \mathbb{1} - \lambda\mathcal{K}$ with $\lambda \neq 0$. As $\mathcal{K}$ is a compact operator there is at most a denumerable set of λ for which the second part of the Fredholm alternative holds: apart from this set the inverse operator $(\mathbb{1} - \lambda\mathcal{T})^{-1}$ exists and is bounded (in the operator sense). When λ is sufficiently small we may look for a perturbative expression for such an inverse, as a geometric series

explain why "alternative"

$$(\mathbb{1} - \lambda\mathcal{K})^{-1} \;=\; \mathbb{1} + \lambda\mathcal{K} + \lambda^2\mathcal{K}^2 + \cdots \;=\; \mathbb{1} + \lambda\mathcal{W}\,, \tag{16.19}$$

where $\mathcal{K}^n$ is a compact integral operator with kernel

$$\mathcal{K}^n(x,y) \;=\; \int_{\mathcal{M}^{n-1}} dz_1 \,\ldots dz_{n-1}\, \mathcal{K}(x,z_1)\cdots\mathcal{K}(z_{n-1},y)\,,$$

and $\mathcal{W}$ is also compact, as it is given by the convergent sum of compact operators. The problem with (16.19) is that the series has a finite radius of convergence, while apart from a denumerable set of λ's the inverse operator is well defined. A fundamental result in the theory of integral equations consists in rewriting the resolving kernel $\mathcal{W}$ as a ratio of two *analytic* functions of λ

$$\mathcal{W}(x,y) \;=\; \frac{\mathcal{D}(x,y;\lambda)}{D(\lambda)}.$$

essential!spectrum

If we introduce the notation

$$\mathcal{K}\begin{pmatrix} x_1 \dots x_n \\ y_1 \dots y_n \end{pmatrix} = \begin{vmatrix} \mathcal{K}(x_1,y_1) & \dots & \mathcal{K}(x_1,y_n) \\ \dots & \dots & \dots \\ \mathcal{K}(x_n,y_1) & \dots & \mathcal{K}(x_n,y_n) \end{vmatrix}$$

we may write the explicit expressions

$$\begin{aligned} D(\lambda) &= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} \int_{\mathcal{M}^n} dz_1 \dots dz_n \, \mathcal{K}\begin{pmatrix} z_1 \dots z_n \\ z_1 \dots z_n \end{pmatrix} \\ &= \exp\left(-\sum_{m=1}^{\infty} \frac{\lambda^m}{m} \mathrm{tr}\, \mathcal{K}^m \right) \qquad (16.20) \\ \mathcal{D}(x,y;\lambda) &= \mathcal{K}\begin{pmatrix} x \\ y \end{pmatrix} + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_{\mathcal{M}^n} dz_1 \dots dz_n \, \mathcal{K}\begin{pmatrix} x & z_1 & \dots & z_n \\ y & z_1 & \dots & z_n \end{pmatrix} \end{aligned}$$

The quantity $D(\lambda)$ is known as the Fredholm determinant (see (15.25) and appendix M): it is an entire analytic function of λ, and $D(\lambda) = 0$ if and only if $1/\lambda$ is an eigenvalue of $\mathcal{K}$.

say error $\simeq e^{-nInn}$

Worth emphasizing again: the Fredholm theory is based on the compactness of the integral operator, i.e., on the functional properties (summability) of its kernel. As the Perron-Frobenius operator is not compact, there is a bit of wishful thinking involved here.

16.5 Analyticity of spectral determinants

insert text from "Beyond periodic orbit theory"

> They savored the strange warm glow of being much more ignorant than ordinary people, who were only ignorant of ordinary things.
> Terry Pratchett

Spaces of functions integrable L^1, or square-integrable L^2 on interval $[0,1]$ are mapped into themselves by the Perron-Frobenius operator, and in both cases the constant function $\phi_0 \equiv 1$ is an eigenfunction with eigenvalue 1. If we focus our attention on L^1 we also have a family of L^1 eigenfunctions,

$$\phi_\theta(y) = \sum_{k\neq 0} \exp(2\pi i k y) \frac{1}{|k|^\theta} \qquad (16.21)$$

with complex eigenvalue $2^{-\theta}$, parametrized by complex θ with Re $\theta > 0$. By varying θ one realizes that such eigenvalues fill out the entire unit disk. Such *essential spectrum*, the case $k = 0$ of figure 16.4, hides all fine details of the spectrum.

What's going on? Spaces L^1 and L^2 contain arbitrarily ugly functions, allowing any singularity as long as it is (square) integrable - and there is no way that expanding dynamics can smooth a kinky function with a non-differentiable singularity, let's say a discontinuous step, and that is why the eigenspectrum is dense rather than discrete. Mathematicians love to wallow in this kind of muck, but there is no way to prepare a nowhere differentiable L^1 initial density in a laboratory. The only thing we can prepare and measure are piecewise smooth (real-analytic) density functions.

spectral!radius
essential!spectral radius

move this somewhere; define Banach space

For a bounded linear operator $\mathcal{A}$ on a Banach space $\mathbf{\Omega}$, the spectral radius is the smallest positive number ρ_{spec} such that the spectrum is inside the disk of radius ρ_{spec}, while the essential spectral radius is the smallest positive number ρ_{ess} such that outside the disk of radius ρ_{ess} the spectrum consists only of isolated eigenvalues of finite multiplicity (see figure 16.4).

16.5
page 316

We may shrink the essential spectrum by letting the Perron-Frobenius operator act on a space of smoother functions, exactly as in the one-branch repeller case of sect. 16.1. We thus consider a smaller space, $\mathbb{C}^{k+\alpha}$, the space of k times differentiable functions whose k'th derivatives are Hölder continuous with an exponent $0 < \alpha \leq 1$: the expansion property guarantees that such a space is mapped into itself by the Perron-Frobenius operator. In the strip $0 < \mathrm{Re}\,\theta < k+\alpha$ most ϕ_θ will cease to be eigenfunctions in the space $\mathbb{C}^{k+\alpha}$; the function ϕ_n survives only for integer valued $\theta = n$. In this way we arrive at a finite set of *isolated* eigenvalues 1, 2^{-1}, $\cdots$, 2^{-k}, and an essential spectral radius $\rho_{ess} = 2^{-(k+\alpha)}$.

this is not Bernuolli specific

We follow a simpler path and restrict the function space even further, namely to a space of analytic functions, i.e., functions for which the Taylor expansion is convergent at each point of the interval $[0,1]$. With this choice things turn out easy and elegant. To be more specific, let ϕ be a holomorphic and bounded function on the disk $D = B(0,R)$ of radius $R > 0$ centered at the origin. Our Perron-Frobenius operator preserves the space of such functions provided $(1+R)/2 < R$ so all we need is to choose $R > 1$. If F_s, $s \in \{0,1\}$, denotes the s inverse branch of the Bernoulli shift (16.6), the corresponding action of the Perron-Frobenius operator is given by $\mathcal{L}_s h(y) = \sigma\ F_s'(y)\ h \circ F_s(y)$, using the Cauchy integral formula along the ∂D boundary contour:

copy Driebe p. 19, draw figure

$$\mathcal{L}_s h(y) = \sigma \oint_{\partial D} \frac{dw}{2\pi i} \frac{h(w) F_s'(y)}{w - F_s(y)}\,. \tag{16.22}$$

MAP: this exposition is confusing.

For reasons that will be made clear later we have introduced a sign $\sigma = \pm 1$ of the given real branch $|F'(y)| = \sigma\, F'(y)$. For both branches of the Bernoulli shift $s = 1$, but in general one is not allowed to take absolute values as this could destroy analyticity. In the above formula one may also replace the domain D by *any domain* containing $[0,1]$ such that the inverse branches maps the closure of D into the interior of D. Why? simply because the kernel remains non-singular under this condition, i.e., $w - F(y) \neq 0$

whenever $w \in \partial D$ and $y \in \mathrm{Cl}\ D$. The problem is now reduced to the standard theory for Fredholm determinants, sect. 16.4. The integral kernel is no longer singular, traces and determinants are well-defined, and we can evaluate the trace of $\mathcal{L}_F$ by means of the Cauchy contour integral formula:

MAP:explain C notation

$$\mathrm{tr}\ \mathcal{L}_F = \oint \frac{dw}{2\pi i} \frac{\sigma F'(w)}{w - F(w)}\,.$$

Elementary complex analysis shows that since F maps the closure of D into its own interior, F has a unique (real-valued) fixed point x^* with a multiplier strictly smaller than one in absolute value. Residue calculus therefore yields

16.6 page 316

$$\mathrm{tr}\ \mathcal{L}_F = \frac{\sigma F'(x^*)}{1 - F'(x^*)} = \frac{1}{|f'(x^*) - 1|}\,,$$

justifying our previous *ad hoc* calculations of traces using Dirac delta functions.

compare to the Dettmann? papers

rewrite in Cauchy int form

Example 16.8 Perron-Frobenius operator in a matrix representation: *As in example 16.1, we start with a map with a single fixed point, but this time with a nonlinear piecewise analytic map f with a nonlinear inverse $F = f^{-1}$, sign of the derivative $\sigma = \sigma(F') = F'/|F'|$*

$$\mathcal{L}\phi(z) = \int dx\ \delta(z - f(x))\,\phi(x) = \sigma\ F'(z)\ \phi(F(z))\,.$$

Assume that F is a contraction of the unit disk, i.e.,

$$|F(z)| < \theta < 1 \quad \textit{and} \quad |F'(z)| < C < \infty \quad \textit{for} \quad |z| < 1\,, \tag{16.23}$$

and expand ϕ in a polynomial basis by means of the Cauchy formula

$$\phi(z) = \sum_{n\geq 0} z^n \phi_n = \oint \frac{dw}{2\pi i}\ \frac{\phi(w)}{w - z}\,, \quad \phi_n = \oint \frac{dw}{2\pi i}\ \frac{\phi(w)}{w^{n+1}}$$

Combining this with (16.22), we see that in this basis $\mathcal{L}$ is represented by the matrix

$$\mathcal{L}\phi(w) = \sum_{m,n} w^m L_{mn} \phi_n\,, \quad L_{mn} = \oint \frac{dw}{2\pi i}\ \frac{\sigma\ F'(w)(F(w))^n}{w^{m+1}}\,. \tag{16.24}$$

this is repeat of (16.15)?

Taking the trace and summing we get:

$$\mathrm{tr}\ \mathcal{L} = \sum_{n\geq 0} L_{nn} = \oint \frac{dw}{2\pi i}\ \frac{\sigma\ F'(w)}{w - F(w)}\,.$$

do it for several poles

This integral has but one simple pole at the unique fixed point $w^* = F(w^*) = f(w^*)$. Hence

16.6 page 316

$$\mathrm{tr}\ \mathcal{L} = \frac{\sigma\ F'(w^*)}{1 - F'(w^*)} = \frac{1}{|f'(w^*) - 1|}\,.$$

Bernuolli
nues here

three-disk@3-disk!convergence

We worked out a very specific example, yet our conclusions can be generalized, provided a number of restrictive requirements are met by the dynamical system under investigation:

1) the evolution operator is *multiplicative* along the flow,
2) the symbolic dynamics is a *finite subshift*,
3) all cycle eigenvalues are *hyperbolic* (exponentially bounded in magnitude away from 1),
4) the map (or the flow) is *real analytic*, i.e., it has a piecewise analytic continuation to a complex extension of the state space.

These assumptions are romantic expectations not satisfied by the dynamical systems that we actually desire to understand. Still, they are not devoid of physical interest; for example, nice repellers like our 3-disk game of pinball do satisfy the above requirements.

Properties 1 and 2 enable us to represent the evolution operator as a finite matrix in an appropriate basis; properties 3 and 4 enable us to bound the size of the matrix elements and control the eigenvalues. To see what can go wrong, consider the following examples:

Property 1 is violated for flows in 3 or more dimensions by the following weighted evolution operator

$$\mathcal{L}^t(y,x) = |\Lambda^t(x)|^\beta \delta\big(y - f^t(x)\big) \ ,$$

where $\Lambda^t(x)$ is an eigenvalue of the fundamental matrix transverse to the flow. Semiclassical quantum mechanics suggest operators of this form with $\beta = 1/2$, (see chapter 36). The problem with such operators arises from the fact that when considering the fundamental matrices $J_{ab} = J_a J_b$ for two successive trajectory segments a and b, the corresponding eigenvalues are in general *not* multiplicative, $\Lambda_{ab} \neq \Lambda_a \Lambda_b$ (unless a, b are iterates of the same prime cycle p, so $J_a J_b = J_p^{r_a+r_b}$). Consequently, this evolution operator is not multiplicative along the trajectory. The theorems require that the evolution be represented as a matrix in an appropriate polynomial basis, and thus cannot be applied to non-multiplicative kernels, i.e., kernels that do not satisfy the semi-group property $\mathcal{L}^{t'}\mathcal{L}^t = \mathcal{L}^{t'+t}$. The cure for this problem in this particular case is given in appendix I.1.

Property 2 is violated by the 1-d tent map (see figure 16.2 (a))

$$f(x) = \alpha(1 \ - \ |1-2x|) \ , \quad 1/2 < \alpha < 1 \ .$$

define pre-periodic

All cycle eigenvalues are hyperbolic, but in general the critical point $x_c = 1/2$ is not a pre-periodic point, so there is no finite Markov partition and the symbolic dynamics does not have a finite grammar (see sect. 12.4 for

intermittency
marginal!stability

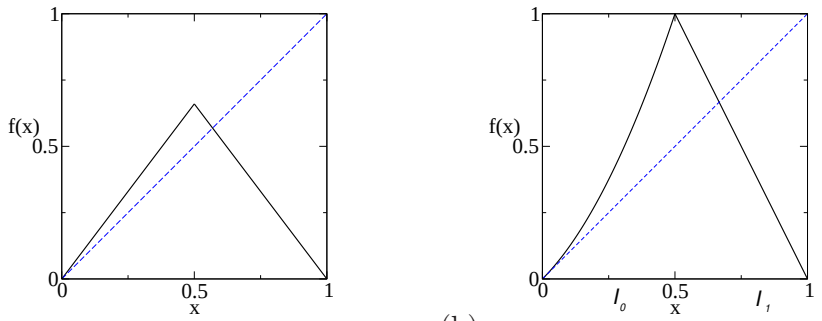


Figure 16.2: (a) A (hyperbolic) tent map without a finite Markov partition. (b) A Markov map with a marginal fixed point.

definitions). In practice, this means that while the leading eigenvalue of $\mathcal{L}$ might be computable, the rest of the spectrum is very hard to control; as the parameter α is varied, the non-leading zeros of the spectral determinant move wildly about.

Property 3 is violated by the map (see figure 16.2 (b))

$$f(x) = \begin{cases} x + 2x^2 & , \quad x \in I_0 = [0, \frac{1}{2}] \\ 2 - 2x & , \quad x \in I_1 = [\frac{1}{2}, 1] \end{cases} .$$

draw figure

Here the interval $[0, 1]$ has a Markov partition into two subintervals I_0 and I_1, and f is monotone on each. However, the fixed point at $x = 0$ has marginal stability $\Lambda_0 = 1$, and violates condition 3. This type of map is called "intermittent" and necessitates much extra work. The problem is that the dynamics in the neighborhood of a marginal fixed point is very slow, with correlations decaying as power laws rather than exponentially. We will discuss such flows in chapter 21.

Property 4 is required as the heuristic approach of chapter 14 faces two major hurdles:

1. The trace (14.7) is not well defined because the integral kernel is singular.

2. The existence and properties of eigenvalues are by no means clear.

Actually, property 4 is quite restrictive, but we need it in the present approach, so that the Banach space of analytic functions in a disk is preserved by the Perron-Frobenius operator.

In attempting to generalize the results, we encounter several problems. First, in higher dimensions life is not as simple. Multi-dimensional residue calculus is at our disposal but in general requires that we find poly-domains (direct product of domains in each coordinate) and this need not be the

case. Second, and perhaps somewhat surprisingly, the 'counting of periodic orbits' presents a difficult problem. For example, instead of the Bernoulli shift consider the doubling map of the circle, $x \mapsto 2x \bmod 1$, $x \in R/Z$. Compared to the shift on the interval $[0,1]$ the only difference is that the endpoints 0 and 1 are now glued together. Because these endpoints are fixed points of the map, the number of cycles of length n decreases by 1. The determinant becomes:

doubling map

add a cat-map exercise

$$\det(1 - z\mathcal{L}) = \exp\left(-\sum_{n=1} \frac{z^n}{n}\, \frac{2^n - 1}{2^n - 1}\right) = 1 - z. \tag{16.25}$$

The value $z = 1$ still comes from the constant eigenfunction, but the Bernoulli polynomials no longer contribute to the spectrum (as they are not periodic). Proofs of these facts, however, are difficult if one sticks to the space of analytic functions.

Third, our Cauchy formulas *a priori* work only when considering purely expanding maps. When stable and unstable directions co-exist we have to resort to stranger function spaces, as shown in the next section.

16.6 Hyperbolic maps

> I can give you a definion of a Banach space, but I do not know what that means.
> Federico Bonnetto, *Banach space*
>
> (H.H. Rugh)

Proceeding to hyperbolic systems, one faces the following paradox: If f is an area-preserving hyperbolic and real-analytic map of, for example, a 2-dimensional torus then the Perron-Frobenius operator is unitary on the space of L^2 functions, and its spectrum is confined to the unit circle. On the other hand, when we compute determinants we find eigenvalues scattered around inside the unit disk. Thinking back to the Bernoulli shift example 16.5 [2] one would like to imagine these eigenvalues as popping up from the L^2 spectrum by shrinking the function space. Shrinking the space, however, can only make the spectrum smaller so this is obviously not what happens. Instead one needs to introduce a 'mixed' function space where in the unstable direction one resorts to analytic functions, as before, but in the stable direction one instead considers a 'dual space' of distributions on analytic functions. Such a space is neither included in nor includes L^2 and we have thus resolved the paradox. However, it still remains to be seen how traces and determinants are calculated.

The linear hyperbolic fixed point example 16.6 is somewhat misleading, as we have made explicit use of a map that acts independently along the

[2] Zazy: check this!

Axiom A!systems

stable and unstable directions. For a more general hyperbolic map, there is no way to implement such direct product structure, and the whole argument falls apart. Her comes an idea; use the analyticity of the map to rewrite the Perron-Frobenius operator acting as follows (where σ denotes the sign of the derivative in the unstable direction):

$$\mathcal{L}h(z_1,z_2) = \oint\oint \frac{\sigma\ h(w_1,w_2)}{(z_1-f_1(w_1,w_2)(f_2(w_1,w_2)-z_2)} \frac{dw_1}{2\pi i}\frac{dw_2}{2\pi i}. \quad (16.26)$$

Here the function ϕ should belong to a space of functions analytic respectively *outside* a disk and *inside* a disk in the first and the second coordinates; with the additional property that the function decays to zero as the first coordinate tends to infinity. The contour integrals are along the boundaries of these disks. It is an exercise in multi-dimensional residue calculus to verify that for the above linear example this expression reduces to (16.9). Such operators form the building blocks in the calculation of traces and determinants. One can prove the following:

Theorem: *The spectral determinant for 2-d hyperbolic analytic maps is entire.*

☞ remark 16.12

The proof, apart from the Markov property that is the same as for the purely expanding case, relies heavily on the analyticity of the map in the explicit construction of the function space. The idea is to view the hyperbolicity as a cross product of a contracting map in forward time and another contracting map in backward time. In this case the Markov property introduced above has to be elaborated a bit. Instead of dividing the state space into intervals, one divides it into rectangles. The rectangles should be viewed as a direct product of intervals (say horizontal and vertical), such that the forward map is contracting in, for example, the horizontal direction, while the inverse map is contracting in the vertical direction. For Axiom A systems (see remark 16.12) one may choose coordinate axes close to the stable/unstable manifolds of the map. With the state space divided into N rectangles $\{\mathcal{M}_1,\mathcal{M}_2,\ldots,\mathcal{M}_N\}$, $\mathcal{M}_i = I_i^h \times I_i^v$ one needs a complex extension $D_i^h \times D_i^v$, with which the hyperbolicity condition (which simultaneously guarantees the Markov property) can be formulated as follows:

Analytic hyperbolic property: Either $f(\mathcal{M}_i)\cap \mathrm{Int}(\mathcal{M}_j) = \emptyset$, or for each pair $w_h \in \mathrm{Cl}(D_i^h)$, $z_v \in \mathrm{Cl}(D_j^v)$ there exist unique analytic functions of w_h, z_v: $w_v = w_v(w_h,z_v) \in \mathrm{Int}(D_i^v)$, $z_h = z_h(w_h,z_v) \in \mathrm{Int}(D_j^h)$, such that $f(w_h,w_v) = (z_h,z_v)$. Furthermore, if $w_h \in I_i^h$ and $z_v \in I_j^v$, then $w_v \in I_i^v$ and $z_h \in I_j^h$ (see figure 16.3).

In plain English, this means for the iterated map that one replaces the coordinates z_h, z_v at time n by the contracting pair z_h, w_v, where w_v is the contracting coordinate at time $n+1$ for the 'partial' inverse map.

Perron-Frobenius!theorem
natural measure

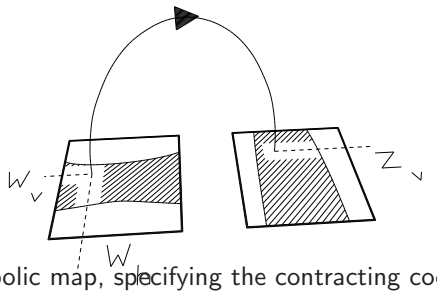


Figure 16.3: For an analytic hyperbolic map, specifying the contracting coordinate w_h at the initial rectangle and the expanding coordinate z_v at the image rectangle defines a unique trajectory between the two rectangles. In particular, w_v and z_h (not shown) are uniquely specified.

In two dimensions the operator in (16.26) acts on functions analytic outside D_i^h in the horizontal direction (and tending to zero at infinity) and inside D_i^v in the vertical direction. The contour integrals are precisely along the boundaries of these domains.

A map f satisfying the above condition is called *analytic hyperbolic* and the theorem states that the associated spectral determinant is entire, and that the trace formula (14.7) is correct.

Examples of analytic hyperbolic maps are provided by small analytic perturbations of the cat map, the 3-disk repeller, and the 2-d baker's map.

check linear horseshoes in Nitecki book

16.7 The physics of eigenvalues and eigenfunctions

We appreciate by now that any honest attempt to look at the spectral properties of the Perron-Frobenius operator involves hard mathematics, but the effort is rewarded by the fact that we are finally able to control the analyticity properties of dynamical zeta functions and spectral determinants, and thus substantiate the claim that these objects provide a powerful and well-founded perturbation theory.

Often (see chapter 10) physically important part of the spectrum is just the leading eigenvalue, which gives us the escape rate from a repeller, or, for a general evolution operator, formulas for expectation values of observables and their higher moments. Also the eigenfunction associated to the leading eigenvalue has a physical interpretation (see chapter 9): it is the density of the natural measures, with singular measures ruled out by the proper choice of the function space. This conclusion is in accord with the generalized Perron-Frobenius theorem for evolution operators. In the finite dimensional setting, such a theorem is formulated as follows:

replace "coherent"

☞ remark 16.8

- **Perron-Frobenius theorem:** Let L_{ij} be a nonnegative matrix, such that some n exists for which $(L^n)_{ij} > 0 \ \forall i,j$: then
  1. The maximal modulus eigenvalue is non-degenerate real, and positive
  2. The corresponding eigenvector (defined up to a constant) has nonnegative coordinates

explain $L_{ij} > 0$ means that (L^k) is growing with no cancellations, maximal growth

We may ask what physical information is contained in eigenvalues beyond the leading one: suppose that we have a probability conserving system (so that the dominant eigenvalue is 1), for which the essential spectral radius satisfies $0 < \rho_{ess} < \theta < 1$ on some Banach space $\mathcal{B}$. Denote by $\mathbf{P}$ the projection corresponding to the part of the spectrum inside a disk of radius θ. We denote by λ_1, $\lambda_2 \ldots, \lambda_M$ the eigenvalues outside of this disk, ordered by the size of their absolute value, with $\lambda_1 = 1$. Then we have the following decomposition

$$\mathcal{L}\varphi \ = \ \sum_{i=1}^{M} \lambda_i \psi_i L_i \psi_i^* \varphi \ + \ \mathbf{P}\mathcal{L}\varphi \tag{16.27}$$

when L_i are (finite) matrices in Jordan canomical form ($L_0 = 0$ is a $[1\times 1]$ matrix, as λ_0 is simple, due to the Perron-Frobenius theorem), whereas ψ_i is a row vector whose elements form a basis on the eigenspace corresponding to λ_i, and ψ_i^* is a column vector of elements of $\mathcal{B}^*$ (the dual space of linear functionals over $\mathcal{B}$) spanning the eigenspace of $\mathcal{L}^*$ corresponding to λ_i. For iterates of the Perron-Frobenius operator, (16.27) becomes

$$\mathcal{L}^n\varphi \ = \ \sum_{i=1}^{M} \lambda_i^n \psi_i L_i^n \psi_i^* \varphi \ + \ \mathbf{P}\mathcal{L}^n\varphi \,. \tag{16.28}$$

If we now consider, for example, correlation between initial φ evolved n steps and final ξ,

$$\langle \xi | \mathcal{L}^n | \varphi \rangle = \int_{\mathcal{M}} dy\, \xi(y) \left(\mathcal{L}^n\varphi\right)(y) \ = \ \int_{\mathcal{M}} dw\, (\xi \circ f^n)(w)\varphi(w)\,, \tag{16.29}$$

it follows that

$$\langle \xi | \mathcal{L}^n | \varphi \rangle = \ \lambda_1^n \omega_1(\xi,\varphi) + \sum_{i=2}^{L} \lambda_i^n \omega_i^{(n)}(\xi,\varphi) + \mathcal{O}(\theta^n)\,, \tag{16.30}$$

where

$$\omega_i^{(n)}(\xi,\varphi) \ = \ \int_{\mathcal{M}} dy\, \xi(y) \psi_i L_i^n \psi_i^* \varphi \,.$$

ce λ with s

16.7
316

correlation!function
correlation!spectrum
exponential!decay rate of correlations
decay!rate of correlations
Euler-MacLaurin!formula

The eigenvalues beyond the leading one provide two pieces of information: they rule the convergence of expressions containing high powers of the evolution operator to leading order (the λ_1 contribution). Moreover if $\omega_1(\xi,\varphi) = 0$ then (16.29) defines a correlation function: as each term in (16.30) vanishes exponentially in the $n \to \infty$ limit, the eigenvalues $\lambda_2, \ldots, \lambda_M$ determine the exponential decay of correlations for our dynamical system. The prefactors ω depend on the choice of functions, whereas the exponential decay rates (given by logarithms of λ_i) do not: the correlation spectrum is thus a *universal* property of the dynamics (once we fix the overall functional space on which the Perron-Frobenius operator acts).

Example 16.9 Bernoulli shift eigenfunctions: *Let us revisit the Bernoulli shift example (16.6) on the space of analytic functions on a disk: apart from the origin we have only simple eigenvalues $\lambda_k = 2^{-k}$, $k = 0, 1, \ldots$. The eigenvalue $\lambda_0 = 1$ corresponds to probability conservation: the corresponding eigenfunction $B_0(x) = 1$ indicates that the natural measure has a constant density over the unit interval. If we now take any analytic function $\eta(x)$ with zero average (with respect to the Lebesgue measure), it follows that $\omega_1(\eta,\eta) = 0$, and from (16.30) the asymptotic decay of the correlation function is (unless also $\omega_1(\eta,\eta) = 0$)*

$$C_{\eta,\eta}(n) \sim \exp(-n \log 2)\,. \tag{16.31}$$

Thus, $-\log\lambda_1$ gives the exponential decay rate of correlations (with a prefactor that depends on the choice of the function). Actually the Bernoulli shift case may be treated exactly, as for analytic functions we can employ the Euler-MacLaurin summation formula

$$\eta(z) = \int_0^1 dw\,\eta(w) + \sum_{m=1}^{\infty} \frac{\eta^{(m-1)}(1) - \eta^{(m-1)}(0)}{m!} B_m(z)\,. \tag{16.32}$$

As we are considering functions with zero average, we have from (16.29) and the fact that Bernoulli polynomials are eigenvectors of the Perron-Frobenius operator that

$$C_{\eta,\eta}(n) = \sum_{m=1}^{\infty} \frac{(2^{-m})^n(\eta^{(m)}(1) - \eta^{(m)}(0))}{m!} \int_0^1 dz\,\eta(z) B_m(z)\,.$$

The decomposition (16.32) is also useful in realizing that the linear functionals ψ_i^* are singular objects: if we write it as

$$\eta(z) = \sum_{m=0}^{\infty} B_m(z)\,\psi_m^*[\eta]\,,$$

we see that these functionals are of the form

might use Driebe sect 3.5.2 here; also say it as a left eigenstate $\hat{B}_j(x) = \frac{1}{j!}\frac{d^j}{dx^j}$

$$\psi_i^*[\varepsilon] = \int_0^1 dw\,\Psi_i(w)\varepsilon(w)\,,$$

where

$$\Psi_i(w) = \frac{(-1)^{i-1}}{i!}\left(\delta^{(i-1)}(w-1) - \delta^{(i-1)}(w)\right)\,, \tag{16.33}$$

when $i \geq 1$ and $\Psi_0(w) = 1$. This representation is only meaningful when the function ε is analytic in neighborhoods of $w, w-1$.

space!analytic functions
functions!analytic
essential spectral radius
spectral!radius!essential
functions!L^2 square-integrable
L2@L^2 function space

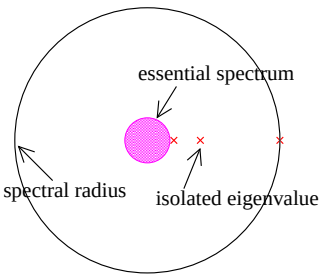


Figure 16.4: Spectrum of the Perron-Frobenius operator acting on the space of $\mathbb{C}^{k+\alpha}$ Hölder-continuous functions: only k isolated eigenvalues remain between the spectral radius, and the essential spectral radius which bounds the "essential", continuous spectrum.

16.8 Troubles ahead

The above discussion confirms that for a series of examples of increasing generality formal manipulations with traces and determinants are justified: the Perron-Frobenius operator has isolated eigenvalues, the trace formulas are explicitly verified, and the spectral determinant is an entire function whose zeroes yield the eigenvalues. Real life is harder, as we may appreciate through the following considerations:

- Our discussion tacitly assumed something that is physically entirely reasonable: our evolution operator is acting on the space of analytic functions, i.e., we are allowed to represent the initial density $\rho(x)$ by its Taylor expansions in the neighborhoods of periodic points. This is however far from being the only possible choice: mathematicians often work with the function space $\mathbb{C}^{k+\alpha}$, i.e., the space of k times differentiable functions whose k'th derivatives are Hölder continuous with an exponent $0 < \alpha \leq 1$: then every y^η with $\mathrm{Re}\,\eta > k$ is an eigenfunction of the Perron-Frobenius operator and we have

  16.1 page 316

  define Hölder continuity

  $$\mathcal{L}y^\eta = \frac{1}{|\Lambda|\Lambda^\eta} y^\eta\,, \qquad \eta \in \mathbb{C}\,.$$

  This spectrum differs markedly from the analytic case: only a small number of isolated eigenvalues remain, enclosed between the spectral radius and a smaller disk of radius $1/|\Lambda|^{k+1}$, see figure 16.4. In literature the radius of this disk is called the *essential spectral radius*.

  define the essential spectral radius?

  In sect. 16.5 we discussed this point further, with the aid of a less trivial 1-dimensional example. The physical point of view is complementary to the standard setting of ergodic theory, where many chaotic properties of a dynamical system are encoded by the presence of a *continuous* spectrum, used to prove asymptotic decay of correlations in the space of L^2 square-integrable functions.

  rewrite!

  OK in QM, dumb in classical dynamics

- A deceptively innocent assumption is hidden beneath many features discussed so far: that (16.1) maps a given function space into itself. This is strictly related to the *expanding* property of the map: if $f(x)$

  16.2 page 316

is smooth in a domain D then $f(x/\Lambda)$ is smooth on a *larger* domain, provided $|\Lambda| > 1$. This is not obviously the case for hyperbolic systems in higher dimensions, and, as we saw in sect. 16.6, extensions of the results obtained for expanding maps are highly nontrivial.

- It is not at all clear that the above analysis of a simple one-branch, one fixed point repeller can be extended to dynamical systems with a Cantor set of periodic points: we showed this in sect. 16.5.

Commentary

Remark 16.1 Surveys of rigorous theory. We recommend the references listed in sect. 1.8 for an introduction to the mathematical literature on this subject. For a physicist, Driebe's monograph [1.32] might be the most accessible introduction into mathematics discussed briefley in this chapter. There are a number of reviews of the mathematical approach to dynamical zeta functions and spectral determinants, with pointers to the original references, such as refs. [16.1, 16.2]. An alternative approach to spectral properties of the Perron-Frobenius operator is given in ref. [16.3].

Ergodic theory, as presented by Sinai [16.13] and others, tempts one to describe the densities on which the evolution operator acts in terms of either integrable or square-integrable functions. For our purposes, as we have already seen, this space is not suitable. An introduction to ergodic theory is given by Sinai, Kornfeld and Fomin [16.14]; more advanced old-fashioned presentations are Walters [**?**] and Denker, Grillenberger and Sigmund [16.15]; and a more formal one is given by Peterson [16.16].

give credit to Prigopeople + ...(/)

Remark 16.2 Fredholm theory. Our brief summary of Fredholm theory is based on the exposition of ref. [16.4]. A technical introduction of the theory from an operator point of view is given in ref. [16.5]. The theory is presented in a more general form in ref. [16.6].

MAP: add sjoötrend's book for a non-undertandable exposition, Rudin's functional analysis book

Remark 16.3 Bernoulli shift. For a more detailed discussion, consult chaper 3 of ref. [1.32]. The extension of Fredholm theory to the case or Bernoulli shift on $\mathbb{C}^{k+\alpha}$ (in which the Perron-Frobenius operator is *not* compact – technically it is only *quasi-compact*. That is, the essential spectral radius is strictly smaller than the spectral radius) has been given by Ruelle [16.7]: a concise and readable statement of the results is contained in ref. [16.8].

⇓PRELIMINARY

fix this

Remark 16.4 Higher dimensions and generalized Fredholm theory. When studying the Bernoulli shift in higher dimensions, extensions of Fredholm theory [16.6] which avoid problems with associated multi-dimensional residue calculus may be used - see ref. [16.9].

⇑PRELIMINARY

smooth!dynamics!Spectral determinant
ceiling function
Perron-Frobenius!theorem
Koopman operator
operator!Koopman

Remark 16.5 Hyperbolic dynamics. When dealing with hyperbolic systems one might try to reduce to the expanding case by projecting the dynamics along the unstable directions. As mentioned in the text this can be quite involved technically, as such unstable foliations are not characterized by strong smoothness properties. For such an approach, see ref. [16.3].

Remark 16.6 Spectral determinants for smooth flows. The theorem on page 306 also applies to hyperbolic analytic maps in d dimensions and smooth hyperbolic analytic flows in $(d+1)$ dimensions, provided that the flow can be reduced to a piecewise analytic map by a suspension on a Poincaré section, complemented by an analytic "ceiling" function (3.5) that accounts for a variation in the section return times. For example, if we take as the ceiling function $g(x) = e^{sT(x)}$, where $T(x)$ is the next Poincaré section time for a trajectory staring at x, we reproduce the flow spectral determinant (15.14). Proofs are beyond the scope of this chapter; details are discussed in ref.(?).

⇓PRIVATE
⇑PRIVATE

Remark 16.7 Explicit diagonalization. For 1-d repellers a diagonalization of an explicit truncated L_{mn} matrix evaluated in a judiciously chosen basis may yield many more eigenvalues than a cycle expansion (see refs. [16.10, 16.11]). The reasons why one persists in using periodic orbit theory are partially aesthetic and partially pragmatic. The explicit calculation of L_{mn} demands an explicit choice of a basis and is thus non-invariant, in contrast to cycle expansions which utilize only the invariant information of the flow. In addition, we usually do not know how to construct L_{mn} for a realistic high-dimanensional flow, such as the hyperbolic 3-disk game of pinball flow of sect. 1.3, whereas periodic orbit theory is true in higher dimensions and straightforward to apply.

Remark 16.8 Perron-Frobenius theorem. A proof of the Perron-Frobenius theorem may be found in ref. [**?**]. For positive transfer operators, this theorem has been generalized by Ruelle [16.12].

⇓PRELIMINARY
move into the Predrag appendix

Remark 16.9 Fried estimates. The form of the decay of the coefficients in the $F(z)$ expansion, as $u^{n^{1+1/d}}$, agree with the estimates of Fried [16.17] for the spectral determinants of d-dimensional expanding flows.

Remark 16.10 The choice of observables We have been quite sloppy on the mathematical side, as in discussing the spectral features of $\mathcal{L}$ the choice of the function space is crucial (especially when one is looking beyond the dominant eigenvalue). As a matter of fact in the function space where usually ergodic properties are defined, $L^2(d\mu)$ there is no gap, due to unitarity property of the Koopman operator: this means that there exist (ugly yet summable) functions for which no exponential decay is present even if the Fredholm determinant has isolated zeroes. A particularly nice example is worked out in [I.20], and a more mathematical argument is presented in [I.21].

Remark 16.11 Lattice models The relationship between the spectral gap and exponential decay properties is very well known in the statistical mechanical framework, where one deals with spatial correlations in lattice systems and links them to the gap of the transfer matrix.

Axiom A

⇑PRELIMINARY

Remark 16.12 Axiom A systems. The proofs in sect. 16.6 follow the thesis work of H.H. Rugh [16.9, 16.17, 16.18]. For a mathematical introduction to the subject, consult the excellent review by V. Baladi [16.1]. It would take us too far afield to give and explain the definition of Axiom A systems (see refs. [1.21, 1.22]). Axiom A implies, however, the existence of a Markov partition of the state space from which the properties 2 and 3 assumed on page 293 follow.

move ref. [16.18]
MAP define Axiam A
find refs

Remark 16.13 Exponential mixing speed of the Bernoulli shift. We see from (16.31) that for the Bernoulli shift the exponential decay rate of correlations coincides with the Lyapunov exponent: while such an identity holds for a number of systems, it is by no means a general result, and there exist explicit counterexamples.

give references!

Remark 16.14 Left eigenfunctions. We shall never use an explicit form of left eigenfunctions, corresponding to highly singular kernels like (16.33). Many details have been elaborated in a number of papers, such as ref. [16.19], with a daring physical interpretation.

refer to Dettmann-Vattay

Remark 16.15 Ulam's idea. The approximation of Perron-Frobenius operator defined by (10.37) has been shown to reproduce the spectrum for expanding maps, once finer and finer Markov partitions are used [16.20]. The subtle point of choosing a state space partitioning for a "generic case" is discussed in ref. [16.21].

Résumé

Examples of analytic eigenfunctions for 1-d maps are seductive, and make the problem of evaluating ergodic averages appears easy; just integrate over the desired observable weightes by the natural measure, right? No, generic natural measure sits on a fractal set and is singular everywhere. The point of this book is that you never need to construct the natural measure, cycle expansions will do that job.

A theory of evaluation of dynamical averages by means of trace formulas and spectral determinants requires a deep understanding of their analyticity and convergence.

We work here through a series of examples:

1. exact spectrum (but for a single fixed point of a linear map)

2. exact spectrum for a locally analytic map, matix representation

3. rigorous proof of existence of dicrete spectrum for 2-d hyperbolic maps

In the case of especially well-behaved "Axiom A" systems, where both the symbolic dynamics and hyperbolicity are under control, it is possible to treat traces and determinants in a rigorous fashion, and strong results about the analyticity properties of dynamical zeta functions and spectral determinants outlined above follow.

this paragraph is already used elsewhere?

Most systems of interest are *not* of the "axiom A" category; they are neither purely hyperbolic nor (as we have seen in chapters 11 and 12) do they have finite grammar. The importance of symbolic dynamics is generally grossly unappreciated; the crucial ingredient for nice analyticity properties of zeta functions is the existence of a finite grammar (coupled with uniform hyperbolicity). The dynamical systems which are *really* interesting - for example, smooth bounded Hamiltonian potentials - are presumably never fully chaotic, and the central question remains: How do we attack this problem in a systematic and controllable fashion?

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update

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Exercises

Exercise 16.1 What space does $\mathcal{L}$ act on? *Show that (16.2) is a complete basis on the space of analytic functions on a disk (and thus that we found the* complete *set of eigenvalues).*

Exercise 16.2 What space does $\mathcal{L}$ act on? *What can be said about the spectrum of (16.1) on $L^1[0,1]$? Compare the result with figure 16.4.*

Exercise 16.3 Euler formula. *Derive the Euler formula (16.5)*

$$\begin{aligned}\prod_{k=0}^{\infty}(1+tu^k) &= 1+\frac{t}{1-u}+\frac{t^2u}{(1-u)(1-u^2)}+\frac{t^3u^3}{(1-u)(1-u^2)(1-u^3)}\cdots \\ &= \sum_{k=0}^{\infty} t^k \frac{u^{\frac{k(k-1)}{2}}}{(1-u)\cdots(1-u^k)}\,, \qquad |u|<1. \end{aligned} \tag{16.34}$$

Exercise 16.4 2-d product expansion**. *We conjecture that the expansion corresponding to (16.34) is in this case*

$$\begin{aligned}\prod_{k=0}^{\infty}(1+tu^k)^{k+1} &= \sum_{k=0}^{\infty}\frac{F_k(u)}{(1-u)^2(1-u^2)^2\cdots(1-u^k)^2}t^k \\ &= 1+\frac{1}{(1-u)^2}t+\frac{2u}{(1-u)^2(1-u^2)^2}t^2 \\ &\quad +\frac{u^2(1+4u+u^2)}{(1-u)^2(1-u^2)^2(1-u^3)^2}t^3+\cdots \end{aligned} \tag{16.35}$$

$F_k(u)$ is a polynomial in u, and the coefficients fall off asymptotically as $C_n \approx u^{n^{3/2}}$. Verify; if you have a proof to all orders, e-mail it to the authors. (See also solution 16.3).

Exercise 16.5 Bernoulli shift on L spaces. *Check that the family (16.21) belongs to $L^1([0,1])$. What can be said about the essential spectral radius on $L^2([0,1])$? A useful reference is [16.23].*

Exercise 16.6 Cauchy integrals. *Rework all complex analysis steps used in the Bernoulli shift example on analytic functions on a disk.*

Exercise 16.7 Escape rate. *Consider the escape rate from a strange repeller: find a choice of trial functions ξ and φ such that (16.29) gives the fraction on particles surviving after n iterations, if their initial density distribution is $\rho_0(x)$. Discuss the behavior of such an expression in the long time limit.*

periodic!orbit extraction—(
periodic!orbit condition
prime cycle
cycle!prime
fixed point

Chapter 17

Fixed points, and how to get them

(F. Christiansen)

Having set up the dynamical context, now we turn to the key and unavoidable piece of numerics in this subject; search for the solutions (x, T), $x \in \mathbb{R}^d$, $T \in \mathbb{R}$ of the *periodic orbit condition*

$$f^{t+T}(x) = f^t(x)\,, \qquad T > 0 \tag{17.1}$$

for a given flow or mapping.

We know from chapter 14 that cycles are the necessary ingredient for evaluation of spectra of evolution operators. In chapter 11 we have developed a qualitative theory of how these cycles are laid out topologically.

This chapter is intended as a hands-on guide to extraction of periodic orbits, and should be skipped on first reading - you can return to it whenever the need for finding actual cycles arises. Sadly, searching for periodic orbits will never become as popular as a week on Côte d'Azur, or publishing yet another log-log plot in *Phys. Rev. Letters*. A serious cyclist might want to also learn about the variational methods to find cycles, chapter 37. They are particularly useful when little is understood about the topology of a flow, such as in high-dimensional periodic orbit searches.

☞ chapter 37

fast track:
chapter 18, p. 337

A *prime* cycle p of period T_p is a single traversal of the periodic orbit, so our task will be to find a cycle point $x \in p$ and the shortest time T_p for which (17.1) has a solution. A cycle point of a flow f^t which crosses a Poincaré section n_p times is a fixed point of the P^{n_p} iterate of the Poincaré section return map P, hence we shall refer to all cycles as "fixed points" in

cyclic!invariance
invariance!cyclic
Rossler@Rössler system!cycles
cycle!Rössler system

this chapter. By cyclic invariance, stability eigenvalues and the period of the cycle are independent of the choice of the initial point, so it will suffice to solve (17.1) at a single cycle point.

☞ sect. 8.2

If the cycle is an attracting limit cycle with a sizable basin of attraction, it can be found by integrating the flow for sufficiently long time. If the cycle is unstable, simple integration forward in time will not reveal it, and methods to be described here need to be deployed. In essence, any method for finding a cycle is based on devising a new dynamical system which possesses the same cycle, but for which this cycle is attractive. Beyond that, there is a great freedom in constructing such systems, and many different methods are used in practice.

Due to the exponential divergence of nearby trajectories in chaotic dynamical systems, fixed point searches based on direct solution of the fixed-point condition (17.1) as an initial value problem can be numerically very unstable. Methods that start with initial guesses for a number of points along the cycle, such as the multipoint shooting method described here in sect. 17.3, and the variational methods of chapter 37, are considerably more robust and safer.

☞ chapter 37

A prerequisite for any exhaustive cycle search is a good understanding of the topology of the flow: a preliminary step to any serious periodic orbit calculation is preparation of a list of all distinct admissible prime periodic symbol sequences, such as the list given in table 11.1. The relations between the temporal symbol sequences and the spatial layout of the topologically distinct regions of the state space discussed in chapters 11 and 12 should enable us to guess location of a series of periodic points along a cycle. Armed with such informed guess we proceed to improve it by methods such as the Newton-Raphson iteration; we illustrate this by considering 1-dimensional and d-dimensional maps.

perhaps include also Kai's method

17.1 Where are the cycles?

Ergodic exploration of recurrences that we turn to now sometimes performs admirably well in getting us started.

emphasize importance of equilibrium points!

In the Rössler flow example we sketched the attractors by running a long chaotic trajectory, and noted that the attractors are very thin, but otherwise the return maps that we plotted were disquieting – figure 3.3 did not appear to be a 1-to-1 map. In this section we show how to use such information to approximately locate cycles. In the remainder of this chapter and in chapter 37 we shall learn how to turn such guesses into highly accurate cycles.

Example 17.1 Rössler attractor *(G. Simon and P. Cvitanović)*

Run a long simulation of the Rössler flow f^t, plot a Poincaré section, as in figure 3.2, and extract the corresponding Poincaré return map P, as in figure 3.3. Luck is with

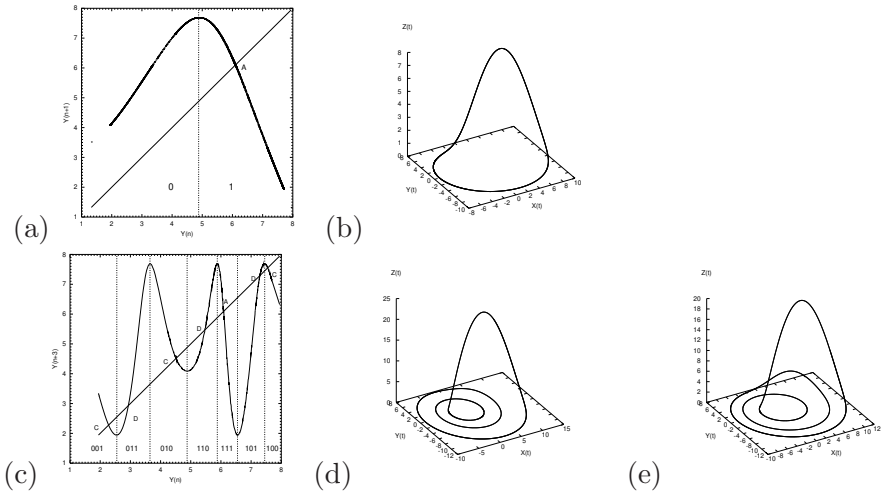


Figure 17.1: (a) $y \to P_1(y,z)$ return map for $x=0$, $y>0$ Poincaré section of the Rössler flow figure 2.3. (b) The $\overline{1}$-cycle found by taking the fixed point $y_{k+n} = y_k$ together with the fixed point of the $z \to z$ return map (not shown) an initial guess $(0, y^{(0)}, z^{(0)})$ for the Newton-Raphson search. (c) $y_{k+3} = P_1^3(y_k, z_k)$, the third iterate of Poincaré return map (3.1) together with the corresponding plot for $z_{k+3} = P_2^3(y_k, z_k)$, is used to pick starting guesses for the Newton-Raphson searches for the two 3-cycles: (d) the $\overline{001}$ cycle, and (e) the $\overline{011}$ cycle. (G. Simon)

us; the figure 17.1 (a) return map $y \to P_1(y,z)$ looks much like a parabola, so we take the unimodal map symbolic dynamics, sect. 11.3.1, as our guess for the covering dynamics. Strictly speaking, the attractor is "fractal", but for all practical purposes the return map is 1-dimensional; your printer will need a resolution better than 10^{14} dots per inch to start resolving its structure.

Periodic points of a prime cycle p of cycle length n_p for the $x=0$, $y>0$ Poincaré section of the Rössler flow figure 2.3 are fixed points $(y,z) = P^n(y,z)$ of the nth Poincaré return map.

Using the fixed point $y_{k+1} = y_k$ in figure 17.1 (a) together with the simultaneous fixed point of the $z \to P_1(y,z)$ return map (not shown) as a starting guess $(0, y^{(0)}, z^{(0)})$ for the Newton-Raphson search for the cycle p with symbolic dynamics label $\overline{1}$, we find the cycle figure 17.1 (b) with the Poincaré section point $(0, y_p, z_p)$, period T_p, expanding, marginal, contracting stability eigenvalues $(\Lambda_{p,e}, \Lambda_{p,m}, \Lambda_{p,c})$, and Lyapunov exponents $(\lambda_{p,e}, \lambda_{p,m}, \lambda_{p,c})$:

$$\begin{aligned}
\overline{1}\text{-cycle:} \qquad (x,y,z) &= (0, 6.09176832, 1.2997319) \\
T_1 &= 5.88108845586 \\
(\Lambda_{1,e}, \Lambda_{1,m}, \Lambda_{1,c}) &= (-2.40395353, 1+10^{-14}, -1.29\times 10^{-14}) \\
(\lambda_{1,e}, \lambda_{1,m}, \lambda_{1,c}) &= (0.149141556, 10^{-14}, -5.44)\,. \qquad (17.2)
\end{aligned}$$

The Newton-Raphson method that we used is described in sect. 17.5.

17.8 page 334

As an example of a search for longer cycles, we use $y_{k+3} = P_1^3(y_k, z_k)$, the third iterate of Poincaré return map (3.1) plotted in figure 17.1 (c), together with a corresponding plot for $z_{k+3} = f^3(y_k, z_k)$, to pick starting guesses for the Newton-Raphson searches for the two 3-cycles plotted in figure 17.1 (d), (e). For a listing of the short cycles of the Rössler flow, consult table 17.1.

recheck $\Lambda_c = -1.29 \times 10^{-14}$

The numerical evidence suggests (but a proof is lacking) that all cycles that comprise the strange attractor of the Rössler system are hyperbolic, each with an

template
inverse iteration
iteration!inverse
periodic!orbit extraction!inverse iteration
periodic!orbit extraction!Newton's method—(
Newton's method

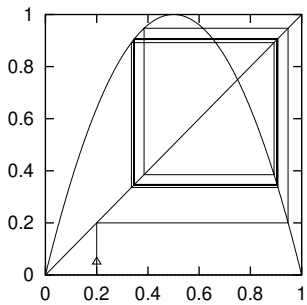


Figure 17.2: The inverse time path to the $\overline{01}$-cycle of the logistic map $f(x) = 4x(1-x)$ from an initial guess of $x = 0.2$. At each inverse iteration we chose the 0, respectively 1 branch.

expanding eigenvalue $|\Lambda_e| > 1$, a contracting eigenvalue $|\Lambda_c| < 1$, and a marginal eigenvalue $|\Lambda_m| = 1$ corresponding to displacements along the direction of the flow.

For the Rössler system the contracting eigenvalues turn out to be insanely contracting, a factor of e^{-32} per one par-course of the attractor, so their numerical determination is quite difficult. Fortunately, they are irrelevant; for all practical purposes the strange attractor of the Rössler system is 1-dimensional, a very good realization of a horseshoe template.

17.2 One-dimensional mappings

17.2.1 Inverse iteration

draw a figure of a unimodal repeller, its inverse

Let us first consider a very simple method to find unstable cycles of a 1-dimensional map such as the logistic map. Unstable cycles of 1-d maps are attracting cycles of the inverse map. The inverse map is not single valued, so at each backward iteration we have a choice of branch to make. By choosing branch according to the symbolic dynamics of the cycle we are trying to find, we will automatically converge to the desired cycle. The rate of convergence is given by the stability of the cycle, i.e., the convergence is exponentially fast. Figure 17.2 shows such path to the $\overline{01}$-cycle of the logistic map.

17.11
page 334

The method of inverse iteration is fine for finding cycles for 1-d maps and some 2-d systems such as the repeller of exercise 17.11. It is not particularly fast, especially if the inverse map is not known analytically. However, it completely fails for higher dimensional systems where we have both stable and unstable directions. Inverse iteration will exchange these, but we will still be left with both stable and unstable directions. The best strategy is to directly attack the problem of finding solutions of $f^T(x) = x$.

17.2.2 Newton's method

Newton's method for determining a zero x^* of a function $F(x)$ of one variable is based on a linearization around a starting guess x_0:

$$F(x) \approx F(x_0) + F'(x_0)(x - x_0). \tag{17.3}$$

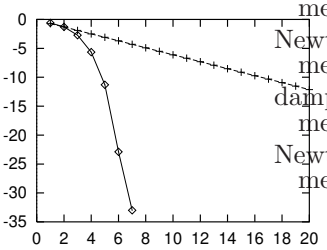


Figure 17.3: Convergence of Newton's method ($\diamondsuit$) vs. inverse iteration ($+$). The error after n iterations searching for the $\overline{01}$-cycle of the logistic map $f(x) = 4x(1-x)$ with an initial starting guess of $x_1 = 0.2, x_2 = 0.8$. y-axis is $\log_{10}$ of the error. The difference between the exponential convergence of the inverse iteration method and the super-exponential convergence of Newton's method is dramatic.

periodic!orbit extraction!Newton's method—)
Newton's method!convergence
damped Newton's method
Newton's method!damped

An approximate solution x_1 of $F(x) = 0$ is

$$x_1 = x_0 - F(x_0)/F'(x_0). \tag{17.4}$$

The approximate solution can then be used as a new starting guess in an iterative process. A fixed point of a map f is a solution to $F(x) = x - f(x) = 0$. We determine x by iterating

$$\begin{aligned} x_m &= g(x_{m-1}) = x_{m-1} - F(x_{m-1})/F'(x_{m-1}) \\ &= x_{m-1} - \frac{1}{1 - f'(x_{m-1})}(x_{m-1} - f(x_{m-1}))\,. \end{aligned} \tag{17.5}$$

Provided that the fixed point is not marginally stable, $f'(x) \neq 1$ at the fixed point x, a fixed point of f is a super-stable fixed point of the Newton-Raphson map g, $g'(x) = 0$, and with a sufficiently good initial guess, the Newton-Raphson iteration will converge super-exponentially fast.

To illustrate the efficiency of the Newton's method we compare it to the inverse iteration method in figure 17.3. Newton's method wins hands down: the number of significant digits of the accuracy of x estimate doubles with each iteration.

In order to avoid jumping too far from the desired x^* (see figure 17.4), one often initiates the search by the *damped Newton's method*,

$$\Delta x_m = x_{m+1} - x_m = -\frac{F(x_m)}{F'(x_m)}\,\Delta\tau\,, \qquad 0 < \Delta\tau \leq 1\,,$$

takes small $\Delta\tau$ steps at the beginning, reinstating to the full $\Delta\tau = 1$ jumps only when sufficiently close to the desired x^*.

17.3 Multipoint shooting method

Periodic orbits of length n are fixed points of f^n so in principle we could use the simple Newton's method described above to find them. However,

periodic!orbit extraction!multipoint shooting
multipoint shooting method

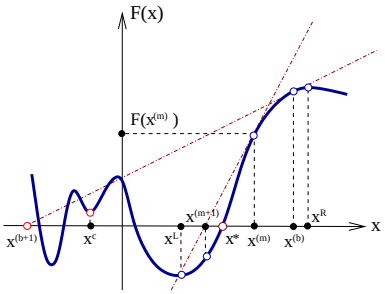


Figure 17.4: Newton method: bad initial guess $x^{(b)}$ leads to the Newton estimate $x^{(b+1)}$ far away from the desired zero of $F(x)$. Sequence $\cdots$, $x^{(m)}$, $x^{(m+1)}, \cdots$, starting with a good guess converges super-exponentially to x^*. The method diverges if it iterates into the basin of attraction of a local minimum x^c.

this is not an optimal strategy. f^n will be a highly oscillating function with perhaps as many as 2^n or more closely spaced fixed points, and finding a specific periodic point, for example one with a given symbolic sequence, requires a *very* good starting guess. For binary symbolic dynamics we must expect to improve the accuracy of our initial guesses by at least a factor of 2^n to find orbits of length n. A better alternative is the *multipoint shooting method*. While it might very hard to give a precise initial point guess for a long periodic orbit, if our guesses are informed by a good state space partition, a rough guess for each point along the desired trajectory might suffice, as for the individual short trajectory segments the errors have no time to explode exponentially.

recheck the name!

A cycle of length n is a zero of the n-dimensional vector function F:

$$F(x) \;=\; F\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 - f(x_n) \\ x_2 - f(x_1) \\ \cdots \\ x_n - f(x_{n-1}) \end{pmatrix} .$$

The relations between the temporal symbol sequences and the spatial layout of the topologically distinct regions of the state space discussed in chapter 11 enable us to guess location of a series of periodic points along a cycle. Armed with such informed initial guesses we can initiate a Newton-Raphson iteration. The iteration in the Newton's method now takes the form of

$$\frac{d}{dx}F(x)(x' - x) = -F(x), \tag{17.6}$$

where $\frac{d}{dx}F(x)$ is an $[n \times n]$ matrix:

$$\tfrac{d}{dx}F(x) \;=\; \begin{pmatrix} 1 & & & & -f'(x_n) \\ -f'(x_1) & 1 & & & \\ & \cdots & 1 & & \\ & & \cdots & 1 & \\ & & & -f'(x_{n-1}) & 1 \end{pmatrix} . \tag{17.7}$$

make inversion in "weak noise, diagrammatic" article an exercise here

This matrix can easily be inverted numerically by first eliminating the elements below the diagonal. This creates non-zero elements in the nth

column. We eliminate these and are done. Let us take it step by step for a period 3 cycle. Initially the setup for the Newton step looks like this:

$$\begin{pmatrix} 1 & 0 & -f'(x_3) \\ -f'(x_1) & 1 & 0 \\ 0 & -f'(x_2) & 1 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} -F_1 \\ -F_2 \\ -F_3 \end{pmatrix}, \qquad (17.8)$$

where $\delta_i = x'_i - x_i$ is the correction of our guess for a solution and where $F_i = x_i - f(x_{i-1})$. First we eliminate the below diagonal elements by adding $f'(x_1)$ times the first row to the second row, then adding $f'(x_2)$ times the second row to the third row. We then have

$$\begin{pmatrix} 1 & 0 & -f'(x_3) \\ 0 & 1 & -f'(x_1)f'(x_3) \\ 0 & 0 & 1 - f'(x_2)f'(x_1)f'(x_3) \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} -F_1 \\ -F_2 - f'(x_1)F_1 \\ -F_3 - f'(x_2)F_2 - f'(x_2)f'(x_1)F_1 \end{pmatrix}. \qquad (17.9)$$

The next step is to invert the last element in the diagonal, i.e., divide the third row by $1 - f'(x_2)f'(x_1)f'(x_3)$. It is clear that if this element is zero at the periodic orbit this step might lead to problems. In many cases this will just mean a slower convergence, but it might throw the Newton iteration completely off. We note that $f'(x_2)f'(x_1)f'(x_3)$ is the stability of the cycle (when the Newton iteration has converged) and that this therefore is not a good method to find marginally stable cycles. We now have

$$\begin{pmatrix} 1 & 0 & -f'(x_3) \\ 0 & 1 & -f'(x_1)f'(x_3) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} -F_1 \\ -F_2 - f'(x_1)F_1 \\ \frac{-F_3 - f'(x_2)F_2 - f'(x_2)f'(x_1)F_1}{1 - f'(x_2)f'(x_1)f'(x_3)} \end{pmatrix}. \qquad (17.10)$$

Finally we add $f'(x_3)$ times the third row to the first row and $f'(x_1)f'(x_3)$ times the third row to the second row. On the left hand side the matrix is now the unit matrix, on the right hand side we have the corrections to our initial guess for the cycle, i.e., we have gone through one step of the Newton iteration scheme.

Can one write the final result as a formula?

When one sets up the Newton iteration on the computer it is not necessary to write the left hand side as a matrix. All one needs is a vector containing the $f'(x_i)$'s, a vector containing the n'th column, that is the cumulative product of the $f'(x_i)$'s and a vector containing the right hand side. After the iteration the vector containing the right hand side should be the correction to the initial guess.

17.1 page 332

Newton's method!flows

17.4 d-dimensional mappings

(F. Christiansen)

Armed with symbolic dynamics informed initial guesses we can utilize the Newton-Raphson iteration in d-dimensions as well.

17.4.1 Newton's method for d-dimensional mappings

Newton's method for 1-dimensional mappings is easily extended to higher dimensions. In this case $f'(x_i)$ is a $[d \times d]$ matrix. $\frac{d}{dx}F(x)$ is then an $[nd \times nd]$ matrix. In each of the steps that we went through above we are then manipulating d rows of the left hand side matrix. (Remember that matrices do not commute - always multiply from the left.) In the inversion of the n'th element of the diagonal we are inverting a $[d\times d]$ matrix $(1-\prod f'(x_i))$ which can be done if none of the eigenvalues of $\prod f'(x_i)$ equals 1, i.e., the cycle must not have any marginally stable directions.

Some d-dimensional mappings (such as the Hénon map (3.15)) can be written as 1-dimensional time delay mappings of the form

$$f(x_i) = f(x_{i-1}, x_{i-2}, \ldots, x_{i-d}). \tag{17.11}$$

In this case $\frac{d}{dx}F(x)$ is an $[n\times n]$ matrix as in the case of usual 1-dimensional maps but with non-zero matrix elements on d off-diagonals. In the elimination of these off-diagonal elements the last d columns of the matrix will become non-zero and in the final cleaning of the diagonal we will need to invert a $[d \times d]$ matrix. In this respect, nothing is gained numerically by looking at such maps as 1-dimensional time delay maps.

17.5 Flows

(F. Christiansen)

Further complications arise for flows due to the fact that for a periodic orbit the stability eigenvalue corresponding to the flow direction of necessity equals unity; the separation of any two points along a cycle remains unchanged after a completion of the cycle. More unit eigenvalues can arise if the flow satisfies conservation laws, such as the energy invariance for Hamiltonian systems. We now show how such problems are solved by increasing the number of fixed point conditions.

☞ sect. 8.2.1

17.5.1 Newton's method for flows

flow!autonomous
autonomous flow

A flow is equivalent to a mapping in the sense that one can reduce the flow to a mapping on the Poincaré surface of section. An autonomous flow (2.5) is given as

$$\dot{x} = v(x), \tag{17.12}$$

The corresponding fundamental matrix M (4.32) is obtained by integrating the linearized equation (4.34)

$$\dot{M} = \mathbf{A}M\,, \qquad A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j}$$

along the trajectory. The flow and the corresponding fundamental matrix are integrated simultaneously, by the same numerical routine. Integrating an initial condition on the Poincaré surface until a later crossing of the same and linearizing around the flow we can write

$$f(x') \approx f(x) + M(x' - x). \tag{17.13}$$

Notice here, that, even though all of x', x and $f(x)$ are on the Poincaré surface, $f(x')$ is usually not. The reason for this is that M corresponds to a specific integration time and has no explicit relation to the arbitrary choice of Poincaré section. This will become important in the extended Newton's method described below.

To find a fixed point of the flow near a starting guess x we must solve the linearized equation

$$(1 - M)(x' - x) = -(x - f(x)) = -F(x) \tag{17.14}$$

where $f(x)$ corresponds to integrating from one intersection of the Poincaré surface to another and M is integrated accordingly. Here we run into problems with the direction along the flow, since - as shown in sect. 8.2.1 - this corresponds to a unit eigenvector of M. The matrix $(1 - M)$ does therefore not have full rank. A related problem is that the solution x' of (17.14) is not guaranteed to be in the Poincaré surface of section. The two problems are solved simultaneously by adding a small vector along the flow plus an extra equation demanding that x be in the Poincaré surface. Let us for the sake of simplicity assume that the Poincaré surface is a (hyper)-plane, i.e., it is given by the linear equation

should prove this?

draw figure

$$(x - x_0) \cdot a = 0, \tag{17.15}$$

where a is a vector normal to the Poincaré section and x_0 is any point in the Poincaré section. (17.14) then becomes

use Sune's identity?

$$\begin{pmatrix} 1-M & v(x) \\ a & 0 \end{pmatrix} \begin{pmatrix} x'-x \\ \delta T \end{pmatrix} = \begin{pmatrix} -F(x) \\ 0 \end{pmatrix} . \tag{17.16}$$

The last row in this equation ensures that x will be in the surface of section, and the addition of $v(x)\delta T$, a small vector along the direction of the flow, ensures that such an x can be found at least if x is sufficiently close to a solution, i.e., to a fixed point of f.

To illustrate this little trick let us take a particularly simple example; consider a 3-d flow with the $(x, y, 0)$-plane as Poincaré section. Let all trajectories cross the Poincaré section perpendicularly, i.e., with $v = (0, 0, v_z)$, which means that the marginally stable direction is also perpendicular to the Poincaré section. Furthermore, let the unstable direction be parallel to the x-axis and the stable direction be parallel to the y-axis. In this case the Newton setup looks as follows

draw figure

$$\begin{pmatrix} 1-\Lambda & 0 & 0 & 0 \\ 0 & 1-\Lambda_s & 0 & 0 \\ 0 & 0 & 0 & v_z \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \delta_x \\ \delta_y \\ \delta_z \\ \delta t \end{pmatrix} = \begin{pmatrix} -F_x \\ -F_y \\ -F_z \\ 0 \end{pmatrix} . \tag{17.17}$$

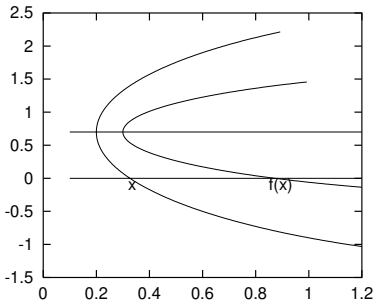
If you consider only the upper-left $[3 \times 3]$ matrix (which is what we would have without the extra constraints that we have introduced) then this matrix is clearly not invertible and the equation does not have a unique solution. However, the full $[4\times4]$ matrix is invertible, as $\det(\cdot) = v_z \det(1 - M_\perp)$, where $M_\perp$ is the monodromy matrix for a surface of section transverse to the orbit, see for ex. (36.2).

For periodic orbits (17.16) generalizes in the same way as (17.7), but with n additional equations – one for each point on the Poincaré surface. The Newton setup looks like this

$$\left(\begin{array}{ccccc|ccc} 1 & & & & -J_n & v_1 & & \\ -J_1 & 1 & & & & & \ddots & \\ & \cdots & 1 & & & & & v_n \\ & & \cdots & 1 & & & & \\ & & & -J_{n-1} & 1 & & & \\ \hline & a & & & & 0 & & \\ & & \ddots & & & & \ddots & \\ & & & & a & & & 0 \end{array}\right) \begin{pmatrix} \delta_1 \\ \delta_2 \\ \cdot \\ \cdot \\ \delta_n \\ \delta t_1 \\ \cdot \\ \delta t_n \end{pmatrix} = \begin{pmatrix} -F_1 \\ -F_2 \\ \cdot \\ \cdot \\ -F_n \\ 0 \\ \cdot \\ 0 \end{pmatrix} .$$

Solving this equation resembles the corresponding task for maps. However, in the process we will need to invert an $[(d+1)n \times (d+1)n]$ matrix rather than a $[d \times d]$ matrix. The task changes with the length of the cycle.

This method can be extended to take care of the same kind of problems if other eigenvalues of the fundamental matrix equal 1. This happens if the flow has an invariant of motion, the most obvious example being energy conservation in Hamiltonian systems. In this case we add an extra equation



Newton's method!optimal surface of section
surface of section!optimal

Figure 17.5: Illustration of the optimal Poincaré surface. The original surface $y = 0$ yields a large distance $x - f(x)$ for the Newton iteration. A much better choice is $y = 0.7$.

$\nabla H(x)$ looks g - perhaps $x)/R$ as in 41.3 needed, extra matrix $(\nabla H(x)00)$, with tional correction ne linearization?

for x to be on the energy shell plus and extra variable corresponding to adding a small vector along the gradient of the Hamiltonian. We then have to solve

$$\begin{pmatrix} 1-M & v(x) & \nabla H(x) \\ a & 0 & 0 \end{pmatrix} \begin{pmatrix} x'-x \\ \delta t \\ \delta E \end{pmatrix} = \begin{pmatrix} -(x-f(x)) \\ 0 \\ 0 \end{pmatrix} \quad (17.18)$$

simultaneously with

$$H(x') - H(x) = 0. \quad (17.19)$$

This last equation is nonlinear. It is often best to treat this separately in the sense that we really solve this equation in each Newton step. This might mean putting in an additional Newton routine to solve the single step of (17.18) and (17.19) together. One might be tempted to linearize (17.19) and put it into (17.18) to do the two different Newton routines simultaneously, but this will not guarantee a solution on the energy shell. In fact, it may not even be possible to find any solution of the combined linearized equations, if the initial guess is not very good.

17.5.2 Newton's method with optimal surface of section

(F. Christiansen)

In some systems it might be hard to find a good starting guess for a fixed point, something that could happen if the topology and/or the symbolic dynamics of the flow is not well understood. By changing the Poincaré section one might get a better initial guess in the sense that x and $f(x)$ are closer together. In figure 17.5 there is an illustration of this. The figure shows a Poincaré section, $y = 0$, an initial guess x, the corresponding $f(x)$ and pieces of the trajectory near these two points.

If the Newton iteration does not converge for the initial guess x we might have to work very hard to find a better guess, particularly if this is in a high-dimensional system (high-dimensional might in this context mean a Hamiltonian system with 3 degrees of freedom.) But clearly we could easily have a much better guess by simply shifting the Poincaré section to $y = 0.7$ where the distance $x - f(x)$ would be much smaller. Naturally, one cannot see by eye the best surface in higher dimensional systems. The way to proceed is as follows: We want to have a minimal distance between our initial guess x and the image of this $f(x)$. We therefore integrate the flow looking for a minimum in the distance $d(t) = |f^t(x) - x|$. $d(t)$ is now a minimum with respect to variations in $f^t(x)$, but not necessarily with respect to x. We therefore integrate x either forward or backward in time. Doing this we minimize d with respect to x, but now it is no longer minimal with respect to $f^t(x)$. We therefore repeat the steps, alternating between correcting x and $f^t(x)$. In most cases this process converges quite rapidly. The result is a trajectory for which the vector $(f(x) - x)$ connecting the two end points is perpendicular to the flow at both points. We can now choose to define a Poincaré surface of section as the hyper-plane that goes through x and is normal to the flow at x. In other words the surface of section is determined by

$$(x' - x) \cdot v(x) = 0. \tag{17.20}$$

Note that $f(x)$ lies on this surface. This surface of section is optimal in the sense that a close return on the surface is really a local minimum of the distance between x and $f^t(x)$. But more importantly, the part of the stability matrix that describes linearization perpendicular to the flow is exactly the stability of the flow in the surface of section when $f(x)$ is close to x. In this method, the Poincaré surface changes with each iteration of the Newton scheme. Should we later want to put the fixed point on a specific Poincaré surface it will only be a matter of moving along the trajectory.

⇓PRELIMINARY

17.6 Sundry methods for cycle extraction

Newton-Raphson iteration is a bit of work to implement. In practice we might be satisfied with any rampaging robot that finds "the most important" cycles. Ergodic exploration of recurrences that we turn to now, and the relaxation methods described in chapter 37 sometimes perform admirably well.

17.6.1 Cycles from long time series

(L. Rondoni)

For ergodic systems we can rely on near recurrence as a way to search for cycles up to a given period. consists of following a long trajectory

use Gregor Tanner, L. Rondoni, G. Morris, C.P. Dettmann results; how one gets both admissible itineraries and their weights without working

refer to Tanner etc in a remark concerning this method, proposed in ref. [24.2].

in state space, and in looking for close returns of the trajectory itself. Whenever the trajectory almost closes in a loop (within a given tolerance), any point of this near miss of a cycle can be taken as the approximate initial condition. A slight change of such approximate initial condition may indeed lead to closing a cycle. This method has one disadvantage: it does not systematically scan for all possible periodic symbol sequences. However, it does preferentially find all orbits with high stability weights, while missing those which, having low stability weights, contribute little to the cycle expansions.

triple PIM method
periodic!orbit extraction!triple PIM method
Kuramoto-Sivashinsky system

in depth:
sect. 18.6, p. 350

17.6.2 Saddle-straddle methods

3.5
page 60

A simple alternative to periodic orbit exploration is the *triple PIM method* [17.5] which uses the instability of the repeller to find trajectories that stay arbitrarily close to the non–wandering set for a given time interval. The method is efficient and useful in exploration of repelling sets.

17.6.3 Bijection searches

One can keep halving the intervals and checking whether the desired cycle visits them. This approach, called a bijection search, is rather slow, not very insightful, and hard to use in dimensions higher than one, so we will not discuss it here.

include Kai's method
Baker-Moore-Spiegel eigenvalue method

17.6.4 Symmetry lines

discuss symmetry lines

17.6.5 Braids, templates and simplicial triangulations

⇑PRELIMINARY

Commentary

Prepare Lozi map exercise

Remark 17.1 Piece-wise linear maps. The Lozi map (3.17) is linear, and 100,000's of cycles can be easily computed by [2x2] matrix multiplication and inversion.

⇓PRELIMINARY

link to Dahlqvist-Russberg paper

Remark 17.2 Kuramoto-Sivashinsky system cycles. We have found all cycles up to topological length 10 (the redefined topological length in the case of the period-3 window), 92 cycles in the chaotic regime and 228 in the period-3 window, by using

periodic!orbit condition
natural measure
periodic!orbit extraction—)

the 1-dimensional parametrization $f(s)$ to find initial guesses for periodic points of the full $N = 16$ Fourier modes truncation and then determining the cycles by a multi-shooting Newton routine. It is worth noting that the effectiveness of using the 1-dimensional $f(s)$ approximation to the dynamics to determine initial guesses is such that for a typical cycle it takes only 2-3 Newton iterations to find the cycle with an accuracy of 10^{-10}.

⇑PRELIMINARY

Résumé

There is no general computational algorithm that is guaranteed to find all solutions (up to a given period $T_{\max}$) to the periodic orbit condition

$$f^{t+T}(x) = f^t(x)\,, \qquad T > 0$$

for a general flow or mapping. Due to the exponential divergence of nearby trajectories in chaotic dynamical systems, direct solution of the periodic orbit condition can be numerically very unstable.

A prerequisite for a systematic and complete cycle search is a good (but hard to come by) understanding of the topology of the flow. Usually one starts by - possibly analytic - determination of the equilibria of the flow. Their locations, stabilities, stability eigenvectors and invariant manifolds offer skeletal information about the topology of the flow. Next step is numerical long-time evolution of "typical" trajectories of the dynamical system under investigation. Such numerical experiments build up the "natural measure", and reveal regions most frequently visited. The periodic orbit searches can then be initialized by taking nearly recurring orbit segments and deforming them into a closed orbits. With a sufficiently good initial guess the Newton-Raphson formula (17.16)

☞ sect. 9.3.1

$$\begin{pmatrix} 1-M & v(x) \\ a & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta T \end{pmatrix} = \begin{pmatrix} f(x)-x \\ 0 \end{pmatrix}$$

yields improved estimate $x' = x + \delta x, T' = T + \delta T$. Iteration then yields the period T and the location of a periodic point x_p in the Poincaré surface $(x_p - x_0) \cdot a = 0$, where a is a vector normal to the Poincaré section at x_0.

The problem one faces with high-dimensional flows is that their topology is hard to visualize, and that even with a decent starting guess for a point on a periodic orbit, methods like the Newton-Raphson method are likely to fail. Methods that start with initial guesses for a number of points along the cycle, such as the multipoint shooting method of sect. 17.3, are more robust. The relaxation (or variational) methods take this strategy to its logical extreme, and start by a guess of not a few points along a periodic orbit, but a guess of the entire orbit. As these methods are intimately related to variational principles and path integrals, we postpone their introduction to chapter 37.

☞ chapter 37

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Exercises

Ulam map
alternating binary tree
binary!tree, alternating
later recheck for other appearances of the same Ulam equation

Exercise 17.1 Cycles of the Ulam map. *Test your cycle-searching routines by computing a bunch of short cycles and their stabilities for the Ulam map*

$$f(x) = 4x(1-x)\,. \tag{17.21}$$

Exercise 17.2 Cycles stabilities for the Ulam map, exact. *In exercise 17.1 you should have observed that the numerical results for the cycle stability eigenvalues (4.39) are exceptionally simple: the stability eigenvalue of the $x_0 = 0$ fixed point is 4, while the eigenvalue of any other n-cycle is $\pm 2^n$. Prove this. (Hint: the Ulam map can be conjugated to the tent map (11.8). This problem is perhaps too hard, but give it a try - the answer is in many introductory books on nolinear dynamics.)*

⇓PRELIMINARY

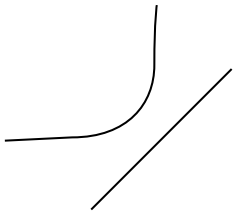
Exercise 17.3 Unimodal map cycles. *Knowledge of the topological coordinate (??) is very useful when searching for periodic orbits. Assume that we have already determined all periodic points $x_a, x_b, \cdots$ of period n, and would like to have a good initial guess for the period $(n+1)$ periodic point x_d with prescribed itinerary $S_d^+ := S^+(x_d)$. It is easy to determine the two closest $\gamma(S_a^+)$, $\gamma(S_b^+)$ that bracket $\gamma(S_d^+)$. If $\gamma(S_a^+) < \gamma(S_d^+) < \gamma(S_b^+)$, then we know that we can restrict the search for x_c into the $x_c \in [x_a, x_b]$ interval. For example, relative ordering of all unimodal map periodic points up to $n = 5$ is given in figure 17.6. Appendix F.1.1 contains further details of the symbolics dynamics for periodic point of unimodal maps.*

⇑PRELIMINARY

Exercise 17.4 Stability of billiard cycles. *Compute stabilities of few simple cycles.*

(a) *A simple scattering billiard is the two-disk billiard. It consists of a disk of radius one centered at the origin and another disk of unit radius located at $L+2$. Find all periodic orbits for this system and compute their stabilities. (You might have done this already in exercise 1.2; at least now you will be able to see where you went wrong when you knew nothing about cycles and their extraction.)*

(b) *Find all periodic orbits and stabilities for a billiard ball bouncing between the diagonal $y = x$ and one of the hyperbola branches $y = 1/x$.*



Exercise 17.5 Cycle stability. *Add to the pinball simulator of exercise 6.1 a routine that evaluates the expanding eigenvalue (4.19) for a given cycle.*

⇓PRELIMINARY
⇑PRELIMINARY

Exercise 17.6 Pinball cycles. *Determine the stability and length of all fundamental domain prime cycles of the binary symbol string lengths up to 5 (*or longer*) for $R : a = 6$ 3-disk pinball.*

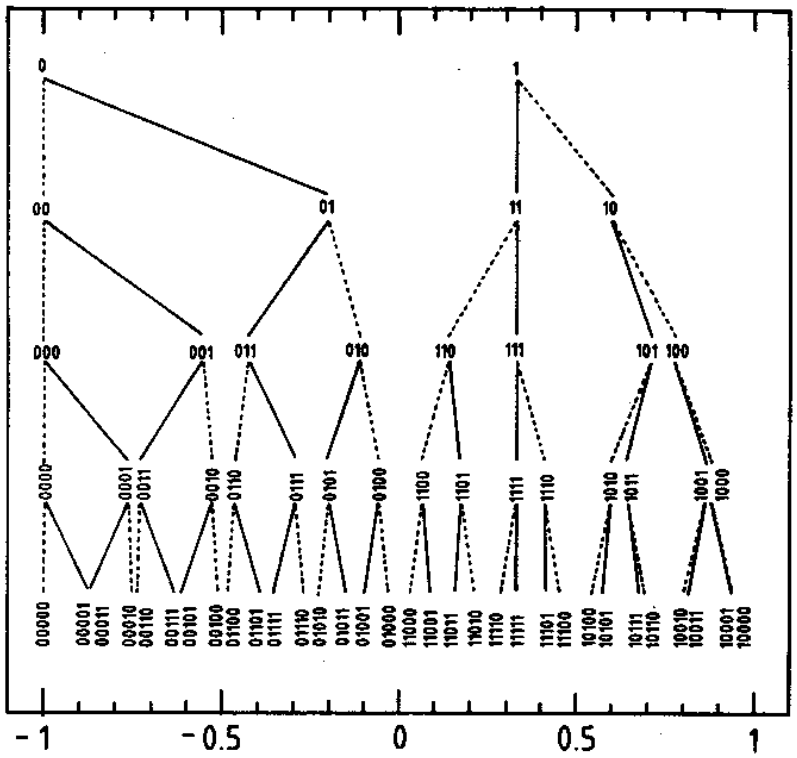


Figure 17.6: The alternating binary tree organization of the periodic points of a unimodal map (in this example, the Ulam map (17.21)). The itinerary of a point is read off the tree by starting at the root and following the branches down to x; relative ordering of points along the x axis is given by the relative ordering of the corresponding nodes. (specify the MAP)

Rossler@Rössler system!cycles
cycles!Rössler system
helium!collinear!stability
helium!collinear!cycles
inverse iteration!Hamiltonian repeller
iteration!inverse!Hamiltonian repeller
periodic!orbit extraction!Hamiltonian repeller
Hamiltonian!repeller, periodic orbits

n_p	p	y_p	z_p	Λ_e
1	1	6.091768319056803	1.299731937639821	-2.4039535318268
2	01	3.915804049621049	3.692833386542665	-3.5120069815161
3	001	2.278281031720258	7.416480984019008	-2.3419235232340
	011	2.932877559129124	5.670805943881501	5.3449081538885
4	0111	3.466758713211455	4.506217531477667	-16.6967406980700
5	01011	4.162798782914948	3.303903338609633	-23.1995830097831
	01111	3.278914359770783	4.890452922955567	36.8863297988981
6	001011	2.122093931936202	7.886172854283211	-6.8576654190825
	010111	4.059210605826523	3.462265228606606	61.6490940089068
	011111	3.361494458061049	4.718206217035575	-92.0825560711089
7	0101011	3.842769382372052	3.815493592299824	77.7611048852412
	0110111	3.025956697151134	5.451444475664179	-95.1838846735358
	0101111	4.102255295518855	3.395643547170646	-142.2379888163439
	0111111	3.327986189581191	4.787462810306583	218.0283602810993

Table 17.1: The Rössler system (2.14): The itinerary p, a periodic point $x_p = (0, y_p, z_p)$ and the expanding eigenvalue Λ_p for all cycles up to the topological length 7. (Joachim Mathiesen)

Exercise 17.7 **Newton-Raphson method.** *Implement the Newton-Raphson method in 2-d and apply it to determination of pinball cycles.*

Exercise 17.8 **Rössler system cycles.** *(continuation of exercise 4.4)*

Determine all cycles up to 5 Poincaré sections returns for the Rössler system (2.14), as well as their stabilities. (Hint: implement (17.16), the multipoint shooting methods for flows; you can cross-check your shortest cycles against the ones listed in table 17.1.)

add hyperlinks to Mathiasen, Simon projects

⇓PRELIMINARY
⇑PRELIMINARY

Exercise 17.9 **Cycle stability, helium.** *Add to the helium integrator of exercise 2.11 a routine that evaluates the expanding eigenvalue (4.19) for a given cycle.*

Exercise 17.10 **Colinear helium cycles.** *Determine the stability and length of all fundamental domain prime cycles up to symbol sequence length 5 or longer for collinear helium of figure 41.5.*

Exercise 17.11 **Uniqueness of unstable cycles***.** *Prove that there exists only one 3-disk prime cycle for a given finite admissible prime cycle symbol string. Hints: look at the Poincaré section mappings; can you show that there is exponential contraction to a unique periodic point with a given itinerary? Exercise 37.1 might be helpful in this effort.*

Exercise 17.12 **Inverse iteration method for a Hamiltonian repeller.** *Consider the Hénon map (3.15) for area-preserving ("Hamiltonian") parameter value $b = -1$. The coordinates of a periodic orbit of length n_p satisfy the equation*

$$x_{p,i+1} + x_{p,i-1} = 1 - a x_{p,i}^2 \,, \quad i = 1, ..., n_p \,, \tag{17.22}$$

with the periodic boundary condition $x_{p,0} = x_{p,n_p}$. Verify that the itineraries and the stabilities of the short periodic orbits for the Hénon repeller (17.22) at $a = 6$ are as listed in table 17.2.

center of mass

p	Λ_p	$\sum x_{p,i}$
0	$0.71516752438\times10^{1}$	-0.6076252185107
1	$-0.29528463259\times10^{1}$	0.2742918851774
10	$-0.98989794855\times10^{1}$	0.3333333333333
100	$-0.13190727397\times10^{3}$	-0.2060113295833
110	$0.55896964996\times10^{2}$	0.5393446629166
1000	$-0.10443010730\times10^{4}$	-0.8164965809277
1100	$0.57799826989\times10^{4}$	0.0000000000000
1110	$-0.10368832509\times10^{3}$	0.8164965809277
10000	$-0.76065343718\times10^{4}$	-1.4260322065792
11000	$0.44455240007\times10^{4}$	-0.6066540777738
10100	$0.77020248597\times10^{3}$	0.1513755016405
11100	$-0.71068835616\times10^{3}$	0.2484632276044
11010	$-0.58949885284\times10^{3}$	0.8706954728949
11110	$0.39099424812\times10^{3}$	1.0954854155465
100000	$-0.54574527060\times10^{5}$	-2.0341342556665
110000	$0.32222060985\times10^{5}$	-1.2152504370215
101000	$0.51376165109\times10^{4}$	-0.4506624359329
111000	$-0.47846146631\times10^{4}$	-0.3660254037844
110100	$-0.63939998436\times10^{4}$	0.3333333333333
101100	$-0.63939998436\times10^{4}$	0.3333333333333
111100	$0.39019387269\times10^{4}$	0.5485837703548
111010	$0.10949094597\times10^{4}$	1.1514633582661
111110	$-0.10433841694\times10^{4}$	1.3660254037844

Table 17.2: All periodic orbits up to 6 bounces for the Hamiltonian Hénon mapping (17.22) with $a = 6$. Listed are the cycle itinerary, its expanding eigenvalue Λ_p, and its "center of mass". The "center of mass" is listed because it turns out the "center of mass" is often a simple rational or a quadratic irrational.

Hint: *you can use any cycle-searching routine you wish, but for the complete repeller case (all binary sequences are realized), the cycles can be evaluated simply by inverse iteration, using the inverse of (17.22)*

$$x''_{p,i} = S_{p,i}\sqrt{\frac{1-x'_{p,i+1}-x'_{p,i-1}}{a}},\quad i = 1, ..., n_p\,.$$

Here $S_{p,i}$ are the signs of the corresponding cycle point coordinates, $S_{p,i} = x_{p,i}/|x_{p,i}|$.

(G. Vattay)

Exercise 17.13 "Center of mass" puzzle****.** *Why is the "center of mass", listed in table 17.2, a simple rational every so often?*

Chapter 18

Cycle expansions

> Recycle... It's the Law!
> Poster, New York City Department of Sanitation

The Euler product representations of spectral determinants (15.10) and dynamical zeta functions (15.16) are really only a shorthand notation - the zeros of the individual factors are *not* the zeros of the zeta function, and convergence of such objects is far from obvious. Now we shall give meaning to the dynamical zeta functions and spectral determinants by expanding them as cycle expansions, series representations ordered by increasing topological cycle length, with products in (15.10), (15.16) expanded as sums over *pseudocycles*, products of t_p's. The zeros of correctly truncated cycle expansions yield the desired eigenvalues, and the expectation values of observables are given by the cycle averaging formulas obtained from the partial derivatives of dynamical zeta functions (or spectral determinants).

18.1 Pseudocycles and shadowing

How are periodic orbit formulas such as (15.16) evaluated? We start by computing the lengths and stability eigenvalues of the shortest cycles. This always requires numerical work, such as the Newton's method searches for periodic solutions; we shall assume that the numerics is under control, and that *all* short cycles up to a given (topological) length have been found. Examples of the data required for application of periodic orbit formulas are the lists of cycles given in tables 37.3 and 17.2. It is important not to miss any short cycles, as the calculation is as accurate as the shortest cycle dropped - including cycles longer than the shortest omitted does not improve the accuracy (more precisely, improves it, but painfully slowly). write the errors section, refer to it

Expand the dynamical zeta function (15.16) as a formal power series,

$$1/\zeta = \prod_p (1 - t_p) = 1 - \sum_{\{p_1 p_2 \ldots p_k\}}{}' (-1)^{k+1} t_{p_1} t_{p_2} \ldots t_{p_k} \qquad (18.1)$$

pseudocycle
periodic!orbit

where the prime on the sum indicates that the sum is over all distinct non-repeating combinations of prime cycles. As we shall frequently use such sums, let us denote by $t_\pi = (-1)^{k+1} t_{p_1} t_{p_2} \dots t_{p_k}$ an element of the set of all distinct products of the prime cycle weights t_p. The formal power series (18.1) is now compactly written as

$$1/\zeta = 1 - \sum_\pi{}' \, t_\pi \, . \tag{18.2}$$

For $k > 1$, t_π are weights of *pseudocycles*; they are sequences of shorter cycles that shadow a cycle with the symbol sequence $p_1 p_2 \dots p_k$ along segments p_1, $p_2, \dots$, p_k. $\sum'$ denotes the restricted sum, for which any given prime cycle p contributes at most once to a given pseudocycle weight t_π.

refer to the shadowing figure

The pseudocycle weight

$$t_\pi = (-1)^{k+1} \frac{1}{|\Lambda_\pi|} e^{\beta A_\pi - s T_\pi} z^{n_\pi} \, . \tag{18.3}$$

depends on the pseudocycle topological length n_π, integrated observable A_π, period T_π, and stability Λ_π indexorbit!periodic

$$\begin{aligned} n_\pi &= n_{p_1} + \ldots + n_{p_k} \, , \qquad T_\pi = T_{p_1} + \ldots + T_{p_k} \\ A_\pi &= A_{p_1} + \ldots + A_{p_k} \, , \qquad \Lambda_\pi = \Lambda_{p_1} \Lambda_{p_2} \cdots \Lambda_{p_k} \, . \end{aligned} \tag{18.4}$$

Throughout this text, the terms "periodic orbit" and "cycle" are used interchangeably; while "periodic orbit" is more precise, "cycle" (which has many other uses in mathematics) is easier on the ear than "pseudo-periodic-orbit." While in Soviet times acronyms were a rage, we shy away from acronyms such as UPOs (Unstable Periodic Orbits).

18.1.1 Curvature expansions

The simplest example is the pseudocycle sum for a system described by a complete binary symbolic dynamics. In this case the Euler product (15.16) is given by

$$\begin{aligned} 1/\zeta &= (1-t_0)(1-t_1)(1-t_{01})(1-t_{001})(1-t_{011}) \\ &\quad (1-t_{0001})(1-t_{0011})(1-t_{0111})(1-t_{00001})(1-t_{00011}) \\ &\quad (1-t_{00101})(1-t_{00111})(1-t_{01011})(1-t_{01111}) \dots \end{aligned}$$

(see table 11.1), and the first few terms of the expansion (18.2) ordered by increasing total pseudocycle length are:

$$\begin{aligned} 1/\zeta &= 1 - t_0 - t_1 - t_{01} - t_{001} - t_{011} - t_{0001} - t_{0011} - t_{0111} - \dots \\ &\quad + t_0 t_1 + t_0 t_{01} + t_{01} t_1 + t_0 t_{001} + t_0 t_{011} + t_{001} t_1 + t_{011} t_1 \\ &\quad - t_0 t_{01} t_1 - \dots \end{aligned}$$

We refer to such series representation of a dynamical zeta function or a spectral determinant, expanded as a sum over pseudocycles, and ordered by increasing cycle length and instability, as a *cycle expansion*.

cycle!expansion
cycle!fundamental
curvature!correction
curvature!expansion

The next step is the key step: regroup the terms into the dominant *fundamental* contributions t_f and the decreasing *curvature* corrections $\hat{c}_n$, each $\hat{c}_n$ split into prime cycles p of length $n_p{=}n$ grouped together with pseudocycles whose full itineraries build up the itinerary of p. For the binary case this regrouping is given by

$$\begin{aligned} 1/\zeta &= 1-t_0-t_1-[(t_{01}-t_1t_0)]-[(t_{001}-t_{01}t_0)+(t_{011}-t_{01}t_1)] \\ &\quad -[(t_{0001}-t_0t_{001})+(t_{0111}-t_{011}t_1) \\ &\quad\quad +(t_{0011}-t_{001}t_1-t_0t_{011}+t_0t_{01}t_1)]-\dots \\ &= 1-\sum_f t_f-\sum_n \hat{c}_n\ . \end{aligned} \tag{18.5}$$

All terms in this expansion up to length $n_p = 6$ are given in table 18.1. We refer to such regrouped series as *curvature expansions*. .

Shadowing figure here
explain curvature
explain why "curvature"

Such separation into "fundamental" and "curvature" parts of cycle expansions is possible *only* for dynamical systems whose symbolic dynamics has finite grammar. The fundamental cycles t_0, t_1 have no shorter approximants; they are the "building blocks" of the dynamics in the sense that all longer orbits can be approximately pieced together from them. The fundamental part of a cycle expansion is given by the sum of the products of all non-intersecting loops of the associated Markov graph. The terms grouped in brackets are the curvature corrections; the terms grouped in parenthesis are combinations of longer cycles and corresponding sequences of "shadowing" pseudocycles. If all orbits are weighted equally ($t_p = z^{n_p}$), such combinations cancel exactly, and the dynamical zeta function reduces to the topological polynomial (13.21). If the flow is continuous and smooth, orbits of similar symbolic dynamics will traverse the same neighborhoods and will have similar weights, and the weights in such combinations will almost cancel. The utility of cycle expansions of dynamical zeta functions and spectral determinants, in contrast to direct averages over periodic orbits such as the trace formulas discussed in sect. ?? and sect. ??, lies precisely in this organization into nearly cancelling combinations: cycle expansions are dominated by short cycles, with long cycles giving exponentially decaying corrections.

☞ sect. 13.3
☞ sect. 18.4

⇓PRELIMINARY
⇑PRELIMINARY

In the case where we know of no finite grammar symbolic dynamics that would help us organize the cycles, the best thing to use is a *stability cutoff* which we shall discuss in sect. 18.6. The idea is to truncate the cycle expansion by including only the pseudocycles such that $|\Lambda_{p_1}\cdots\Lambda_{p_k}| \leq \Lambda_{\max}$, with the cutoff $\Lambda_{\max}$ equal to or greater than the most unstable Λ_p in the data set.

- t_0			
- t_1			
- t_{10}	+ t_1t_0		
- t_{100}	+ $t_{10}t_0$		
- t_{101}	+ $t_{10}t_1$		
- t_{1000}	+ $t_{100}t_0$		
- t_{1001}	+ $t_{100}t_1$	+ $t_{101}t_0$	- $t_1t_{10}t_0$
- t_{1011}	+ $t_{101}t_1$		
- t_{10000}	+ $t_{1000}t_0$		
- t_{10001}	+ $t_{1001}t_0$	+ $t_{1000}t_1$	- $t_0t_{100}t_1$
- t_{10010}	+ $t_{100}t_{10}$		
- t_{10101}	+ $t_{101}t_{10}$		
- t_{10011}	+ $t_{1011}t_0$	+ $t_{1001}t_1$	- $t_0t_{101}t_1$
- t_{10111}	+ $t_{1011}t_1$		
- t_{100000}	+ $t_{10000}t_0$		
- t_{100001}	+ $t_{10001}t_0$	+ $t_{10000}t_1$	- $t_0t_{1000}t_1$
- t_{100010}	+ $t_{10010}t_0$	+ $t_{1000}t_{10}$	- $t_0t_{100}t_{10}$
- t_{100011}	+ $t_{10011}t_0$	+ $t_{10001}t_1$	- $t_0t_{1001}t_1$
- t_{100101}	- t_{100110}	+ $t_{10010}t_1$	+ $t_{10110}t_0$
	+ $t_{10}t_{1001}$	+ $t_{100}t_{101}$	- $t_0t_{10}t_{101}$ - $t_1t_{10}t_{100}$
- t_{101110}	+ $t_{10110}t_1$	+ $t_{1011}t_{10}$	- $t_1t_{101}t_{10}$
- t_{100111}	+ $t_{10011}t_1$	+ $t_{10111}t_0$	- $t_0t_{1011}t_1$
- t_{101111}	+ $t_{10111}t_1$		

Table 18.1: The binary curvature expansion (18.5) up to length 6, listed in such way that the sum of terms along the pth horizontal line is the curvature $\hat{c}_p$ associated with a prime cycle p, or a combination of prime cycles such as the $t_{100101} + t_{100110}$ pair.

18.2 Construction of cycle expansions

18.2.1 Evaluation of dynamical zeta functions

Cycle expansions of dynamical zeta functions are evaluated numerically by first computing the weights $t_p = t_p(\beta, s)$ of all prime cycles p of topological length $n_p \leq N$ for given fixed β and s. Denote by subscript (i) the ith prime cycle computed, ordered by the topological length $n_{(i)} \leq n_{(i+1)}$. The dynamical zeta function $1/\zeta_N$ truncated to the $n_p \leq N$ cycles is computed recursively, by multiplying

$$1/\zeta_{(i)} = 1/\zeta_{(i-1)}(1 - t_{(i)}z^{n_{(i)}})\,,$$

and truncating the expansion at each step to a finite polynomial in z^n, $n \leq N$. The result is the Nth order polynomial approximation

$$1/\zeta_N = 1 - \sum_{n=1}^{N} \hat{c}_n z^n\,. \tag{18.6}$$

In other words, a cycle expansion is a Taylor expansion in the dummy variable z raised to the topological cycle length. If both the number of

cycles and their individual weights grow not faster than exponentially with the cycle length, and we multiply the weight of each cycle p by a factor z^{n_p}, the cycle expansion converges for sufficiently small $|z|$.

If the dynamics is given by iterated mapping, the leading zero of (18.6) as function of z yields the leading eigenvalue of the appropriate evolution operator. For continuous time flows, z is a dummy variable that we set to $z = 1$, and the leading eigenvalue of the evolution operator is given by the leading zero of (18.6) as function of s.

refer to $1\zeta(s,\beta)$ here

18.2.2 Evaluation of traces, spectral determinants

add
! to det definition

Due to the lack of factorization of the full pseudocycle weight,

$$\det(\mathbf{1} - M_{p_1 p_2}) \neq \det(\mathbf{1} - M_{p_1}) \det(\mathbf{1} - M_{p_2}) \ ,$$

the cycle expansions for the spectral determinant (15.10) are somewhat less transparent than is the case for the dynamical zeta functions.

We commence the cycle expansion evaluation of a spectral determinant by computing recursively the trace formula (14.9) truncated to all prime cycles p and their repeats such that $n_p r \leq N$:

$$\begin{aligned}
\left.\operatorname{tr}\frac{z\mathcal{L}}{1-z\mathcal{L}}\right|_{(i)} &= \left.\operatorname{tr}\frac{z\mathcal{L}}{1-z\mathcal{L}}\right|_{(i-1)} + n_{(i)} \sum_{r=1}^{n_{(i)}r\leq N} \frac{e^{(\beta\cdot A_{(i)} - sT_{(i)})r}}{\left|\prod\left(1-\Lambda_{(i),j}^{r}\right)\right|} z^{n_{(i)}r} \\
\left.\operatorname{tr}\frac{z\mathcal{L}}{1-z\mathcal{L}}\right|_{N} &= \sum_{n=1}^{N} C_n z^n \,, \qquad C_n = \operatorname{tr}\mathcal{L}^n \,. \qquad (18.7)
\end{aligned}$$

This is done numerically: the periodic orbit data set consists of the list of the cycle periods T_p, the cycle stability eigenvalues $\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}$, and the cycle averages of the observable A_p for all prime cycles p such that $n_p \leq N$. The coefficient of $z^{n_p r}$ is then evaluated numerically for the given (β, s) parameter values. Now that we have an expansion for the trace formula (14.8) as a power series, we compute the Nth order approximation to the spectral determinant (15.3),

find whatever was to be inserted here!

For Markov graphs modify the $n_p \leq N$ criterion to something that groups the curvature contributions optimally

$$\det(1 - z\mathcal{L})|_N = 1 - \sum_{n=1}^{N} Q_n z^n \,, \qquad Q_n = n\text{th cumulant} \,, \qquad (18.8)$$

as follows. The logarithmic derivative relation (15.5) yields

$$\begin{aligned}
\left(\operatorname{tr}\frac{z\mathcal{L}}{1-z\mathcal{L}}\right)\det(1-z\mathcal{L}) &= -z\frac{d}{dz}\det(1-z\mathcal{L}) \\
(C_1 z + C_2 z^2 + \cdots)(1 - Q_1 z - Q_2 z^2 - \cdots) &= Q_1 z + 2Q_2 z^2 + 3Q_3 z^3 \cdots
\end{aligned}$$

cumulant!expansion
escape!rate

so the nth order term of the spectral determinant cycle (or in this case, the cumulant) expansion is given recursively by the trace formula expansion coefficients

$$Q_n = \frac{1}{n}\left(C_n - C_{n-1}Q_1 - \cdots C_1 Q_{n-1}\right) , \qquad Q_1 = C_1 . \tag{18.9}$$

Given the trace formula (18.7) truncated to z^N, we now also have the spectral determinant truncated to z^N.

The same program can also be reused to compute the dynamical zeta function cycle expansion (18.6), by replacing $\prod\left(1-\Lambda_{(i),j}^{r}\right)$ in (18.7) by the product of expanding eigenvalues $\Lambda_{(i)} = \prod_e \Lambda_{(i),e}$ (see sect. 15.3).

note: due to large cancelations, accuracy drops

⇓PRELIMINARY

A conceptually more elegant spectral determinant cycle expansion is based on the property of unique factorization of symbol sequences. This expansion has been relegated to appendix F.3.1, as its computational implementation is probably not worth the effort as far as the numerical evaluation of spectral determinants is concerned.

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The calculation of the leading eigenvalue of a given continuous flow evolution operator is now straightforward. After the prime cycles and the pseudocycles have been grouped into subsets of equal topological length, the dummy variable can be set equal to $z = 1$. With $z = 1$, expansion (18.8) is the cycle expansion for (15.7), the spectral determinant $\det(s-\mathcal{A})$. We vary s in cycle weights, and determine the eigenvalue s_α by finding $s = s_\alpha$ for which (18.8) vanishes. As an example, the convergence of a leading eigenvalue for a nice hyperbolic system is illustrated in table 18.2 by the listing of pinball escape rate γ estimates computed from truncations of (18.5) and (18.8) to different maximal cycle lengths.

Figure: plot of $1/\zeta(s)$ and $\det(s-\mathcal{A})$ with first few zeros, perhaps from ref. [18.6]

Fill table 18.2 out with Per's numbers.

probably should supplement with a graph, where one notes that the 3-disk convergence is not monotone.

Rosenqvist has also T_p = mean number of bounces. Is there a simple relation?

refer to chapter 16

☞ chapter 16

The pleasant surprise is that the coefficients in these cycle expansions can be proven to fall off exponentially or even faster, due to analyticity of $\det(s-\mathcal{A})$ or $1/\zeta(s)$ for s values well beyond those for which the corresponding trace formula diverges.

18.2.3 Newton algorithm for determination of the evolution operator eigenvalues

The cycle expansions of spectral determinants yield the eigenvalues of the evolution operator beyond the leading one. A convenient way to search for these is by plotting either the absolute magnitude $\ln|\det(s-\mathcal{A})|$ or the phase of spectral determinants and dynamical zeta functions as functions of the complex variable s. The eye is guided to the zeros of spectral determinants and dynamical zeta functions by means of complex s plane contour plots, with different intervals of the absolute value of the function under investigation assigned different colors; zeros emerge as

$R{:}a$	N	. $\det(s-\mathcal{A})$	$1/\zeta(s)$	$1/\zeta(s)_{\text{3-disk}}$
	1	0.39	0.407	
	2	0.4105	0.41028	0.435
	3	0.410338	0.410336	0.4049
6	4	0.4103384074	0.4103383	0.40945
	5	0.4103384077696	0.4103384	0.410367
	6	0.41033840776934648 2	0.4103383	0.410338
	7	0.4103384077693464892		0.4103396
	8	0.41033840776934648933846 8		
	9	0.41033840776934648933846 13074		
	10	0.41033840776934648933846 13078192		
	1	0.41		
	2	0.72		
	3	0.675		
	4	0.67797		
3	5	0.677921		
	6	0.6779227		
	7	0.6779226894		
	8	0.6779226896002		
	9	0.677922689599532		
	10	0.67792268959953606		

Table 18.2: 3-disk repeller escape rates computed from the cycle expansions of the spectral determinant (15.7) and the dynamical zeta function (15.16), as function of the maximal cycle length N. The first column indicates the disk-disk center separation to disk radius ratio $R{:}a$, the second column gives the maximal cycle length used, and the third the estimate of the classical escape rate from the fundamental domain spectral determinant cycle expansion. As for larger disk-disk separations the dynamics is more uniform, the convergence is better for $R{:}a = 6$ than for $R{:}a = 3$. For comparison, the fourth column lists a few estimates from from the fundamental domain dynamical zeta function cycle expansion (18.5), and the fifth from the full 3-disk cycle expansion (18.37). The convergence of the fundamental domain dynamical zeta function is significantly slower than the convergence of the corresponding spectral determinant, and the full (unfactorized) 3-disk dynamical zeta function has still poorer convergence. (P.E. Rosenqvist.)

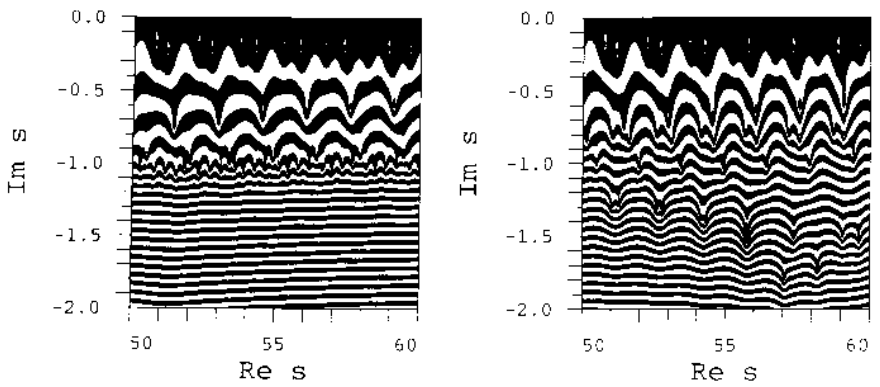

Figure 18.1: Examples of the complex s plane scans: contour plots of the logarithm of the absolute values of (a) $1/\zeta(s)$, (b) spectral determinant $\det(s-\mathcal{A})$ for the 3-disk system, separation $a:R=6$, A_1 subspace are evaluated numerically. The eigenvalues of the evolution operator $\mathcal{L}$ are given by the centers of elliptic neighborhoods of the rapidly narrowing rings. While the dynamical zeta function is analytic on a strip $\mathrm{Im}\, s \geq -1$, the spectral determinant is entire and reveals further families of zeros. (P.E. Rosenqvist)

redraw figure 18.1 so elliptic regions can be seen. Interchange $\mathrm{Re}\, s$, $\mathrm{Im}\, s$. Replace $\mathrm{Im}\, s \geq -1$ by correct value.

⇓PRELIMINARY

18.7

page 360

⇑PRELIMINARY

centers of elliptic neighborhoods of rapidly changing colors. Detailed scans of the whole area of the complex s plane under investigation and searches for the zeros of spectral determinants, figure 18.1, reveal complicated patterns of resonances even for something so simple as the 3-disk game of pinball. As we shall see in sect. **??**, this classical spectrum is closely related to the quantum resonances for the corresponding quantum system. With a good starting guess (such as a location of a zero suggested by the complex s scan of figure 18.1), a zero $1/\zeta(s)=0$ can now be easily determined by standard numerical methods, such as the iterative Newton algorithm (17.4), with the mth Newton estimate given by

$$s_{m+1} = s_m - \left(\zeta(s_m)\frac{\partial}{\partial s}\zeta^{-1}(s_m)\right)^{-1} = s_m - \frac{1/\zeta(s_m)}{\langle T\rangle_\zeta}\,. \qquad (18.10)$$

The dominator $\langle T\rangle_\zeta$ required for the Newton iteration is given below, by the cycle expansion (18.19). We need to evaluate it anyhow, as $\langle T\rangle_\zeta$ enters our cycle averaging formulas.

18.3 Cycle formulas for dynamical averages

forward reference

The eigenvalue condition in any of the three forms that we have given so far - the level sum (19.18), the dynamical zeta function (18.2), the spectral determinant (18.8):

$$1 \;=\; \sum_i^{(n)} t_i\,, \qquad t_i = t_i(\beta, s(\beta))\,, \qquad n_i = n\,, \qquad (18.11)$$

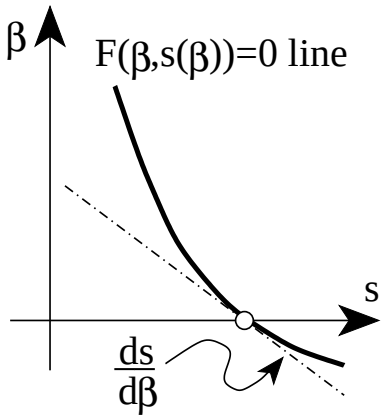

Figure 18.2: The eigenvalue condition is satisfied on the curve $F = 0$ the (β, s) plane. The expectation value of the observable (10.12) is given by the slope of the curve.

$$0 = 1 - \sum_{\pi}{}' t_\pi \,, \qquad t_\pi = t_\pi(z, \beta, s(\beta)) \tag{18.12}$$

$$0 = 1 - \sum_{n=1}^{\infty} Q_n \,, \qquad Q_n = Q_n(\beta, s(\beta)) \,, \tag{18.13}$$

is an implicit equation for the eigenvalue $s = s(\beta)$ of form $F(\beta, s(\beta)) = 0$. The eigenvalue $s = s(\beta)$ as a function of β is sketched in figure 18.2; the eigenvalue condition is satisfied on the curve $F = 0$. The cycle averaging formulas for the slope and the curvature of $s(\beta)$ are obtained as in (10.12) by taking derivatives of the eigenvalue condition. Evaluated along $F = 0$, the first derivative leads to

in move figure 18.2 the $s = 0$ axis

$$\begin{aligned} 0 &= \frac{d}{d\beta} F(\beta, s(\beta)) \\ &= \frac{\partial F}{\partial \beta} + \frac{ds}{d\beta} \frac{\partial F}{\partial s}\Big|_{s=s(\beta)} \quad \Longrightarrow \quad \frac{ds}{d\beta} = -\frac{\partial F}{\partial \beta} / \frac{\partial F}{\partial s} \,, \end{aligned} \tag{18.14}$$

and the second derivative of $F(\beta, s(\beta)) = 0$ yields

$$\frac{d^2 s}{d\beta^2} = -\left[\frac{\partial^2 F}{\partial \beta^2} + 2 \frac{ds}{d\beta} \frac{\partial^2 F}{\partial \beta \partial s} + \left(\frac{ds}{d\beta} \right)^2 \frac{\partial^2 F}{\partial s^2} \right] / \frac{\partial F}{\partial s} \,. \tag{18.15}$$

Denoting by

$$\begin{aligned} \langle A \rangle_F &= -\frac{\partial F}{\partial \beta}\Big|_{\beta, s=s(\beta)} \,, \qquad \langle T \rangle_F = \frac{\partial F}{\partial s}\Big|_{\beta, s=s(\beta)} \,, \\ \left\langle (A - \langle A \rangle)^2 \right\rangle_F &= \frac{\partial^2 F}{\partial \beta^2}\Big|_{\beta, s=s(\beta)} \end{aligned} \tag{18.16}$$

respectively the mean cycle expectation value of A, the mean cycle period, and the second derivative of F computed for $F(\beta, s(\beta)) = 0$, we obtain the cycle averaging formulas for the expectation value of the observable (10.12), and its variance:

Roberto says LHS is wrong, fix the second one!

escape!rate, vanishing

$$\langle a \rangle = \frac{\langle A \rangle_F}{\langle T \rangle_F} \tag{18.17}$$

$$\left\langle (a - \langle a \rangle)^2 \right\rangle = \frac{1}{\langle T \rangle_F} \left\langle (A - \langle A \rangle)^2 \right\rangle_F . \tag{18.18}$$

forward reference

These formulas are the central result of the periodic orbit theory. As we shall now show, for each choice of the eigenvalue condition function $F(\beta, s)$ in (19.18), (18.2) and (18.8), the above quantities have explicit cycle expansions.

18.3.1 Dynamical zeta function cycle expansions

For the dynamical zeta function condition (18.12), the cycle averaging formulas (18.14), (18.18) require evaluation of the derivatives of dynamical zeta function at a given eigenvalue. Substituting the cycle expansion (18.2) for dynamical zeta function we obtain

$$\begin{aligned} \langle A \rangle_\zeta &:= -\frac{\partial}{\partial \beta}\frac{1}{\zeta} = \sum{}' A_\pi t_\pi \\ \langle T \rangle_\zeta &:= \frac{\partial}{\partial s}\frac{1}{\zeta} = \sum{}' T_\pi t_\pi , \quad \langle n \rangle_\zeta := -z\frac{\partial}{\partial z}\frac{1}{\zeta} = \sum{}' n_\pi t_\pi , \end{aligned} \tag{18.19}$$

where the subscript in $\langle \cdots \rangle_\zeta$ stands for the dynamical zeta function average over prime cycles, A_π, T_π, and n_π are evaluated on pseudocycles (18.4), and pseudocycle weights $t_\pi = t_\pi(z, \beta, s(\beta))$ are evaluated at the eigenvalue $s(\beta)$. In most applications $\beta = 0$, and $s(\beta)$ of interest is typically the leading eigenvalue $s_0 = s_0(0)$ of the evolution generator $\mathcal{A}$.

For bounded flows the leading eigenvalue (the escape rate) vanishes, $s(0) = 0$, the exponent $\beta A_\pi - s T_\pi$ in (18.3) vanishes, so the cycle expansions take a simple form

$$\langle A \rangle_\zeta = \sum_\pi{}' (-1)^{k+1} \frac{A_{p_1} + A_{p_2} \cdots + A_{p_k}}{|\Lambda_{p_1} \cdots \Lambda_{p_k}|} , \tag{18.20}$$

and similarly for $\langle T \rangle_\zeta$, $\langle n \rangle_\zeta$. For example, for the complete binary symbolic dynamics the mean cycle period $\langle T \rangle_\zeta$ is given by

$$\begin{aligned} \langle T \rangle_\zeta &= \frac{T_0}{|\Lambda_0|} + \frac{T_1}{|\Lambda_1|} + \left(\frac{T_{01}}{|\Lambda_{01}|} - \frac{T_0 + T_1}{|\Lambda_0 \Lambda_1|} \right) \\ &\quad + \left(\frac{T_{001}}{|\Lambda_{001}|} - \frac{T_{01} + T_0}{|\Lambda_{01}\Lambda_0|} \right) + \left(\frac{T_{011}}{|\Lambda_{011}|} - \frac{T_{01} + T_1}{|\Lambda_{01}\Lambda_1|} \right) + \dots . \end{aligned} \tag{18.21}$$

find FastTrack to shadowing section

Note that the cycle expansions for averages are grouped into the same

shadowing combinations as the dynamical zeta function cycle expansion (18.5), with nearby pseudocycles nearly cancelling each other.

three-disk@3-disk!shadowing
three-disk@3-disk!escape rate
escape!rate, 3-disk
shadowing!3-disk

The cycle averaging formulas for the expectation value of the observable $\langle a \rangle$ follow by substitution into (18.18). Assuming zero mean drift $\langle a \rangle = 0$, the cycle expansion (18.8) for the variance $\langle (A - \langle A \rangle)^2 \rangle_\zeta$ is given by

$$\langle A^2 \rangle_\zeta = \sum{}' (-1)^{k+1} \frac{(A_{p_1} + A_{p_2} \cdots + A_{p_k})^2}{|\Lambda_{p_1} \cdots \Lambda_{p_k}|} . \tag{18.22}$$

18.3.2 Spectral determinant cycle expansions

expand this section

The dynamical zeta function cycle expansions have a particularly simple structure, with the shadowing apparent already by a term-by-term inspection of table 18.2. For "nice" hyperbolic systems the shadowing ensures exponential convergence of the dynamical zeta function cycle expansions. This, however, is not the best achievable convergence. As has been explained in chapter 16, for such systems the spectral determinant constructed from the same cycle data base is entire, and its cycle expansion converges faster than exponentially. In practice, the best convergence is attained by the spectral determinant cycle expansion (18.13) and its derivatives. The $\partial/\partial s$, $\partial/\partial\beta$ derivatives are in this case computed recursively, by taking derivatives of the spectral determinant cycle expansion contributions (18.9) and (18.7).

The cycle averaging formulas are exact, and highly convergent for nice hyperbolic dynamical systems. An example of its utility is the cycle expansion formula for the Lyapunov exponent of example 18.1. Further applications of cycle expansions will be discussed in chapter 19.

expand

18.3.3 Continuous vs. discrete mean return time

Sometimes it is convenient to compute an expectation value along a flow, in continuous time, and sometimes it might be easier to compute it in [1] discrete time, from a Poincaré return map. Return times (3.1) might vary wildly, and it is not at all clear that the continuous and discrete time averages are related in any simple way. The relationship turns on to be both elegantly simple, and totally general.

rewrite this section, expand, emphasize

[2] The mean cycle period $\langle T \rangle_\zeta$ fixes the normalization of the unit of time; it can be interpreted as the average near recurrence or the average first return time. For example, if we have evaluated a billiard expectation value $\langle a \rangle$ in terms of continuous time, and would like to also have the

[1] Whelan: does $< T > s$ average always exist?
[2] Whelan: find NW comment

Lyapunov exponent!cycle expansion
cycle!expansion!Lyapunov exponent
escape!rate
cycle!expansion!finite subshift
finite subshift!cycle expansion
symbolic dynamics!coding!"MarkGraph

corresponding average $\langle a\rangle_{\rm dscr}$ measured in discrete time, given by the number of reflections off billiard walls, the two averages are related by

$$\langle a\rangle_{\rm dscr} = \langle a\rangle \langle T\rangle_\zeta / \langle n\rangle_\zeta \ , \tag{18.23}$$

where $\langle n\rangle_\zeta$ is the average of the number of bounces n_p along the cycle p.

Example 18.1 Cycle expansion formula for Lyapunov exponents:

In sect. 10.3 we defined the Lyapunov exponent for a 1-d mapping, related it to the leading eigenvalue of an evolution operator and promised to evaluate it. Now we are finally in position to deliver on our promise.

The cycle averaging formula (18.20) yields an exact explict expression for the Lyapunov exponent in terms of prime cycles:

$$\lambda = \frac{1}{\langle n\rangle_\zeta} \sum{}' (-1)^{k+1} \frac{\log|\Lambda_{p_1}| + \cdots + \log|\Lambda_{p_k}|}{|\Lambda_{p_1}\cdots\Lambda_{p_k}|} \ . \tag{18.24}$$

For a repeller, the $1/|\Lambda_p|$ weights are replaced by normalized measure (19.10) $\exp(\gamma n_p)/|\Lambda_p|$, where γ is the escape rate.

refer to open systems section

We mention here without proof that for 2-d Hamiltonian flows such as our game of pinball there is only one expanding eigenvalue and (18.24) applies as it stands.

in depth:
chapter I.1, p. 945

18.4 Cycle expansions for finite alphabets

A finite Markov graph like the one given in figure 13.3 (d) is a compact encoding of the transition or the Markov matrix for a given subshift. It is a sparse matrix, and the associated determinant (13.17) can be written down by inspection: it is the sum of all possible partitions of the graph into products of non-intersecting loops, with each loop carrying a minus sign:

$$\det(1-T) = 1 - t_0 - t_{0011} - t_{0001} - t_{00011} + t_0 t_{0011} + t_{0011} t_{0001} \tag{18.25}$$

The simplest application of this determinant is to the evaluation of the topological entropy; if we set $t_p = z^{n_p}$, where n_p is the length of the p-cycle, the determinant reduces to the topological polynomial (13.18).

The determinant (18.25) is exact for the finite graph figure 13.3 (e), as well as for the associated finite-dimensional transfer operator of example 10.1.

For the associated (infinite dimensional) evolution operator, it is the beginning of the cycle expansion of the corresponding dynamical zeta function:

$$1/\zeta = 1 - t_0 - t_{0011} - t_{0001} + t_{0001}t_{0011} - (t_{00011} - t_0 t_{0011} + \dots \text{curvatures}) \dots \quad (18.26)$$

The cycles $\overline{0}$, $\overline{0001}$ and $\overline{0011}$ are the *fundamental* cycles introduced in (18.5); they are not shadowed by any combinations of shorter cycles, and are the basic building blocks of the dynamicsgenerated by iterating the pruning rules (12.11). All other cycles appear together with their shadows (for example, the $t_{00011} - t_0 t_{0011}$ combination) and yield exponentially small corrections for hyperbolic systems.

⇓PRELIMINARY
⇑PRELIMINARY
restore misplaced

For the cycle counting purposes both t_{ab} and the pseudocycle combination $t_{a+b} = t_a t_b$ in (18.2) have the same weight $z^{n_a+n_b}$, so all curvature combinations $t_{ab} - t_a t_b$ vanish exactly, and the topological polynomial (13.21) offers a quick way of checking the fundamental part of a cycle expansion.

Since for finite grammars the topological zeta functions reduce to polynomials, we are assured that there are just a few fundamental cycles and that all long cycles can be grouped into curvature combinations. For example, the fundamental cycles in exercise 11.5 are the three 2-cycles which bounce back and forth between two disks and the two 3-cycles which visit every disk. It is only after these fundamental cycles have been included that a cycle expansion is expected to start converging smoothly, i.e., only for n larger than the lengths of the fundamental cycles are the curvatures $\hat{c}_n$ (in expansion (18.5)), a measure of the deviations between long orbits and their short cycle approximants, expected to fall off rapidly with n.

⇓PRELIMINARY

18.5 Perturbation expansions in terms of cycles

The eigenvalue condition of sect. 18.3 is an implicit equation for the eigenvalue $s = s(\beta)$ of form $F(\beta, s(\beta)) = 0$. As the eigenvalue condition is satisfied on the curve $F = 0$, all $d/d\beta$ total derivatives of the eigenvalue condition vanish, leading to

move into an appendix

$$0 = \frac{d}{d\beta}F(\beta, s(\beta)) = \frac{ds}{d\beta}\frac{\partial F}{\partial s} + \frac{\partial F}{\partial \beta}, \quad (18.27)$$

$$0 = \frac{d^2 s}{d\beta^2}\frac{\partial F}{\partial s} + \left(\frac{ds}{d\beta}\right)^2 \frac{\partial^2 F}{\partial s^2} + 2\frac{ds}{d\beta}\frac{\partial^2 F}{\partial \beta \partial s} + \frac{\partial^2 F}{\partial \beta^2}, \quad (18.28)$$

$$0 = \frac{d^3 s}{d\beta^3}\frac{\partial F}{\partial s} + 3\frac{d^2 s}{d\beta^2}\frac{ds}{d\beta}\frac{\partial^2 F}{\partial s^2} + \left(\frac{ds}{d\beta}\right)^3 \frac{\partial^3 F}{\partial s^3} + 3\frac{d^2 s}{d\beta^2}\frac{\partial^2 F}{\partial \beta \partial s} + 3\left(\frac{ds}{d\beta}\right)^2 \frac{\partial^3 F}{\partial \beta \partial s^2} + 3\frac{ds}{d\beta}\frac{\partial^3 F}{\partial \beta^2 \partial s} + \frac{\partial^3 F}{\partial \beta^3} \quad (18.29)$$

cycle!perturbation expansions
perturbation expansions
linear response theory
cycle!expansion!stability ordered
stability!ordering!cycle expansions

and so on. If $s(0)$ is known, and $\beta \neq 0$ is parametrizing a weak perturbation to the evolution operator $\mathcal{L}$, the above formulas enable us to compute recursively, order by order in β^n, the perturbative corrections to the eigenvalues of $\mathcal{A}$ in terms of partial derivatives of the eigenvalue condition as the series

$$s(\beta) = \sum_{n=0}^{\infty} \frac{s_n}{n!}\beta^n , \qquad s_n = \left.\frac{d^n s(\beta)}{d\beta^n}\right|_{\beta=0} \tag{18.30}$$

forward reference

As we have seen in sect. 18.3.1, for each choice of the eigenvalue condition function $F(\beta, s)$ in (19.18), (18.2) and (18.8), the partial derivatives of $F(\beta, s(\beta))$ are given by explicit cycle expansions. The simplest application of the above formulas is the linear response theory of eigenvalue shifts due to small perturbations. In the case of stochastic flows and semiclassical quantum mechanics, the small parameter β is the noise width, $\hbar$ respectively, and the formulas (18.29) yield the full perturbative corrections.

⇑PRELIMINARY

18.6 Stability ordering of cycle expansions

> There is never a second chance. Most often there is not even the first chance.
>
> John Wilkins

(C.P. Dettmann and P. Cvitanović)

As we sawin sect. ?

Most dynamical systems of interest have no finite grammar, so at any order in z a cycle expansion may contain unmatched terms which do not fit neatly into the almost cancelling curvature corrections. Similarly, for intermittent systems that we shall discuss in chapter 21, curvature corrections are in general not small, so again the cycle expansions may converge slowly. For such systems schemes which collect the pseudocycle terms according to some criterion other than the topology of the flow may converge more quickly than expansions based on the topological length.

All chaotic systems exhibit some degree of shadowing, and a good truncation criterion should do its best to respect the shadowing at least approximately. If a long cycle is shadowed by two or more shorter cycles and the flow is smooth, the period and the action will be additive in sense that the period of the longer cycle is approximately the sum of the shorter cycle periods. Similarly, stability is multiplicative, so shadowing is approximately preserved by including all terms with pseudocycle stability

$$|\Lambda_{p_1} \cdots \Lambda_{p_k}| \leq \Lambda_{\max} \tag{18.31}$$

and ignoring all more unstable pseudocycles.

ELIMINARY
ELIMINARY

Two such schemes for ordering cycle expansions which approximately respect shadowing are truncations by the pseudocycle period (or action) discussed in remark A.7, and the stability ordering that we shall discuss here. In these schemes a dynamical zeta function or a spectral determinant is expanded keeping all terms for which the period, action or stability for a combination of cycles (pseudocycle) is less than a given cutoff.

The two settings in which the stability ordering may be preferable to the ordering by topological cycle length are the cases of bad grammar and of intermittency.

add Berry-Keating somewhere

18.6.1 Stability ordering for bad grammars

For generic flows it is often not clear what partition of the state space generates the "optimal" symbolic dynamics. Stability ordering does not require understanding dynamics in such detail: if you can find the cycles, you can use stability ordered cycle expansions. Stability truncation is thus easier to implement for a generic dynamical system than the curvature expansions (18.5) which rely on finite subshift approximations to a given flow.

Cycles can be detected numerically by searching a long trajectory for near recurrences. The long trajectory method for detecting cycles discussed in sect. 17.6.1 preferentially finds the least unstable cycles, regardless of their topological length. Another practical advantage of the method (in contrast to Newton method searches) is that it only finds cycles in a given connected ergodic component of state space, ignoring isolated cycles or other ergodic regions elsewhere in the state space.

⇓PRELIMINARY
⇑PRELIMINARY

Why should stability ordered cycle expansion of a dynamical zeta function converge better than the rude trace formula (19.9)? The argument has essentially already been laid out in sect. 13.7: in truncations that respect shadowing most of the pseudocycles appear in shadowing combinations and nearly cancel, while only the relatively small subset affected by the longer and longer pruning rules is not shadowed. So the error is typically of the order of $1/\Lambda$, smaller by factor e^{hT} than the trace formula (19.9) error, where h is the entropy and T typical cycle length for cycles of stability Λ.

explain better
killed by Gabor: (**??**)?
forward ref
forward ref

Incorporate Grassberger's dimension of the Cantor set of prunning rules

18.6.2 Smoothing

The breaking of exact shadowing cancellations deserves further comment. Partial shadowing which may be present can be (partially) restored by smoothing the stability ordered cycle expansions by replacing the $1/\Lambda$ weigth for each term with pseudocycle stability $\Lambda = \Lambda_{p_1} \cdots \Lambda_{p_k}$ by $f(\Lambda)/\Lambda$. Here, $f(\Lambda)$ is a monotonically decreasing function from $f(0) = 1$ to $f(\Lambda_{\max}) = 0$. No smoothing corresponds to a step function.

I do not believe yet that this is signifcant improvement

stability!ordering!intermittent flows
intermittency!stability ordering

A typical "shadowing error" induced by the cutoff is due to two pseudocycles of stability Λ separated by $\Delta\Lambda$, and whose contribution is of opposite signs. Ignoring possible weighting factors the magnitude of the resulting term is of order $1/\Lambda - 1/(\Lambda + \Delta\Lambda) \approx \Delta\Lambda/\Lambda^2$. With smoothing there is an extra term of the form $f'(\Lambda)\Delta\Lambda/\Lambda$, which we want to minimise. A reasonable guess might be to keep $f'(\Lambda)/\Lambda$ constant and as small as possible, that is

$$f(\Lambda) = 1 - \left(\frac{\Lambda}{\Lambda_{\max}}\right)^2$$

The results of a stability ordered expansion (18.31) should always be tested for robustness by varying the cutoff $\Lambda_{\max}$. If this introduces significant variations, smoothing is probably necessary.

18.6.3 Stability ordering for intermittent flows

Longer but less unstable cycles can give larger contributions to a cycle expansion than short but highly unstable cycles. In such situation truncation by length may require an exponentially large number of very unstable cycles before a significant longer cycle is first included in the expansion. This situation is best illustrated by intermittent maps that we shall study in detail in chapter 21, the simplest of which is the Farey map

add figure

$$f(x) = \begin{cases} f_0 = x/(1-x) & 0 \le x \le 1/2 \\ f_1 = (1-x)/x & 1/2 \le x \le 1 \end{cases}, \tag{18.32}$$

a map which will reappear in the intermittency chapter 21, and in chapter 28, in context of circle maps.

For this map the symbolic dynamics is of complete binary type, so lack of shadowing is not due to lack of a finite grammar, but rather to the intermittency caused by the existence of the marginal fixed point $x_0 = 0$, for which the stability equals $\Lambda_0 = 1$. This fixed point does not participate directly in the dynamics and is omitted from cycle expansions. Its presence is felt in the stabilities of neighboring cycles with n consecutive repeats of the symbol 0's whose stability falls of only as $\Lambda \sim n^2$, in contrast to the most unstable cycles with n consecutive 1's which are exponentially unstable, $|\Lambda_{01^n}| \sim [(\sqrt{5}+1)/2]^{2n}$.

The symbolic dynamics is of complete binary type. A quick count in the style of sect. 13.5.2 leads to a total of 74,248,450 prime cycles of length 30 or less, not including the marginal point $x_0 = 0$. Evaluating a cycle expansion to this order would be no mean computational feat. However, the least unstable cycle omitted has stability of roughly $\Lambda_{10^{30}} \sim 30^2 = 900$, and so amounts to a 0.1% correction. The situation may be much worse than

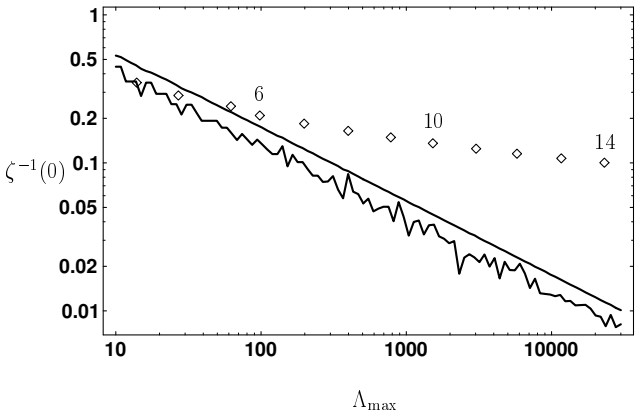

Figure 18.3: Comparison of cycle expansion truncation schemes for the Farey map (18.32); the deviation of the truncated cycles expansion for $|1/\zeta_N(0)|$ from the exact flow conservation value $1/\zeta(0) = 0$ is a measure of the accuracy of the truncation. The jagged line is logarithm of the stability ordering truncation error; the smooth line is smoothed according to sect. 18.6.2; the diamonds indicate the error due the topological length truncation, with the maximal cycle length N shown. They are placed along the stability cutoff axis at points determined by the condition that the total number of cycles is the same for both truncation schemes.

this estimate suggests, because the next, 10^{31} cycle contributes a similar amount, and could easily reinforce the error. Adding up all such omitted terms, we arrive at an estimated error of about 3%, for a cycle-length truncated cycle expansion based on more than 10^9 pseudocycle terms! On the other hand, truncating by stability at say $\Lambda_{\max} = 3000$, only 409 prime cycles suffice to attain the same accuracy of about 3% error, figure 18.3.

As the Farey map maps the unit interval onto itself, the leading eigenvalue of the Perron-Frobenius operator should equal $s_0 = 0$, so $1/\zeta(0) = 0$. Deviation from this exact result serves as an indication of the convergence of a given cycle expansion. The errors of different truncation schemes are indicated in figure 18.3. We see that topological length truncation schemes are hopelessly bad in this case; stability length truncations are somewhat better, but still rather bad. As we shall show in sect. 21.3.1, in simple cases like this one, where intermittency is caused by a single marginal fixed point, the convergence can be improved by going to infinite alphabets. A deeper understanding of why this seemingly trivial example should be so difficult to control requires introduction of thermodynamic formalism and an investigation of its phase transitions, topic that we postpone to sect. **??**.

why is error in figure 18.3larger when smoothed?

⇓PRIVATE

⇑PRIVATE

talk about intermittency chapter?

⇓PRELIMINARY

⇑PRELIMINARY

⇓PRELIMINARY

18.7 Equilibrium points

In sect. 15.7 we have worked out the contribution of an isolated equilibrium point to the trace and spectral determinant formulas. As they differ in form from the cycle weights for periodic orbits, we need to show

equilibrium!point
^PRELIMINARY
Dirichlet series
Riemann zeta function
abscissa!absolute conv.
abscissa!conditional conv.
convergence!abscissa of

that they are shadowed by close-by families of cycles.

18.8 Dirichlet series

> The most patient reader will thank me for compressing so much nonsense and falsehood into a few lines.
>
> Gibbon

motivate this section

A Dirichlet series is defined as

$$f(s) = \sum_{j=1}^{\infty} a_j e^{-\lambda_j s} \tag{18.33}$$

do $\lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots$ have to be real, or just their real parts should be ordered?.

where s, a_j are complex numbers, and $\{\lambda_j\}$ is a monotonically increasing series of real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots$. A classical example of a Dirichlet series is the Riemann zeta function for which $a_j = 1$, $\lambda_j = \ln j$. In the present context, formal series over individual pseudocycles such as (18.2) ordered by the increasing pseudocycle periods are often Dirichlet series. For example, for the pseudocycle weight (18.3), the Dirichlet series is obtained by ordering pseudocycles by increasing periods $\lambda_\pi = T_{p_1} + T_{p_2} + \ldots + T_{p_k}$, with the coefficients

$$a_\pi = \frac{e^{\beta\cdot(A_{p_1}+A_{p_2}+\ldots+A_{p_k})}}{|\Lambda_{p_1}\Lambda_{p_2}\ldots\Lambda_{p_k}|} d_\pi \,,$$

where d_π is a degeneracy factor, in the case that d_π pseudocycles have the same weight.

If the series $\sum |a_j|$ diverges, the Dirichlet series is absolutely convergent for $\mathrm{Re}\, s > \sigma_a$ and conditionally convergent for $\mathrm{Re}\, s > \sigma_c$, where σ_a is the *abscissa of absolute convergence*

$$\sigma_a = \lim_{N\to\infty} \sup \frac{1}{\lambda_N} \ln \sum_{j=1}^{N} |a_j| \,, \tag{18.34}$$

and σ_c is the *abscissa of conditional convergence*

$$\sigma_c = \lim_{N\to\infty} \sup \frac{1}{\lambda_N} \ln \left| \sum_{j=1}^{N} a_j \right| \,. \tag{18.35}$$

define:
spectral radius
essential spectral radius
spectral gap

backlink chapter 34 to here.

We shall encounter another example of a Dirichlet series in the semiclassical quantization chapter 34, where the inverse Planck constant is a complex variable $s = i/\hbar$, $\lambda_\pi = S_{p_1} + S_{p_2} + \ldots + S_{p_k}$ is the pseudocycle action, and $a_\pi = 1/\sqrt{|\Lambda_{p_1}\Lambda_{p_2}\ldots\Lambda_{p_k}|}$ (times possible degeneracy and topological phase factors). As the action is in general not a linear function of energy (except for billiards and for scaling potentials, where a variable s can be extracted from S_p), semiclassical cycle expansions are Dirichlet series in variable $s = i/\hbar$ but not in E, the complex energy variable.

add Period doubling Lyapunovs vs 1/n of paper II.

Commentary

Remark 18.1 Pseudocycle expansions. Bowen's introduction of shadowing ϵ-pseudoorbits [1.22] was a significant contribution to Smale's theory. Expression "pseudoorbits" seems to have been introduced in the Parry and Pollicott's 1983 paper [15.5]. Following them M. Berry [18.9] had used the expression "pseudoorbits" in his 1986 paper on Riemann zeta and quantum chaos. Cycle and curvature expansions of dynamical zeta functions and spectral determinants were introduced in refs. [18.10, 18.2]. Some literature [15.14] refers to the pseudoorbits as "composite orbits", and to the cycle expansions as "Dirichlet series" (see also remark 18.7 and sect. 18.8).

Remark 18.2 Cumulant expansion. To a statistical mechanician the curvature expansions are very reminiscent of cumulant expansions. Indeed, (18.9) is the standard Plemelj-Smithies cumulant formula (M.28) for the Fredholm determinant, discussed in more detail in appendix M. The difference is that in cycle expansions each Q_n coefficient is expressed as a sum over exponentially many cycles.

Remark 18.3 Exponential growth of the number of cycles. Going from $N_n \approx N^n$ periodic points of length n to M_n prime cycles reduces the number of computations from N_n to $M_n \approx N^{n-1}/n$. Use of discrete symmetries (chapter 22) reduces the number of nth level terms by another factor. While the reformulation of the theory from the trace (14.26) to the cycle expansion (18.5) thus does not eliminate the exponential growth in the number of cycles, in practice only the shortest cycles are used, and for them the computational labor saving can be significant.

Remark 18.4 Shadowing cycle-by-cycle. A glance at the low order curvatures in the table 18.1 leads to the temptation of associating curvatures to individual cycles, such as $\hat{c}_{0001} = t_{0001} - t_0 t_{001}$. Such combinations tend to be numerically small (see for example ref. [18.3], table 1). However, splitting $\hat{c}_n$ into individual cycle curvatures is not possible in general [11.14]; the first example of such ambiguity in the binary cycle expansion is given by the t_{100101}, t_{100110} $0 \leftrightarrow 1$ symmetric pair of 6-cycles; the counterterm $t_{001}t_{011}$ in table 18.1 is shared by the two cycles.

include this table:

Remark 18.5 Stability ordering. The stability ordering was introduced by Dahlqvist and Russberg [18.12] in a study of chaotic dynamics for the $(x^2y^2)^{1/a}$ potential.

entropy!barrier

The presentation here runs along the lines of Dettmann and Morriss [18.13] for the Lorentz gas which is hyperbolic but the symbolic dynamics is highly pruned, and Dettmann and Cvitanović [18.14] for a family of intermittent maps. In the applications discussed in the above papers, the stability ordering yields a considerable improvement over the topological length ordering. In quantum chaos applications cycle expansion cancellations are affected by the phases of pseudocycles (their actions), hence *period ordering* rather than stability is frequently employed.

⇓PRELIMINARY

Remark 18.6 A homoclinic web with a equilibrium point. We still need to show that the periodic orbits that come close to the equilibrium point shadow it.

⇑PRELIMINARY

Remark 18.7 Are cycle expansions Dirichlet series? Even though some literature [15.14] refers to cycle expansions as "Dirichlet series", they are not Dirichlet series. Cycle expansions collect contributions of individual cycles into groups that correspond to the coefficients in cumulant expansions of spectral determinants, and the convergence of cycle expansions is controlled by general properties of spectral determinants. Dirichlet series order cycles by their periods or actions, and are only conditionally convergent in regions of interest. The abscissa of absolute convergence is in this context called the "entropy barrier"; contrary to the frequently voiced anxieties, this number does not necessarily has much to do with the actual convergence of the theory.

find F. Steiner references

refer to Aurell

link to chapter 16

Résumé

A *cycle expansion* is a series representation of a dynamical zeta function, trace formula or a spectral determinant, with products in (15.16), (36.19) expanded as sums over *pseudocycles*, products of the prime cycle weigths t_p.

(36.19) forw ref?

If a flow is hyperbolic and has a topology of a Smale horseshoe (a subshift of finite type), the dynamical zeta functions are holomorphic, the spectral determinants are entire, and the spectrum of the evolution operator is discrete. The situation is considerably more reassuring than what practitioners of quantum chaos fear; there is no "abscissa of absolute convergence" and no "entropy barier", the exponential proliferation of cycles is no problem, spectral determinants are entire and converge everywhere, and the topology dictates the choice of cycles to be used in cycle expansion truncations.

define holomorphic

In that case, the basic observation is that the motion in dynamical systems of few degrees of freedom is in this case organized around a few *fundamental* cycles, with the cycle expansion of the Euler product

$$1/\zeta = 1 - \sum_f t_f - \sum_n \hat{c}_n,$$

regrouped into dominant *fundamental* contributions t_f and decreasing *curvature* corrections $\hat{c}_n$. The fundamental cycles t_f have no shorter approximants; they are the "building blocks" of the dynamics in the sense that all longer orbits can be approximately pieced together from them. A typical curvature contribution to $\hat{c}_n$ is a *difference* of a long cycle $\{ab\}$ minus its shadowing approximation by shorter cycles $\{a\}$ and $\{b\}$:

refer to shadowing figure

$$t_{ab} - t_a t_b = t_{ab}(1 - t_a t_b / t_{ab})$$

The orbits that follow the same symbolic dynamics, such as $\{ab\}$ and a "pseudocycle" $\{a\}\{b\}$, lie close to each other, have similar weights, and for longer and longer orbits the curvature corrections fall off rapidly. Indeed, for systems that satisfy the "axiom A" requirements, such as the 3-disk billiard, curvature expansions converge very well.

explained where?

link to Axion A

Once a set of the shortest cycles has been found, and the cycle periods, stabilities and integrated observable computed, the cycle averaging formulas such as the ones associated with the dynamical zeta function

$$\begin{aligned} \langle a \rangle &= \langle A \rangle_\zeta \,/\, \langle T \rangle_\zeta \\ \langle A \rangle_\zeta &= -\frac{\partial}{\partial \beta}\frac{1}{\zeta} = \sum{}' A_\pi t_\pi \,, \qquad \langle T \rangle_\zeta = \frac{\partial}{\partial s}\frac{1}{\zeta} = \sum{}' T_\pi t_\pi \end{aligned}$$

yield the expectation value (the chaotic, ergodic average over the non–wandering set) of the observable $a(x)$.

References

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Exercises

three-disk@3-disk!cycle!expansion
cycle!expansion!3-disk
escape!rate

Exercise 18.1 **Cycle expansions.** *Write programs that implement* binary *symbolic dynamics cycle expansions for (a) dynamical zeta functions, (b) spectral determinants. Combined with the cycles computed for a 2-branch repeller or a 3-disk system they will be useful in problem that follow.*

Exercise 18.2 **Escape rate for a 1-d repeller.** *(Continuation of exercise 15.1 - easy, but long)*
Consider again the quadratic map (15.41)

$$f(x) = Ax(1-x)$$

on the unit interval, for definitivness take either $A = 9/2$ or $A = 6$. Describing the itinerary of any trajectory by the binary alphabet $\{0, 1\}$ ('0' if the iterate is in the first half of the interval and '1' if is in the second half), we have a repeller with a complete binary symbolic dynamics.

(a) *Sketch the graph of f and determine its two fixed points $\overline{0}$ and $\overline{1}$, together with their stabilities.*

(b) *Sketch the two branches of f^{-1}. Determine all the prime cycles up to topological length 4 using your pocket calculator and backwards iteration of f (see sect. 17.2.1).*

(c) *Determine the leading zero of the zeta function (15.16) using the weigths $t_p = z^{n_p}/|\Lambda_p|$ where Λ_p is the stability of the p cycle.*

(d) *Show that for $A = 9/2$ the escape rate of the repeller is $0.361509\ldots$ using the spectral determinant, with the same cycle weight. If you have taken $A = 6$, the escape rate is in $0.83149298\ldots$, as shown in solution 18.2. Compare the coefficients of the spectral determinant and the zeta function cycle expansions. Which expansion converges faster?*

(Per Rosenqvist)

⇓PRELIMINARY

Exercise 18.3 **Tent map spectral determinant zeros, Per's pp.31.**

⇑PRELIMINARY

Exercise 18.4 **Escape rate for the Ulam map.** *(medium) We will try to compute the escape rate for the Ulam map (17.21)*

$$f(x) = 4x(1-x),$$

using the method of cycle expansions. The answer should be zero, as nothing escapes.

(a) *Compute a few of the stabilities for this map. Show that $\Lambda_0 = 4$, $\Lambda_1 = -2$, $\Lambda_{01} = -4$, $\Lambda_{001} = -8$ and $\Lambda_{011} = 8$.*

three-disk@3-disk!escape rate
escape!rate,3-disk

(b) *Show that*

$$\Lambda_{\epsilon_1 \ldots \epsilon_n} = \pm 2^n$$

and determine a rule for the sign.

(c) *(hard) Compute the dynamical zeta function for this system*

$$\zeta^{-1} = 1 - t_0 - t_1 - (t_{01} - t_0 t_1) - \cdots$$

You might note that the convergence as function of the truncation cycle length is slow. Try to fix that by treating the $\Lambda_0 = 4$ cycle separately.

Exercise 18.5 **Pinball escape rate, semi-analytical.** *Estimate the 3-disk pinball escape rate for $R : a = 6$ by substituting analytical cycle stabilities and periods (exercise 8.1 and exercise 8.2) into the appropriate binary cycle expansion. Compare with the numerical estimate exercise 10.4*

Exercise 18.6 **Pinball escape rate, from numerical cycles.** *Compute the escape rate for $R : a = 6$ 3-disk pinball by substituting list of numerically computed cycle stabilities of exercise 17.6 into the binary cycle expansion.*

Exercise 18.7 **Pinball resonances, in the complex plane.** *Plot the logarithm of the absolute value of the dynamical zeta function and/or the spectral determinant cycle expansion (18.6) as contour plots in the complex s plane. Do you find zeros other than the one corresponding to the complex one? Do you see evidence for a finite radius of convergence for either cycle expansion?*

where did the counterterms section go in the main text?

Exercise 18.8 **Counting the 3-disk pinball counterterms.** *Verify that the number of terms in the 3-disk pinball curvature expansion (18.36) is given by*

$$\begin{aligned} \prod_p (1+t_p) &= \frac{1-3z^4-2z^6}{1-3z^2-2z^3} = 1+3z^2+2z^3+\frac{z^4(6+12z+2z^2)}{1-3z^2-2z^3} \\ &= 1+3z^2+2z^3+6z^4+12z^5+20z^6+48z^7+84z^8+184z^9+\ldots \end{aligned}$$

This means that, for example, c_6 has a total of 20 terms, in agreement with the explicit 3-disk cycle expansion (18.37).

Exercise 18.9 **3–disk unfactorized zeta cycle expansions.** *Check that the curvature expansion (18.2) for the 3-disk pinball, assuming no symmetries between disks, is given by*

$$\begin{aligned} 1/\zeta &= (1-z^2t_{12})(1-z^2t_{13})(1-z^2t_{23})(1-z^3t_{123})(1-z^3t_{132}) \\ &\quad (1-z^4t_{1213})(1-z^4t_{1232})(1-z^4t_{1323})(1-z^5t_{12123})\cdots \\ &= 1-z^2t_{12}-z^2t_{23}-z^2t_{31}-z^3t_{123}-z^3t_{132} \\ &\quad -z^4[(t_{1213}-t_{12}t_{13})+(t_{1232}-t_{12}t_{23})+(t_{1323}-t_{13}t_{23})] \\ &\quad -z^5[(t_{12123}-t_{12}t_{123})+\cdots]-\cdots \end{aligned} \tag{18.36}$$

The symmetrically arranged 3-disk pinball cycle expansion of the Euler product (18.2) (see table 13.4 and figure 22.2) is given by:

$$\begin{aligned}
1/\zeta &= (1-z^2t_{12})^3(1-z^3t_{123})^2(1-z^4t_{1213})^3 \\
&\quad (1-z^5t_{12123})^6(1-z^6t_{121213})^6(1-z^6t_{121323})^3\cdots \\
&= 1-3z^2\,t_{12}-2z^3\,t_{123}-3z^4\,(t_{1213}-t_{12}^2)-6z^5\,(t_{12123}-t_{12}t_{123}) \\
&\quad -z^6\,(6\,t_{121213}+3\,t_{121323}+t_{12}^3-9\,t_{12}t_{1213}-t_{123}^2) \\
&\quad -6z^7\,(t_{1212123}+t_{1212313}+t_{1213123}+t_{12}^2t_{123}-3\,t_{12}t_{12123}-t_{123}t_{1213}) \\
&\quad -3z^8\,(2\,t_{12121213}+t_{12121313}+2\,t_{12121323}+2\,t_{12123123} \\
&\qquad +2\,t_{12123213}+t_{12132123}+3\,t_{12}^2t_{1213}+t_{12}t_{123}^2 \\
&\qquad -6\,t_{12}t_{121213}-3\,t_{12}t_{121323}-4\,t_{123}t_{12123}-t_{1213}^2)-\cdots
\end{aligned} \tag{18.37}$$

Remark 18.8 Unsymmetrized cycle expansions. The above 3-disk cycle expansions might be useful for cross-checking purposes, but, as we shall see in chapter 22, they are not recommended for actual computations, as the factorized zeta functions yield much better convergence.

Exercise 18.10 4–disk unfactorized dynamical zeta function cycle expansions *For the symmetriclly arranged 4-disk pinball the symmetry group is C_{4v}, of order 8. The degenerate cycles can have multiplicities 2, 4 or 8 (see table 13.2):*

$$\begin{aligned}
1/\zeta &= (1-z^2t_{12})^4(1-z^2t_{13})^2(1-z^3t_{123})^8(1-z^4t_{1213})^8(1-z^4t_{1214})^4 \\
&\quad (1-z^4t_{1234})^2(1-z^4t_{1243})^4(1-z^5t_{12123})^8(1-z^5t_{12124})^8(1-z^5t_{12134})^8 \\
&\quad (1-z^5t_{12143})^8(1-z^5t_{12313})^8(1-z^5t_{12413})^8\cdots
\end{aligned} \tag{18.38}$$

and the cycle expansion is given by

$$\begin{aligned}
1/\zeta &= 1-z^2(4\,t_{12}+2\,t_{13})-8z^3\,t_{123} \\
&\quad -z^4(8\,t_{1213}+4\,t_{1214}+2\,t_{1234}+4\,t_{1243}-6\,t_{12}^2-t_{13}^2-8\,t_{12}t_{13}) \\
&\quad -8z^5(t_{12123}+t_{12124}+t_{12134}+t_{12143}+t_{12313}+t_{12413}-4\,t_{12}t_{123}-2\,t_{13}t_{123}) \\
&\quad -4z^6(2\,S_8+\,S_4+\,t_{12}^3+3\,t_{12}^2\,t_{13}+\,t_{12}t_{13}^2-8\,t_{12}t_{1213}-4\,t_{12}t_{1214} \\
&\quad -2\,t_{12}t_{1234}-4\,t_{12}t_{1243}-4\,t_{13}t_{1213}-2\,t_{13}t_{1214}-\,t_{13}t_{1234} \\
&\quad -2\,t_{13}t_{1243}-7\,t_{123}^2)-\cdots
\end{aligned} \tag{18.39}$$

where in the coefficient to z^6 the abbreviations S_8 and S_4 stand for the sums over the weights of the 12 orbits with multiplicity 8 and the 5 orbits of multiplicity 4, respectively; the orbits are listed in table 13.4.

add Freddy p. 25.

refer to the golden mean exercise 13.7

Exercise 18.11 Tail resummations. *A simple illustration of such tail resummation is the ζ function for the Ulam map (17.21) for which the cycle structure is exceptionally simple: the eigenvalue of the $x_0 = 0$ fixed point is 4, while the eigenvalue of any other n-cycle is $\pm 2^n$. Typical cycle weights used in thermodynamic averaging are $t_0 = 4^\tau z$, $t_1 = t = 2^\tau z$, $t_p = t^{n_p}$ for $p \neq 0$. The simplicity of the cycle eigenvalues enables us to evaluate the ζ function by a simple trick: we note that if the value of*

any n-cycle eigenvalue were t^n, (15.22) would yield $1/\zeta = 1-2t$. There is only one cycle, the x_0 fixed point, that has a different weight $(1-t_0)$, so we factor it out, multiply the rest by $(1-t)/(1-t)$, and obtain a rational ζ function

$$1/\zeta(z) \;=\; \frac{(1-2t)(1-t_0)}{(1-t)} \tag{18.40}$$

Consider how we would have detected the pole at $z = 1/t$ without the above trick. As the $\overline{0}$ fixed point is isolated in its stability, we would have kept the factor $(1-t_0)$ in (18.5) unexpanded, and noted that all curvature combinations in (18.5) which include the t_0 factor are unbalanced, so that the cycle expansion is an infinite series:

$$\prod_p (1-t_p) \;=\; (1-t_0)(1-t-t^2-t^3-t^4-\ldots) \tag{18.41}$$

(we shall return to such infinite series in chapter 21). The geometric series in the brackets sums up to (18.40). Had we expanded the $(1-t_0)$ factor, we would have noted that the ratio of the successive curvatures is exactly $c_{n+1}/c_n = t$; summing we would recover the rational ζ function (18.40).

⇓PRELIMINARY

Exercise 18.12 Cycle expansions of Fredholm determinants. *Use ref. [2.13] table - derive it here.*

⇑PRELIMINARY

paradise!this side of
three-disk@3-disk!escape rate
escape!rate, 3-disk

Chapter 19

Why cycle?

> "Progress was a labyrinth ... people plunging blindly in and then rushing wildly back, shouting that they had found it ... the invisible king the élan vital the principle of evolution ... writing a book, starting a war, founding a school...."
>
> F. Scott Fitzgerald, *This Side of Paradise*

In the preceding chapters we have moved rather briskly through the evolution operator formalism. Here we slow down in order to develop some fingertip feeling for the traces of evolution operators. We start out by explaining qualitatively how local exponential instability in topologically distinct trajectories leads to a global exponential instability.

⇓PRELIMINARY

⇑PRELIMINARY

this chapter makes no sense...

[1]

19.1 Escape rates

in the lecture this section was used as a verification that physically measurable escape rate is *the leading* eigenvalue

We start by verifying the claim (10.11) that for a nice hyperbolic flow the trace of the evolution operator grows exponentially with time. Consider again the game of pinball of figure 1.1. Designate by $\mathcal{M}$ a state space region that encloses the three disks, say the surface of the table $\times$ all pinball directions. The fraction of initial points whose trajectories start out within the state space region $\mathcal{M}$ and recur within that region at the time t is given by

$$\hat{\Gamma}_{\mathcal{M}}(t) = \frac{1}{|\mathcal{M}|}\int\int_{\mathcal{M}} dxdy\, \delta\big(y - f^t(x)\big)\ . \tag{19.1}$$

This quantity is eminently measurable and physically interesting in a variety of problems spanning nuclear physics to celestial mechanics. The integral

[1] Artuso: Roberto froze this text on Aug 31, 00: "I'll try to work a little on correlations in getused chapter ". Did he? Were they incorporated before Ronnie's rearrangements of Sep 20?

over x takes care of all possible initial pinballs; the integral over y checks whether they are still within $\mathcal{M}$ by the time t. If the dynamics is bounded, and $\mathcal{M}$ envelops the entire accessible state space, $\hat{\Gamma}_{\mathcal{M}}(t) = 1$ for all t. However, if trajectories exit $\mathcal{M}$ the recurrence fraction decreases with time. For example, any trajectory that falls off the pinball table in figure 1.1 is gone for good.

These observations can be made more concrete by examining the pinball phase space of figure 1.9. With each pinball bounce the initial conditions that survive get thinned out, each strip yielding two thiner strips within it. The total fraction of survivors (1.2) after n bounces is given by

$$\hat{\Gamma}_n = \frac{1}{|\mathcal{M}|} \sum_i^{(n)} |\mathcal{M}_i| \,, \tag{19.2}$$

where i is a binary label of the ith strip, and $|\mathcal{M}_i|$ is the area of the ith strip. The phase space volume is preserved by the flow, so the strips of survivors are contracted along the stable eigendirections, and ejected along the unstable eigendirections. As a crude estimate of the number of survivors in the ith strip, assume that the spreading of a ray of trajectories per bounce is given by a factor Λ, the mean value of the expanding eigenvalue of the corresponding fundamental matrix of the flow, and replace $|\mathcal{M}_i|$ by the phase space strip width estimate $|\mathcal{M}_i|/|\mathcal{M}| \sim 1/\Lambda_i$. This estimate of a size of a neighborhood (given already on p. 121) is right in spirit, but not without drawbacks. One problem is that in general the eigenvalues of a fundamental matrix for a finite segment of a trajectory have no invariant meaning; they depend on the choice of coordinates. However, we saw in chapter 14 that the sizes of neighborhoods are determined by stability eigenvalues of periodic points, and those are invariant under smooth coordinate transformations.

rewrite this sentence

In the approximation $\hat{\Gamma}_n$ receives 2^n contributions of equal size

$$\hat{\Gamma}_1 \sim \frac{1}{\Lambda} + \frac{1}{\Lambda} \,, \cdots \qquad , \hat{\Gamma}_n \sim \frac{2^n}{\Lambda^n} = e^{-n(\lambda - h)} = e^{-n\gamma} \,, \tag{19.3}$$

up to preexponential factors. We see here the interplay of the two key ingredients of chaos first alluded to in sect. 1.3.1: the escape rate γ equals local expansion rate (the Lyapunov exponent $\lambda = \ln \Lambda$), minus the rate of global reinjection back into the system (the topological entropy $h = \ln 2$). As we shall see in (20.16), with correctly defined "entropy" this result is exact.

As at each bounce one loses routinely the same fraction of trajectories, one expects the sum (19.2) to fall off exponentially with n. More precisely, by the hyperbolicity assumption of sect. 14.1.1 the expanding eigenvalue of the fundamental matrix of the flow is exponentially bounded from both above and below,

$$1 < |\Lambda_{min}| \leq |\Lambda(x)| \leq |\Lambda_{max}| \,, \tag{19.4}$$

and the area of each strip in (19.2) is bounded by $|\Lambda_{max}^{-n}| \leq |\mathcal{M}_i| \leq |\Lambda_{min}^{-n}|$. Replacing $|\mathcal{M}_i|$ in (19.2) by its over (under) estimates in terms of $|\Lambda_{max}|$, $|\Lambda_{min}|$ immediately leads to exponential bounds $(2/|\Lambda_{max}|)^n \leq \hat{\Gamma}_n \leq (2/|\Lambda_{min}|)^n$, i.e., escape!rate

$$\ln|\Lambda_{max}| - \ln 2 \geq -\frac{1}{n}\ln\hat{\Gamma}_n \geq \ln|\Lambda_{min}| - \ln 2\,. \tag{19.5}$$

The argument based on (19.5) establishes only that the sequence $\gamma_n = -\frac{1}{n}\ln\Gamma_n$ has a lower and an upper bound for any n. In order to prove that γ_n converge to the limit γ, we first show that for hyperbolic systems the sum over survivor intervals (19.2) can be replaced by the sum over periodic orbit stabilities. By (19.4) the size of $\mathcal{M}_i$ strip can be bounded by the stability Λ_i of ith periodic point:

$$\mathcal{C}_1\frac{1}{|\Lambda_i|} < \frac{|\mathcal{M}_i|}{|\mathcal{M}|} < \mathcal{C}_2\frac{1}{|\Lambda_i|}\,, \tag{19.6}$$

for any periodic point i of period n, with constants $\mathcal{C}_j$ dependent on the dynamical system but independent of n. The meaning of these bounds is that for longer and longer cycles in a system of bounded hyperbolicity, the shrinking of the ith strip is better and better approximated by the derivaties evaluated on the periodic point within the strip. Hence the survival probability can be bounded close to the cycle point stability sum make Per's proof into an exercise?

$$\hat{\mathcal{C}}_1\,\Gamma_n < \sum_i^{(n)}\frac{|\mathcal{M}_i|}{|\mathcal{M}|} < \hat{\mathcal{C}}_2\,\Gamma_n \quad, \tag{19.7}$$

where $\Gamma_n = \sum_i^{(n)} 1/|\Lambda_i|$ is the asymptotic trace sum (14.24). In this way we have established that for hyperbolic systems the survival probability sum (19.2) can be replaced by the periodic orbit sum (14.24). abrupt change to continuous time

We conclude that for hyperbolic, locally unstable flows the fraction (19.1) of initial x whose trajectories remain trapped within $\mathcal{M}$ up to time t is expected to decay exponentially,

$$\Gamma_{\mathcal{M}}(t) \;\propto\; e^{-\gamma t}\,,$$

where γ is the asymptotic *escape rate* defined by

$$\gamma = -\lim_{t\to\infty}\frac{1}{t}\ln\Gamma_{\mathcal{M}}(t)\,. \tag{19.8}$$

19.1 page 376
9.4 page 147

19.2 Natural measure in terms of periodic orbits

rewrite this sectio
in the lecture
section was use
interpret trace
"Euler" integral

We now refine the reasoning of sect. 19.1. Consider the trace (14.6) in the asymptotic limit (14.23):

$$\operatorname{tr}\mathcal{L}^n = \int dx\, \delta(x - f^n(x))\; e^{\beta A^n(x)} \approx \sum_i^{(n)} \frac{e^{\beta A^n(x_i)}}{|\Lambda_i|}\,.$$

The factor $1/|\Lambda_i|$ was interpreted in (19.2) as the area of the ith phase space strip. Hence $\operatorname{tr}\mathcal{L}^n$ is a discretization of the *integral* $\int dx e^{\beta A^n(x)}$ approximated by a tessellation into strips centered on periodic points x_i, figure 1.10, with the volume of the ith neighborhood given by estimate $|\mathcal{M}_i| \sim 1/|\Lambda_i|$, and $e^{\beta A^n(x)}$ estimated by $e^{\beta A^n(x_i)}$, its value at the ith periodic point. If the symbolic dynamics is a complete, any rectangle $[s_{-m}\cdots s_0.s_1 s_2\cdots s_n]$ of sect. 12.3.1 always contains the cycle point $\overline{s_{-m}\cdots s_0 s_1 s_2\cdots s_n}$; hence even though the periodic points are of measure zero (just like rationals in the unit interval), they are dense on the non–wandering set. Equiped with a measure for the associated rectangle, periodic orbits suffice to cover the entire non–wandering set. The average of $e^{\beta A^n}$ evaluated on the non–wandering set is therefore given by the trace, properly normalized so $\langle 1\rangle = 1$:

$$\left\langle e^{\beta A^n}\right\rangle_n \approx \frac{\sum_i^{(n)} e^{\beta A^n(x_i)}/|\Lambda_i|}{\sum_i^{(n)} 1/|\Lambda_i|} = \sum_i^{(n)} \mu_i\, e^{\beta A^n(x_i)}\,. \tag{19.9}$$

Here μ_i is the *normalized natural measure*

$$\sum_i^{(n)} \mu_i = 1\,, \qquad \mu_i = e^{n\gamma}/|\Lambda_i|\,, \tag{19.10}$$

correct both for the closed systems as well as the open systems of sect. 10.1.3.

Unlike brute numerical slicing of the integration space into an arbitrary lattice (for a critique, see sect. 10.4), the periodic orbit theory is smart, as it automatically partitions integrals by the intrinsic topology of the flow, and assigns to each tile the invariant natural measure μ_i.

19.2.1 Unstable periodic orbits are dense

(L. Rondoni and P. Cvitanović)

Our goal in sect. 10.1 was to evaluate the space and time averaged expectation value (10.9). An average over all periodic orbits can accomplish the job only if the periodic orbits fully explore the asymptotically accessible state space.

Why should the unstable periodic points end up being dense? The cycles are intuitively expected to be *dense* because on a connected chaotic set a typical trajectory is expected to behave ergodically, and pass infinitely many times arbitrarily close to any point on the set, including the initial point of the trajectory itself. The argument is more or less the following. Take a partition of $\mathcal{M}$ in arbitrarily small regions, and consider particles that start out in region $\mathcal{M}_i$, and return to it in n steps after some peregrination in state space. In particular, a particle might return a little to the left of its original position, while a close neighbor might return a little to the right of its original position. By assumption, the flow is continuous, so generically one expects to be able to gently move the initial point in such a way that the trajectory returns precisely to the initial point, i.e., one expects a periodic point of period n in cell i. As we diminish the size of regions $\mathcal{M}_i$, aiming a trajectory that returns to $\mathcal{M}_i$ becomes increasingly difficult. Therefore, we are guaranteed that unstable orbits of larger and larger period are densely interspersed in the asymptotic non–wandering set.

The above argument is heuristic, by no means guaranteed to work, and it must be checked for the particular system at hand. A variety of ergodic but insufficiently mixing counter-examples can be constructed - the most familiar being a quasiperiodic motion on a torus.

19.3 Flow conservation sum rules

If the dynamical system is bounded, all trajectories remain confined for all times, escape rate (19.8) vanishes $\gamma = -s_0 = 0$, and the leading eigenvalue of the Perron-Frobenius operator (9.10) is simply $\exp(-t\gamma) = 1$. Conservation of material flow thus implies that for bound flows cycle expansions of dynamical zeta functions and spectral determinants satisfy exact *flow conservation* sum rules:

ref to (K.2) is wrong, say someplace escape rate = leading eig

$$
\begin{aligned}
1/\zeta(0,0) &= 1 + \sum_{\pi}{}' \frac{(-1)^k}{|\Lambda_{p_1}\cdots\Lambda_{p_k}|} = 0 \\
F(0,0) &= 1 - \sum_{n=1}^{\infty} c_n(0,0) = 0
\end{aligned}
\tag{19.11}
$$

obtained by setting $s = 0$ in (18.12), (18.13) cycle weights $t_p = e^{-sT_p}/|\Lambda_p| \to 1/|\Lambda_p|$. These sum rules depend neither on the cycle periods T_p nor on the observable $a(x)$ under investigation, but only on the cycle stabilities $\Lambda_{p,1}$, $\Lambda_{p,2}$, $\cdots$, $\Lambda_{p,d}$, and their significance is purely geometric: they are a measure of how well periodic orbits tessellate the state space. Conservation of material flow provides the first and very useful test of the quality of finite cycle length truncations, and is something that you should always check first when constructing a cycle expansion for a bounded flow.

The trace formula version of the flow conservation flow sum rule comes

correlation!time

in two varieties, one for the maps, and another for the flows. By flow conservation the leading eigenvalue is $s_0 = 0$, and for maps (18.11) yields

$$\mathrm{tr}\,\mathcal{L}^n = \sum_{i\in \mathrm{Fix} f^n} \frac{1}{|\det\,(\mathbf{1} - M^n(x_i))\,|} = 1 + e^{s_1 n} + \dots\ . \tag{19.12}$$

restore the discussion

For flows one can apply this rule by grouping together cycles from $t = T$ to $t = T + \Delta T$

$$\begin{aligned}\frac{1}{\Delta T}\sum_{p,r}^{T\le rT_p\le T+\Delta T} \frac{T_p}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|} &= \frac{1}{\Delta T}\int_T^{T+\Delta T} dt\left(1 + e^{s_1 t} + \dots\right) \\ &= 1 + \frac{1}{\Delta T}\sum_{\alpha=1}^{\infty}\frac{e^{s_\alpha T}}{s_\alpha}\left(e^{s_\alpha \Delta T} - 1\right) \approx 1 + e^{s_1 T}\,. \end{aligned} \tag{19.13}$$

As is usual for the the fixed level trace sums, the convergence of (19.12) is controled by the gap between the leading and the next-to-leading eigenvalues of the evolution operator.

19.4 Correlation functions

The *time correlation function* $C_{AB}(t)$ of two observables A and B along the trajectory $x(t) = f^t(x_0)$ is defined as

$$C_{AB}(t;x_0) \;=\; \lim_{T\to\infty}\frac{1}{T}\int_0^T d\tau A(x(\tau+t))B(x(\tau))\,, \qquad x_0 = x(0)\,. \tag{19.14}$$

If the system is ergodic, with invariant continuous measure $\rho_0(x)dx$, then correlation functions do not depend on x_0 (apart from a set of zero measure), and may be computed by a phase average as well

$$C_{AB}(t) \;=\; \int_{\mathcal{M}} dx_0\,\rho_0(x_0)A(f^t(x_0))B(x_0)\,. \tag{19.15}$$

For a chaotic system we expect that time evolution will loose the information contained in the initial conditions, so that $C_{AB}(t)$ will approach the *uncorrelated* limit $\langle A\rangle\cdot\langle B\rangle$. As a matter of fact the asymptotic decay of correlation functions

$$\hat{C}_{AB} := C_{AB} - \langle A\rangle\,\langle B\rangle \tag{19.16}$$

for any pair of observables coincides with the definition of *mixing*, a fundamental property in ergodic theory. We now assume $\langle B\rangle = 0$ (otherwise we may

define a new observable by $B(x) - \langle B \rangle$). Our purpose is now to connect the asymptotic behavior of correlation functions with the spectrum of the Perron-Frobenius operator $\mathcal{L}$. We can write (19.15) as

$$\tilde{C}_{AB}(t) = \int_{\mathcal{M}} dx \int_{\mathcal{M}} dy\, A(y)B(x)\rho_0(x)\delta(y - f^t(x))\,,$$

and recover the evolution operator

$$\tilde{C}_{AB}(t) = \int_{\mathcal{M}} dx \int_{\mathcal{M}} dy\, A(y)\mathcal{L}^t(y,x)B(x)\rho_0(x)$$

We recall that in sect. 9.1 we showed that $\rho(x)$ is the eigenvector of $\mathcal{L}$ corresponding to probability conservation

$$\int_{\mathcal{M}} dy\, \mathcal{L}^t(x,y)\rho(y) = \rho(x)\,.$$

Now, we can expand the x dependent part in terms of the eigenbasis of $\mathcal{L}$:

$$B(x)\rho_0(x) = \sum_{\alpha=0}^{\infty} c_\alpha \rho_\alpha(x)\,,$$

where $\rho_0(x)$ is the natural measure. Since the average of the left hand side is zero the coefficient c_0 must vanish. The action of $\mathcal{L}$ then can be written as

$$\tilde{C}_{AB}(t) = \sum_{\alpha\neq 0} e^{-s_\alpha t} c_\alpha \int_{\mathcal{M}} dy\, A(y)\rho_\alpha(y). \tag{19.17}$$

19.2
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We see immediately that if the spectrum has a *gap*, i.e., if the second largest leading eigenvalue is isolated from the largest eigenvalue ($s_0 = 0$) then (19.17) implies *exponential* decay of correlations

$$\tilde{C}_{AB}(t) \sim e^{-\nu t}\,.$$

The correlation decay rate $\nu = s_1$ then depends only on intrinsic properties of the dynamical system (the position of the next-to-leading eigenvalue of the Perron-Frobenius operator), while the choice of a particular observable influences only the prefactor.

Correlation functions are often accessible from time series measurable in laboratory experiments and numerical simulations: moreover they are linked to transport exponents.

⇓PRELIMINARY

19.5 The most chaotic of billiards?

Suppose you are an experimentalist, you can build a chaotic billiard, and in order to explore consequences of chaos, you would like to build the *most* chaotic of billiards. What to do?

Roberto Artuso, 6 Apr 1998

ahem: for overlapping 4 disc (and other geometry billiards Lyapunov exp + mean field should be in Benettin Physica D13, 211 ('84)), i think that the overall behavior should not be much different than the expression for shortest periodic orbit (at least i think i saw that in ancient times).

maybe also quantum stadium and rough billiards could be of interest? (prl 77, 4744 (96) 78, 1440 (97)?

Sune F Nielsen, 6 Apr 1998

Lyapunov is zero in the integrable case (triangular/square billiard) and decreases in the other end due to intermittency, so there is a maximum. I expect the Lyapunov is highest near the ratio of a:R=1.9 although I have not made any scans.

Stephanie Reimann, 6 Apr 1998

How about just a simple Sinai? This has the advantage that the square is done already, and he can compare his results to M.'s. Perhaps one could design a sample where the inner ring could be turned on gradually, to see the transition.

Niall Whelan, 3 Apr 1998

What exactly are they proposing to measure? If it is something like a magnetic susceptibility, I am not sure that the Lyapunov exponent is really all that relevant a consideration. In that event it is rather the distinction between having families of orbits or isolated orbits which is crucial. If they are measuring conductances, I think a similar statement applies. In that event ergodicity is important since you are interested in the transport properties across the device and I don't think the Lyapunov exponent is of direct importance.

I would think that a more relevant concern would be structural stability of the system. The confining potential is typically applied to gate voltages which sit above the electron gas by a fair amount. I have forgotten the

exact numbers but presumably experimentalists know. The point is that whatever sharply defined potential you think you are working with on the gate level is considerably softened on the electron gas level. So you want something which maintains the generic features of chaotic systems under such a "perturbation". A stadium whose straight bits were too short, for example, would presumably be bad since it might look like a more or less integrable oval or ellipse down below. The shapes you suggest are probably safe in this regard. I know that for the cardioid, you can lop off the cuspy bit and it is still chaotic, for example. Except it may end up looking like a normal circle down below.

No one has ever done this for mesoscopic systems that I know of, but a Sinai-type billiard is presumably quite robust in its chaotic properties, except that it has the bouncing ball modes. I suppose there may be engineering problems associated with trying to leave holes behind or putting tiny circular gates in the middle of the device. But an eighth Sinai billiard (its fundamental domain) might not be a bad choice if the presence of the bb modes is not a concern.

⇑PRELIMINARY

19.6 Trace formulas *vs.* level sums

Trace formulas (14.9) and (14.21) diverge precisely where one would like to use them, at s equal to eigenvalues s_α. Instead, one can proceed as follows; according to (14.25) the "level" sums (all symbol strings of length n) are asymptotically going like $e^{s_0 n}$

$$\sum_{i\in \mathrm{Fix} f^n} \frac{e^{\beta A^n(x_i)}}{|\Lambda_i|} \to e^{s_0 n}\,,$$

so an nth order estimate $s_{(n)}$ of the leading eigenvalue is given by

$$1 = \sum_{i\in \mathrm{Fix} f^n} \frac{e^{\beta A^n(x_i)} e^{-s_{(n)} n}}{|\Lambda_i|} \tag{19.18}$$

which generates a "normalized measure". The difficulty with estimating this $n \to \infty$ limit is at least twofold:

1. due to the exponential growth in number of intervals, and the exponential decrease in attainable accuracy, the maximal n attainable experimentally or numerically is in practice of order of something between 5 to 20.

2. the preasymptotic sequence of finite estimates $s_{(n)}$ is not unique, because the sums Γ_n depend on how we define the escape region, and because in general the areas $\mathcal{M}_i$ in the sum (19.2) should be weighted by the density of initial conditions x_0. For example, an overall measuring unit

rescaling $\mathcal{M}_i \to \alpha\mathcal{M}_i$ introduces $1/n$ corrections in $s_{(n)}$ defined by the log of the sum (19.8): $s_{(n)} \to s_{(n)} - \ln\alpha/n$. This can be partially fixed by defining a level average

$$\left\langle e^{\beta A(s)} \right\rangle_{(n)} := \sum_{i\in \mathrm{Fix} f^n} \frac{e^{\beta A^n(x_i)} e^{sn}}{|\Lambda_i|} \tag{19.19}$$

and requiring that the ratios of successive levels satisfy

$$1 = \frac{\left\langle e^{\beta A(s_{(n)})} \right\rangle_{(n+1)}}{\left\langle e^{\beta A(s_{(n)})} \right\rangle_{(n)}} .$$

This avoids the worst problem with the formula (19.18), the inevitable $1/n$ corrections due to its lack of rescaling invariance. However, even though much published pondering of "chaos" relies on it, there is no need for such gymnastics: the dynamical zeta functions and spectral determinants are already invariant not only under linear rescalings, but under *all* smooth nonlinear conjugacies $x \to h(x)$, and require no $n \to \infty$ extrapolations to asymptotic times. Comparing with the cycle expansions (18.5) we see what the difference is; while in the level sum approach we keep increasing exponentially the number of terms with no reference to the fact that most are already known from shorter estimates, in the cycle expansions short terms dominate, longer ones enter only as exponentially small corrections.

The beauty of the trace formulas is that they are coordinatization independent: both $\left|\det\left(\mathbf{1}-M_p\right)\right| = \left|\det\left(\mathbf{1}-M^{T_p}(x)\right)\right|$ and $e^{\beta A_p} = e^{\beta A^{T_p}(x)}$ contribution to the cycle weight t_p are independent of the starting periodic point point x. For the fundamental matrix M_p this follows from the chain rule for derivatives, and for $e^{\beta A_p}$ from the fact that the integral over $e^{\beta A^t(x)}$ is evaluated along a closed loop. In addition, $\left|\det\left(\mathbf{1}-M_p\right)\right|$ is invariant under smooth coordinate transformations.

⇓PRELIMINARY

19.6.1 Equipartition measures

> Benoit B. Mandelbrot: "I would be perfectly happy being Kepler" [to a coming fractals' Newton]. Referring to the broad array of things now described by fractals, he added, "I have been Kepler many times over."
> J. Gleick, New York Times science section, Jan. 22 1985

There exist many strange sets which cannot be partitioned by the topology of a dynamical flow: some well known examples are the Mandelbrot set, the period doubling repeller and the probabilistically generated

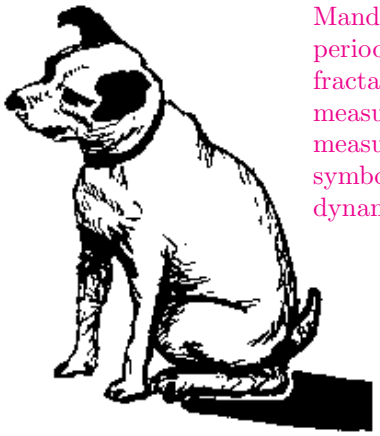

Figure 19.1: Johannes Kepler contemplating the bust of Mandelbrot, after Rembrandt's "Aristotle contemplating the bust of Homer" (Metropolitan Museum, New York).

Mandelbrot set
period!doubling!repeller
fractal!aggregates
measure!equipartition
measure!cylinder
symbolic!dynamics
dynamics!symbolic

fractal aggregates. In such cases the choice of measure is wide open. One easy choice is the *equipartition* or *cylinder* measure: given a symbolic dynamics partition, weigh all symbol sequences of length n equally. Given a symbolic dynamics, the equipartition measure is easy to implement: the rate of growth of the number of admissible symbol sequences K_n with the sequence length n is given by the topological entropy h (discussed in sect. 13.1) and the equipartition measure for the ith region $\mathcal{M}_i$ is simply

forward reference sect. 13.1?

$$\Delta\mu_i = 1/K_n \to e^{-nh} \,. \tag{19.20}$$

The problem with the equipartition measure is twofold: it usually has no physical basis, and it is not an intrinsic invariant property of the strange set, as it depends on the choice of a partition. One is by no means forced to use either the natural or the equipartition measure; there is a variety of other choices, depending on the problem. Also the stability eigenvalues Λ_i need not refer to motion in the dynamical space; in more general settings it can be a renormalization scaling function(sect. 30.2), or even a scaling function describing a non–wandering set in the parameter space (sect. 28.3).

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Commentary

Remark 19.1 Nonhyperbolic measures. $\mu_i = 1/|\Lambda_i|$ is the natural measure only for the strictly hyperbolic systems. For non-hyperbolic systems, the measure might develop cusps. For example, for Ulam type maps (unimodal maps with quadratic critical point mapped onto the "left" unstable fixed point x_0, discussed in more detail in chapter 21), the measure develops a square-root singularity on the $\overline{0}$ cycle:

$$\mu_0 = \frac{1}{|\Lambda_0|^{1/2}} \,. \tag{19.21}$$

The thermodynamics averages are still expected to converge in the "hyperbolic" phase where the positive entropy of unstable orbits dominates over the marginal orbits, but they fail in the "non-hyperbolic" phase. The general case remains unclear [12.22, I.18, I.14, I.11].

Remark 19.2 Trace formula periodic orbit averaging. The cycle averaging formulas are not the first thing that one would intuitively write down; the approximate trace formulas are more accessibly heuristically. The trace formula averaging (19.13) seems to have be discussed for the first time by Hannay and Ozorio de Almeida [I.1, 7.11]. Another novelty of the cycle averaging formulas and one of their main virtues, in contrast to the explicit analytic results such as those of ref. [18.4], is that their evaluation *does not* require any explicit construction of the (coordinate dependent) eigenfunctions of the Perron-Frobenius operator (i.e., the natural measure ρ_0).

Remark 19.3 Role of noise in dynamical systems. In any physical application the dynamics is always accompanied by additional external noise. The noise can be characterized by its strength σ and distribution. Lyapunov exponents, correlation decay and dynamo rate can be defined in this case the same way as in the deterministic case. You might fear that noise completely destroys the results derived here. However, one can show that the deterministic formulas remain valid to accuracy comparable with noise width if the noise level is small. A small level of noise even helps as it makes the dynamics more ergodic, with deterministically non-communicating parts of the state space now weakly connected due to the noise, making periodic orbit theory applicable to non-ergodic systems. For small amplitude noise one can expand

$$\overline{a} = \overline{a}_0 + \overline{a_1}\sigma^2 + \overline{a_2}\sigma^4 + \dots ,$$

around the deterministic averages a_0. The expansion coefficients $\overline{a_1}, \overline{a_2}, \dots$ can also be expressed via periodic orbit formulas. The calculation of these coefficients is one of the challenges facing periodic orbit theory, discussed in refs. [9.5, 9.6, 9.7].

Résumé

expand this to cover the chapter

We conclude this chapter by a general comment on the relation of the finite trace sums such as (19.2) to the spectral determinants and dynamical zeta functions. One might be tempted to believe that given a deterministic rule, a sum like (19.2) could be evaluated to any desired precision. For short finite times this is indeed true: every region $\mathcal{M}_i$ in (19.2) can be accurately delineated, and there is no need for fancy theory. However, if the dynamics is unstable, local variations in initial conditions grow exponentially and in finite time attain the size of the system. The difficulty with estimating the $n \to \infty$ limit from (19.2) is then at least twofold:

repat of the text already used, fix!

1. due to the exponential growth in number of intervals, and the exponential decrease in attainable accuracy, the maximal n attainable experimentally or numerically is in practice of order of something between 5 to 20;

2. the preasymptotic sequence of finite estimates γ_n is not unique, because the sums $\hat{\Gamma}_n$ depend on how we define the escape region, and because in general the areas $|\mathcal{M}_i|$ in the sum (19.2) should be weighted by the density of initial x_0.

In contrast, the dynamical zeta functions and spectral determinants are invariant under *all* smooth nonlinear conjugacies $x \to h(x)$, not only linear rescalings, and require no $n \to \infty$ extrapolations.

References

[19.1] F. Christiansen, G. Paladin and H.H. Rugh, *Phys. Rev. Lett.* **65**, 2087 (1990).

Exercises

Exercise 19.1 **Escape rate of the logistic map.**

(a) *Calculate the fraction of trajectories remaining trapped in the interval* $[0,1]$ *for the logistic map*

$$f(x) = a(1-4(x-0.5)^2), \tag{19.22}$$

and determine the a *dependence of the escape rate* $\gamma(a)$ *numerically.*

(b) *Work out a numerical method for calculating the lengths of intervals of trajectories remaining stuck for* n *iterations of the map.*

(c) *What is your expectation about the* a *dependence near the critical value* $a_c = 1$*?*

Exercise 19.2 **Four scale map decay.** *Compute the second largest eigenvalue of the Perron-Frobenius operator for the four scale map*

$$f(x) = \begin{cases} a_1 x & \text{if } \ 0 < x < b/a_1, \\ (1-b)((x-b/a_1)/(b-b/a_1)) + b & \text{if } \ b/a_1 < x < b, \\ a_2(x-b) & \text{if } \ b < x < b + b/a_2, \\ (1-b)((x-b-b/a_2)/(1-b-b/a_2)) + b & \text{if } \ b + b/a_2 < x < 1. \end{cases} \tag{19.23}$$

Exercise 19.3 **Lyapunov exponents for 1-dimensional maps.** *Extend your cycle expansion programs so that the first and the second moments of observables can be computed. Use it to compute the Lyapunov exponent for some or all of the following maps:*

(a) the piecewise-linear flow conserving map, the skew tent map

$$f(x) = \begin{cases} ax & \text{if } \ 0 \le x \le a^{-1}, \\ \frac{a}{a-1}(1-x) & \text{if } \ a^{-1} \le x \le 1. \end{cases}$$

(b) the Ulam map $f(x) = 4x(1-x)$

(c) the skew Ulam map

$$f(x) = 0.1218x(1-x)(1-0.6x)$$

with a peak at 0.7*.*

(d) the repeller of $f(x) = Ax(1-x)$*, for either* $A = 9/2$ *or* $A = 6$ *(this is a continuation of exercise 18.2).*

(e) *for the 2-branch flow conserving map*

$$f_0(x) = \frac{h-p+\sqrt{(h-p)^2+4hx}}{2h}, \qquad x \in [0,p] \tag{19.24}$$
$$f_1(x) = \frac{h+p-1+\sqrt{(h+p-1)^2+4h(x-p)}}{2h}, \qquad x \in [p,1]$$

This is a nonlinear perturbation of ($h = 0$) Bernoulli map (16.6); the first 15 eigenvalues of the Perron-Frobenius operator are listed in ref. [19.1] for $p = 0.8$, $h = 0.1$. Use these parameter values when computing the Lyapunov exponent.

Cases (a) and (b) can be computed analytically; cases (c), (d) and (e) require numerical computation of cycle stabilities. Just to see whether the theory is worth the trouble, also cross check your cycle expansions results for cases (c) and (d) with Lyapunov exponent computed by direct numerical averaging along trajectories of randomly chosen initial points:

(f) *trajectory-trajectory separation (10.27) (hint: rescale δx every so often, to avoid numerical overflows),*

(g) *iterated stability (10.31).*

How good is the numerical accuracy compared with the periodic orbit theory predictions?

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Exercise 19.4 Sequence of orbital measures for the Ulam map. *(L. Rondoni)*

As an example, let us develop the sequence of orbital measures for the quadratic map, even if it is not precisely of the form we discussed above. Indeed, this illustrates how some of our assumptions could be relaxed, and still get correct results. Moreover, although for such a simple case it is possible to explicitly find all periodic points up to the desired period, we will only use those that are found by looking at close returns of a generic chaotic trajectory. Consider the Ulam map

$$x_{n+1} = 4x_n(1-x_n) \ . \tag{19.25}$$

This is quadratic, and it has two fixed points, four period two periodic points (including the two repeats of the fixed points) and, in general, 2^n periodic points of period n, as explained in exercise 11.6. How these are structured is given in table 13.1.

make exercise, draw the figure

n	$\langle x \rangle$	$\langle x^2 \rangle$	$\sum N_k$
2	0.62500	0.46875	2
3	0.54167	0.40625	6
4	0.50000	0.37500	12
5	0.50833	0.38125	30
6	0.50354	0.37766	52(54)
7	0.51240	0.38430	120(126)
8	0.51017	0.38263	230(240)
9	0.50914	0.38185	482(504)
10	0.50904	0.38178	940(990)
11	0.50795	0.38097	1962(2046)
12	0.50731	0.38049	3853(4020)
exact	0.5	0.375	

Table 19.1: An approximate cycle expansion using only periodic points of prime orbits, and only those found by a probabilistic exploration of the state space. (In the parenthesis the correct total number of such periodic points, see table 13.2).

The natural measure of the Ulam map (19.25), figure B.2b is explicitly expressed by

need to derive somewhere, refe Ulam

$$d\mu(x) = \frac{dx}{\pi\sqrt{x - x^2}} \ . \tag{19.26}$$

this explicitely shows that periodic points do not have equal weights? But smooth conjugacy to tent map justifies it.

We can now calculate the average values of $\langle x \rangle$ and of $\langle x^2 \rangle$ exactly using the known analytic distribution function (19.26), and also numerically using the cycle expansion. This exercise illustrates well the use of orbital measures to calculate averages. The results are given in table 19.1, where we have used the periodic points of order 2 to 12 which are found by a generic chaotic trajectory, and we have neglected the cycles of a given period which are repeats of cycles of smaller period. This because the contributions of non prime (repeat) cycles become less and less important as the period grows, as one can see also from table 13.1. Notice also, that the number of cycles found numerically differs from the exact number (given in parenthesis for period 6 and larger). All this mimics what actually can be done, when sampling periodic points from the state space of more realistic systems. Indeed, not being able to find all the existing cycles is a common numerical difficulty, linked to their instability. That can easily be remedied for the quadratic map, but in general one does not even know how many cycles of a certain period should be found. Notice the convergence of the method, despite these difficulties.

terrible; compare with the AACII calculation

yes, I note no convergence whatsoever

We have introduced the idea of periodic orbit expansions for simple maps, as an example of a theory of much wider generality. The reason why the method is expected to work in more general situations, is that the relevant attractors can be proven, or have been observed to be hierarchically approximated through sets of progressively longer and longer unstable periodic orbits. Indeed, the weak limit of these measures has been rigorously proved to be the natural measure for Axiom-A systems, independently of the (finite) dimensions of the relevant state spaces.

plot $\log_{10}$ error from table 19.1, numbers are useless. Also plot the error of the correct cycle computation

Remark 19.4 Nonhyperbolicity. In the above example we have succeeded in developing a sequence of orbital measures for the Ulam map, even though the Ulam

map does not satisfy the hyperbolicity hypothesis (the critical point is superstable) invoked in the derivation. This illustrates how some of our assumptions could be relaxed, and still get sensible results. To solve this example *exactly* the cyclist way, consult refsect.?!

include the text from ref. [18.3]

(L. Rondoni)

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Chapter 20

Thermodynamic formalism

(G. Vattay)

In the preceding chapters we characterized chaotic systems via global quantities such as averages. It turned out that these are closely related to very fine details of the dynamics like stabilities and time periods of individual periodic orbits. In statistical mechanics a similar duality exists. Macroscopic systems are characterized with thermodynamic quantities (pressure, temperature and chemical potential) which are averages over fine details of the system called microstates. One of the greatest achievements of the theory of dynamical systems was when in the sixties and seventies Bowen, Ruelle and Sinai made the analogy between these two subjects explicit. Later this "Thermodynamic Formalism" of dynamical systems became widely used when the concept of fractals and multifractals has been introduced. The formalism made it possible to calculate various fractal dimensions in an elegant way and become a standard instrument in a wide range of scientific fields. Next we sketch the main ideas of this theory and show how periodic orbit theory helps to carry out calculations.

20.1 Rényi entropies

As we have already seen trajectories in a dynamical system can be characterized by their symbolic sequences from a generating Markov partition. We can locate the set of starting points $\mathcal{M}_{s_1 s_2 \ldots s_n}$ of trajectories whose symbol sequence starts with a given set of n symbols $s_1 s_2 \ldots s_n$. We can associate many different quantities to these sets. There are geometric measures such as the volume $V(s_1 s_2 \ldots s_n)$, the area $A(s_1 s_2 \ldots s_n)$ or the length $l(s_1 s_2 \ldots s_n)$ of this set. Or in general we can have some measure $\mu(\mathcal{M}_{s_1 s_2 \ldots s_n}) = \mu(s_1 s_2 \ldots s_n)$ of this set. As we have seen in (19.10) the most important is the natural measure, which is the probability that a non-periodic trajectory visits the set $\mu(s_1 s_2 \ldots s_n) = P(s_1 s_2 \ldots s_n)$. The natural measure is additive. Summed up for all possible symbol sequences of length n it gives the

measure of the whole state space:

$$\sum_{s_1 s_2 \ldots s_n} \mu(s_1 s_2 \ldots s_n) = 1 \tag{20.1}$$

expresses probability conservation. Also, summing up for the last symbol we get the measure of a one step shorter sequence

$$\sum_{s_n} \mu(s_1 s_2 \ldots s_n) = \mu(s_1 s_2 \ldots s_{n-1}).$$

As we increase the length (n) of the sequence the measure associated with it decreases typically with an exponential rate. It is then useful to introduce the exponents

$$\lambda(s_1 s_2 \ldots s_n) = -\frac{1}{n} \log \mu(s_1 s_2 \ldots s_n). \tag{20.2}$$

To get full information on the distribution of the natural measure in the symbolic space we can study the distribution of exponents. Let the number of symbol sequences of length n with exponents between λ and $\lambda + d\lambda$ be given by $N_n(\lambda)d\lambda$. For large n the number of such sequences increases exponentially. The rate of this exponential growth can be characterized by $g(\lambda)$ such that

$$N_n(\lambda) \sim \exp(n g(\lambda)).$$

The knowledge of the distribution $N_n(\lambda)$ or its essential part $g(\lambda)$ fully characterizes the microscopic structure of our dynamical system.

As a natural next step we would like to calculate this distribution. However it is very time consuming to calculate the distribution directly by making statistics for millions of symbolic sequences. Instead, we introduce auxiliary quantities which are easier to calculate and to handle. These are called partition sums

$$Z_n(\beta) = \sum_{s_1 s_2 \ldots s_n} \mu^\beta(s_1 s_2 \ldots s_n), \tag{20.3}$$

as they are obviously motivated by Gibbs type partition sums of statistical mechanics. The parameter β plays the role of inverse temperature $1/k_B T$ and $E(s_1 s_2 \ldots s_n) = -\log \mu(s1 s_2 \ldots s_n)$ is the energy associated with the microstate labeled by $s_1 s_2 \ldots s_n$ We are tempted also to introduce something analogous with the Free energy. In dynamical systems this is called the Rényi entropy [I.5] defined by the growth rate of the partition sum

$$K_\beta = \lim_{n\to\infty} \frac{1}{n} \frac{1}{1-\beta} \log \left(\sum_{s_1 s_2 \ldots s_n} \mu^\beta(s_1 s_2 \ldots s_n) \right). \tag{20.4}$$

In the special case $\beta \to 1$ we get Kolmogorov's entropy

$$K_1 = \lim_{n\to\infty} \frac{1}{n} \sum_{s_1 s_2 \ldots s_n} -\mu(s_1 s_2 \ldots s_n) \log \mu(s_1 s_2 \ldots s_n),$$

while for $\beta = 0$ we recover the topological entropy

$$h_{top} = K_0 = \lim_{n\to\infty} \frac{1}{n} \log N(n),$$

where $N(n)$ is the number of existing length n sequences. To connect the partition sums with the distribution of the exponents, we can write them as averages over the exponents

$$Z_n(\beta) = \int d\lambda N_n(\lambda) \exp(-n\lambda\beta),$$

where we used the definition (20.2). For large n we can replace $N_n(\lambda)$ with its asymptotic form

$$Z_n(\beta) \sim \int d\lambda \exp(ng(\lambda)) \exp(-n\lambda\beta).$$

For large n this integral is dominated by contributions from those λ^* which maximize the exponent

$$g(\lambda) - \lambda\beta.$$

The exponent is maximal when the derivative of the exponent vanishes

$$g'(\lambda^*) = \beta. \tag{20.5}$$

From this equation we can determine $\lambda^*(\beta)$. Finally the partition sum is

$$Z_n(\beta) \sim \exp(n[g(\lambda^*(\beta)) - \lambda^*(\beta)\beta]).$$

Using the definition (20.4) we can now connect the Rényi entropies and $g(\lambda)$

$$(\beta - 1)K_\beta = \lambda^*(\beta)\beta - g(\lambda^*(\beta)). \tag{20.6}$$

Equations (20.5) and (20.6) define the Legendre transform of $g(\lambda)$. This equation is analogous with the thermodynamic equation connecting the entropy and the free energy. As we know from thermodynamics we can invert the Legendre transform. In our case we can express $g(\lambda)$ from the Rényi entropies via the Legendre transformation

$$g(\lambda) = \lambda\beta^*(\lambda) - (\beta^*(\lambda) - 1)K_{\beta^*(\lambda)}, \tag{20.7}$$

where now $\beta^*(\lambda)$ can be determined from

$$\frac{d}{d\beta^*}[(\beta^* - 1)K_{\beta^*}] = \lambda. \tag{20.8}$$

Obviously, if we can determine the Rényi entropies we can recover the distribution of probabilities from (20.7) and (20.8).

The periodic orbit calculation of the Rényi entropies can be carried out by approximating the natural measure corresponding to a symbol sequence by the expression (19.10)

$$\mu(s_1,...,s_n) \approx \frac{e^{n\gamma}}{|\Lambda_{\overline{s_1 s_2 ... s_n}}|}. \tag{20.9}$$

The partition sum (20.3) now reads

$$Z_n(\beta) \approx \sum_i \frac{e^{n\beta\gamma}}{|\Lambda_i|^\beta}, \tag{20.10}$$

where the summation goes for periodic orbits of length n. We can define the characteristic function

$$\Omega(z,\beta) = \exp\left(-\sum_n \frac{z^n}{n} Z_n(\beta)\right). \tag{20.11}$$

According to (20.4) for large n the partition sum behaves as

$$Z_n(\beta) \sim e^{-n(\beta-1)K_\beta}. \tag{20.12}$$

Substituting this into (20.11) we can see that the leading zero of the characteristic function is

$$z_0(\beta) = e^{(\beta-1)K_\beta}.$$

On the other hand substituting the periodic orbit approximation (20.10) into (20.11) and introducing prime and repeated periodic orbits as usual we get

$$\Omega(z,\beta) = \exp\left(-\sum_{p,r} \frac{z^{n_p r} e^{\beta\gamma n_p r}}{r|\Lambda_p^r|^\beta}\right).$$

We can see that the characteristic function is the same as the zeta function we introduced for Lyapunov exponents (I.14) except we have $ze^{\beta\gamma}$ instead of z. Then we can conclude that the Rényi entropies can be expressed with the pressure function directly as

$$P(\beta) = (\beta-1)K_\beta + \beta\gamma, \tag{20.13}$$

since the leading zero of the zeta function is the pressure. The Rényi entropies K_β, hence the distribution of the exponents $g(\lambda)$ as well, can be calculated via finding the leading eigenvalue of the operator (I.4).

boredom

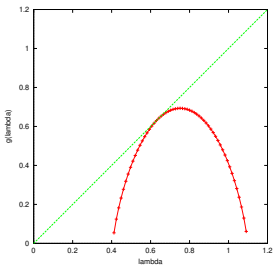

Figure 20.1:

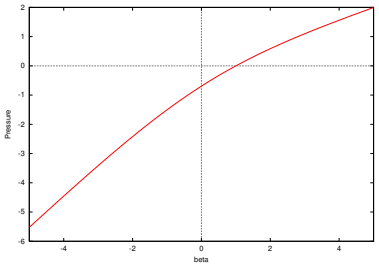

Figure 20.2: $g(\lambda)$ and $P(\beta)$ for the map of exercise 20.4 at $a = 3$ and $b = 3/2$. See solution R for details.

From (20.13) we can get all the important quantities of the thermodynamic formalism. For $\beta = 0$ we get the topological entropy

$$P(0) = -K_0 = -h_{top}. \tag{20.14}$$

For $\beta = 1$ we get the escape rate

$$P(1) = \gamma. \tag{20.15}$$

Taking the derivative of (20.13) in $\beta = 1$ we get Pesin's formula [I.2] connecting Kolmogorov's entropy and the Lyapunov exponent

$$P'(1) = \overline{\lambda} = K_1 + \gamma. \tag{20.16}$$

20.1
page 391

It is important to note that, as always, these formulas are strictly valid for nice hyperbolic systems only. At the end of this Chapter we discuss the important problems we are facing in non-hyperbolic cases.

On figure 20.2 we show a typical pressure and $g(\lambda)$ curve computed for the two scale tent map of Exercise 20.4. We have to mention, that all typical hyperbolic dynamical system produces a similar parabola like curve. Although this is somewhat boring we can interpret it like a sign of a high level of universality: The exponents λ have a sharp distribution around the most probable value. The most probable value is $\lambda = P'(0)$ and $g(\lambda) = h_{top}$ is the topological entropy. The average value in closed systems is where $g(\lambda)$ touches the diagonal: $\overline{\lambda} = g(\overline{\lambda})$ and $1 = g'(\overline{\lambda})$.

Next, we are looking at the distribution of trajectories in real space.

three-disk@3-disk!state space
three-disk@3-disk!fractal dimension
fractal

20.2 Fractal dimensions

By looking at the repeller we can recognize an interesting spatial structure. In the 3-disk case the starting points of trajectories not leaving the system after the first bounce form two strips. Then these strips are subdivided into an infinite hierarchy of substrings as we follow trajectories which do not leave the system after more and more bounces. The finer strips are similar to strips on a larger scale. Objects with such self similar properties are called *fractals*.

We can characterize fractals via their local scaling properties. The first step is to draw a uniform grid on the surface of section. We can look at various measures in the square boxes of the grid. The most interesting measure is again the natural measure located in the box. By decreasing the size of the grid ϵ the measure in a given box will decrease. If the distribution of the measure is smooth then we expect that the measure of the ith box is proportional with the dimension of the section

$$\mu_i \sim \epsilon^d.$$

If the measure is distributed on a hairy object like the repeller we can observe unusual scaling behavior of type

$$\mu_i \sim \epsilon^{\alpha_i},$$

where α_i is the local "dimension" or Hölder exponent of the the object. As α is not necessarily an integer here we are dealing with objects with fractional dimensions. We can study the distribution of the measure on the surface of section by looking at the distribution of these local exponents. We can define

$$\alpha_i = \frac{\log \mu_i}{\log \epsilon},$$

the local Hölder exponent and then we can count how many of them are between α and $\alpha + d\alpha$. This is $N_\epsilon(\alpha)d\alpha$. Again, in smooth objects this function scales simply with the dimension of the system

$$N_\epsilon(\alpha) \sim \epsilon^{-d},$$

while for hairy objects we expect an α dependent scaling exponent

$$N_\epsilon(\alpha) \sim \epsilon^{-f(\alpha)}.$$

$f(\alpha)$ can be interpreted [I.7] as the dimension of the points on the surface of section with scaling exponent α. We can calculate $f(\alpha)$ with the help of partition sums as we did for $g(\lambda)$ in the previous section. First we define

$$Z_\epsilon(q) = \sum_i \mu_i^q. \tag{20.17}$$

Then we would like to determine the asymptotic behavior of the partition sum characterized by the $\tau(q)$ exponent

$$Z_\epsilon(q) \sim \epsilon^{-\tau(q)}.$$

fractal!dimension
dimension!fractal
dimension!box counting
information!dimension
dimension!information

The partition sum can be written in terms of the distribution function of α-s

$$Z_\epsilon(q) = \int d\alpha N_\epsilon(\alpha)\epsilon^{q\alpha}.$$

Using the asymptotic form of the distribution we get

$$Z_\epsilon(q) \sim \int d\alpha \epsilon^{q\alpha - f(\alpha)}.$$

As ϵ goes to zero the integral is dominated by the term maximizing the exponent. This α^* can be determined from the equation

$$\frac{d}{d\alpha^*}(q\alpha^* - f(\alpha^*)) = 0,$$

leading to

$$q = f'(\alpha^*).$$

Finally we can read off the scaling exponent of the partition sum

$$\tau(q) = \alpha^* q - f(\alpha^*).$$

In a uniform fractal characterized by a single dimension both α and $f(\alpha)$ collapse to $\alpha = f(\alpha) = D$. The scaling exponent then has the form $\tau(q) = (q-1)D$. In case of non uniform fractals we can introduce generalized dimensions [I.9] D_q via the definition

$$D_q = \tau(q)/(q-1).$$

Some of these dimensions have special names. For $q = 0$ the partition sum (20.17) counts the number of non empty boxes $\bar{N}_\epsilon$. Consequently

$$D_0 = -\lim_{\epsilon\to 0} \frac{\log \bar{N}_\epsilon}{\log \epsilon},$$

is called the box counting dimension. For $q = 1$ the dimension can be determined as the limit of the formulas for $q \to 1$ leading to

$$D_1 = \lim_{\epsilon\to 0} \sum_i \mu_i \log \mu_i / \log \epsilon.$$

This is the scaling exponent of the Shannon information entropy [I.16] of the distribution, hence its name is *information dimension*.

Using equisize grids is impractical in most of the applications. Instead, we can rewrite (20.17) into the more convenient form

$$\sum_i \frac{\mu_i^q}{\epsilon^\tau(q)} \sim 1. \tag{20.18}$$

If we cover the ith branch of the fractal with a grid of size l_i instead of ϵ we can use the relation [20.5]

$$\sum_i \frac{\mu_i^q}{{l_i}^\tau(q)} \sim 1, \tag{20.19}$$

the non-uniform grid generalization of 20.18. Next we show how can we use the periodic orbit formalism to calculate fractal dimensions. We have already seen that the width of the strips of the repeller can be approximated with the stabilities of the periodic orbits situating in them

$$l_i \sim \frac{1}{|\Lambda_i|}.$$

Then using this relation and the periodic orbit expression of the natural measure we can write (20.19) into the form

$$\sum_i \frac{e^{q\gamma n}}{|\Lambda_i|^{q-\tau(q)}} \sim 1, \tag{20.20}$$

where the summation goes for periodic orbits of length n. The sum for stabilities can be expressed with the pressure function again

$$\sum_i \frac{1}{|\Lambda_i|^{q-\tau(q)}} \sim e^{-nP(q-\tau(q))},$$

and (20.20) can be written as

$$e^{q\gamma n} e^{-nP(q-\tau(q))} \sim 1,$$

for large n. Finally we get an implicit formula for the dimensions

$$P(q-(q-1)D_q) = q\gamma. \tag{20.21}$$

Solving this equation directly gives us the partial dimensions of the multifractal repeller along the stable direction. We can see again that the pressure function alone contains all the relevant information. Setting $q = 0$ in (20.21) we can prove that the zero of the pressure function is the box-counting dimension of the repeller

$$P(D_0) = 0.$$

Taking the derivative of (20.21) in $q = 1$ we get

$$P'(1)(1-D_1) = \gamma.$$

This way we can express the information dimension with the escape rate and the Lyapunov exponent

$$D_1 = 1 - \gamma/\overline{\lambda}. \tag{20.22}$$

If the system is bound ($\gamma = 0$) the information dimension and all other dimensions are $D_q = 1$. Also since $D_1 0$ is positive (20.22) proves that the Lyapunov exponent must be larger than the escape rate $\overline{\lambda} > \gamma$ in general.

20.4 page 392
20.5 page 392
20.6 page 392

Commentary

Remark 20.1 Mild phase transition. In non-hyperbolic systems the formulas derived in this chapter should be modified. As we mentioned in 19.1 in non-hyperbolic systems the periodic orbit expression of the measure can be

$$\mu_0 = e^{\gamma n}/|\Lambda_0|^{\delta},$$

where δ can differ from 1. Usually it is 1/2. For sufficiently *negative* β the corresponding term $1/|\Lambda_0|^{\beta}$ can dominate (20.10) while in (20.3) $e^{\gamma n}/|\Lambda_0|^{\delta\beta}$ plays no dominant role. In this case the pressure as a function of β can have a kink at the critical point $\beta = \beta_c$ where $\beta_c \log|\Lambda_0| = (\beta_c - 1)K_{\beta_c} + \beta_c\gamma$. For $\beta < \beta_c$ the pressure and the Rényi entropies differ

$$P(\beta) \neq (\beta - 1)K_{\beta} + \beta\gamma.$$

This phenomena is called phase transition. This is however not a very deep problem. We can fix the relation between pressure and the entropies by replacing $1/|\Lambda_0|$ with $1/|\Lambda_0|^{\delta}$ in (20.10).

Remark 20.2 Hard phase transition The really deep trouble of thermodynamics is caused by intermittency. In that case we have periodic orbits with $|\Lambda_0| \to 1$ as $n \to \infty$. Then for $\beta > 1$ the contribution of these orbits dominate both (20.10) and (20.3). Consequently the partition sum scales as $Z_n(\beta) \to 1$ and both the pressure and the entropies are zero. In this case quantities connected with $\beta \leq 1$ make sense only. These are for example the topological entropy, Kolmogorov entropy, Lyapunov exponent, escape rate, D_0 and D_1. This phase transition cannot be fixed. It is probably fair to say that quantities which depend on this phase transition are only of mathematical interest and not very useful for characterization of realistic dynamical systems.

Résumé

In this chapter we have shown that thermodynamic quantities and various fractal dimensions can be expressed in terms of the pressure function. The pressure function is the leading eigenvalue of the operator which generates the Lyapunov exponent. In the Lyapunov case β is just an auxiliary variable. In thermodynamics it plays an essential role. The good news of the chapter is that the distribution of locally fluctuating exponents should not be computed via making statistics. We can use cyclist formulas for determining the pressure. Then the pressure can be found using short cycles + curvatures. Here the head reaches the tail of the snake. We just argued that the statistics of long trajectories coded in $g(\lambda)$ and $P(\beta)$ can be calculated from short cycles. To use this intimate relation between long and short trajectories effectively is still a research level problem.

need to extract more refsThermo.tex from refsAppApplic?

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[20.3] Even though the thermodynamic formalism is of older vintage (we refer the reader to ref. [20.4] for a comprehensive overview), we adhere here to the notational conventions of ref. [20.5] which are more current in the physics literature: we strongly recommend also ref. [20.6], dealing with period doubling universality.

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Exercises

Exercise 20.1 Thermodynamics in higher dimensions *Introduce the time averages of the eigenvalues of the Jacobian*

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log |\Lambda_i^t(x_0)|, \tag{20.23}$$

as a generalization of (10.31).

Show that in higher dimensions Pesin's formula is

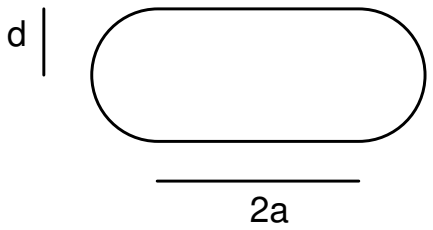
$$K_1 = \sum_i \lambda_i - \gamma, \tag{20.24}$$

where the summation goes for the positive λ_i-s only. (Hint: Use the higher dimensional generalization of (19.10)

$$\mu_i = e^{n\gamma} / |\prod_j \Lambda_{i,j}|,$$

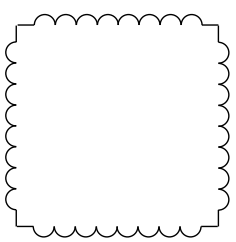
where the product goes for the expanding eigenvalues of the Jacobian of the periodic orbit.

Exercise 20.2 Bunimovich stadium Kolmogorov entropy. *Take for definitiveness $a = 1.6$ and $d = 1$ in the Bunimovich stadium of exercise 6.4,*



estimate the Lyapunov exponent by averaging over a very long trajectory. Biham and Kvale [20.14] estimate the discrete time Lyapunov to $\lambda \approx 1.0 \pm .1$, the continuous time Lyapunov to $\lambda \approx 0.43 \pm .02$, the topological entropy (for their symbolic dynamics) $h \approx 1.15 \pm .03$.

Exercise 20.3 Entropy of rugged-edge billiards. *Take a semi-circle of diameter ε and replace the sides of a unit square by $\lfloor 1/\varepsilon \rfloor$ catenated copies of the semi-circle.*



(a) *Is the billiard ergodic as $\varepsilon \to 0$?*

(b) *(hard) Show that the entropy of the billiard map is*

$$K_1 \to -\frac{2}{\pi}\ln\varepsilon + \text{const}\,,$$

as $\varepsilon \to 0$. *(Hint: do not write return maps.)*

(c) *(harder) Show that when the semi-circles of the Bunimovich stadium are far apart, say* L*, the entropy for the flow decays as*

$$K_1 \to \frac{2\ln L}{\pi L}\,.$$

Exercise 20.4 Two scale map *Compute all those quantities - dimensions, escape rate, entropies, etc. - for the repeller of the one dimensional map*

$$f(x) = \begin{cases} 1+ax & \text{if} \quad x<0, \\ 1-bx & \text{if} \quad x>0. \end{cases} \tag{20.25}$$

where a *and* b *are larger than* 2. *Compute the fractal dimension, plot the pressure and compute the* $f(\alpha)$ *spectrum of singularities. Observe how* K_1 *may be obtained directly from (??).*

⇓PRELIMINARY
⇑PRELIMINARY

Exercise 20.5 Four scale map *Compute the Rényi entropies and* $g(\lambda)$ *for the four scale map*

$$f(x) = \begin{cases} a_1 x & \text{if} \quad 0<x<b/a_1, \\ (1-b)((x-b/a_1)/(b-b/a_1))+b & \text{if} \quad b/a_1<x<b, \\ a_2(x-b) & \text{if} \quad b<x<b+b/a_2, \\ (1-b)((x-b-b/a_2)/(1-b-b/a_2))+b & \text{if} \quad b+b/a_2<x<1. \end{cases} \tag{20.26}$$

Hint: Calculate the pressure function and use (20.13).

Exercise 20.6 Transfer matrix *Take the unimodal map* $f(x)=\sin(\pi x)$ *of the interval* $I=[0,1]$. *Calculate the four preimages of the intervals* $I_0=[0,1/2]$ *and* $I_1=[1/2,1]$. *Extrapolate* $f(x)$ *with piecewise linear functions on these intervals. Find* a_1, a_2 *and* b *of the previous exercise. Calculate the pressure function of this linear extrapolation. Work out higher level approximations by linearly extrapolating the map on the* 2^n*-th preimages of* I.

Chapter 21

Intermittency

Sometimes They Come Back
Stephen King

(R. Artuso, P. Dahlqvist, G. Tanner and P. Cvitanović)

must incorporate the article version

In the theory of chaotic dynamics developed so far we assumed that the evolution operators have discrete spectra $\{z_0, z_1, z_2, \ldots\}$ given by the zeros of

$$1/\zeta(z) = (\cdots)\prod_k (1 - z/z_k)\,.$$

The assumption was based on the tacit premise that the dynamics is everywhere exponentially unstable. Real life is nothing like that - state spaces are generically infinitely interwoven patterns of stable and unstable behaviors. The stable (in the case of Hamiltonian flows, integrable) orbits do not communicate with the ergodic components of the phase space, and can be treated by classical methods. In general, one is able to treat the dynamics near stable orbits as well as chaotic components of the phase space dynamics well within a periodic orbit approach. Problems occur at the borderline between chaos and regular dynamics where marginally stable orbits and manifolds present difficulties and still unresolved challenges.

We shall use the simplest example of such behavior - intermittency in 1-dimensional maps - to illustrate effects of marginal stability. The main message will be that spectra of evolution operators are no longer discrete, dynamical zeta functions exhibit branch cuts of the form

$$1/\zeta(z) = (\cdots) + (1 - z)^{\alpha}(\cdots)\,,$$

and correlations decay no longer exponentially, but as power laws.

laminar states
intermittency
Lorentz gas
standard map
KAM!tori

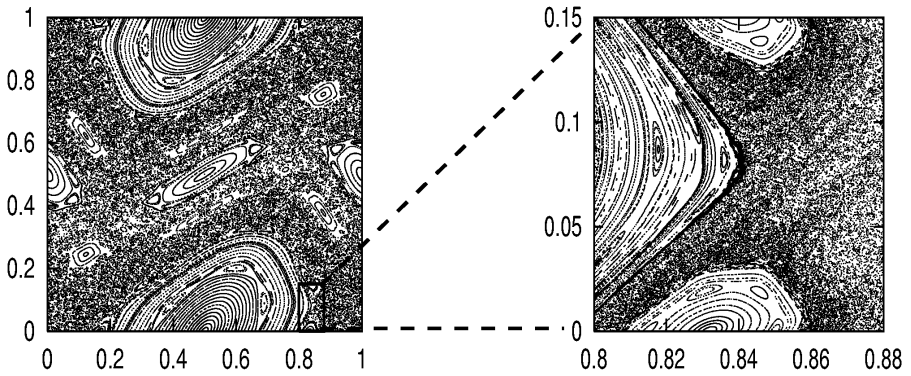


Figure 21.1: Typical phase space for an area-preserving map with mixed phase space dynamics; here the standard map for k=1.2.

21.1 Intermittency everywhere

In many fluid dynamics experiments one observes transitions from regular behaviors to behaviors where long time intervals of regular behavior ("laminar phases") are interrupted by fast irregular bursts. The closer the parameter is to the onset of such bursts, the longer are the intervals of regular behavior. The distributions of laminar phase intervals are well described by power laws.

This phenomenon is called *intermittency*, and it is a very general aspect of dynamics, a shadow cast by non-hyperbolic, marginally stable state space regions. Complete hyperbolicity assumed in (14.5) is the exception rather than the rule, and for almost any dynamical system of interest (dynamics in smooth potentials, billiards with smooth walls, the infinite horizon Lorentz gas, etc.) one encounters mixed state spaces with islands of stability coexisting with hyperbolic regions, see figure 21.1. Wherever stable islands are interspersed with chaotic regions, trajectories which come close to the stable islands can stay 'glued' for arbitrarily long times. These intervals of regular motion are interrupted by irregular bursts as the trajectory is re-injected into the chaotic part of the phase space. How the trajectories are precisely 'glued' to the marginally stable region is often hard to describe. What coarsely looks like a border of an island will under magnification dissolve into infinities of island chains of decreasing sizes, broken tori and bifurcating orbits, as illustrated in figure 21.1.

Lorenz or Lorentz define standard map

Intermittency is due to the existence of fixed points and cycles of marginal stability (8.3), or (in studies of the onset of intermittency) to the proximity of a nearly marginal complex or unstable orbits. In Hamiltonian systems intermittency goes hand in hand with the existence of (marginally stable) KAM tori. In more general settings, the existence of marginal or nearly marginal orbits is due to incomplete intersections of stable and unstable manifolds in a Smale horseshoe type dynamics (see figure 13.2). Following the stretching and folding of the invariant manifolds in time one will inevitably

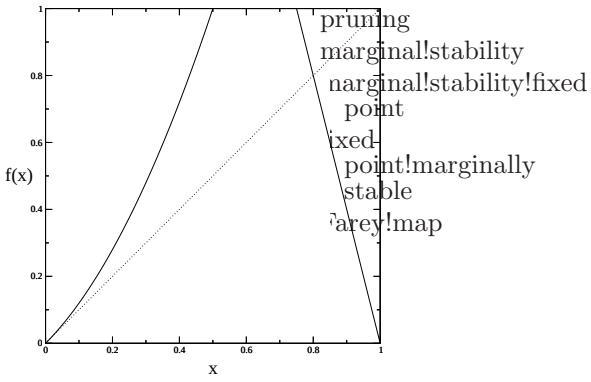


Figure 21.2: A complete binary repeller with a marginal fixed point.

find state space points at which the stable and unstable manifolds are almost or exactly tangential to each other, implying non-exponential separation of nearby points in state space or, in other words, marginal stability. Under small parameter perturbations such neighborhoods undergo tangent bifurcations - a stable/unstable pair of periodic orbits is destroyed or created by coalescing into a marginal orbit, so the pruning which we shall encounter in chapter 12, and the intermittency discussed here are two sides of the same coin.

☞ sect. 12.4

redraw figure **??** so axes are possible to read

How to deal with the full complexity of a typical Hamiltonian system with mixed phase space is a very difficult, still open problem. Nevertheless, it is possible to learn quite a bit about intermittency by considering rather simple examples. Here we shall restrict our considerations to 1-dimensional maps which in the neighborhood of a single marginally stable fixed point at x=0 take the form

$$x \mapsto f(x) = x + O(x^{1+s}) \,, \tag{21.1}$$

and are expanding everywhere else. Such a map may allow for escape, like the map shown in figure 21.2 or the dynamics may be bounded, like the Farey map (18.32) 163,164c153,154

$$x \mapsto f(x) = \begin{cases} x/(1-x) & x \in [0, 1/2[\\ (1-x)/x & x \in [1/2, 1] \end{cases}$$

introduced in sect. 18.6. [1] [2]

Figure 21.3 compares a trajectory of the tent map (11.8) side by side with a trajectory of the Farey map. In a stark contrast to the uniformly chaotic trajectory of the tent map, the Farey map trajectory alternates intermittently between slow regular motion close to the marginally stable fixed point, and chaotic bursts.

why does Figure 21.3 take whole page?

labels (a), (b) missing in figure **??**

☞ sect. 18.6.3

The presence of marginal stability has striking dynamical consequences: correlation decay may exhibit long range power law asymptotic behavior

⇓PRELIMINARY

☞ sect. 24.3

⇑PRELIMINARY

[1] Vattay: why can one see $\overline{1}$ fixed point in the time series (a)? Measure is constant - is this due to numerical errors?

[2] Tanner: I checked it again; it just looks like accumulating at the period one orbit because I connect the points; when plotting only points, it seems uniform.

power
law!correlation
decay
correlation!decay!power
law

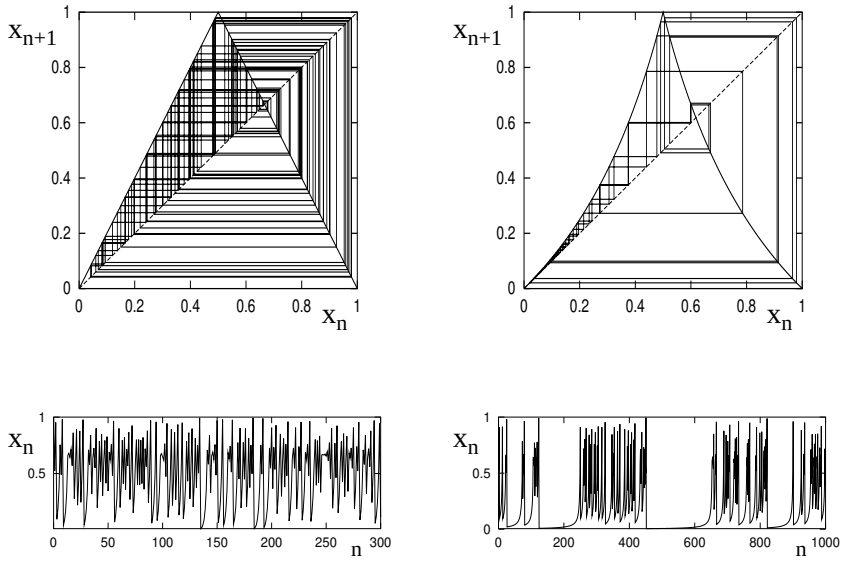


Figure 21.3: (a) A tent map trajectory. (b) A Farey map trajectory.

and diffusion processes can assume anomalous character. Escape from a repeller of the form figure 21.2 may be algebraic rather than exponential. In long time explorations of the dynamics intermittency manifests itself by enhancement of natural measure in the proximity of marginally stable cycles.

The questions we shall address here are: how does marginal stability affect zeta functions or spectral determinants? And, can we deduce power law decays of correlations from cycle expansions?

In example 16.5 we saw that marginal stability violates one of the conditions which ensure that the spectral determinant is an entire function. Already the simple fact that the cycle weight $1/|1-\Lambda_p^r|$ in the trace (14.3) or the spectral determinant (15.3) diverges for marginal orbits with $|\Lambda_p| = 1$ tells us that we have to treat these orbits with care.

In the following we will incorporate marginal stability orbits into cycle-expansions in a systematic manner. To get to know the difficulties lying ahead, we will start in sect. 21.2 with a piecewise linear map, with the asymptotics (21.1). We will construct a dynamical zeta function in the usual way without worrying too much about its justification and show that it has a branch cut singularity. We will calculate the rate of escape from our piecewise linear map and find that it is characterized by decay, rather than exponential decay, a power law. We will show that dynamical zeta functions in the presence of marginal stability can still be written in terms of periodic orbits, exactly as in chapters 10 and 19, with one exception: the marginally stable orbits have to be explicitly excluded. This innocent looking step has far reaching consequences; it forces us to change the symbolic dynamics from a finite to an infinite alphabet, and entails a reorganization of the

order of summations in cycle expansions, sect. 21.2.4.

enemy!thy
intermittency!piecewise linear model
piecewise linear map!intermittency

Branch cuts are typical also for smooth intermittent maps with isolated marginally stable fixed points and cycles. In sect. 21.3, we discuss the cycle expansions and curvature combinations for zeta functions of smooth maps tailored to intermittency. The knowledge of the type of singularity one encounters enables us to develop the efficient resummation method presented in sect. 21.3.2.

Finally, in sect. 21.4, we discuss a probabilistic approach to intermittency that yields approximate dynamical zeta functions and provides valuable information about more complicated systems, such as billiards.

21.2 Intermittency for pedestrians

Intermittency does not only present us with a large repertoire of interesting dynamics, it is also at the root of many sorrows such as slow convergence of cycle expansions. In order to get to know the kind of problems which arise when studying dynamical zeta functions in the presence of marginal stability we will consider an artfully concocted piecewise linear model first. From there we will move on to the more general case of smooth intermittant maps, sect. 21.3.

21.2.1 A toy map

The Bernoulli shift map (16.6) is an idealized, but highly instructive, example of a hyperbolic map. To study intermittency we will now construct a likewise piecewise linear model, an intermittent map stripped down to its bare essentials.

Consider a map $x \mapsto f(x)$ on the unit interval $\mathcal{M} = [0,1]$ with two monotone branches

$$f(x) = \begin{cases} f_0(x) & \text{for } x \in \mathcal{M}_0 = [0,a] \\ f_1(x) & \text{for } x \in \mathcal{M}_1 = [b,1] \end{cases} . \tag{21.2}$$

The two branches are assumed complete, that is $f_0(\mathcal{M}_0) = f_1(\mathcal{M}_1) = \mathcal{M}$. The map allows escape if $a < b$ and is bounded if $a = b$ (see figure 21.2 and figure 21.4). We take the right branch to be expanding and linear:

$$f_1(x) = \frac{1}{1-b}(x-b)\,.$$

Next, we will construct the left branch in a way, which will allow us to model the intermittent behavior (21.1) near the origin. We chose a

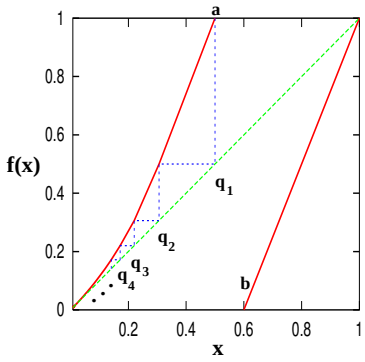


Figure 21.4: A piecewise linear intermittent map of (21.2) type: more specifically, the map piecewise linear over intervals (21.8) of the toy example studied below, $a = .5$, $b = .6$, $s = 1.0$.

monotonically decreasing sequence of points q_n in $[0, a]$ with $q_1 = a$ and $q_n \to 0$ as $n \to \infty$. This sequence defines a partition of the left interval $\mathcal{M}_0$ into an infinite number of connected intervals $\mathcal{M}_n$, $n \geq 2$ with

$$\mathcal{M}_n =]q_n, q_{n-1}] \qquad \text{and} \qquad \mathcal{M}_0 = \bigcup_{n=2}^{\infty} \mathcal{M}_n . \tag{21.3}$$

The map $f_0(x)$ is now specified by the following requirements

- $f_0(x)$ is continuous.
- $f_0(x)$ is linear on the intervals $\mathcal{M}_n$ for $n \geq 2$.
- $f_0(q_n) = q_{n-1}$, that is $\mathcal{M}_n = f_0^{-n+1}([a, 1])$.

This fixes the map for any given sequence $\{q_n\}$. The last condition ensures the existence of a simple Markov partition. The slopes of the various linear segments are

define Markov chain/partition somewhere

$$\begin{array}{rcccll} f_0'(x) & = & \frac{f_0(q_{n-1})-f_0(q_n)}{q_{n-1}-q_n} & = & \frac{|\mathcal{M}_{n-1}|}{|\mathcal{M}_n|} & \text{for } x \in \mathcal{M}_n,\ n \geq 3 \\ f_0'(x) & = & \frac{f_0(q_1)-f_0(q_2)}{q_1-q_2} & = & \frac{1-a}{|\mathcal{M}_2|} & \text{for } x \in \mathcal{M}_2 \\ f_0'(x) & = & \frac{1}{1-b} & = & \frac{|\mathcal{M}|}{|\mathcal{M}_1|} & \text{for } x \in \mathcal{M}_1 \end{array} \tag{21.4}$$

with $|\mathcal{M}_n| = q_{n-1} - q_n$ for $n \geq 2$. Note that we do not require as yet that the map exhibit intermittent behavior.

We will see that the family of periodic orbits with code 10^n plays a key role for intermittent maps of the form (21.1). An orbit 10^n enters the intervals $\mathcal{M}_1 \to \mathcal{M}_{n+1} \to \mathcal{M}_n \to \ldots \to \mathcal{M}_2$ successively and the family approaches the marginal stable fixed point at $x = 0$ for $n \to \infty$. The stability of a cycle 10^n for $n \geq 1$ is given by the chain rule (4.39),

$$\Lambda_{10^n} = f_0'(x_{n+1}) f_0'(x_n) \ldots f_0'(x_2) f_1'(x_1) = \frac{1}{|\mathcal{M}_{n+1}|} \frac{1-a}{1-b} , \tag{21.5}$$

with $x_i \in \mathcal{M}_i$.

Jonqui"ere function
polylogarithm

The properties of the map (21.2) are completely determined by the sequence $\{q_n\}$. By choosing $q_n = 2^{-n}$, for example, we recover the uniformly hyperbolic Bernoulli shift map (16.6). An intermittent map of the form (21.3) having the asymptotic behavior (21.1) can be constructed by choosing an algebraically decaying sequence $\{q_n\}$ behaving asymptotically like

$$q_n \sim \frac{1}{n^{1/s}}, \tag{21.6}$$

where s is the intermittency exponent in (21.1). Such a partition leads to intervals whose length decreases asymptotically like a power-law, that is,

$$|\mathcal{M}_n| \sim \frac{1}{n^{1+1/s}}\,. \tag{21.7}$$

remind reader of (18.1)

As can be seen from (21.5), the stability eigenvalues of periodic orbit families approaching the marginal fixed point, such as the 10^n family increase in turn only algebraically with the cycle length.

It may now seem natural to construct an intermittent toy map in terms of a partition $|\mathcal{M}_n| = 1/n^{1+1/s}$, that is, a partition which follows (21.7) exactly. Such a choice leads to a dynamical zeta function which can be written in terms of so-called Jonquière functions (or polylogarithms) which arise naturally also in the context of the Farey map (18.32), and the anomalous diffusion of sect. 24.3. We will, however, not go along this route here; instead, we will engage in a bit of reverse engineering and construct a less obvious partition which will simplify the algebra considerably later without loosing any of the key features typical for intermittent systems. We fix the intermittent toy map by specifying the intervals $\mathcal{M}_n$ in terms of Gamma functions according to

☞ remark 24.8

$$|\mathcal{M}_n| = \mathcal{C}\,\frac{\Gamma(n+m-1/s-1)}{\Gamma(n+m)} \qquad \text{for} \quad n \geq 2, \tag{21.8}$$

where $m = [1/s]$ denotes the integer part of $1/s$ and $\mathcal{C}$ is a normalization constant fixed by the condition $\sum_{n=2}^{\infty} |\mathcal{M}_n| = q_1 = a$, that is,

$$\mathcal{C} = a\left[\sum_{n=m+1}^{\infty} \frac{\Gamma(n-1/s)}{\Gamma(n+1)}\right]^{-1}. \tag{21.9}$$

Using Stirling's formula for the Gamma function

$$\Gamma(z) \sim e^{-z} z^{z-1/2}\sqrt{2\pi}\,(1 + 1/12z + \ldots)\,,$$

we verify that the intervals decay asymptotically like $n^{-(1+1/s)}$, as required by the condition (21.7).

Next, let us write down the dynamical zeta function of the toy map in terms of its periodic orbits, that is

$$1/\zeta(z) = \prod_p \left(1 - \frac{z^{n_p}}{|\Lambda_p|}\right)$$

One may be tempted to expand the dynamical zeta function in terms of the binary symbolic dynamics of the map; we saw, however, in sect. 18.6 that such cycle expansion converges extremely slowly. The shadowing mechanism between orbits and pseudo-orbits fails for orbits of the form 10^n with stabilities given by (21.5), due to the marginal stability of the fixed point $\overline{0}$. It is therefore advantageous to choose as the fundamental cycles the family of orbits with code 10^n or, equivalently, switch from the finite (binary) alphabet to an infinite alphabet given by

$$10^{n-1} \to n.$$

Due to the piecewise-linear form of the map which maps intervals $\mathcal{M}_n$ exactly onto $\mathcal{M}_{n-1}$, all periodic orbits entering the left branch at least twice are canceled exactly by pseudo cycles, and the cycle expanded dynamical zeta function depends only on the fundamental series $1, 10, 100, \ldots$:

$$\begin{aligned} 1/\zeta(z) &= \prod_{p\neq 0}\left(1 - \frac{z^{n_p}}{|\Lambda_p|}\right) = 1 - \sum_{n=1}^{\infty} \frac{z^n}{|\Lambda_{10^{n-1}}|} \\ &= 1 - (1-b)z - \mathcal{C}\frac{1-b}{1-a}\sum_{n=2}^{\infty} \frac{\Gamma(n+m-1/s-1)}{\Gamma(n+m)} z^n \end{aligned} \quad (21.10)$$

refer to cyc-exp-formula

The fundamental term (18.5) consists here of an infinite sum over algebraically decaying cycle weights. The sum is divergent for $|z| \geq 1$. We will see that this behavior is due to a branch cut of $1/\zeta$ starting at $z = 1$. We need to find analytic continuations of sums over algebraically decreasing terms in (21.10). Note also that we omitted the fixed point $\overline{0}$ in the above Euler product; we will discussed this point as well as a proper derivation of the zeta function in more detail in sect. 21.2.4.

21.2.2 Branch cuts

Starting from the dynamical zeta function (21.10), we first have to worry about finding an analytical continuation of the sum for $|z| \geq 1$. We do, however, get this part for free here due to the particular choice of interval lengths made in (21.8). The sum over ratios of Gamma functions in (21.10) can be evaluated analytically by using the following identities valid for $1/s = \alpha > 0$ (the famed binomial theorem in disguise),

- α non-integer branch cut

$$(1-z)^\alpha = \sum_{n=0}^{\infty} \frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)\Gamma(n+1)} z^n \tag{21.11}$$

- α integer

$$\begin{aligned}(1-z)^\alpha \log(1-z) &= \sum_{n=1}^{\alpha} (-1)^n c_n z^n \\ &+ (-1)^{\alpha+1}\alpha! \sum_{n=\alpha+1}^{\infty} \frac{(n-\alpha-1)!}{n!} z^n\end{aligned} \tag{21.12}$$

with

$$c_n = \left(\begin{array}{c} \alpha \\ n \end{array} \right) \sum_{k=0}^{n-1} \frac{1}{\alpha-k}\,.$$

In order to simplify the notation, we restrict the intermittency parameter to the range $1 \le 1/s < 2$ with $[1/s] = m = 1$. All what follows can easily be generalized to arbitrary $s > 0$ using equations (21.11) and (21.12). The infinite sum in (21.10) can now be evaluated with the help of (21.11) or (21.12), that is,

$$\sum_{n=2}^{\infty} \frac{\Gamma(n-1/s)}{\Gamma(n+1)} z^n = \begin{cases} \Gamma(-\frac{1}{s}) \left[(1-z)^{1/s} - 1 + \frac{1}{s}z\right] & \text{for} \quad 1 < 1/s < 2; \\ (1-z)\log(1-z) + z & \text{for} \quad s = 1\,. \end{cases}$$

The normalization constant $\mathcal{C}$ in (21.8) can be evaluated explicitly using (21.9) and the dynamical zeta function can be given in closed form. We obtain for $1 < 1/s < 2$

$$1/\zeta(z) = 1 - (1-b)z - \frac{a}{1/s-1}\frac{1-b}{1-a}\left((1-z)^{1/s} - 1 + \frac{1}{s}z\right). \tag{21.13}$$

and for $s = 1$,

$$1/\zeta(z) = 1 - (1-b)z - a\frac{1-b}{1-a}\left((1-z)\log(1-z) + z\right). \tag{21.14}$$

It now becomes clear why the particular choice of intervals $\mathcal{M}_n$ made in the last section is useful; by summing over the infinite family of periodic orbits 0^n1 explicitly, we have found the desired analytical continuation for the dynamical zeta function for $|z| \ge 1$. The function has a branch cut starting at the branch point $z = 1$ and running along the positive real axis. That means, the dynamical zeta function takes on different values when

approaching the positive real axis for $\mathrm{Re}\, z > 1$ from above and below. The dynamical zeta function for general $s > 0$ takes on the form

$$1/\zeta(z) = 1-(1-b)z - \frac{a}{g_s(1)}\frac{1-b}{1-a}\frac{1}{z^{m-1}}\left((1-z)^{1/s} - g_s(z)\right) \quad (21.15)$$

for non-integer s with $m = [1/s]$ and

$$1/\zeta(z) = 1-(1-b)z - \frac{a}{g_m(1)}\frac{1-b}{1-a}\frac{1}{z^{m-1}}\left((1-z)^m \log(1-z) - g_m(z)\right) \quad (21.16)$$

for $1/s = m$ integer and $g_s(z)$ are polynomials of order $m = [1/s]$ which can be deduced from (21.11) or (21.12). We thus find algebraic branch cuts for non integer intermittency exponents $1/s$ and logarithmic branch cuts for $1/s$ integer. We will see in sect. 21.3 that branch cuts of that form are generic for 1-dimensional intermittent maps.

Branch cuts are the all important new feature of dynamical zeta functions due to intermittency. So, how do we calculate averages or escape rates of the dynamics of the map from a dynamical zeta function with branch cuts? We take 'a learning by doing' approach and calculate the escape from our toy map for $a < b$.

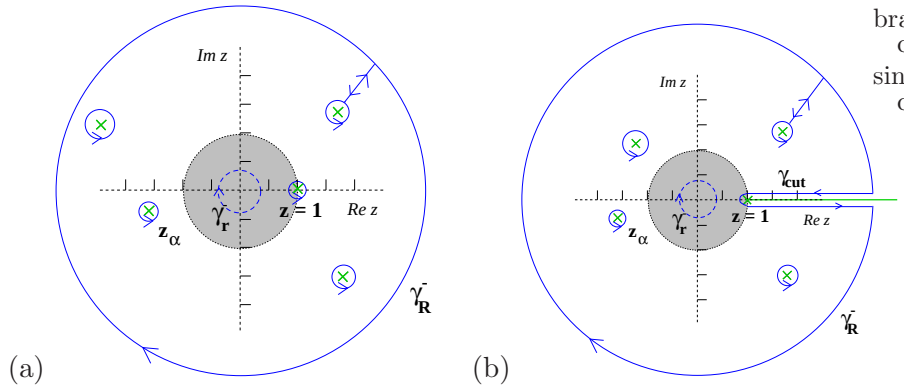
21.2.3 Escape rate

Our starting point for the calculation of the fraction of survivors after n time steps, is the integral representation (15.20)

$$\Gamma_n = \frac{1}{2\pi i}\oint_{\gamma_r^-} z^{-n}\left(\frac{d}{dz}\log\zeta^{-1}(z)\right)dz\,, \quad (21.17)$$

where the contour encircles the origin in the clockwise direction. If the contour lies inside the unit circle $|z| = 1$, we may expand the logarithmic derivative of $\zeta^{-1}(z)$ as a convergent sum over all periodic orbits. Integrals and sums can be interchanged, the integrals can be solved term by term, and the formula (14.24) is recovered. For hyperbolic maps, cycle expansion methods or other techniques may provide an analytic extension of the dynamical zeta function beyond the leading zero; we may therefore deform the original contour into a larger circle with radius R which encircles both poles and zeros of $\zeta^{-1}(z)$, see figure 21.5 (a). Residue calculus turns this into a sum over the zeros z_α and poles z_β of the dynamical zeta function, that is

$$\Gamma_n = \sum_{|z_\alpha|<R}^{\text{zeros}} \frac{1}{z_\alpha^n} - \sum_{|z_\beta|<R}^{\text{poles}} \frac{1}{z_\beta^n} + \frac{1}{2\pi i}\oint_{\gamma_R^-} dz\, z^{-n}\frac{d}{dz}\log\zeta^{-1}, \quad (21.18)$$



branch cut!singularity
singularity!branch cut

Figure 21.5: The survival probability Γ_n calculated by contour integration; integrating (21.17) inside the domain of convergence $|z| < 1$ (shaded area) of $1/\zeta(z)$ in periodic orbit representation yields (14.24). A deformation of the contour γ_r^- (dashed line) to a larger circle γ_R^- gives contributions from the poles and zeros (x) of $1/\zeta(z)$ between the two circles. These are the only contributions for hyperbolic maps (a), for intermittent systems additional contributions arise, given by the contour γ_{cut} running along the branch cut (b).

where the last term gives a contribution from a large circle γ_R^-. We thus find exponential decay of Γ_n dominated by the leading zero or pole of $\zeta^{-1}(z)$, see chapter 20.1 for more details.

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Things change considerably in the intermittent case. The point $z = 1$ is a branch cut singularity and there exists no Taylor series expansion of ζ^{-1} around $z = 1$. Secondly, the path deformation that led us to (21.18) requires more care, as it must not cross the branch cut. When expanding the contour to large $|z|$ values, we have to deform it along the branch $\mathrm{Re}\,(z) \geq 1$, $\mathrm{Im}\,(z) = 0$ encircling the branch cut in anti-clockwise direction, see figure 21.5 (b). We will denote the detour around the cut as γ_{cut}. We may write symbolically

$$\oint_{\gamma_r} = \sum^{\text{zeros}} - \sum^{\text{poles}} + \oint_{\gamma_R} + \oint_{\gamma_{cut}}$$

where the sums include only the zeros and the poles in the area enclosed by the contours. The asymptotics is controlled by the zero, pole or cut closest to the origin.

Let us now go back to our intermittent toy map. The asymptotics of the survival probability of the map is here governed by the behavior of the integrand $\frac{d}{dz}\log\zeta^{-1}$ in (21.17) at the branch point $z = 1$. We restrict ourselves again to the case $1 < 1/s < 2$ first and write the dynamical zeta function (21.13) in the form

$$1/\zeta(z) = a_0 + a_1(1-z) + b_0(1-z)^{1/s} \equiv G(1-z)$$

and

$$a_0 = \frac{b-a}{1-a}, \quad b_0 = \frac{a}{1-1/s}\frac{1-b}{1-a}.$$

Setting $u = 1 - z$, we need to evaluate

$$\frac{1}{2\pi i}\oint_{\gamma_{cut}} (1-u)^{-n}\frac{d}{du}\log G(u)du \tag{21.19}$$

where γ_{cut} goes around the cut (i.e., the negative u axis). Expanding the integrand $\frac{d}{du}\log G(u) = G'(u)/G(u)$ in powers of u and $u^{1/s}$ at $u = 0$, one obtains

$$\frac{d}{du}\log G(u) = \frac{a_1}{a_0} + \frac{1}{s}\frac{b_0}{a_0}u^{1/s-1} + O(u)\,. \tag{21.20}$$

The integrals along the cut may be evaluated using the general formula

$$\frac{1}{2\pi i}\oint_{\gamma_{cut}} u^{\alpha}(1-u)^{-n}du = \frac{\Gamma(n-\alpha-1)}{\Gamma(n)\Gamma(-\alpha)} \sim \frac{1}{n^{\alpha+1}}(1+O(1/n)) \tag{21.21}$$

which can be obtained by deforming the contour back to a loop around the point $u = 1$, now in positive (anti-clockwise) direction. The contour integral then picks up the $(n-1)$st term in the Taylor expansion of the function u^{α} at $u = 1$, cf. (21.11). For the continuous time case the corresponding formula is

$$\frac{1}{2\pi i}\oint_{\gamma_{cut}} z^{\alpha}e^{zt}dz = \frac{1}{\Gamma(-\alpha)}\frac{1}{t^{\alpha+1}}. \tag{21.22}$$

Plugging (21.20) into (21.19) and using (21.21) we get the asymptotic result

$$\Gamma_n \sim \frac{b_0}{a_0}\frac{1}{s}\frac{1}{\Gamma(1-1/s)}\frac{1}{n^{1/s}} = \frac{a}{s-1}\frac{1-b}{b-a}\frac{1}{\Gamma(1-1/s)}\frac{1}{n^{1/s}}\,. \tag{21.23}$$

We see that, asymptotically, the escape from an intermittent repeller is described by power law decay rather than the exponential decay we are familiar with for hyperbolic maps; a numerical simulation of the power-law escape from an intermittent repeller is shown in figure 21.6.

For general non-integer $1/s > 0$, we write

$$1/\zeta(z) = A(u) + (u)^{1/s}B(u) \equiv G(u)$$

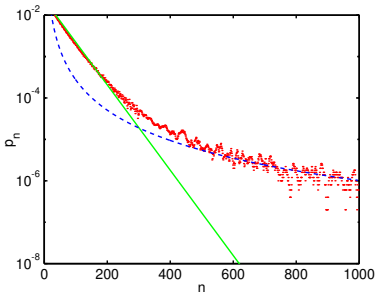

Figure 21.6: The asymptotic escape from an intermittent repeller is a power law. Normally it is preceded by an exponential, which can be related to zeros close to the cut but beyond the branch point $z = 1$, as in figure 21.5 (b).

with $u = 1 - z$ and $A(u)$, $B(u)$ are functions analytic in a disc of radius 1 around $u = 0$. The leading terms in the Taylor series expansions of $A(u)$ and $B(u)$ are

$$a_0 = \frac{b-a}{1-a}, \quad b_0 = \frac{a}{g_s(1)}\frac{1-b}{1-a},$$

see (21.15). Expanding $\frac{d}{du}\log G(u)$ around $u = 0$, one again obtains leading order contributions according to (21.20) and the general result follows immediately using (21.21), that is,

$$\Gamma_n \sim \frac{a}{s g_s(1)}\frac{1-b}{b-a}\frac{1}{\Gamma(1-1/s)}\frac{1}{n^{1/s}}\,. \tag{21.24}$$

Applying the same arguments for integer intermittency exponents $1/s = m$, one obtains

$$\Gamma_n \sim (-1)^{m+1}\frac{a}{s g_m(1)}\frac{1-b}{b-a}\frac{m!}{n^m}\,. \tag{21.25}$$

So far, we have considered the survival probability for a repeller, that is we assumed $a < b$. The formulas (21.24) and (21.25) do obviously not apply for the case $a = b$, that is, for the bounded map. The coefficient $a_0 = (b-a)/(1-a)$ in the series representation of $G(u)$ is zero, and the expansion of the logarithmic derivative of $G(u)$ (21.20) is no longer valid. We get instead

$$\frac{d}{du}\log G(u) = \begin{cases} \frac{1}{u}\left(1 + O(u^{1/s-1})\right) & s < 1 \\ \frac{1}{u}\left(\frac{1}{s} + O(u^{1-1/s})\right) & s > 1 \end{cases},$$

assuming non-integer $1/s$ for convenience. One obtains for the survival probability.

$$\Gamma_n \sim \begin{cases} 1 + O(n^{1-1/s}) & s < 1 \\ 1/s + O(n^{1/s-1}) & s > 1 \end{cases}.$$

escape!intermittency
intermittency!escape rate

For $s > 1$, this is what we expect. There is no escape, so the survival probability is equal to 1, which we get as an asymptotic result here. The result for $s > 1$ is somewhat more worrying. It says that Γ_n defined as sum over the instabilities of the periodic orbits as in (19.12) does not tend to unity for large n. However, the case $s > 1$ is in many senses anomalous. For instance, the invariant density cannot be normalized. It is therefore not reasonable to expect that periodic orbit theories will work without complications.

21.2.4 Why does it work (anyway)?

Due to the piecewise linear nature of the map constructed in the previous section, we had the nice property that interval lengths did exactly coincide with the inverse of the stability of periodic orbits of the system, that is

$$|\mathcal{M}_n| = 1/|\Lambda_{10}|^{n-1}.$$

There is thus no problem in replacing the survival probability Γ_n given by (1.2), (19.2), that is the fraction of state space $\mathcal{M}$ surviving n iterations of the map,

$$\Gamma_n = \frac{1}{|\mathcal{M}|} \sum_i^{(n)} |\mathcal{M}_i| \,.$$

by a sum over periodic orbits of the form (14.24). The only orbit to worry about is the marginal fixed point $\overline{0}$ itself which we excluded from the zeta function (21.10).

For smooth intermittent maps, things are less clear and the fact that we had to prune the marginal fixed point is a warning sign that interval estimates by periodic orbit stabilities might go horribly wrong. The derivation of the survival probability in terms of cycle stabilities in chapter 19 did indeed rely heavily on a hyperbolicity assumption which is clearly not fulfilled for intermittent maps. We therefore have to carefully reconsider this derivation in order to show that periodic orbit formulas are actually valid for intermittent systems in the first place.

We will for simplicity consider maps, which have a finite number of say s branches defined on intervals $\mathcal{M}_s$ and we assume that the map maps each interval $\mathcal{M}_s$ onto $\mathcal{M}$, that is $f(\mathcal{M}_s) = \mathcal{M}$. This ensures the existence of a complete symbolic dynamics - just to make things easy (see figure 21.2).

replace blah-blah up to (21.26) by DasBuch symbolic dynamics

The generating partition is composed of the domains $\mathcal{M}_s$. The nth level partition $\mathcal{C}^{(n)} = \{\mathcal{M}_i\}$ can be constructed iteratively. Here i's are words $i = s_2 s_2 \dots s_n$ of length n, and the intervals $\mathcal{M}_i$ are constructed recursively

$$\mathcal{M}_{sj} = f_s^{-1}(\mathcal{M}_j) \,, \tag{21.26}$$

where sj is the concatenation of letter s with word j of length $n_j < n$.

averages!chaotic

In what follows we will concentrate on the survival probability Γ_n, postponing other quantities of interest, such as averages, to later considerations. In establishing the equivalence of the survival probability and the periodic orbit formula for the escape rate for hyperbolic systems we have assumed that the map is expanding, with a minimal expansion rate $|f'(x)| \geq \Lambda_{\min} > 1$. This enabled us to bound the size of every survivor strip $\mathcal{M}_i$ by (19.6), the stability Λ_i of the periodic orbit i within the $\mathcal{M}_i$, and bound the survival probability by the periodic orbit sum (19.7).

⇓PRELIMINARY
21.1
page 422
⇑PRELIMINARY

The bound (19.6)

$$\mathcal{C}_1 \frac{1}{|\Lambda_i|} < \frac{|\mathcal{M}_i|}{|\mathcal{M}|} < \mathcal{C}_2 \frac{1}{|\Lambda_i|}$$

relies on hyperbolicity, and is thus indeed violated for intermittent systems. The problem is that now there is no lower bound on the expansion rate, the minimal expansion rate is $\Lambda_{\min} = 1$. The survivor strip $\mathcal{M}_{0^n}$ which includes the marginal fixed point is thus completely overestimated by $1/|\Lambda_{0^n}| = 1$ which is constant for all n.

move exercise 15.6 to chapter 15

15.6
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However, bounding survival probability strip by strip is not what is required for establishing the bound (19.7). For intermittent systems a somewhat weaker bound can be established, saying that the average size of intervals *along a periodic orbit* can be bounded close to the stability of the periodic orbit for all but the interval $\mathcal{M}_{0^n}$. The weaker bound applies to averaging over each prime cycle p separately

prove this? move after (21.7)?

$$\mathcal{C}_1 \frac{1}{|\Lambda_p|} < \frac{1}{n_p} \sum_{i \in p} \frac{|\mathcal{M}_i|}{|\mathcal{M}|} < \mathcal{C}_2 \frac{1}{|\Lambda_p|}, \tag{21.27}$$

where the word i represents a code of the periodic orbit p and all its cyclic permutations. It can be shown that one can find positive constants $\mathcal{C}_1$, $\mathcal{C}_2$ independent of p. Summing over all periodic orbits leads then again to (19.7).

To study averages of multiplicative weights we follow sect. 10.1 and introduce a state space observable $a(x)$ and the integrated quantity

$$A^n(x) = \sum_{k=0}^{n-1} a(f^k(x)).$$

This leads us to introduce the generating function (10.10)

$$\langle e^{\beta\, A^n(x)} \rangle,$$

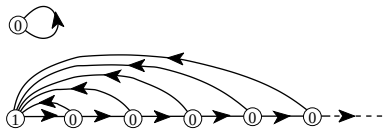

Figure 21.7: Markov graph corresponding to the alphabet $\{0^{k-1}1;\overline{0},\ k\geq 1\}$

where $\langle . \rangle$ denote some averaging over the distribution of initial points, which we choose to be uniform (rather than the *a priori* unknown invariant density). Again, all we have to show is, that constants $\mathcal{C}_1$, $\mathcal{C}_2$ exist, such that

$$\mathcal{C}_1\frac{e^{\beta A_p}}{|\Lambda_p|} < \frac{1}{n_p}\sum_{i\in p}\frac{1}{|\mathcal{M}|}\int_{\mathcal{M}_Q} e^{\beta A^n(x)}dx < \mathcal{C}_2\frac{e^{\beta A_p}}{|\Lambda_p|}, \tag{21.28}$$

is valid for all p. After performing the above average one gets

$$\mathcal{C}_1\Gamma_n(\beta) < \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} e^{\beta A(x,n)}dx < \mathcal{C}_2\Gamma_n(\beta), \tag{21.29}$$

with

$$\Gamma_n(\beta) = \sum_p^n \frac{e^{\beta A_p}}{|\Lambda_p|}. \tag{21.30}$$

and a dynamical zeta function can be derived. In the intermittent case one can expect that the bound (21.28) holds using an averaging argument similar to the one discussed in (21.27). This justifies the use of dynamical zeta functions for intermittent systems.

One lesson we should have learned so far is that the natural alphabet to use is not $\{0,1\}$ but rather the infinite alphabet $\{0^{k-1}1,\overline{0};\ k\geq 1\}$. The symbol 0 occurs unaccompanied by any 1's only in the $\overline{0}$ marginal fixed point which is disconnected from the rest of the Markov graph see figure 21.7.

☞ chapter 12

What happens if we remove a single prime cycle from a dynamical zeta function? In the hyperbolic case such a removal introduces a pole in the $1/\zeta$ and slows down the convergence of cycle expansions. The heuristic interpretation of such a pole is that for a subshift of finite type removal of a single prime cycle leads to unbalancing of cancellations within the infinity of of shadowing pairs. Nevertheless, removal of a single prime cycle is an exponentially small perturbation of the trace sums, and the asymptotics of the associated trace formulas is unaffected.

☞ chapter 16

In the intermittent case, the fixed point $\overline{0}$ does not provide any shadowing (cf. sect. K.1), and a statement such as

???

$$\Lambda_{1\cdot 0^{k+1}} \approx \Lambda_{1\cdot 0^k}\Lambda_0,$$

is meaningless. It seems therefore sensible to take out the factor $(1-t_0) = 1-z$ from the product representation of the dynamical zeta function (15.16), that is, to consider a pruned dynamical zeta function $1/\zeta_{inter}(z)$ defined by

$$1/\zeta(z) = (1-z)1/\zeta_{inter}(z)\,.$$

We saw in the last sections, that the zeta function $1/\zeta_{inter}(z)$ has all the nice properties we know from the hyperbolic case, that is, we can find a cycle expansion with - in the toy model case - vanishing curvature contributions and we can calculate dynamical properties like escape after having understood, how to handle the branch cut. But you might still be worried about leaving out the extra factor $1-z$ all together. It turns out, that this is not only a matter of convenience, omitting the marginal $\overline{0}$ cycle is a dire necessity. The cycle weight $\Lambda_0^n = 1$ overestimates the corresponding interval length of $\mathcal{M}_{0^n}$ in the partition of the phase space $\mathcal{M}$ by an increasing amount thus leading to wrong results when calculating escape. By leaving out the $\overline{0}$ cycle (and thus also the $\mathcal{M}_{0^n}$ contribution), we are guaranteed to get at least the right asymptotical behavior.

Note also, that if we are working with the spectral determinant (15.3), given in product form as

$$\det(1-z\mathcal{L}) = \prod_p \prod_{m=0}^{\infty} \left(1 - \frac{z^{n_p}}{|\Lambda_p|\Lambda_p^m}\right) \quad ,$$

for intermittent maps the marginal stable cycle has to be excluded. It introduces an (unphysical) essential singularity at $z=1$ due the presence of a factor $(1-z)^\infty$ stemming from the $\overline{0}$ cycle.

21.3 Intermittency for cyclists

Admittedly, the toy map is what is says - a toy model. The piece wise linearity of the map led to exact cancellations of the curvature contributions leaving only the fundamental terms. There are still infinitely many orbits included in the fundamental term, but the cycle weights were chosen in such a way that the zeta function could be written in closed form. For a smooth intermittent map this all will not be the case in general; still, we will argue that we have already seen almost all the fundamentally new features due to intermittency. What remains are technicalities - not necessarily easy to handle, but nothing very surprise any more.

In the following we will sketch, how to make cycle expansion techniques work for general 1-dimensional maps with a single isolated marginal fixed point. To keep the notation simple, we will consider two-branch maps with a complete binary symbolic dynamics as for example the Farey map, figure 21.3, or the repeller depicted in figure 21.2. The necessary modifications

⇓PRELIMINARY

induced map

for multi-branch maps will briefly be discussed in the remark **??**. We again assume that the behavior near the fixed point is given by (21.1). This implies that the stability of a family of periodic orbits approaching the marginally stable orbit, as for example the family 10^n, will increase only algebraically, that is we find again for large n

missing remark?
⇑PRELIMINARY

$$\frac{1}{\Lambda_{10^n}} \sim \frac{1}{n^{1+1/s}} ,$$

move the following to infinite symbolic dynamics?

where s denotes the intermittency exponent.

When considering zeta functions or trace formulas, we again have to take out the marginal orbit $\overline{0}$; periodic orbit contributions of the form t_{0^n1} are now unbalanced and we arrive at a cycle expansion in terms of infinitely many fundamental terms as for our toy map. This corresponds to moving from our binary symbolic dynamics to an infinite symbolic dynamics by making the identification

$$10^{n-1} \to n; \quad 10^{n-1}10^{m-1} \to nm; \quad 10^{n-1}10^{m-1}10^{k-1} \to nmk; \ldots$$

see also table 21.1. The topological length of the orbit is thus no longer determined by the iterations of our two-branch map, but by the number of times the cycle goes from the right to the left branch. Equivalently, one may define a new map, for which all the iterations on the left branch are done in one step. Such a map is called an *induced map* and the topological length of orbits in the [3] infinite alphabet corresponds to the iterations of this induced map, see also remark **??**.

⇓PRIVATE
find the remark
⇑PRIVATE
12.1
page 220

For generic intermittent maps, curvature contributions in the cycle expanded zeta function will not vanish exactly. The most natural way to organize the cycle expansion is to collect orbits and pseudo orbits of the same topological length with respect to the infinite alphabet. Denoting cycle weights in the new alphabet as $t_{nm\ldots} = t_{10^{n-1}10^{m-1}\ldots}$, one obtains

$$\begin{aligned}
\zeta^{-1} &= \prod_{p\neq 0}(1-t_p) = 1 - \sum_{n=1}^{\infty} ce \qquad (21.31)\\
&= 1 - \sum_{n=1}^{\infty} t_n - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{2}(t_{mn} - t_m t_n)\\
&- \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\frac{1}{3}t_{kmn} - \frac{1}{2}t_{km}t_n + \frac{1}{6}t_k t_m t_n) - \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\ldots .
\end{aligned}$$

[4] The first sum is the fundamental term, which we have already seen in the toy model, (21.10). The curvature terms c_n in the expansion are now e-fold

[3] Tanner: Reference for induced map
[4] Tanner: Check how curvature, fundamental term etc defined

∞ – alphabet		binary alphabet				
		$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
1-cycles	n	1	10	100	1000	10000
2-cycles	mn					
	$1n$	11	110	1100	11000	110000
	$2n$	101	0101	10100	101000	1010000
	$3n$	1001	10010	100100	1001000	10010000
	$4n$	10001	100010	1000100	10001000	100010000
3-cycles	kmn					
	$11n$	111	1110	11100	111000	1110000
	$12n$	1101	11010	110100	1101000	11010000
	$13n$	11001	110010	1100100	11001000	110010000
	$21n$	1011	10110	101100	1011000	10110000
	$22n$	10101	101010	1010100	10101000	101010000
	$23n$	101001	1010010	10100100	101001000	1010010000
	$31n$	10011	100110	1001100	10011000	100110000
	$32n$	100101	1001010	10010100	100101000	1001010000
	$33n$	1001001	10010010	100100100	1001001000	10010010000

Table 21.1: Infinite alphabet versus the original binary alphabet for the shortest periodic orbit families. Repetitions of prime cycles ($11 = 1^2, 0101 = 01^2, \ldots$) and their cyclic repeats ($110 = 101, 1110 = 1101, \ldots$) are accounted for by cancellations and combination factors in the cycle expansion (21.31).

infinite sums where the prefactors take care of double counting of prime periodic orbits.

Let us consider the fundamental term first. For generic intermittent maps, we can not expect to obtain an analytic expression for the infinite sum of the form [5]

$$f(z) = \sum_{n=0}^{\infty} h_n z^n. \tag{21.32}$$

with algebraically decreasing coefficients

$$h_n \sim \frac{1}{n^\alpha} \quad \text{with} \quad \alpha > 0$$

To evaluate the sum, we face the same problem as for our toy map: the power series diverges for $z > 1$, that is, exactly in the 'interesting' region where poles, zeros or branch cuts of the zeta function are to be expected. By carefully subtracting the asymptotic behavior with the help of (21.11) or (21.12), one can in general construct an analytic continuation of $f(z)$ around $z = 1$ of the form

$$f(z) \quad \sim \quad A(z) + (1-z)^{\alpha-1} B(z) \qquad \alpha \notin \mathbb{N} \tag{21.33}$$

[5] Tanner: Do we need Tauberian theorem; make into remark

intermittency!resummation
resummation!intermittency

$$f(z) \quad \sim \quad A(z) + (1-z)^{\alpha-1}\ln(1-z) \qquad \alpha \in \mathbb{N}\,,$$

where $A(z)$ and $B(z)$ are functions analytic in a disc around $z = 1$. We thus again find that the zeta function (21.31) has a branch cut along the real axis $\mathrm{Re}\, z \geq 1$. From here on we can switch to auto-pilot and derive algebraic escape, decay of correlation and all the rest. We find in particular that the asymptotic behavior derived in (21.24) and (21.25) is a general result, that is, the survival probability is given asymptotically by

$$\Gamma_n \sim C\frac{1}{n^{1/s}} \tag{21.34}$$

for all 1-dimensional maps of the form (21.1). We have to work a bit harder if we want more detailed information like the prefactor C, exponential precursors given by zeros or poles of the dynamical zeta function or higher order corrections. This information is buried in the functions $A(z)$ and $B(z)$ or more generally in the analytically continued zeta function. To get this analytic continuation, one may follow either of the two different strategies which we will sketch next.

⇓PRELIMINARY

21.3.1 Infinite alphabets

The meat of this section will be provided by Gregor Tanner, who promises (7 Feb 1996): "I hope to have something different and new the next weeks. So, I would like to wait a little bit until I have figured out, what I can do. But you get something, ... versprochen ist versprochen!"

Explain intermitent map - maybe the Farey map?

If plan GT fails, we follow refs. [18.2, 18.3] in which it is explained that for infinite alphabets the curvature expansions looks something like table 21.2.

table 21.2: in the application of AACII, sect. ref GAUSS, the left hand side labels are the corresponding continued fraction entries, and the right hand side are the corresponding binary Farey labels.

⇑PRELIMINARY

21.3.2 Resummation

One way to get information about the zeta function near the branch cut is to derive the leading coefficients in the Taylor series of the functions $A(z)$ and $B(z)$ in (21.33) at $z = 1$. This can be done in principle, if the coefficients h_n in sums like (21.32) are known (as for our toy model). One then considers a resummation of the form

$$\sum_{j=0}^{\infty} h_j z^j = \sum_{j=0}^{\infty} a_j (1-z)^j + (1-z)^{\alpha-1} \sum_{j=0}^{\infty} b_j (1-z)^j, \tag{21.35}$$

and the coefficients a_j and b_j are obtained in terms of the h_j's by expanding $(1-z)^j$ and $(1-z)^{j+\alpha-1}$ on the right hand side around $z = 0$ using (21.11) and equating the coefficients.

n	$=1$		2		3		4		5		6		7
$\hat{t}_1$	$=t_1$	$+$	t_{10}	$+$	t_{100}	$+$	t_{1000}	$+$	t_{10000}	$+$	t_{100000}	$+$	$t_{1000000}$
$\hat{t}_{12}$	$=$				t_{110}	$+$	t_{1100}	$+$	t_{11000}	$+$	t_{110000}	$+$	$t_{1100000}$
$\hat{t}_{112}$	$=$						t_{1110}	$+$	t_{11100}	$+$	t_{111000}	$+$	$t_{1110000}$
$\hat{t}_{23}$	$=$								t_{10100}	$+$	t_{101000}	$+$	$t_{1010000}$
$\hat{t}_{122}$	$=$								t_{11010}	$+$	t_{110100}	$+$	$t_{1101000}$
$\hat{t}_{1112}$	$=$								t_{11110}	$+$	t_{111100}	$+$	$t_{1111000}$
$\hat{t}_{132}$	$=$										t_{110010}	$+$	$t_{1100100}$
$\hat{t}_{1122}$	$=$										t_{111010}	$+$	$t_{1110100}$
$\hat{t}_{11112}$	$=$										t_{111110}	$+$	$t_{1111100}$
$\hat{t}_{34}$	$=$												$t_{1001000}$
$\hat{t}_{142}$	$=$												$t_{1100010}$
$\hat{t}_{223}$	$=$												$t_{1010100}$
$\hat{t}_{1213}$	$=$												$t_{1101100}$
$\hat{t}_{1132}$	$=$												$t_{1110010}$
$\hat{t}_{1222}$	$=$												$t_{1101010}$
$\hat{t}_{11122}$	$=$												$t_{1111010}$
$\hat{t}_{11212}$	$=$												$t_{1110110}$
$\hat{t}_{111112}$	$=$												$t_{1111110}$

Table 21.2: The infinite sequences of cycles used in computing the curvature corrections for binary dynamics with the $\overline{0}$ fixed point pruned. The left hand side is labeled by the integer labels of eq. (?!), $\hat{t}_{a_1\cdots a_k} = \sum_{a=a_k}^{\infty} t_{a_1\cdots a_{k-1},a}$; the right hand side by the corresponding binary labels.

In practical calculations one often has only a finite number of coefficients h_j, $0 \le j \le N$, which may have been obtained by finding periodic orbits and their stabilities numerically. One can still design a resummation scheme for the computation of the coefficients a_j and b_j in (21.35). We replace the infinite sums in (21.35) by finite sums of increasing degrees n_a and n_b, and require that

$$\sum_{i=0}^{n_a} a_i(1-z)^i + (1-z)^{\alpha-1} \sum_{i=0}^{n_b} b_i(1-z)^i = \sum_{i=0}^{N} h_i z^i + O(z^{N+1}) \quad . (21.36)$$

One proceeds again by expanding the right hand side around $z = 0$, skipping all powers z^{N+1} and higher, and then equating coefficients. It is natural to require that $|n_b + \alpha - 1 - n_a| < 1$, so that the maximal powers of the two sums in (21.36) are adjacent. If one chooses $n_a + n_b + 2 = N + 1$, then, for each cutoff length N, the integers n_a and n_b are uniquely determined from a linear system of equations. The price we pay is that the so obtained coefficients depend on the cutoff N. One can now study convergence of the coefficients a_j, and b_j, with respect to increasing values of N, or various quantities derived from a_j and b_j. Note that the leading coefficients a_0 and b_0 determine the prefactor C in (21.34), cf. (21.23). The resummed expression can also be used to compute zeros, inside or outside the radius of convergence of the cycle expansion $\sum h_j z^j$.

The scheme outlined in this section tacitly assumes that a representation of form (21.33) holds in a disc of radius 1 around $z = 1$. Convergence is

Poisson!resummation

improved further if additional information about the asymptotics of sums like (21.32) is used to improve the ansatz (21.35).

21.3.3 Analytical continuation by integral transformations

We will now introduce a method which provides an analytic continuation of sums of the form (21.32) without explicitly relying on an ansatz (21.35). The main idea is to rewrite the sum (21.32) as a sum over integrals with the help of the Poisson summation formula and find an analytic continuation of each integral by contour deformation. In order to do so, we need to know the n dependence of the coefficients $h_n \equiv h(n)$ explicitly for all n. If the coefficients are not known analytically, one may proceed by approximating the large n behavior in the form

$$h(n) = n^{-\alpha}(C_1 + C_2 n^{-1} + \ldots)\,, \qquad n \neq 0\,,$$

and determine the constants C_i numerically from periodic orbit data. By using the Poisson resummation identity

$$\sum_{n=-\infty}^{\infty} \delta(x-n) = \sum_{m=-\infty}^{\infty} \exp(2\pi i m x)\,, \tag{21.37}$$

we may write the sum as (21.32)

$$f(z) = \frac{1}{2}h(0) + \sum_{m=-\infty}^{\infty} \int_0^{\infty} dx\, e^{2\pi i m x} h(x) z^x. \tag{21.38}$$

The continuous variable x corresponds to the discrete summation index n and it is convenient to write $z = r\exp(i\sigma)$ from now on. The integrals are still not convergent for $r > 0$, but an analytical continuation can be found by considering the contour integral, where the contour goes out along the real axis, makes a quarter circle to either the positive or negative imaginary axis and goes back to zero. By letting the radius of the circle go to infinity, we essentially rotate the line of integration from the real onto the imaginary axis. For the $m = 0$ term in (21.38), we transform $x \to ix$ and the integral takes on the form

$$\int_0^{\infty} dx\, h(x)\, r^x\, e^{ix\sigma} = i \int_0^{\infty} dx\, h(ix)\, r^{ix} e^{-x\sigma}.$$

The integrand is now exponentially decreasing for all $r > 0$ and $\sigma \neq 0$ or 2π. The last condition reminds us again of the existence of a branch cut at $\mathrm{Re}\, z \geq 1$. By the same technique, we find the analytic continuation for

all the other integrals in (21.38). The real axis is then rotated according to $x \to \mathrm{sign}(m) ix$ where $\mathrm{sign}(m)$ refers to the sign of m.

$$\int_0^\infty dx\, e^{\pm 2\pi i|m|x} h(x)\, r^x e^{ix\sigma} = \pm \mathrm{i} \int_0^\infty dx\, h(\pm ix)\, r^{\pm ix} e^{-x(2\pi|m|\pm\sigma)}.$$

Changing summation and integration, we can carry out the sum over $|m|$ explicitly and one finally obtains the compact expression

$$\begin{aligned} f(z) &= \frac{1}{2}h(0) + i\int_0^\infty dx\, h(ix)\, r^{ix} e^{-x\sigma} \qquad (21.39) \\ &+ i\int_0^\infty dx\, \frac{e^{-2\pi x}}{1-e^{-2\pi x}} \left[h(ix) r^{ix} e^{-x\sigma} - h(-ix) r^{-ix} e^{x\sigma} \right]. \end{aligned}$$

need explicit example, problem sets

The transformation from the original sum to the two integrals in (21.39) is exact for $r \leq 1$, and provides an analytic continuation for $r > 0$. The expression (21.39) is especially useful for an efficient numerical calculations of a dynamical zeta function for $|z| > 1$, which is essential when searching for its zeros and poles.

21.3.4 Curvature contributions

So far, we have discussed only the fundamental term $\sum_{n=1}^\infty t_n$ in (21.31), and showed how to deal with such power series with algebraically decreasing coefficients. The fundamental term determines the main structure of the zeta function in terms of the leading order branch cut. Corrections to both the zeros and poles of the dynamical zeta function as well as the leading and subleading order terms in expansions like (21.33) are contained in the curvature terms in (21.31). The first curvature correction is the 2-cycle sum

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{1}{2}(t_{mn} - t_m t_n),$$

with algebraically decaying coefficients which again diverge for $|z| > 1$. The analytically continued curvature terms have as usual branch cuts along the positive real z axis. Our ability to calculate the higher order curvature terms depends on how much we know about the cycle weights t_{mn}. The form of the cycle stability (21.5) suggests that t_{mn} decrease asymptotically as

$$t_{mn} \sim \frac{1}{(nm)^{1+1/s}} \qquad (21.40)$$

for 2-cycles, and in general for n-cycles as

$$t_{m_1 m_2 \ldots m_n} \sim \frac{1}{(m_1 m_2 \ldots m_n)^{1+1/s}}.$$

first return time
probabilistic zeta function
zeta function!probabilistic
return time

in AACII I ar
for exponent
decreasing curva
here as well, may
was wrong?

If we happen to know the cycle weights $t_{m_1 m_2 \ldots m_n}$ analytically, we may proceed as in sect. 21.3.3, transform the multiple sums into multiple integrals and rotate the integration contours.

We have reached the edge of what has been accomplished so far in computing and what is worth the dynamical zeta functions from periodic orbit data. In the next section, we describe a probabilistic method applicable to intermittent maps which does not rely on periodic orbits.

21.4 BER zeta functions

So far we have focused on 1-d models as the simplest setting in which to investigate dynamical implications of marginal fixed points. We now take an altogether different track and describe how probabilistic methods may be employed in order to write down approximate dynamical zeta functions for intermittent systems.

We will discuss the method in a very general setting, for a flow in arbitrary dimension. The key idea is to introduce a surface of section $\mathcal{P}$ such that all trajectories traversing this section will have spent some time both near the marginal stable fixed point and in the chaotic phase. An important quantity in what follows is (3.5), the *first return time* $\tau(x)$, or the time of flight of a trajectory starting in x to the next return to the surface of section $\mathcal{P}$. The period of a periodic orbit p intersecting the $\mathcal{P}$ section n_p times is

$$T_p = \sum_{k=0}^{n_p-1} \tau(f^k(x_p)),$$

where $f(x)$ is the Poincaré map, and $x_p \in \mathcal{P}$ is a cycle point. The dynamical zeta function (15.16)

$$1/\zeta(z,s,\beta) = \prod_p \left(1 - \frac{z^{n_p} e^{\beta A_p - s T_p}}{|\Lambda_p|}\right), \qquad A_p = \sum_{k=0}^{n_p-1} a(f^k(x_p)), \quad (21.41)$$

☞ chapter 10

associated with the observable $a(x)$ captures the dynamics of both the flow *and* the Poincaré map. The dynamical zeta function for the flow is obtained as $1/\zeta(s,\beta) = 1/\zeta(1,s,\beta)$, and the dynamical zeta function for the discrete time Poincaré map is $1/\zeta(z,\beta) = 1/\zeta(z,0,\beta)$.

Our basic assumption will be *probabilistic*. We assume that the chaotic interludes render the consecutive *return* (or *recurrence*) *times* $T(x_i)$, $T(x_{i+1})$ and observables $a(x_i)$, $a(x_{i+1})$ effectively uncorrelated. Consider

to some earlier
ition of $T(x_0, n)$

the quantity $e^{\beta A(x_0,n)-sT(x_0,n)}$ averaged over the surface of section $\mathcal{P}$. With the above probabilistic assumption the large n behavior is

$$\langle e^{\beta A(x_0,n)-sT(x_0,n)}\rangle_{\mathcal{P}} \sim \left(\int_{\mathcal{P}} e^{\beta a(x)-s\tau}\rho(x)dx\right)^n ,$$

where $\rho(x)$ is the invariant density of the Poincaré map. This type of behavior is equivalent to there being only one zero $z_0(s,\beta) = \int e^{\beta a(x)-s\tau(x)}\rho(x)dx$ of $1/\zeta(z,s,\beta)$ in the z-β plane. In the language of Ruelle-Pollicott resonances this means that there is an infinite gap to the first resonance. This in turn implies that $1/\zeta(z,s,\beta)$ may be written as

☞ remark 10.1

$$1/\zeta(z,s,\beta) = z - \int_{\mathcal{P}} e^{\beta a(x)-s\tau(x)}\rho(x)dx \quad , \tag{21.42}$$

where we have neglected a possible analytic and non-zero prefactor. The dynamical zeta function of the flow is now

$$1/\zeta(s,\beta) = 1/\zeta(1,s,\beta) = 1 - \int_{\mathcal{P}} e^{\beta a(x)}\rho(x)e^{-s\tau(x)}dx \quad . \tag{21.43}$$

Normally, the best one can hope for is a finite gap to the leading resonance of the Poincaré map. with the above dynamical zeta function only approximatively valid. As it is derived from an approximation due to Baladi, Eckmann, and Ruelle, we shall refer to it as the BER zeta function $1/\zeta_{\rm BER}(s,\beta)$ in what follows.

A central role is played by the probability distribution of return times

$$\psi(\tau) = \int_{\mathcal{P}} \delta(\tau-\tau(x))\rho(x)dx \tag{21.44}$$

24.7
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The BER zeta function at $\beta = 0$ is then given in terms of the Laplace transform of this distribution

$$1/\zeta_{\rm BER}(s) = 1 - \int_0^\infty \psi(\tau)e^{-s\tau}d\tau.$$

21.7
page 423

Example 21.1 Return times for the Bernoulli map. *For the Bernoulli shift map (16.6)*

$$x \mapsto f(x) = 2x \ \mathit{mod}\ 1,$$

one easily derives the distribution of return times

$$\psi_n = \frac{1}{2^n} \qquad n \geq 1.$$

Ruelle!-Pollicott resonances
resonance!Ruelle-Pollicott
Pollicott, M
Ruelle, D
return time!distribution
Bernoulli!shift
shift!Bernoulli

Laplace!transform, discrete

The BER zeta function becomes (by the discrete Laplace transform (14.8))

$$1/\zeta_{BER}(z) = 1 - \sum_{n=1}^{\infty} \psi_n z^n = 1 - \sum_{n=1}^{\infty} \frac{z^n}{2^n}$$
$$= \frac{1-z}{1-z/2} = \zeta^{-1}(z)/(1-z/\Lambda_0)\,. \tag{21.45}$$

Thanks to the uniformity of the piecewise linear map measure (10.19) the "approximate" zeta function is in this case the exact *dynamical zeta function, with the cycle point* $\overline{0}$ *pruned.*

Example 21.2 Return times for the model of sect. 21.2.1. *For the toy model of sect. 21.2.1 one gets* $\psi_1 = |\mathcal{M}_1|$, *and* $\psi_n = |\mathcal{M}_n|(1-b)/(1-a)$, *for* $n \geq 2$, *leading to a BER zeta function*

$$1/\zeta_{BER}(z) = 1 - z|\mathcal{M}_1| - \sum_{n=2}^{\infty} |\mathcal{M}_n| z^n,$$

which again coincides with the exact result, (21.10). [6]

wrong! it is because they are linear

It may seem surprising that the BER approximation produces exact results in the two examples above. The reason for this peculiarity is that both these systems are piecewise linear and have complete Markov partitions. As long as the map is piecewise linear and complete, and the probabilistic approximation is exactly fulfilled, the cycle expansion curvature terms vanish. The BER zeta function and the fundamental part of a cycle expansion discussed in sect. 18.1.1 are indeed intricately related, but not identical in general. In particular, note that the BER zeta function obeys the flow conservation sum rule (19.11) by construction, whereas the fundamental part of a cycle expansion as a rule does not.

Commentary

Remark 21.1 What about the evolution operator formalism? The main virtue of evolution operators was their semigroup property (10.25). This was natural for hyperbolic systems where instabilities grow exponentially, and evolution operators capture this behavior due to their multiplicative nature. Whether the evolution operator formalism is a good way to capture the slow, power law instabilities of intermittent dynamics is less clear. The approach taken here leads us to a formulation in terms of *dynamical zeta functions* rather than spectral determinants, circumventing evolution operators altogether. It is not known if the spectral determinants formulation would yield any benefits when applied to intermittent chaos. Some results on spectral determinants and intermittency can be found in [21.2]. A useful mathematical technique to deal with isolated marginally stable

[6] Tanner: Check formula for ψ_n; differs from PD, but PD's result does not give the exact zeta function

fixed point is that of *inducing*, that is, replacing the intermittent map by a completely hyperbolic map with infinite alphabet and redefining the discrete time; we have used this method implicitly by changing from a finite to an infinite alphabet. We refer to refs. [21.3, 21.19] for detailed discussions of this technique, as well as applications to 1-dimensional maps.

piecewise linear map
Gauss map
Farey!map
Lorentz gas
Riemann zeta function
Tauberian theorems etc.
Kaufmann references
BER!approximation

Remark 21.2 Intermittency. Intermittency was discovered by Manneville and Pomeau [21.1] in their study of the Lorentz system. They demonstrated that in neighborhood of parameter value $r_c = 166.07$ the mean duration of the periodic motion scales as $(r-r_c)^{1/2}$. In ref. [21.5] they explained this phenomenon in terms of a 1-dimensional map (such as (21.1)) near tangent bifurcation, and classified possible types of intermittency.

Piecewise linear models like the one considered here have been studied by Gaspard and Wang [21.6]. The escape problem has here been treated following ref. [21.7], resummations following ref. [21.8]. The proof of the bound (21.27) can be found in P. Dahlqvist's notes on `ChaosBook.org/PDahlqvistEscape.ps.gz`.

Farey map (18.32) has been studied widely in the context of intermittent dynamics, for example in refs. [21.16, 21.17, 18.3, 21.18, N.23, 18.14, 21.2]. The Fredholm determinant and the dynamical zeta functions for the Farey map (18.32) and the related Gauss shift map (28.38) have been studied by Mayer [21.16]. He relates the continued fraction transformation to the Riemann zeta function, and constructs a Hilbert space on which the evolution operator is self-adjoint, and its eigenvalues are exponentially spaced, just as for the dynamical zeta functions [21.23] for "Axiom A" hyperbolic systems.

Remark 21.3 Tauberian theorems. In this chapter we used Tauberian theorems for power series and Laplace transforms: Feller's monograph [21.9] is a highly recommended introduction to these methods.

Remark 21.4 Probabilistic methods, BER zeta functions. Probabilistic description of intermittent chaos was introduced by Geisal and Thomae [21.10]. The BER approximation studied here is inspired by Baladi, Eckmann and Ruelle [21.14], with further developments in refs. [21.13, 21.15].

Résumé

The presence of marginally stable fixed points and cycles changes the analytic structure of dynamical zeta functions and the rules for constructing cycle expansions. The marginal orbits have to be omitted, and the cycle expansions now need to include families of infinitely many longer and longer unstable orbits which accumulate toward the marginally stable cycles. Correlations for such non-hyperbolic systems may decay algebraically with the decay

rates controlled by the branch cuts of dynamical zeta functions. Compared to pure hyperbolic systems, the physical consequences are drastic: exponential decays are replaced by slow power-law decays, and transport properties, such as the diffusion may become anomalous.

link to different chapters

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Exercises

⇓PRELIMINARY

Exercise 21.1 Scaling of covering intervals for intermittent map. *The bound (19.6) is violated for intermittent systems. Show that the problem is that and the survivor strip $\mathcal{M}_i$ which includes the marginal fixed scales with symol string length differently from its estimate by $1/|\Lambda_i|$.*

⇑PRELIMINARY

Exercise 21.2 Integral representation of Jonquière functions. *Check the integral representation*

$$J(z,\alpha) = \frac{z}{\Gamma(\alpha)} \int_0^\infty d\xi \, \frac{\xi^{\alpha-1}}{e^\xi - z} \qquad \text{for} \quad \alpha > 0\,. \tag{21.46}$$

Note how the denominator is connected to Bose-Einstein distribution. Compute $J(x+i\epsilon) - J(x-i\epsilon)$ for a real $x > 1$.

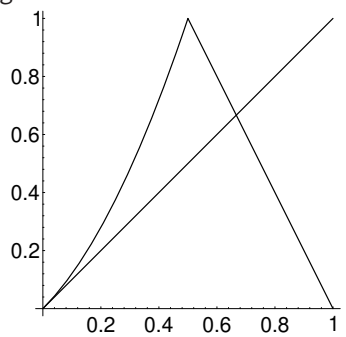
Exercise 21.3 Power law correction to a power law. *Expand (21.20) further and derive the leading power law correction to (21.23).*

⇓PRELIMINARY

Exercise 21.4 Inverse Laplace. *Consider (24.32) in the case of* discrete *time mappings: show that it can be rewritten in a form analogous to (??).*

⇑PRELIMINARY

Exercise 21.5 Power-law fall off. *In cycle expansions the stabilities of orbits do not always behave in a geometric fashion. Consider the map f*



This map behaves as $f \to x$ as $x \to 0$. Define a symbolic dynamics for this map by assigning 0 to the points that land on the interval $[0, 1/2)$ and 1 to the points that land on $(1/2, 1]$. Show that the stability of orbits that spend a long time on the 0 side goes as n^2. In particular, show that

$$\Lambda_{\underbrace{00\cdots0}_{n}1} \sim n^2$$

Exercise 21.6 Power law fall-off of stability eigenvalues in the stadium billiard.** *From the cycle expansions point of view, the most important consequence of the shear in $\mathbf{J}^n$ for long sequences of rotation bounces n_k in (6.21) is that the Λ_n grows only as a power law in number of bounces:*

$$\Lambda_n \propto n_k^2\,. \tag{21.47}$$

Check.

Exercise 21.7 Probabilistic zeta function for maps. *Derive the probabilistic zeta function for a map with recurrence distribution ψ_n.*

Exercise 21.8 Accelerated diffusion. *Consider a map h, such that $\hat{h} = \hat{f}$, but now running branches are turner into standing branches and vice versa, so that $1, 2, 3, 4$ are standing while 0 leads to both positive and negative jumps. Build the corresponding dynamical zeta function and show that*

$$\sigma^2(t) \sim \begin{cases} t & \text{for } \alpha > 2 \\ t \ln t & \text{for } \alpha = 2 \\ t^{3-\alpha} & \text{for } \alpha \in (1,2) \\ t^2/\ln t & \text{for } \alpha = 1 \\ t^2 & \text{for } \alpha \in (0,1) \end{cases}$$

Exercise 21.9 Anomalous diffusion (hyperbolic maps). *Anomalous diffusive properties are associated to deviations from linearity of the variance of the phase variable we are looking at: this means the the diffusion constant (10.13) either vanishes or diverges. We briefly illustrate in this exercise how the local local properties of a map are crucial to account for anomalous behavior even for hyperbolic systems.*

rewrite this

Consider a class of piecewise linear maps, relevant to the problem of the onset of diffusion, defined by

need a figure here

$$f_\epsilon(x) = \begin{cases} \Lambda x & \text{for } x \in \left[0, x_1^+\right] \\ a - \Lambda_{\epsilon,\gamma}|x - x^+| & \text{for } x \in \left[x_1^+, x_2^+\right] \\ 1 - \Lambda'(x - x_2^+) & \text{for } x \in \left[x_2^+, x_1^-\right] \\ 1 - a + \Lambda_{\epsilon,\gamma}|x - x^-| & \text{for } x \in \left[x_1^-, x_2^-\right] \\ 1 + \Lambda(x-1) & \text{for } x \in \left[x_2^-, 1\right] \end{cases} \tag{21.48}$$

where $\Lambda = (1/3 - \epsilon^{1/\gamma})^{-1}$, $\Lambda' = (1/3 - 2\epsilon^{1/\gamma})$, $\Lambda_{\epsilon,\gamma} = \epsilon^{1-1/\gamma}$, $a = 1 + \epsilon$, $x^+ = 1/3$, $x_1^+ = x^+ - \epsilon^{1/\gamma}$, $x_2^+ = x^+ + \epsilon^{1/\gamma}$, and the usual symmetry properties (24.11) are satisfied.

Thus this class of maps is characterized by two escaping windows (through which the diffusion process may take place) of size $2\epsilon^{1/\gamma}$: the exponent γ mimicks the order of the maximum for a continuous map, while piecewise linearity, besides making curvatures vanish and leading to finite cycle expansions, prevents the appearance of stable cycles. The symbolic dynamics is easily described once we consider a sequence of parameter values $\{\epsilon_m\}$, where $\epsilon_m = \Lambda^{-(m+1)}$: we then partition the unit interval though the sequence of points $0, x_1^+, x^+, x_2^+, x_1^-, x^-, x_2^-, 1$ and label the corresponding sub–intervals $1, s_a, s_b, 2, d_b, d_a, 3$: symbolic dynamics is described by an unrestricted grammar over the following set of symbols

$$\{1, 2, 3, s_\# \cdot 1^i, d_\# \cdot 3^k\} \qquad \# = a, b \quad i, k = m, m+1, m+2, \ldots$$

This leads to the following dynamical zeta function:

$$\zeta_0^{-1}(z, \alpha) = 1 - \frac{2z}{\Lambda} - \frac{z}{\Lambda'} - 4\cosh(\alpha)\epsilon_m^{1/\gamma - 1}\frac{z^{m+1}}{\Lambda^m}\left(1 - \frac{z}{\Lambda}\right)^{-1}$$

from which, by (24.8) we get

$$D = \frac{2\epsilon_m^{1/\gamma-1}\Lambda^{-m}(1-1/\Lambda)^{-1}}{1-\frac{2}{\Lambda}-\frac{1}{\Lambda'}-4\epsilon_m^{1/\gamma-1}\left(\frac{m+1}{\Lambda^m(1-1/\Lambda)}+\frac{1}{\Lambda^{m+1}(1-1/\Lambda)^2}\right)} \tag{21.49}$$

The main interest in this expression is that it allows exploring how D vanishes in the $\epsilon \mapsto 0$ ($m \mapsto \infty$) limit: as a matter of fact, from (21.49) we get the asymptotic behavior $D \sim \epsilon^{1/\gamma}$, which shows how the onset of diffusion is governed by the order of the map at its maximum.

Remark 21.5 Onset of diffusion for continuous maps. The zoology of behavior for continuous maps at the onset of diffusion is described in refs. [24.11, 24.12, 21.24]: our treatment for piecewise linear maps was introduced in ref. [21.25].

⇓PRIVATE

Chapter 22

Discrete symmetries

Utility of discrete symmetries in reducing spectrum calculations is familiar from quantum mechanics. Here we show that the classical spectral determinants factor in essentially the same way as in quantum mechanics. In the process we also learn how to simplify the classical dynamics. The main result of this chapter can be stated as follows:

If the dynamics possesses a discrete symmetry, the contribution of a cycle p of multiplicity m_p to a dynamical zeta function factorizes into a product over the d_α-dimensional irreducible representations D_α of the symmetry group,

$$(1-t_p)^{m_p} = \prod_\alpha \det\,(1-D_\alpha(h_{\tilde{p}})t_{\tilde{p}})^{d_\alpha}\,, \quad t_p = t_{\tilde{p}}^{g/m_p}\,,$$

where $t_{\tilde{p}}$ is the cycle weight evaluated on the fundamental domain, g is the dimension of the group, $h_{\tilde{p}}$ is the group element relating the fundamental domain cycle $\tilde{p}$ to a segment of the full space cycle p, and m_p is the multiplicity of the p cycle. As the dynamical zeta functions have particularly simple cycle expansions, a simple geometrical shadowing interpretation of their convergence, and as they suffice for determination of leading eigenvalues, we shall concentrate in this chapter on their factorizations; the full spectral determinants can be factorized by the same techniques. To emphasize the group theoretic structure of zeta functions, we shall combine all the non-group-theory dependence of a p-cycle into a cycle weight t_p.

This chapter is meant to serve as a detailed guide to computation of dynamical zeta functions and spectral determinants for systems with discrete symmetries. Familiarity with basic group-theoretic notions is assumed, with the definitions relegated to appendix J.1. We develop here the cycle expansions for factorized determinants, and exemplify them by working out a series of cases of physical interest: C_2, C_{3v} symmetries in this chapter, and C_{2v}, C_{4v} symmetries in appendix J below.

irreducible!segment
three-disk@3-disk!symmetry
symmetry!3-disk

22.1 Preview

Dynamical systems often come equipped with discrete symmetries, such as the reflection and the rotation symmetries of various potentials. Such symmetries simplify and improve the cycle expansions in a rather beautiful way; they can be exploited to relate classes of periodic orbits and reduce dynamics to a fundamental domain. Furthermore, in classical dynamics, just as in quantum mechanics, the symmetrized subspaces can be probed by linear operators of different symmetries. If a linear operator commutes with the symmetry, it can be block-diagonalized, and, as we shall now show, the associated spectral determinants and dynamical zeta functions factorize.

Invariance of a system under symmetries means that the symmetry image of a cycle is again a cycle, with the same weight. The new orbit may be topologically distinct (in which case it contributes to the multiplicity of the cycle) or it may be the same cycle, shifted in time. A cycle is *symmetric* if some symmetry operations act on it like a shift in time, advancing the starting point to the starting point of a symmetry related segment. A symmetric cycle can thus be subdivided into a sequence of repeats of an *irreducible segment*. The period or any average evaluated along the full orbit is given by the sum over the segments, whereas the stability is given by the product of the stability matrices of the individual segments.

Cycle degeneracies induced by the symmetry are removed by *desymmetrization*, reduction of the full dynamics to the dynamics on a *fundamental domain*.

Give formal definition of the fundamental domain.

The state space can be completely tiled by a fundamental domain and its symmetry images. The irreducible segments of cycles in the full space, folded back into the fundamental domain, are closed orbits in the reduced space.

22.1.1 3-disk game of pinball

We have already exploited a discrete symmetry in our introduction to the 3-disk game of pinball, sect. 1.3. As the three disks are equidistantly spaced, our game of pinball has a sixfold symmetry. The symmetry group of relabeling the 3 disks is the permutation group S_3; however, it is better to think of this group geometrically, as C_{3v}, the group of rotations by $\pm 2\pi/3$ and reflections across the three symmetry axes. Applying an element (identity, rotation by $\pm 2\pi/3$, or one of the three possible reflections) of this symmetry group to any trajectory yields another trajectory. For instance, the cycles $\overline{12}$, $\overline{23}$, and $\overline{13}$, are related to each other by rotation by $\pm 2\pi/3$, or, equivalently, by a relabeling of the disks.

add Figures

An irreducible segment corresponds to a periodic orbit in the *fundamental domain*, a one-sixth slice of the full 3-disk system, with the symmetry axes acting as reflecting mirrors, see figure 11.6. A set of orbits related in the full space by discrete symmetries maps onto a single fundamental domain orbit. The reduction to the fundamental domain desymmetrizes the

dynamics and removes all global discrete symmetry induced degeneracies: rotationally symmetric global orbits (such as the 3-cycles $\overline{123}$ and $\overline{132}$) have degeneracy 2, reflection symmetric ones (such as the 2-cycles $\overline{12}$, $\overline{13}$ and $\overline{23}$) have degeneracy 3, and global orbits with no symmetry are 6-fold degenerate. Table 11.2 lists some of the shortest binary symbols strings, together with the corresponding full 3-disk symbol sequences and orbit symmetries. Some examples of such orbits are shown in figure 22.3.

sawtooth map
map!sawtooth

recheck with Freddy!

We shall return to the 3-disk game of pinball desymmetrization in sects. 22.2.2 and 22.6, but first we develop a feeling for discrete symmetries by working out a simple 1-d example.

22.1.2 Reflection symmetric 1-d maps

Consider f, a map on the interval with reflection symmetry $f(-x) = -f(x)$. A simple example is the piecewise-linear sawtooth map of figure 22.1.

Denote the reflection operation by $\mathbf{C}x = -x$. The symmetry of the map implies that if $\{x_n\}$ is a trajectory, than also $\{\mathbf{C}x_n\}$ is a trajectory because $\mathbf{C}x_{n+1} = \mathbf{C}f(x_n) = f(\mathbf{C}x_n)$. The dynamics can be restricted to a fundamental domain, in this case to one half of the original interval; every time a trajectory leaves this interval, it can be mapped back using $\mathbf{C}$. Furthermore, the evolution operator commutes with $\mathbf{C}$, $\mathcal{L}(y,x) = \mathcal{L}(\mathbf{C}y,\mathbf{C}x)$. $\mathbf{C}$ satisfies $\mathbf{C}^2 = \mathbf{e}$ and can be used to decompose the state space into mutually orthogonal symmetric and antisymmetric subspaces by means of projection operators

$$\begin{aligned} P_{A_1} &= \frac{1}{2}(\mathbf{e}+\mathbf{C})\ , \qquad P_{A_2} = \frac{1}{2}(\mathbf{e}-\mathbf{C})\,, \\ \mathcal{L}_{A_1}(y,x) &= P_{A_1}\mathcal{L}(y,x) = \frac{1}{2}\left(\mathcal{L}(y,x)+\mathcal{L}(-y,x)\right)\ , \\ \mathcal{L}_{A_2}(y,x) &= P_{A_2}\mathcal{L}(y,x) = \frac{1}{2}\left(\mathcal{L}(y,x)-\mathcal{L}(-y,x)\right)\ . \end{aligned} \tag{22.1}$$

To compute the traces of the symmetrization and antisymmetrization projection operators (22.1), we have to distinguish three kinds of cycles: asymmetric cycles a, symmetric cycles s built by repeats of irreducible segments $\tilde{s}$, and boundary cycles b. Now we show that the spectral determinant can be written as the product over the three kinds of cycles: $\det(1-\mathcal{L}) = \det(1-\mathcal{L})_a \det(1-\mathcal{L})_{\tilde{s}} \det(1-\mathcal{L})_b$.

Asymmetric cycles: A periodic orbits is not symmetric if $\{x_a\}\cap\{\mathbf{C}x_a\} = \emptyset$, where $\{x_a\}$ is the set of periodic points belonging to the cycle a. Thus $\mathbf{C}$ generates a second orbit with the same number of points and the same stability properties. Both orbits give the same contribution to the first term and no contribution to the second term in (22.1); as they are degenerate, the prefactor 1/2 cancels. Resuming as in the derivation of (15.16) we find

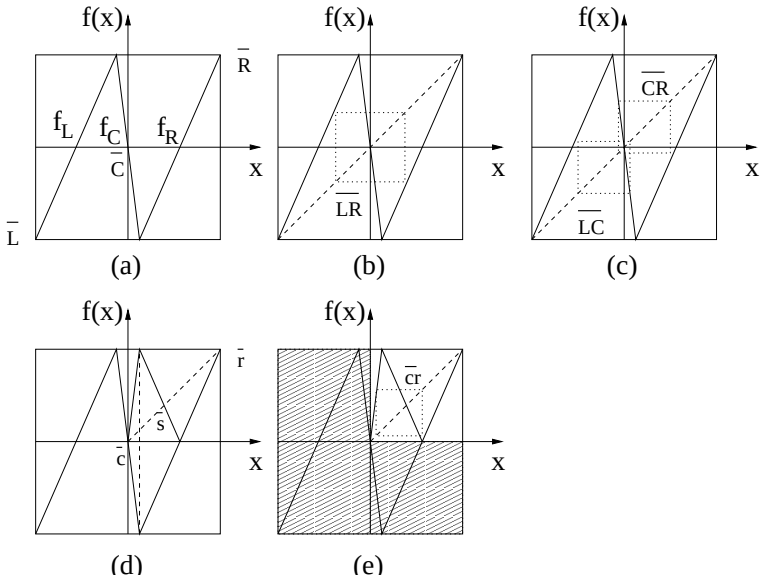


Figure 22.1: The Ulam sawtooth map with the C_2 symmetry $f(-x) = -f(x)$. (a) boundary fixed point $\overline{C}$, (b) symmetric 2-cycle $\overline{LR}$, (c) asymmetric 2-cycles pair $\{\overline{LC},\overline{CR}\}$. The Ulam sawtooth map restricted to the fundamental domain; pieces of the global map (a) are reflected into the upper right quadrant. (d) Boundary fixed point $\overline{C}$ maps into the fixed point $\overline{c}$, symmetric 2-cycle $\overline{LR}$ maps into fixed point $\overline{s}$, and the asymmetric fixed point pair $\{\overline{L},\overline{R}\}$ maps into a single fixed point $\overline{r}$, (e) the asymmetric 2-cycles pair $\{\overline{LC},\overline{CR}\}$ maps into a single 2-cycle $\overline{cr}$.

draw examples of such orbits!

that asymmetric orbits yield the same contribution to the symmetric and the antisymmetric subspaces:

$$\det\left(1-\mathcal{L}_{\pm}\right)_a = \prod_a \prod_{k=0}^{\infty}\left(1-\frac{t_a}{\Lambda_a^k}\right), \quad t_a = \frac{z^{n_a}}{|\Lambda_a|}\,.$$

Symmetric cycles: A cycle s is reflection symmetric if operating with $\mathbf{C}$ on the set of cycle points reproduces the set. The period of a symmetric cycle is always even ($n_s = 2n_{\tilde{s}}$) and the mirror image of the x_s cycle point is reached by traversing the irreducible segment $\tilde{s}$ of length $n_{\tilde{s}}$, $f^{n_{\tilde{s}}}(x_s) = \mathbf{C}x_s$. $\delta(x - f^n(x))$ picks up $2n_{\tilde{s}}$ contributions for every even traversal, $n = rn_{\tilde{s}}$, r even, and $\delta(x + f^n(x))$ for every odd traversal, $n = rn_{\tilde{s}}$, r odd. Absorb the group-theoretic prefactor in the stability eigenvalue by defining the stability computed for a segment of length $n_{\tilde{s}}$,

$$\Lambda_{\tilde{s}} = -\left.\frac{\partial f^{n_{\tilde{s}}}(x)}{\partial x}\right|_{x=x_s}\,.$$

Restricting the integration to the infinitesimal neighborhood $\mathcal{M}_s$ of the s cycle, we obtain the contribution to $\mathrm{tr}\,\mathcal{L}_{\pm}^n$:

$$z^n \mathrm{tr}\,\mathcal{L}_{\pm}^n \;\;\to\;\; \int_{\mathcal{M}_s} dx\, z^n\, \frac{1}{2}\left(\delta(x-f^n(x)) \pm \delta(x+f^n(x))\right)$$

$$
\begin{aligned}
&= n_{\tilde{s}} \left(\sum_{r=2}^{\text{even}} \delta_{n,rn_{\tilde{s}}} \frac{t_{\tilde{s}}^r}{1-1/\Lambda_{\tilde{s}}^r} \pm \sum_{r=1}^{\text{odd}} \delta_{n,rn_{\tilde{s}}} \frac{t_{\tilde{s}}^r}{1-1/\Lambda_{\tilde{s}}^r} \right) \\
&= n_{\tilde{s}} \sum_{r=1}^{\infty} \delta_{n,rn_{\tilde{s}}} \frac{(\pm t_{\tilde{s}})^r}{1-1/\Lambda_{\tilde{s}}^r} \,.
\end{aligned}
$$

Substituting all symmetric cycles s into $\det(1-\mathcal{L}_\pm)$ and resuming we obtain:

recheck $\tilde{s}$ tilde on LHS

$$
\det(1-\mathcal{L}_\pm)_{\tilde{s}} = \prod_{\tilde{s}} \prod_{k=0}^{\infty} \left(1 \mp \frac{t_{\tilde{s}}}{\Lambda_{\tilde{s}}^k}\right)
$$

Boundary cycles: In the example at hand there is only one cycle which is neither symmetric nor antisymmetric, but lies on the boundary of the fundamental domain, the fixed point at the origin. Such cycle contributes simultaneously to both $\delta(x-f^n(x))$ and $\delta(x+f^n(x))$:

$$
\begin{aligned}
z^n \mathrm{tr}\, \mathcal{L}_\pm^n &\to \int_{\mathcal{M}_b} dx\, z^n\, \frac{1}{2}\, (\delta(x-f^n(x)) \pm \delta(x+f^n(x))) \\
&= \sum_{r=1}^{\infty} \delta_{n,r}\, t_b^r\, \frac{1}{2} \left(\frac{1}{1-1/\Lambda_b^r} \pm \frac{1}{1+1/\Lambda_b^r} \right) \\
z^n\, \mathrm{tr}\, \mathcal{L}_+^n &\to \sum_{r=1}^{\infty} \delta_{n,r} \frac{t_b^r}{1-1/\Lambda_b^{2r}}\,; \qquad z^n\, \mathrm{tr}\, \mathcal{L}_-^n \to \sum_{r=1}^{\infty} \delta_{n,r} \frac{1}{\Lambda_b^r} \frac{t_b^r}{1-1/\Lambda_b^{2r}}\,.
\end{aligned}
$$

Boundary orbit contributions to the factorized spectral determinants follow by resummation:

$$
\det(1-\mathcal{L}_+)_b = \prod_{k=0}^{\infty} \left(1 - \frac{t_b}{\Lambda_b^{2k}}\right)\,, \qquad \det(1-\mathcal{L}_-)_b = \prod_{k=0}^{\infty} \left(1 - \frac{t_b}{\Lambda_b^{2k+1}}\right)
$$

Only even derivatives contribute to the symmetric subspace (and odd to the antisymmetric subspace) because the orbit lies on the boundary.

Finally, the symmetry reduced spectral determinants follow by collecting the above results:

$$
F_+(z) = \prod_a \prod_{k=0}^{\infty} \left(1 - \frac{t_a}{\Lambda_a^k}\right) \prod_{\tilde{s}} \prod_{k=0}^{\infty} \left(1 - \frac{t_{\tilde{s}}}{\Lambda_{\tilde{s}}^k}\right) \prod_{k=0}^{\infty} \left(1 - \frac{t_b}{\Lambda_b^{2k}}\right)
$$

$$
F_-(z) = \prod_a \prod_{k=0}^{\infty} \left(1 - \frac{t_a}{\Lambda_a^k}\right) \prod_{\tilde{s}} \prod_{k=0}^{\infty} \left(1 + \frac{t_{\tilde{s}}}{\Lambda_{\tilde{s}}^k}\right) \prod_{k=0}^{\infty} \left(1 - \frac{t_b}{\Lambda_b^{2k+1}}\right) \quad (22.2)
$$

We shall work out the symbolic dynamics of such reflection symmetric systems in some detail in sect. 22.5. As reflection symmetry is essentially the only discrete symmetry that a map of the interval can have, this example completes the group-theoretic factorization of determinants and zeta functions for 1-d maps. We now turn to discussion of the general case.

22.1
page 449

22.2 Discrete symmetries

A dynamical system is invariant under a symmetry group $G = \{e, g_2, \ldots, g_{|G|}\}$ if the equations of motion are invariant under all symmetries $g \in G$. For a map $x_{n+1} = f(x_n)$ and the evolution operator $\mathcal{L}(y, x)$ defined by (10.23) this means

$$\begin{aligned} f(x) &= \mathbf{g}^{-1} f(\mathbf{g}x) \\ \mathcal{L}(y,x) &= \mathcal{L}(\mathbf{g}y, \mathbf{g}x) \ . \end{aligned} \tag{22.3}$$

Bold face letters for group elements indicate a suitable representation on state space. For example, if a 2-dimensional map has the symmetry $x_1 \to -x_1$, $x_2 \to -x_2$, the symmetry group G consists of the identity and C, a rotation by π around the origin. The map f must then commute with rotations by π, $f(\mathbf{C}x) = \mathbf{C}f(x)$, with $\mathbf{C}$ given by the $[2 \times 2]$ matrix

$$\mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{22.4}$$

$\mathbf{C}$ satisfies $\mathbf{C}^2 = e$ and can be used to decompose the state space into mutually orthogonal symmetric and antisymmetric subspaces by means of projection operators (22.1). More generally the projection operator onto the α irreducible subspace of dimension d_α is given by $P_\alpha = (d_\alpha/|G|) \sum \chi_\alpha(h) \mathbf{h}^{-1}$, where $\chi_\alpha(h) = \mathrm{tr}\, D_\alpha(h)$ are the group characters, and the transfer operator $\mathcal{L}$ splits into a sum of inequivalent irreducible subspace contributions $\sum_\alpha \mathrm{tr}\, \mathcal{L}_\alpha$,

where is D_α defined

why h and not g

$$\mathcal{L}_\alpha(y,x) = \frac{d_\alpha}{|G|} \sum_{h \in G} \chi_\alpha(h) \mathcal{L}(\mathbf{h}^{-1}y, x) \ . \tag{22.5}$$

The prefactor d_α in the above reflects the fact that a d_α-dimensional representation occurs d_α times.

add here parts of PER's Appendix B.

22.2.1 Cycle degeneracies

If $g \in G$ is a symmetry of the dynamical problem, the weight of a cycle p and the weight of its image under a symmetry transformation g are equal,

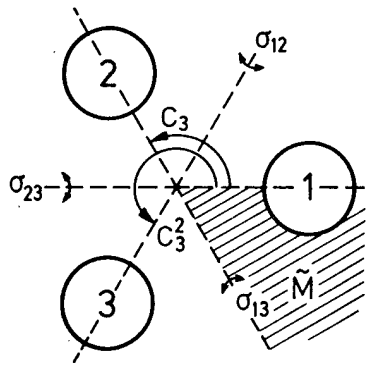


Figure 22.2: The symmetries of three disks on an equilateral triangle. The fundamental domain is indicated by the shaded wedge.

$t_{gp} = t_p$. The number of degenerate cycles (topologically distinct, but mapped into each other by symmetry transformations) depends on the cycle symmetries. Associated with a given cycle p is a maximal subgroup $\mathcal{H}_p \subseteq G$, $\mathcal{H}_p = \{e, b_2, b_3, \ldots, b_h\}$ of order h_p, whose elements leave p invariant. The elements of the quotient space $b \in G/\mathcal{H}_p$ generate the degenerate cycles bp, so the multiplicity of a degenerate cycle is $m_p = g/h_p$.

Taking into account these degeneracies, the Euler product (15.16) takes the form

$$\prod_p (1 - t_p) = \prod_{\hat{p}} (1 - t_{\hat{p}})^{m_{\hat{p}}}. \tag{22.6}$$

Here $\hat{p}$ is one of the m_p degenerate cycles, picked to serve as the label for the entire class. Our labeling convention is usually lexical, i.e., we label a cycle p by the cycle point whose label has the lowest value, and we label a class of degenerate cycles by the one with the lowest label $\hat{p}$. In what follows we shall drop the hat in $\hat{p}$ when it is clear from the context that we are dealing with symmetry distinct classes of cycles.

22.2.2 Example: C_{3v} invariance

An illustration of the above is afforded by C_{3v}, the group of symmetries of a game of pinball with three equal size, equally spaced disks, figure 22.2. The group consists of the identity element e, three reflections across axes $\{\sigma_{12}, \sigma_{23}, \sigma_{13}\}$, and two rotations by $2\pi/3$ and $4\pi/3$ denoted $\{C_3, C_3^2\}$, so its dimension is $g = 6$. On the disk labels $\{1, 2, 3\}$ these symmetries act as permutations which map cycles into cycles. For example, the flip across the symmetry axis going through disk 1 interchanges the symbols 2 and 3; it maps the cycle $\overline{12123}$ into $\overline{13132}$, figure 22.3a.

The subgroups of C_{3v} are C_v, consisting of the identity and any one of the reflections, of dimension $h = 2$, and $C_3 = \{e, C_3, C_3^2\}$, of dimension $h = 3$, so possible cycle multiplicities are $g/h = 2$, 3 or 6.

The C_3 subgroup invariance is exemplified by the cycles $\overline{123}$ and $\overline{132}$ which are invariant under rotations by $2\pi/3$ and $4\pi/3$, but are mapped

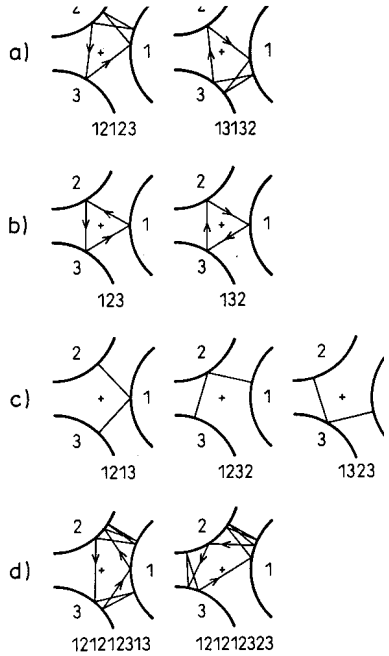


Figure 22.3: Some examples of 3-disk cycles: (a) $\overline{12123}$ and $\overline{13132}$ are mapped into each other by σ_{23}, the flip across 1 axis; this cycle has degeneracy 6 under C_{3v} symmetries. (C_{3v} is the symmetry group of the equilateral triangle.) Similarly (b) $\overline{123}$ and $\overline{132}$ and (c) $\overline{1213}$, $\overline{1232}$ and $\overline{1323}$ are degenerate under C_{3v}. (d) The cycles $\overline{121212313}$ and $\overline{121212323}$ are related by time reversal but not by any C_{3v} symmetry. (from ref. [1.6])

into each other by any reflection, figure 22.3b; $\mathcal{H}_p = \{e, C_3, C_3^2\}$, and the degeneracy is $g/h_{c_3} = 2$.

The C_v type of a subgroup is exemplified by the invariances of $\hat{p} = 1213$. This cycle is invariant under reflection $\sigma_{23}\{\overline{1213}\} = \overline{1312} = \overline{1213}$, so the invariant subgroup is $\mathcal{H}_{\hat{p}} = \{e, \sigma_{23}\}$. Its order is $h_{C_v} = 2$, so the degeneracy is $m_{\hat{p}} = g/h_{C_v} = 3$; the cycles in this class, $\overline{1213}$, $\overline{1232}$ and $\overline{1323}$, are related by $2\pi/3$ rotations, figure 22.3 (c).

A cycle of no symmetry, such as $\overline{12123}$, has $\mathcal{H}_p = \{e\}$ and contributes in all six terms (the remaining cycles in the class are $\overline{12132}$, $\overline{12313}$, $\overline{12323}$, $\overline{13132}$ and $\overline{13232}$), figure 22.3a.

Besides the above discrete symmetries, for Hamiltonian systems cycles may be related by time reversal symmetry. An example are the cycles $\overline{121212313}$ and $\overline{121212323} = \overline{313212121}$ which are related by no space symmetry (figure 22.3 (d)).

The Euler product (15.16) for the C_{3v} symmetric 3-disk problem is given in (18.37).

22.3 Dynamics in the fundamental domain

So far we have used the discrete symmetry to effect a reduction in the number of independent cycles in cycle expansions. The next step achieves much more: the symmetries can be used to restrict all computations to a *fundamental domain*. We show here that to each global cycle p corresponds a fundamental domain cycle $\tilde{p}$. Conversely, each fundamental domain cycle

$\tilde{p}$ traces out a segment of the global cycle p, with the end point of the cycle $\tilde{p}$ mapped into the irreducible segment of p with the group element $h_{\tilde{p}}$.

An important effect of a discrete symmetry is that it tessellates the state space into copies of a fundamental domain, and thus induces a natural partition of phase space. The group elements $g = \{a, b, \cdots, d\}$ which map the fundamental domain $\tilde{M}$ into its copies $g\tilde{M}$, can double in function as letters of a symbolic dynamics alphabet. If the dynamics is symmetric under interchanges of disks, the absolute disk labels $\epsilon_i = 1, 2, \cdots, N$ can be replaced by the symmetry-invariant relative disk→disk increments g_i, where g_i is the discrete group element that maps disk $i-1$ into disk i. We demonstrate the reduction for a series of specific examples in sect. 22.4. An immediate gain arising from symmetry invariant relabeling is that N-disk symbolic dynamics becomes $(N-1)$-nary, with no restrictions on the admissible sequences. However, the main gain is in the close connection between the symbol string symmetries and the state space symmetries which will aid us in the dynamical zeta function factorizations. Once the connection between the full space and the reduced space is established, working in the fundamental domain (*ie.*, with irreducible segments) is so much simpler that we never use the full space orbits in actual computations.

If the dynamics is invariant under a discrete symmetry, the state space M can be completely tiled by the fundamental domain $\tilde{M}$ and its images $a\tilde{M}$, $b\tilde{M}$, ... under the action of the symmetry group $G = \{e, a, b, \ldots\}$,

$$M = \sum_{a \in G} M_a = \sum_{a \in G} a\tilde{M} \, .$$

In the above example (22.4) with symmetry group $G = \{e, C\}$, the state space $M = \{x_1\text{-}x_2 \text{ plane}\}$ can be tiled by a fundamental domain $\tilde{M} = \{\text{half-plane } x_1 \geq 0\}$, and $\mathbf{C}\tilde{M} = \{\text{half-plane } x_1 \leq 0\}$, its image under rotation by π.

If the space M is decomposed into g tiles, a function $\phi(x)$ over M splits into a g-dimensional vector $\phi_a(x)$ defined by $\phi_a(x) = \phi(x)$ if $x \in M_a$, $\phi_a(x) = 0$ otherwise. Let $h = ab^{-1}$ conflicts with be the symmetry operation that maps the endpoint domain M_b into the starting point domain M_a, and let $D(h)_{ba}$, the left regular representation, be the $[g \times g]$ matrix whose b, a-th entry equals unity if $a = hb$ and zero otherwise; $D(h)_{ba} = \delta_{bh,a}$. Since the symmetries act on state space as well, the operation h enters in two guises: as a $[g \times g]$ matrix $D(h)$ which simply permutes the domain labels, and as a $[d \times d]$ matrix representation $\mathbf{h}$ of a discrete symmetry operation on the d state space coordinates. For instance, in the above example (22.4) $h \in C_2$ and $D(h)$ can be either the identity or the interchange of the two domain labels,

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \, , \quad D(C) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \, . \tag{22.7}$$

Note that $D(h)$ is a permutation matrix, mapping a tile M_a into a different tile $M_{ha} \neq M_a$ if $h \neq e$. Consequently only $D(e)$ has diagonal elements, and $\mathrm{tr}\, D(h) = g\delta_{h,e}$. However, the state space transformation $\mathbf{h} \neq \mathbf{e}$ leaves invariant sets of *boundary* points; for example, under reflection σ across a symmetry axis, the axis itself remains invariant. The boundary periodic orbits that belong to such point-wise invariant sets will require special care in $\mathrm{tr}\, \mathcal{L}$ evaluations.

One can associate to the evolution operator (10.23) a $[g \times g]$ matrix evolution operator defined by

$$\mathcal{L}_{ba}(y,x) = D(h)_{ba}\mathcal{L}(y,x) \ ,$$

if $x \in M_a$ and $y \in M_b$, and zero otherwise. Now we can use the invariance condition (22.3) to move the starting point x into the fundamental domain $x = \mathbf{a}\tilde{x}$, $\mathcal{L}(y,x) = \mathcal{L}(\mathbf{a}^{-1}y, \tilde{x})$, and then use the relation $a^{-1}b = h^{-1}$ to also relate the endpoint y to its image in the fundamental domain, $\tilde{\mathcal{L}}(\tilde{y},\tilde{x}) := \mathcal{L}(\mathbf{h}^{-1}\tilde{y},\tilde{x})$. With this operator which is restricted to the fundamental domain, the global dynamics reduces to

$a^{-1}b = h^{-1}$?

$$\mathcal{L}_{ba}(y,x) = D(h)_{ba}\tilde{\mathcal{L}}(\tilde{y},\tilde{x}) \ .$$

While the global trajectory runs over the full space M, the restricted trajectory is brought back into the fundamental domain $\tilde{M}$ any time it crosses into adjoining tiles; the two trajectories are related by the symmetry operation h which maps the global endpoint into its fundamental domain image.

Now the traces (15.3) required for the evaluation of the eigenvalues of the transfer operator can be evaluated on the fundamental domain alone

$$\mathrm{tr}\, \mathcal{L} = \int_M dx \mathcal{L}(x,x) = \int_{\tilde{M}} d\tilde{x} \ \sum_h \mathrm{tr}\, D(h) \ \mathcal{L}(\mathbf{h}^{-1}\tilde{x},\tilde{x}) \qquad (22.8)$$

The fundamental domain integral $\int d\tilde{x}\ \mathcal{L}(\mathbf{h}^{-1}\tilde{x},\tilde{x})$ picks up a contribution from every global cycle (for which $h = e$), but it also picks up contributions from shorter segments of global cycles. The permutation matrix $D(h)$ guarantees by the identity $\mathrm{tr}\, D(h) = 0$, $h \neq e$, that only those repeats of the fundamental domain cycles $\tilde{p}$ that correspond to complete global cycles p contribute. Compare, for example, the contributions of the $\overline{12}$ and $\overline{0}$ cycles of figure 11.6. $\mathrm{tr}\, D(h)\tilde{\mathcal{L}}$ does not get a contribution from the $\overline{0}$ cycle, as the symmetry operation that maps the first half of the $\overline{12}$ into the fundamental domain is a reflection, and $\mathrm{tr}\, D(\sigma) = 0$. In contrast, $\sigma^2 = e$, $\mathrm{tr}\, D(\sigma^2) = 6$ insures that the repeat of the fundamental domain fixed point $\mathrm{tr}\, (D(h)\tilde{\mathcal{L}})^2 = 6t_0^2$, gives the correct contribution to the global trace $\mathrm{tr}\, \mathcal{L}^2 = 3 \cdot 2t_{12}$.

Let p be the full orbit, $\tilde{p}$ the orbit in the fundamental domain and $h_{\tilde{p}}$ an element of $\mathcal{H}_p$, the symmetry group of p. Restricting the volume integrations to the infinitesimal neighborhoods of the cycles p and $\tilde{p}$, respectively, and performing the standard resummations, we obtain the identity

three-disk@3-disk!boundary orbits
boundary orbits

$$(1-t_p)^{m_p} = \det\,(1-D(h_{\tilde{p}})t_{\tilde{p}})\ , \tag{22.9}$$

valid cycle by cycle in the Euler products (15.16) for $\det(1-\mathcal{L})$. Here "det" refers to the $[g\times g]$ matrix representation $D(h_{\tilde{p}})$; as we shall see, this determinant can be evaluated in terms of standard characters, and no explicit representation of $D(h_{\tilde{p}})$ is needed. Finally, if a cycle p is invariant under the symmetry subgroup $\mathcal{H}_p \subseteq G$ of order h_p, its weight can be written as a repetition of a fundamental domain cycle

$$t_p = t_{\tilde{p}}^{h_p} \tag{22.10}$$

computed on the irreducible segment that corresponds to a fundamental domain cycle. For example, in figure 11.6 we see by inspection that $t_{12} = t_0^2$ and $t_{123} = t_1^3$.

22.3.1 Boundary orbits

Before we can turn to a presentation of the factorizations of dynamical zeta functions for the different symmetries we have to discuss an effect that arises for orbits that run on a symmetry line that borders a fundamental domain. In our 3-disk example, no such orbits are possible, but they exist in other systems, such as in the bounded region of the Hénon-Heiles potential and in 1-d maps. For the symmetrical 4-disk billiard, there are in principle two kinds of such orbits, one kind bouncing back and forth between two diagonally opposed disks and the other kind moving along the other axis of reflection symmetry; the latter exists for bounded systems only. While there are typically very few boundary orbits, they tend to be among the shortest orbits, and their neglect can seriously degrade the convergence of cycle expansions, as those are dominated by the shortest cycles.

While such orbits are invariant under some symmetry operations, their neighborhoods are not. This affects the stability matrix M_p of the linearization perpendicular to the orbit and thus the eigenvalues. Typically, *e.g.* if the symmetry is a reflection, some eigenvalues of M_p change sign. This means that instead of a weight $1/\det(\mathbf{1}-M_p)$ as for a regular orbit, boundary cycles also pick up contributions of form $1/\det(\mathbf{1}-\mathbf{h}M_p)$, where $\mathbf{h}$ is a symmetry operation that leaves the orbit pointwise invariant; see for example sect. 22.1.2.

Consequences for the dynamical zeta function factorizations are that sometimes a boundary orbit does not contribute. A derivation of a dynamical zeta function (15.16) from a determinant like (15.10) usually starts

with an expansion of the determinants of the Jacobian. The leading order terms just contain the product of the expanding eigenvalues and lead to the dynamical zeta function (15.16). Next to leading order terms contain products of expanding and contracting eigenvalues and are sensitive to their signs. Clearly, the weights t_p in the dynamical zeta function will then be affected by reflections in the Poincaré surface of section perpendicular to the orbit. In all our applications it was possible to implement these effects by the following simple prescription.

write this up correctly, for arbitrary case!

If an orbit is invariant under a little group $\mathcal{H}_p = \{e, b_2, \ldots, b_h\}$, then the corresponding group element in (22.9) will be replaced by a projector. If the weights are insensitive to the signs of the eigenvalues, then this projector is

$$g_p = \frac{1}{h}\sum_{i=1}^{h} b_i \,. \tag{22.11}$$

In the cases that we have considered, the change of sign may be taken into account by defining a sign function $\epsilon_p(g) = \pm 1$, with the "-" sign if the symmetry element g flips the neighborhood. Then (22.11) is replaced by

$$g_p = \frac{1}{h}\sum_{i=1}^{h} \epsilon(b_i)\, b_i \,. \tag{22.12}$$

We have illustrated the above in sect. 22.1.2 by working out the full factorization for the 1-dimensional reflection symmetric maps.

22.4 Factorizations of dynamical zeta functions

In the above we have shown that a discrete symmetry induces degeneracies among periodic orbits and decomposes periodic orbits into repetitions of irreducible segments; this reduction to a fundamental domain furthermore leads to a convenient symbolic dynamics compatible with the symmetry, and, most importantly, to a factorization of dynamical zeta functions. This we now develop, first in a general setting and then for specific examples.

22.4.1 Factorizations of dynamical dynamical zeta functions

According to (22.9) and (22.10), the contribution of a degenerate class of global cycles (cycle p with multiplicity $m_p = g/h_p$) to a dynamical zeta function is given by the corresponding fundamental domain cycle $\tilde{p}$:

$$(1 - t_{\tilde{p}}^{h_p})^{g/h_p} = \det\,(1 - D(h_{\tilde{p}})t_{\tilde{p}}) \tag{22.13}$$

Let $D(h) = \bigoplus_\alpha d_\alpha D_\alpha(h)$ be the decomposition of the matrix representation $D(h)$ into the d_α dimensional irreducible representations α of a finite group G. Such decompositions are block-diagonal, so the corresponding contribution to the Euler product (15.10) factorizes as

$$\det\left(1 - D(h)t\right) = \prod_\alpha \det\left(1 - D_\alpha(h)t\right)^{d_\alpha} \ , \tag{22.14}$$

where now the product extends over all distinct d_α-dimensional irreducible representations, each contributing d_α times. For the cycle expansion purposes, it has been convenient to emphasize that the group-theoretic factorization can be effected cycle by cycle, as in (22.13); but from the transfer operator point of view, the key observation is that the symmetry reduces the transfer operator to a block diagonal form; this block diagonalization implies that the dynamical zeta functions (15.16) factorize as

$$\frac{1}{\zeta} = \prod_\alpha \frac{1}{\zeta_\alpha^{d_\alpha}} \ , \qquad \frac{1}{\zeta_\alpha} = \prod_{\tilde{p}} \det\left(1 - D_\alpha(h_{\tilde{p}})t_{\tilde{p}}\right) \ . \tag{22.15}$$

Determinants of d-dimensional irreducible representations can be evaluated using the expansion of determinants in terms of traces,

$$\begin{aligned} \det\left(1 + M\right) \ &= \ 1 + \mathrm{tr}\, M + \frac{1}{2}\left((\mathrm{tr}\, M)^2 - \mathrm{tr}\, M^2\right) \\ &\quad + \frac{1}{6}\left((\mathrm{tr}\, M)^3 - 3\,(\mathrm{tr}\, M)(\mathrm{tr}\, M^2) + 2\,\mathrm{tr}\, M^3\right) \\ &\quad + \cdots + \frac{1}{d!}\left((\mathrm{tr}\, M)^d - \cdots\right) \ , \end{aligned} \tag{22.16}$$

(see (M.26), for example) and each factor in (22.14) can be evaluated by looking up the characters $\chi_\alpha(h) = \mathrm{tr}\, D_\alpha(h)$ in standard tables [23.8]. In terms of characters, we have for the 1-dimensional representations

$$\det\left(1 - D_\alpha(h)t\right) = 1 - \chi_\alpha(h)t \ ,$$

for the 2-dimensional representations

$$\det\left(1 - D_\alpha(h)t\right) = 1 - \chi_\alpha(h)t + \frac{1}{2}\left(\chi_\alpha(h)^2 - \chi_\alpha(h^2)\right)t^2,$$

and so forth.

In the fully symmetric subspace $\mathrm{tr}\, D_{A_1}(h) = 1$ for all orbits; hence a straightforward fundamental domain computation (with no group theory weights) always yields a part of the full spectrum. In practice this is the most interesting subspectrum, as it contains the leading eigenvalue of the transfer operator.

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⇑PRELIMINARY

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22.4.2 Factorizations of spectral determinants

Factorization of the full spectral determinant (15.3) proceeds in essentially the same manner as the factorization of dynamical zeta functions outlined above. By (22.5) and (22.8) the trace of the transfer operator $\mathcal{L}$ splits into the sum of inequivalent irreducible subspace contributions $\sum_\alpha \mathrm{tr}\, \mathcal{L}_\alpha$, with

$$\mathrm{tr}\, \mathcal{L}_\alpha = d_\alpha \sum_{h\in G} \chi_\alpha(h) \int_{\tilde{M}} d\tilde{x}\, \mathcal{L}(\mathbf{h}^{-1}\tilde{x}, \tilde{x})\,.$$

This leads by standard manipulations to the factorization of (15.10) into

$$\begin{aligned} F(z) &= \prod_\alpha F_\alpha(z)^{d_\alpha} \\ F_\alpha(z) &= \exp\left(-\sum_{\tilde{p}}\sum_{r=1}^{\infty} \frac{1}{r} \frac{\chi_\alpha(h_{\tilde{p}}^r) z^{n_{\tilde{p}} r}}{|\det\left(\mathbf{1} - \tilde{M}_{\tilde{p}}^r\right)|}\right)\,, \end{aligned} \tag{22.17}$$

where $\tilde{M}_{\tilde{p}} = \mathbf{h}_{\tilde{p}} M_{\tilde{p}}$ is the fundamental domain Jacobian. Boundary orbits require special treatment, discussed in sect. 22.3.1, with examples given in the next section as well as in the specific factorizations discussed below.

The factorizations (22.15), (22.17) are the central formulas of this chapter. We now work out the group theory factorizations of cycle expansions of dynamical zeta functions for the cases of C_2 and C_{3v} symmetries. The cases of the C_{2v}, C_{4v} symmetries are worked out in appendix J below.

22.5 C_2 factorization

As the simplest example of implementing the above scheme consider the C_2 symmetry. For our purposes, all that we need to know here is that each orbit or configuration is uniquely labeled by an infinite string $\{s_i\}$, $s_i = +, -$ and that the dynamics is invariant under the $+ \leftrightarrow -$ interchange, i.e., it is C_2 symmetric. The C_2 symmetry cycles separate into two classes, the self-dual configurations $+-$, $++--$, $+++---$, $+--+-++-$, $\cdots$, with multiplicity $m_p = 1$, and the asymmetric configurations $+$, $-$, $++-$, $--+$, $\cdots$, with multiplicity $m_p = 2$. For example, as there is no absolute distinction between the "up" and the "down" spins, or the "left" or the "right" lobe, $t_+ = t_-$, $t_{++-} = t_{+--}$, and so on.

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The symmetry reduced labeling $\rho_i \in \{0, 1\}$ is related to the standard $s_i \in \{+, -\}$ Ising spin labeling by

$$\begin{aligned} &\text{If} \quad s_i = s_{i-1} \quad \text{then} \quad \rho_i = 1 \\ &\text{If} \quad s_i \neq s_{i-1} \quad \text{then} \quad \rho_i = 0 \end{aligned} \tag{22.18}$$

$\tilde{p}$	p	m_p
1	$+$	2
0	$-+$	1
01	$--\ ++$	1
001	$-++$	2
011	$---\ +++$	1
0001	$-+--\ +-++$	1
0011	$-++++$	2
0111	$----\ ++++$	1
00001	$-+-+-$	2
00011	$-+---\ +-+++$	1
00101	$-+-+-\ +---++$	1
00111	$-+---\ +-+++$	1
01011	$--+++$	2
01111	$-----\ ++++++$	1
001011	$-++---\ +--+++$	1
001101	$-++-+-\ +---++$	1

Table 22.1: Correspondence between the C_2 symmetry reduced cycles $\tilde{p}$ and the standard Ising model periodic configurations p, together with their multiplicities m_p. Also listed are the two shortest cycles (length 6) related by time reversal, but distinct under C_2.

For example, $\overline{+} = \cdots+++\cdots$ maps into $\cdots111\cdots = \overline{1}$ (and so does $\overline{-}$), $\overline{-+} = \cdots-+-+\cdots$ maps into $\cdots000\cdots = \overline{0}$, $\overline{-++-} = \cdots--++--+++\cdots$ maps into $\cdots0101\cdots = \overline{01}$, and so forth. A list of such reductions is given in table 22.1.

Depending on the maximal symmetry group $\mathcal{H}_p$ that leaves an orbit p invariant (see sects. 22.2 and 22.3 as well as sect. 22.1.2), the contributions to the dynamical zeta function factor as

$$\begin{array}{rrcll} & & & A_1 & A_2 \\ \mathcal{H}_p = \{e\}: & (1-t_{\tilde{p}})^2 & = & (1-t_{\tilde{p}})(1-t_{\tilde{p}}) & \\ \mathcal{H}_p = \{e,\sigma\}: & (1-t_{\tilde{p}}^2) & = & (1-t_{\tilde{p}})(1+t_{\tilde{p}})\ , & \end{array} \tag{22.19}$$

For example:

$$\begin{array}{rrcl} \mathcal{H}_{++-} = \{e\}: & (1-t_{++-})^2 & = & (1-t_{001})(1-t_{001}) \\ \mathcal{H}_{+-} = \{e,\sigma\}: & (1-t_{+-}) & = & (1-t_0)\ \ (1+t_0), \quad t_{+-} = t_0^2 \end{array}$$

This yields two binary cycle expansions. The A_1 subspace dynamical zeta function is given by the standard binary expansion (18.5). The antisymmetric A_2 subspace dynamical zeta function ζ_{A_2} differs from ζ_{A_1} only by a minus sign for cycles with an odd number of 0's:

$$\begin{aligned} 1/\zeta_{A_2} &= (1+t_0)(1-t_1)(1+t_{10})(1-t_{100})(1+t_{101})(1+t_{1000}) \\ &\quad (1-t_{1001})(1+t_{1011})(1-t_{10000})(1+t_{10001}) \\ &\quad (1+t_{10010})(1-t_{10011})(1-t_{10101})(1+t_{10111})\dots \\ &= 1+t_0-t_1+(t_{10}-t_1t_0)-(t_{100}-t_{10}t_0)+(t_{101}-t_{10}t_1) \\ &\quad -(t_{1001}-t_1t_{001}-t_{101}t_0+t_{10}t_0t_1)-\dots\dots \end{aligned} \tag{22.20}$$

three-disk@3-disk!symmetry
symmetry!3-disk
symmetry!$C˙3v$
c3v@$C˙3v$ symmetry

Note that the group theory factors do not destroy the curvature corrections (the cycles and pseudo cycles are still arranged into shadowing combinations).

If the system under consideration has a boundary orbit (*cf.* sect. 22.3.1) with group-theoretic factor $\mathbf{h}_p = (\mathbf{e}+\sigma)/2$, the boundary orbit does not contribute to the antisymmetric subspace

$$\begin{array}{lrcl} & & A_1 \quad A_2 & \\ \text{boundary:} & (1-t_p) & = & (1-t_{\tilde{p}})(1-0t_{\tilde{p}}) \end{array} \tag{22.21}$$

This is the $1/\zeta$ part of the boundary orbit factorization of sect. 22.1.2.

22.6 C_{3v} factorization: 3-disk game of pinball

The next example, the C_{3v} symmetry, can be worked out by a glance at figure 11.6a. For the symmetric 3-disk game of pinball the fundamental domain is bounded by a disk segment and the two adjacent sections of the symmetry axes that act as mirrors (see figure 11.6b). The three symmetry axes divide the space into six copies of the fundamental domain. Any trajectory on the full space can be pieced together from bounces in the fundamental domain, with symmetry axes replaced by flat mirror reflections. The binary $\{0,1\}$ reduction of the ternary three disk $\{1,2,3\}$ labels has a simple geometric interpretation: a collision of type 0 reflects the projectile to the disk it comes from (back–scatter), whereas after a collision of type 1 projectile continues to the third disk. For example, $\overline{23} = \cdots 232323\cdots$ maps into $\cdots 000\cdots = \overline{0}$ (and so do $\overline{12}$ and $\overline{13}$), $\overline{123} = \cdots 12312\cdots$ maps into $\cdots 111\cdots = \overline{1}$ (and so does $\overline{132}$), and so forth. A list of such reductions for short cycles is given in table 11.2.

C_{3v} has two 1-dimensional irreducible representations, symmetric and antisymmetric under reflections, denoted A_1 and A_2, and a pair of degenerate 2-dimensional representations of mixed symmetry, denoted E. The contribution of an orbit with symmetry g to the $1/\zeta$ Euler product (22.14) factorizes according to

explain why $\chi_E(h^2) = \chi_{A_2}(h)$

$$\det(1-D(h)t) = (1-\chi_{A_1}(h)t)\,(1-\chi_{A_2}(h)t)\,\left(1-\chi_E(h)t+\chi_{A_2}(h)t^2\right)^2 \tag{22.22}$$

with the three factors contributing to the C_{3v} irreducible representations A_1, A_2 and E, respectively, and the 3-disk dynamical zeta function factorizes into $\zeta = \zeta_{A_1}\zeta_{A_2}\zeta_E^2$. Substituting the C_{3v} characters [23.8]

C_{3v}	A_1	A_2	E
e	1	1	2
C_3, C_3^2	1	1	-1
σ_v	1	-1	0

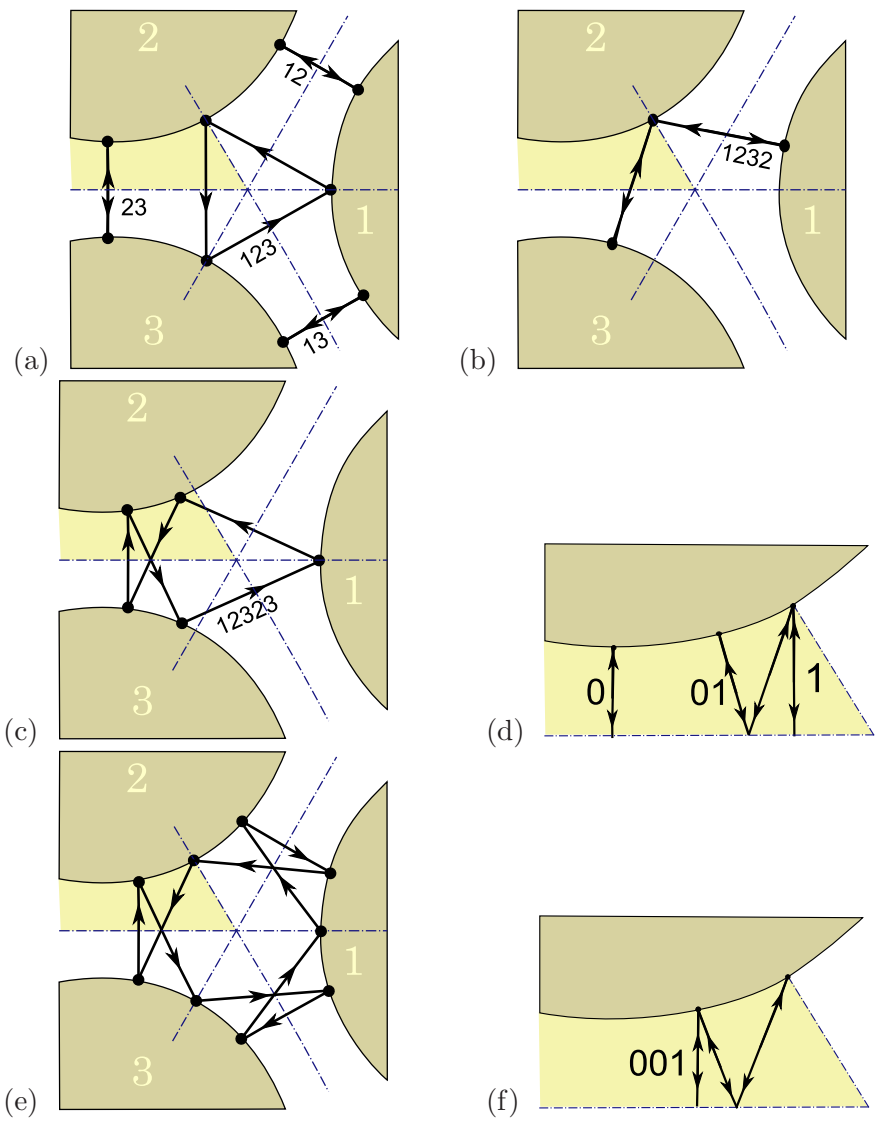

Figure 22.4: The 3-disk pinball cycles: (a) $\overline{12}$, $\overline{13}$, $\overline{23}$, $\overline{123}$. $\overline{132}$ turns clockwise. (b) $\overline{1232}$; $\overline{1213}$ and $\overline{1323}$ not drawn. (c) $\overline{12323}$; $\overline{12123}$, $\overline{12132}$, $\overline{12313}$, $\overline{13131}$ and $\overline{13232}$ not drawn. (d) The fundamental domain, i.e., the 1/6th wedge indicated in (a), consisting of a section of a disk, two segments of symmetry axes acting as straight mirror walls, and the escape gap to the left. The above 13 full-space cycles restricted to the fundamental domain are the two fixed points $\overline{0}$, $\overline{1}$, 2-cycle $\overline{10}$, and 5-cycle $\overline{00111}$ (not drawn). (e) The two full-space 9-cycles $\overline{121232313}$ and the clockwise $\overline{131323212}$ corespond to (f) one fundamental domain 3-cycle $\overline{001}$. Radius : center separation ratio a:R = 1:2.5.

into (22.22), we obtain for the three classes of possible orbit symmetries (indicated in the first column)

$$\begin{array}{rrcll} & \mathbf{h}_{\tilde{p}} & & A_1 \quad A_2 \quad E & \\ e: & (1-t_{\tilde{p}})^6 & = & (1-t_{\tilde{p}})(1-t_{\tilde{p}})(1-2t_{\tilde{p}}+t_{\tilde{p}}^2)^2 & \\ C_3, C_3^2: & (1-t_{\tilde{p}}^3)^2 & = & (1-t_{\tilde{p}})(1-t_{\tilde{p}})(1+\ t_{\tilde{p}}+t_{\tilde{p}}^2)^2 & \\ \sigma_v: & (1-t_{\tilde{p}}^2)^3 & = & (1-t_{\tilde{p}})(1+t_{\tilde{p}})(1+0t_{\tilde{p}}-t_{\tilde{p}}^2)^2. & (22.23) \end{array}$$

where σ_v stands for any one of the three reflections.

The Euler product (15.16) on each irreducible subspace follows from the factorization (22.23). On the symmetric A_1 subspace the ζ_{A_1} is given by the standard binary curvature expansion (18.5). The antisymmetric A_2 subspace ζ_{A_2} differs from ζ_{A_1} only by a minus sign for cycles with an odd number of 0's, and is given in (22.20). For the mixed-symmetry subspace E the curvature expansion is given by

$$\begin{aligned} 1/\zeta_E &= (1+zt_1+z^2t_1^2)(1-z^2t_0^2)(1+z^3t_{100}+z^6t_{100}^2)(1-z^4t_{10}^2) \\ &\quad (1+z^4t_{1001}+z^8t_{1001}^2)(1+z^5t_{10000}+z^{10}t_{10000}^2) \\ &\quad (1+z^5t_{10101}+z^{10}t_{10101}^2)(1-z^5t_{10011})^2\ldots \\ &= 1+zt_1+z^2(t_1^2-t_0^2)+z^3(t_{001}-t_1t_0^2) \\ &\quad +z^4\left[t_{0011}+(t_{001}-t_1t_0^2)t_1-t_{01}^2\right] \\ &\quad +z^5\left[t_{00001}+t_{01011}-2t_{00111}+(t_{0011}-t_{01}^2)t_1+(t_1^2-t_0^2)t_{100}\right]+\ldots \end{aligned} \tag{22.24}$$

We have reinserted the powers of z in order to group together cycles and pseudocycles of the same length. Note that the factorized cycle expansions retain the curvature form; long cycles are still shadowed by (somewhat less obvious) combinations of pseudocycles.

Referring back to the topological polynomial (13.31) obtained by setting $t_p = 1$, we see that its factorization is a consequence of the C_{3v} factorization of the ζ function:

$$1/\zeta_{A_1} = 1-2z\ , \quad 1/\zeta_{A_2} = 1\ , \quad 1/\zeta_E = 1+z\ , \tag{22.25}$$

as obtained from (18.5), (22.20) and (22.24) for $t_p = 1$.

Their symmetry is $K = \{\mathbf{e}, \sigma\}$, so according to (22.11), they pick up the group-theoretic factor $\mathbf{h}_p = (\mathbf{e}+\sigma)/2$. If there is no sign change in t_p, then evaluation of $\det\left(1-\frac{\mathbf{e}+\sigma}{2}t_{\tilde{p}}\right)$ yields

$$\begin{array}{rrcll} & & & A_1 \quad A_2 \quad E & \\ \text{boundary:} & (1-t_p)^3 & = & (1-t_{\tilde{p}})(1-0t_{\tilde{p}})(1-t_{\tilde{p}})^2\,, & t_p = t_{\tilde{p}}^2 \end{array} \tag{22.26}$$

However, if the cycle weight changes sign under reflection, $t_{\sigma\tilde{p}} = -t_{\tilde{p}}$, the boundary orbit does not contribute to the subspace symmetric under reflection across the orbit;

$$\begin{array}{rccc} & A_1 & A_2 & E \\ \text{boundary:} \quad (1-t_p)^3 & = & (1-0t_{\tilde{p}})(1-t_{\tilde{p}})(1-t_{\tilde{p}})^2\,, & t_p = t_{\tilde{p}}^2 \end{array} \tag{22.27}$$

insert here Freddy p.22, Gunnar paper with C_3 symmetry; discuss symmetry splittings as magnetic field is turned on.

Commentary

Remark 22.1 Some examples of systems with discrete symmetries. This chapter is based on ref. [22.1]. One has a C_2 symmetry in the Lorenz system [2.1, 22.16], the Ising model, and in the 3-dimensional anisotropic Kepler potential [36.6, 36.20, 36.21], a C_{3v} symmetry in Hénon-Heiles type potentials [22.2, 22.6, 22.7, 22.5], a C_{4v} symmetry in quartic oscillators [22.9, 22.10], in the pure x^2y^2 potential [22.11, 22.12] and in hydrogen in a magnetic field [22.13], and a $C_{2v} = C_2 \times C_2$ symmetry in the stadium billiard [22.4]. A very nice application of the symmetry factorization is carried out in ref. [22.8].

Remark 22.2 Who did it? This chapter is based on long collaborative effort with B. Eckhardt, ref. [22.1]. The group-theoretic factorizations of dynamical zeta functions that we develop here were first introduced and applied in ref. [6.3]. They are closely related to the symmetrizations introduced by Gutzwiller [36.6] in the context of the semiclassical periodic orbit trace formulas, put into more general group-theoretic context by Robbins [22.4], whose exposition, together with Lauritzen's [22.5] treatment of the boundary orbits, has influenced the presentation given here. A related group-theoretic decomposition in context of hyperbolic billiards was utilized in ref. [22.8].

Mention also the number theoretic zeta's factorizations (Epstein, etc.?).

Remark 22.3 Computations The techniques of this chapter have been applied to computations of the 3-disk classical and quantum spectra in refs. [1.6, 38.15], and to a "Zeeman effect" pinball and the x^2y^2 potentials in refs. [22.3, 18.12]. In a larger perspective, the factorizations developed above are special cases of a general approach to exploiting the group-theoretic invariances in spectra computations, such as those used in enumeration of periodic geodesics [22.8, 15.4, 15.14] for hyperbolic billiards [36.4] and Selberg zeta functions [31.2].

Remark 22.4 Other symmetries. In addition to the symmetries exploited here, time reversal symmetry and a variety of other non-trivial discrete symmetries can induce further relations among orbits; we shall point out several of examples of cycle degeneracies under time reversal. We do not know whether such symmetries can be exploited for further improvements of cycle expansions.

symbolic dynamics!Hénon-Heiles
Henon@Hénon-Heiles!symbolic dynamics

Remark 22.5 Cycles and symmetries. We conclude this section with a few comments about the role of symmetries in actual extraction of cycles. In the example at hand, the N-disk billiard systems, a fundamental domain is a sliver of the N-disk configuration space delineated by a pair of adjoining symmetry axes, with the directions of the momenta indicated by arrows. The flow may further be reduced to a return map on a Poincaré surface of section, on which an appropriate transfer operator may be constructed. While in principle any Poincaré surface of section will do, a natural choice in the present context are crossings of symmetry axes.

In actual numerical integrations only the last crossing of a symmetry line needs to be determined. The cycle is run in global coordinates and the group elements associated with the crossings of symmetry lines are recorded; integration is terminated when the orbit closes in the fundamental domain. Periodic orbits with non-trivial symmetry subgroups are particularly easy to find since their points lie on crossings of symmetry lines.

Remark 22.6 C_2 symmetry The C_2 symmetry arises, for example, in the Lorenz system [22.16], in the 3-dimensional anisotropic Kepler problem [36.6, 36.20, 36.21] or in the cycle expansions treatments of the Ising model [12.65].

Remark 22.7 Hénon-Heiles potential An example of a system with C_{3v} symmetry is provided by the motion of a particle in the Hénon-Heiles potential [22.2]

$$V(r,\theta) = \frac{1}{2}r^2 + \frac{1}{3}r^3\sin(3\theta)\ .$$

Our coding is not directly applicable to this system because of the existence of elliptic islands and because the three orbits that run along the symmetry axis cannot be labeled in our code. However, since these orbits run along the boundary of the fundamental domain, they require the special treatment discussed in sect. 22.3.1.

⇓PRIVATE

Remark 22.8 Literature A summary of discrete groups [23.44].

⇑PRIVATE

Include magnetic symmetry breaking parts of Russberg's unfinished paper.

Résumé

If a dynamical system has a discrete symmetry, the symmetry should be exploited; much is gained, both in understanding of the spectra and ease of their evaluation. Once this is appreciated, it is hard to conceive of a calculation without factorization; it would correspond to quantum mechanical calculations without wave–function symmetrizations.

Reduction to the fundamental domain simplifies symbolic dynamics and eliminates symmetry induced degeneracies. While the resummation of the

three-disk@3-disk!cycle!count

theory from the trace sums to the cycle expansions does not reduce the exponential growth in number of cycles with the cycle length, in practice only the short orbits are used, and for them the labor saving is dramatic. For example, for the 3-disk game of pinball there are 256 periodic points of length 8, but reduction to the fundamental domain non-degenerate prime cycles reduces the number of the distinct cycles of length 8 to 30.

In addition, cycle expansions of the symmetry reduced dynamical zeta functions converge dramatically faster than the unfactorized dynamical zeta functions. One reason is that the unfactorized dynamical zeta function has many closely spaced zeros and zeros of multiplicity higher than one; since the cycle expansion is a polynomial expansion in topological cycle length, accommodating such behavior requires many terms. The dynamical zeta functions on separate subspaces have more evenly and widely spaced zeros, are smoother, do not have symmetry-induced multiple zeros, and fewer cycle expansion terms (short cycle truncations) suffice to determine them. Furthermore, the cycles in the fundamental domain sample state space more densely than in the full space. For example, for the 3-disk problem, there are 9 distinct (symmetry unrelated) cycles of length 7 or less in full space, corresponding to 47 distinct periodic points. In the fundamental domain, we have 8 (distinct) periodic orbits up to length 4 and thus 22 different periodic points in 1/6-th the state space, i.e., an increase in density by a factor 3 with the same numerical effort.

We emphasize that the symmetry factorization (22.23) of the dynamical zeta functionis *intrinsic* to the classical dynamics, and not a special property of quantal spectra. The factorization is not restricted to the Hamiltonian systems, or only to the configuration space symmetries; for example, the discrete symmetry can be a symmetry of the Hamiltonian phase space [22.4]. In conclusion, the manifold advantages of the symmetry reduced dynamics should thus be obvious; full space cycle expansions, such as those of exercise 18.9, are useful only for cross checking purposes.

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Exercises

three-disk@3-disk!symmetry
desymmetrization!3-disk
symmetry!3-disk

Exercise 22.1 **Sawtooth map desymmetrization.** *Work out the some of the shortest global cycles of different symmetries and fundamental domain cycles for the sawtooth map of figure 22.1. Compute the dynamical zeta function and the spectral determinant of the Perron-Frobenius operator for this map; check explicitely the factorization (22.2).*

⇓PRELIMINARY

Exercise 22.2 **Characters.** *Give a bunch of exercises for evaluating simple characters.*

⇑PRELIMINARY

Exercise 22.3 **2-d asymmetric representation.** *The above expressions can sometimes be simplified further using standard group-theoretical methods. For example, the $\frac{1}{2}\left((\mathrm{tr}\, M)^2 - \mathrm{tr}\, M^2\right)$ term in (22.16) is the trace of the antisymmetric part of the $M \times M$ Kronecker product; if α is a 2-dimensional representation, this is the A_2 antisymmetric representation, so*

$$2\text{-dim:} \quad \det\left(1 - D_\alpha(h)t\right) = 1 - \chi_\alpha(h)t + \chi_{A_2}(h)t^2. \tag{22.28}$$

Exercise 22.4 **3-disk desymmetrization.**

a) *Work out the 3-disk symmetry factorization for the 0 and 1 cycles, i.e. which symmetry do they have, what is the degeneracy in full space and how do they factorize (how do they look in the A_1, A_2 and the E representations).*

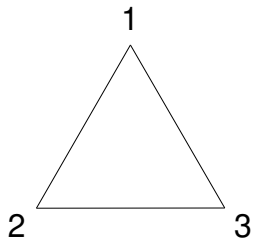
b) *Find the shortest cycle with no symmetries and factorize it like in a)*

c) *Find the shortest cycle that has the property that its time reversal is not described by the same symbolic dynamics.*

d) *Compute the dynamical zeta functions and the spectral determinants (symbolically) in the three representations; check the factorizations (22.15) and (22.17).*

(Per Rosenqvist)

Exercise 22.5 **The group C_{3v}.** *We will compute a few of the properties of the group C_{3v}, the group of symmetries of an equilateral triangle*



(a) *All discrete groups are isomorphic to a permutation group or one of its subgroups, and elements of the permutation group can be expressed as cycles. Express the elements of the group C_{3v} as cycles. For example, one of the rotations is (123), meaning that vertex 1 maps to 2 and 2 to 3 and 3 to 1.*

(b) *Find the subgroups of the group C_{3v}.*

(c) *Find the classes of C_{3v} and the number of elements in them.*

(d) *Their are three irreducible representations for the group. Two are one dimensional and the other one is formed by 2×2 matrices of the form*

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} .$$

Find the matrices for all six group elements.

(e) *Use your representation to find the character table for the group.*

Exercise 22.6 C_2 **factorizations: the Lorenz and Ising systems.** *In the Lorenz system [2.1, 22.16] the labels $+$ and $-$ stand for the left or the right lobe of the attractor and the symmetry is a rotation by π around the z-axis. Similarly, the Ising Hamiltonian (in the absence of an external magnetic field) is invariant under spin flip. Work out the factorizations for some of the short cycles in either system.*

Exercise 22.7 Ising model. *The Ising model with two states $\epsilon_i = \{+, -\}$ per site, periodic boundary condition, and Hamiltonian*

$$H(\epsilon) = -J \sum_i \delta_{\epsilon_i, \epsilon_{i+1}} ,$$

is invariant under spin-flip: $+ \leftrightarrow -$. Take advantage of that symmetry and factorize the dynamical zeta function for the model, i.e., find all the periodic orbits that contribute to each factor and their weights.

Exercise 22.8 One orbit contribution. *If p is an orbit in the fundamental domain with symmetry h, show that it contributes to the spectral determinant with a factor*

$$\det \left(1 - D(h) \frac{t_p}{\lambda_p^k} \right) ,$$

where $D(h)$ is the representation of h in the regular representation of the group.

periodic orbit!relative
relative!periodic orbit

Chapter 23

Continuous symmetries

Hard work builds character.
V.I. Warshavski, Private Investigator

Trace formulas relate short time dynamics (unstable periodic orbits) to long time invariant state space densities (natural measure). Higher dimensional dynamics requires inclusion of higher-dimensional compact invariant sets, such as partially hyperbolic invariant tori, into trace formulas. A trace formula for a partially hyperbolic $(N+1)$-dimensional compact manifold invariant under a global continuous symmetry is derived here. In this extension of "periodic orbit" theory there are no or very few periodic orbits - the relative periodic orbits that the trace formula has support on are almost never eventually periodic.

The classical trace formula for smooth continuous time flows relates the spectrum of the evolution operator

chapter 14

$$\mathcal{L}^t(x',x) \;=\; \delta\big(x'-f^t(x)\big)\; e^{\beta\cdot A^t(x)} \tag{23.1}$$

to the unstable periodic orbits p of the flow $f^t(x)$,

$$\sum_{\alpha=0}^{\infty}\frac{1}{s-s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty}\frac{e^{r(\beta\cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|}\,.$$

This formula (and the associated spectral determinants and cycle expansions) is valid for fully hyperbolic flows.

chapter 15

Here we derive the corresponding formula for dynamics invariant under a compact group of symmetry transformations. In what follows, a familiarity with basic group-theoretic notions is assumed, with the definitions relegated to appendix J.1.

fast track:
chapter 22, p. 427

23.1 Compact groups

The group theory that we shall need here is summarized by the *Peter-Weyl theorem*, and its corrolaries: A compact Lie group $\mathcal{G}$ is completely reducible, its representations can be decomposed like the group algebra of the finite group representation theory, every Lie group is a closed subgroup of $U(n)$ for some n, and every continuous, unitary, irreducible representation of a compact Lie group is finite dimensional.

The theory of semisimple Lie groups is elegant, perhaps too elegant. In what follows, we serve group theoretic nuggets in need-to-know portions, offering a pedestrian route through a series of simple examples of familiar aspects of group theory and Fourier analysis, and a high, cyclist road in the text proper.

But main idea is this: the character $\chi_m(\theta)$ of the Frobenius-Weyl representations theory is a generalization to all compact continuous Lie groups of the weight $e^{i\theta m}$ in the Fourier transform of a smooth function on a circle. So a basis of for a d-dimensional representation $(m_1, m_2, \ldots, m_N)$ of an N-dimensional compact Lie group is a set of d linearly independent eigenfunctions on the N-dimensional compact group manifold, with m_1, m_2, ..., m_N "nodes" along the N directions needed to span the manifold. For a circle this is Fourier analysis; for a sphere these are spherical harmonics, and the Peter-Weyl theorem states that analogous expansion exists for every compact Lie group. We will not need to construct it explicitely.

23.1 page 476
23.2 page 476

23.1.1 Group representations

The vector space q_a.

Repeated indices summed throughout this chapter.

The symmetry group, i.e. the group of all linear transformations

$$q'_a = G_a{}^b q_b \qquad a, b = 1, 2, \ldots, d\,,$$

which leaves invariant the predictions of the theory. The $[d \times d]$ matrices G form a representation of the group $\mathcal{G}$. Vectors in the conjugate space $\overline{q}$ transform as

$$q'^a = G^a{}_b q^b\,.$$

Tensors transform as

$$h'{}_{ab}{}^c = G_a{}^f G_b{}^e G_c{}^d h_{fe}{}^d\,.$$

A function is an invariant function if (and only if) for any transformation $g \in \mathcal{G}$ and for any set of vectors $q, r, s, \ldots$

coordinate reflection
reflection, coordinate
group!integral—(textbf

$$H(\overline{Gq}, \overline{Gr}, \ldots Gs) = H(\overline{q}, \overline{r}, \ldots, s)\,. \tag{23.2}$$

The group transformations can be generated by sequences of infinitesimal transformations

$$G_a{}^b \simeq \delta^a_b + i\epsilon_i (T_i)^b_a \qquad \epsilon_i \in \mathbb{R}\,, T_i \text{ hermitian}\,.$$

In this chapter we consider only infinitesimal transformations of form $G = 1 + iD$, $|D^a_b| \ll 1$, i.e., the transformations connected to the identity. For example, we postpone study of discrete invariances under coordinate reflections to chapter 22.

Consider a multilinear invariant function

$$H(\overline{q}, \overline{r}, \ldots, s) = h_{ab\ldots}{}^{\ldots c} q^a r^b \ldots s_c$$

In terms of the generators T_i, H is invariant if

$$(T_i)^{a'}_a h_{a'b\ldots}{}^{c\ldots} + (T_i)^{b'}_b h_{ab'\ldots}{}^{c\ldots} - (T_i)^c_{c'} h_{ab\ldots}{}^{c'\ldots} + \ldots = 0\,. \tag{23.3}$$

For example, the generators T_i are invariant (one does not want the rules to change at every step)

$$(T_i)\,{}^a_b = G^a{}_{a'} G_b{}^{b'} G_{ii'}\,(T_{i'})\,{}^{a'}_{b'}\,, \tag{23.4}$$

where G_{ij} is the adjoint $[N \times N]$ matrix representation of $g \in \mathcal{G}$.

The $[d \times d]$ matrices T_i are in general non-commuting, and from (23.3) it follows that they close N-element *Lie algebra*

$$T_i T_j - T_j T_i = i C_{ijk} T_k \qquad i, j, k = 1, 2, ..., N$$

where C_{ijk} are the structure constants.

Vector space V is irreducible if the only invariant subspaces of V under the action of G are (0) and V. If every V on which G acts can be written as a direct sum of irreducible subspaces, then G is completely reducible.

Consider a group integral of form

$$\int dg\, G_a{}^b G^c{}_d\,, \tag{23.5}$$

where ${G_a}^b$ is a $[n \times n]$ matrix representation of $g \in \mathcal{G}$, $\mathcal{G}$ a compact Lie group, and ${G_a}^b$ is unitary. $G^\dagger$ is the matrix representation of the action of g on the conjugate vector space, which we write as

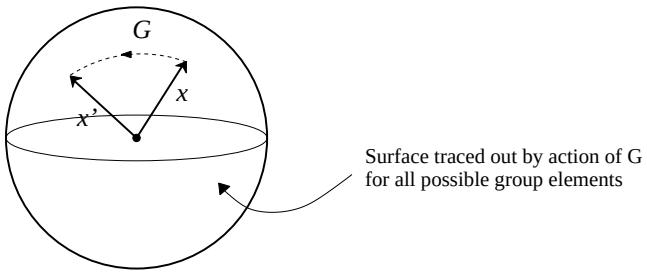
$$ {G^a}_b = {(G^\dagger)_b}^a \,, $$

and the integration is over the entire range of g. The integral (23.5) is defined by two requirements:

(1) Normalization:

$$ \int dg = 1 \,. \tag{23.6} $$

(2) How do we define $\int dg\, {G_a}^b$? The action of $g \in \mathcal{G}$ is to rotate a vector x_a into $x'_a = {G_a}^b x_b$



The averaging smears x in all directions, hence the second integration rule

$$ \int dg\, {G_a}^b = 0 \,, \qquad G \text{ is a } \textit{non-trivial} \text{ representation of } g \,, \tag{23.7} $$

simply states that the average of a vector is zero.

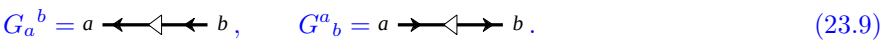
A representation is trivial if $G = 1$ for all group elements g. In this case no averaging is taking place, and the first integration rule (23.6) applies.

What happens if we average a pair of vectors x, y? There is no reason why a pair should average to zero; for example, we know that $|x|^2 = \sum_a x_a x^*_a = x_a x^a$ is invariant (we are considering only unitary representations), so it cannot have a vanishing average. Therefore, in general

$$ \int dg\, {G_a}^b {G^c}_d \neq 0 \,. \tag{23.8} $$

To get a feeling of what the right-hand side looks like, let us work out an example:

Example 23.1 A group integral for $SU(n)$: *Let $G_a{}^b$ be the defining $[n\times n]$ matrix representation of $SU(n)$. The defining representation is non-trivial, so it averages to zero by (23.7). The first non-vanishing average involves $G^\dagger$, the matrix representation of the action of g on the conjugate vector space. To avoid dealing with the multitude of dummy indices, we resort to diagrammatic notation:*

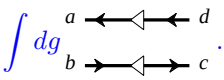


(23.9)

For G the arrows and the triangle point the same way, while for $G^\dagger$ they point the opposite way. Unitarity $G^\dagger G = 1$ is given by

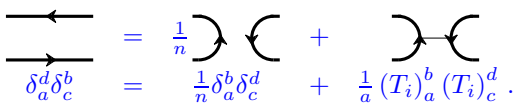
(23.10)

In this notation, the $GG^\dagger$ integral to be evaluated is



(23.11)

For $SU(n)$ the $V\otimes\overline{V}$ tensors decompose into the singlet and the adjoint rep

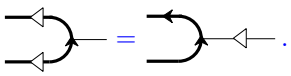


(23.12)

We multiply (23.11) with the above decomposition of the identity. The unitarity relation (23.11) eliminates G's from the singlet:

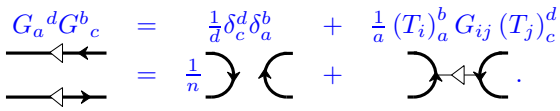
(23.13)

The generators T_i are invariants, and transform under G according to (23.4). Multiplying by G^{-1}_{ii}, we obtain



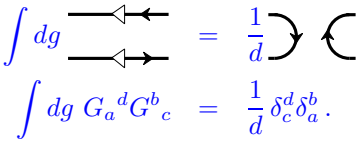
(23.14)

Hence, the pair $GG^\dagger$ in the defining representation can be traded in for a single G in the adjoint rep



(23.15)

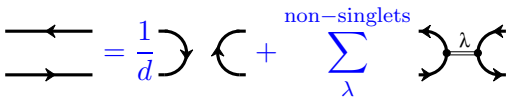
The adjoint representation G_{ij} is non-trivial, so it gets averaged to zero by (23.7). Only the singlet survives



(23.16)

irreducible!representation
representation!irreducible
Schur's Lemma
Clebsch-Gordan!series
character—(textbf
representation!character
character!orthonormality
character—)
group!integral—)

Now let G be any irreducible $[d\times d]$ rep. Irreducibility (known in this context as "Schur's Lemma") means that any invariant tensor A^a_b is proportional to δ^a_b. As the only bilinear invariant is δ^a_b, the Clebsch-Gordan series



(23.17)

contains one and only one singlet. Only the singlet survives the group averaging, and (23.16) is true for any $[d\times d]$ *irreducible* rep. If we take ${G^{(\mu)}}_\alpha{}^\beta$ and ${G^{(\lambda)}}_d{}^c$ in inequivalent representations λ,μ (there is no matrix K such that $G^{(\lambda)} = KG^{(\mu)}K^{-1}$ for any $g\in\mathcal{G}$), then there is no way of forming a singlet, and

$$\int dg\, {G^{(\lambda)}}_a{}^d {G_{(\mu)}}^\beta{}_\alpha = 0 \qquad \text{if} \qquad \lambda\neq\mu\,. \tag{23.18}$$

The trace of an irreducible $[d\times d]$ matrix representation λ of g is called the *character* of the representation:

$$\chi_\lambda(g) = \mathrm{tr}\, G^{(\lambda)} = {G^{(\lambda)}}_a{}^a\,. \tag{23.19}$$

The character of the conjugate representation is

$$\chi^\lambda(g) = \mathrm{tr}\, G^{(\lambda)\dagger} = {G^{(\lambda)a}}_a = \chi_\lambda(g)^*\,. \tag{23.20}$$

Contracting (23.17) with two arbitrary invariant $[d\times d]$ tensors $h_d{}^a$ and $(f^\dagger)_b{}^c$, we obtain the *character orthonormality relation*

$$\int dg\ \chi_\lambda(hg)\,\chi^\mu(gf) = \delta^\mu_\lambda\,\frac{1}{d_\lambda}\,\chi_\lambda(hf^\dagger) \tag{23.21}$$

The character orthonormality tells us that if two group invariant quantities share a $GG^\dagger$ pair, the group averaging sews them into a single group invariant quantity. The replacement of ${G_a}^b$ by the character $\chi_\lambda(h^\dagger g)$ does not mean that any of the tensor index structure is lost; ${G_a}^b$ can be recovered by differentiating

$${G_a}^b = \frac{d}{dh_b{}^a}\chi_\lambda(h^\dagger g)\,. \tag{23.22}$$

The diagrammatic notation and the characters are two equivalent notations for evaluating group integrals.

All the group theory we shall need here is most compactly summarized by the

group!representation
representation!matrix

Group Orthogonality Theorem: Let D_m, $D_{m'}$ be two irreducible matrix representations of a compact group G of dimensions d_m, $d_{m'}$,

$$\int_G dg\ D_m(g)_{ab} D^*_{m'}(g)_{a'b'} = \frac{1}{d_m}\delta_{m,m'}\delta_{aa'}\delta_{bb'}\,.$$

Here dg is the Haar measure, invariant under group transformations

$$\int_G dg \ = \int_G d\mu(x) \ = \int_G d\mu(gx)\,,$$

normalized

$$\int_G dg \ = 1\,,$$

so that the group average of an invariant scalar quantity is the quantity itself. For a finite group G with $|G|$ group elements the normalized Haar measure is a discrete sum.

$$d\mu(x) \ = \ \frac{1}{|G|}\sum_g \delta(gx)\,.$$

For compact groups left- and right-invariant Haar measures coincide. As all we shall do here is integrate over a Dirac delta function, we shall not need here any explicit global formula for $d\mu(x)$ (though it pays to work through a few examples, for one's own comfort).

The new trace formula follows from the full reducibility of representations of a compact group G acting linearly on a vector space V, with irreducible representations labeled by sets of integers $m = (m_1, \cdots, m_N)$, and the vector space V decomposed into invariant subspaces V_m. For a N-dimensional compact Lie group G the fundamental result is the Weyl full reducibility theorem, with projection operator onto the V_m irreducible subspace given by

Weyl reference

$$P_m = d_m \int_G dg\ \chi_m(g) D_m(g^{-1})\,. \tag{23.23}$$

The group elements $g = g(\theta_1, \ldots, \theta_N) = e^{i\theta\cdot T}$ are parametrized by N real numbers $\{\theta_1, \ldots, \theta_N\}$ of finite range, hence designation "compact." The N group generators T_a, $a = 1, \cdots, N$ close the Lie algebra of G.

23.1.2 Transformation operators, projection operators

credit [23.46]

[1] How does the group act on a function $f(x)$ of x? The transformed function should have the same value at gx as the initial function has at x, so, denoting by O_g the desired transformation:

$$O_g f(D(g)x) = f(x)\,.$$

Here the operator O_g acts on functions, whereas $D(g)$ is a matrix representation of the group element $D(g)$ that acts on position vectors $x \in \mathcal{M}$.

Let us replace x with $D(g^{-1})x$, giving

$$O_g f(x) = f(D(g^{-1})x)$$

This is the conventional, Wigner's definition of the effect of transformations on functions.

Now, consider the effect of two successive transformations $g_1 g_2 = g$. We have

$$O_{g_1} O_{g_2} f(x) = O_{g_1} f(g_2^{-1}x) = f(g_2^{-1}[g_1^{-1}x]) = f([g_1 g_2]^{-1}x) = O_g f(x)$$

Hence if $g_1 g_2 = g$, we have $O_{g_1} O_{g_2} = O_g$.

Except in special circumstances, we will not use the O_g notation, but just g when we operate on functions. But be aware of the difference between transformation of functions and of vectors.

Suppose we have an arbitrary function or set of functions. How do we obtain functions with desired symmetry properties? If f is an arbitrary function,

$$P^\alpha_{ij} f = \frac{d_\alpha}{|G|} \sum_G D^\alpha_{ij}(g)^* g f = F^\alpha_{ij}$$

which is either zero or a basis function for the ith row of irrep α: a function of symmetry species (α, i).

[23.46], p. 16 as a soluton to the problem of proving the projection property.

back reference to example 23.7

Example 23.2 Irreducible representations of the $SO(2)^N$ Abelian group: *(Example 23.7 continued) The irreducible representations of the $SO(2)^N$ Abelian group acting on torus T^N are labeled by N integers $m = (m_1, \cdots, m_N)$. All irreducible representations are 1-dimensional, $d_m = 1$. The character of m representation is*

$$\chi_m(g) = e^{-im\cdot\theta}$$

[1] Bartsch: I can't make sense out of the projector (23.23). P_m must operate on the big, reducible representation space. $D_m(g^{-1})$ is a matrix of dimension d_m, which operates on the irreducible representation. Should the formula have the total representation matrix $D(g^{-1})$ instead of D_m?

Example 23.3 Haar measure for $SO(2)$: *The normalized Haar measure is* $d\theta/(2\pi)$.

group!integration
integration!group
Haar measure
symmetry!dynamical system
dynamics!symmetry
flow!generator of
generator!of flow

Example 23.4 Haar measure for $SO(3)$:

$$SO(3): \qquad dg = \frac{1}{2\pi^2}\sin^2(\phi/2)d\Omega_e d\phi$$

with $d\Omega_e$ *solid angle surface element for unit vector* e.

$$\int_{SO(3)} dg = 8\pi^2$$

see ref. [23.20].

The character χ is the trace $\chi_m(g) = \mathrm{tr}\, D_m(g) = \sum_{i=1}^{d_m} D_m(g)_{ii}$, where $D_m(g)$ is a $[d_m \times d_m]$-dimensional matrix representation of action of the group element g on the irreducible subspace V_m. We will sometimes employ notation g as a shorthand for $D(g)$, i.e., by $x' = gx$ we mean the matrix operation $x'_i = \sum_{j=1}^{d} D(g)_{ij}x_j$, and by $f'(x) = g\,f(x) = f(gx)$, $f(x)$ a smooth function over the state space $x \in \mathcal{M}$, we mean $f'(x) = f(D(g)x)$. For an invariant scalar quantity the average over the group in (23.23) must be the quantity itself, so the group integral is weighted by the normalized Haar measure, $\int_G dg = 1$, and d_m is the multiplicity of degenerate eigenvalues in representation m.

23.2 Continuous symmetries of dynamics

reintroduce shorthand g here

If action of every element g of a compact group G commutes with the flow $\dot{x} = v(x)$,

$$D(g)v(x) = v(D(g)x)\,, \;\; D(g)f^t(x) = f^t(D(g)x)\,,$$

G is a global symmetry of the dynamics. The finite time evolution operator (23.1) can be written as $\mathcal{L}^t = e^{t\mathcal{A}}$ in terms of the time evolution generator (9.26)

$$\mathcal{A}\rho(x) = \lim_{\delta\tau\to 0^+}\frac{1}{\delta\tau}\left(\mathcal{L}^{\delta\tau} - I\right)\rho(x) \;=\; -\partial_i(v_i(x)\rho(x))\,. \qquad (23.24)$$

The operator $e^{t\mathcal{A}}$ commutes with all symmetry transformations $e^{i\theta\cdot T}$. For a given state space point x together they sweep out a $(N{+}1)$-dimensional manifold of equivalent orbits.

In other words, the time evolution itself is a (noncompact) 1-parameter Lie group. Thus all continuous symmetries can be considered as being on on the same footing.

stability!marginal
marginal!stability

A symmetry group element acts on $\mathcal{L}(x,y)$, the kernel of $\mathcal{L}^t$ in the state space representation (23.1), as

$$g^{-1}\mathcal{L}(y,x) = \mathcal{L}(D(g^{-1})y,x) = \mathcal{L}(y,D(g)x)\,. \tag{23.25}$$

Here we have used $|\det D(g)|$. The irreducible eigenspaces of G are also eigenspaces of the dynamical evolution operator $\mathcal{L}^t$, with the decomposition of the evolution operator to irreducible subspaces, $\mathcal{L} = \sum_m \mathcal{L}_m$, following immediately by application of the projection operator (23.23):

$$\mathcal{L}^t_m(y,x) = d_m \int_G dg\ \chi_m(g)\mathcal{L}^t(D_m(g^{-1})y,x)\,. \tag{23.26}$$

23.2.1 Relative periodic orbits

Relative periodic orbits are orbits $x(t)$ in state space $\mathcal{M}$ which exactly recur

$$x(t) = gx(t+T) \tag{23.27}$$

for a fixed *relative period* T and a fixed group action $g \in G$ of $\mathcal{M}$. This group action is sometimes referred to as a "phase," or a "shift." Relative periodic orbits are to periodic solutions what relative equilibria (traveling waves) are to equilibria (steady solutions).

For dynamical systems with continuous symmetries relative periodic orbits are almost never eventually periodic, i.e., they almost never lie on periodic trajectories in the full state space. As almost any such orbit explores ergodically the manifold swept by action of G, they are sometimes referred to as "quasiperiodic". However, an orbit can be periodic if it satisfies a special symmetry. If $g^k = 1$ is of finite order k, then the corresponding orbit is periodic with period kT. If g is not of finite order k, orbits can be periodic only after the action of g.

☞ chapter 22

In either case, we refer to the orbits in $\mathcal{M}$ satisfying (23.35) as *relative periodic orbits*.

⇓PRELIMINARY

As G commutes with f^t, all eigenfunctions ρ of $\mathcal{L}^t$ must be invariant under G, $\rho(x) = \rho(gx)$. Infinitesimally, in terms of Lie algebra generators $T_\theta\rho(x) = 0$.

⇑PRELIMINARY

23.2.2 Stability of relative periodic orbits

finish this section

A infinitesimal group transformation maps globally a trajectory in a nearby trajectory, so we expect the initial point perturbations along to group manifold to be marginal, growing at rates slower than exponential. The

argument is akin to (4.7), the proof of marginality of perturbations along the trajectory. Consider two nearby initial points separated by an infinitesimal group rotation $\delta\theta$: $\delta x(0) = f^{\delta\theta}(x_0) - x_0 = v(x_0)\delta\theta$. By the commutativity of the group with the flow, $f^{t+\delta t} = f^{\delta t+t}$. Expanding both sides of $f^t(f^{\delta t}(x_0)) = f^{\delta t}(f^t(x_0))$, keeping the leading term in δt, and using the definition of the fundamental matrix (4.6), we observe that $J^t(x_0)$ transports the velocity vector at x_0 to the velocity vector at $x(t)$ at time t:

Lagrangian!coordinates
Eulerian!coordinates
cycle!marginal stability
marginal stability!cycle

$$v(x(t)) = J^t(x_0)\, v(x_0)\,. \tag{23.28}$$

In nomenclature of page 62, the fundamental matrix maps the initial, Lagrangian coordinate frame into the current, Eulerian coordinate frame.

For relative periodic orbits However, already at this stage we see that if the orbit is periodic, $g_p x(T_p) = x(0)$, at any point along cycle p the velocity v is an eigenvector of the fundamental matrix $J_p = J^{T_p}$ with an eigenvalue of unit magnitude,

$$J^{T_p}(x)\, v(x) = v(x)\,, \qquad x \in p\,. \tag{23.29}$$

Two successive points along the cycle separated by $\delta x(0)$ have the same separation after a completed period $\delta x(T_p) = g_p \delta x(0)$, hence eigenvalue of magnitude 1.

23.3 Symmetry reduced trace formula for flows

not quite true, fix

As any pair of nearby points on a periodic orbit returns to itself exactly at each cycle period, the eigenvalue of the fundamental matrix corresponding to the eigenvector along the flow necessarily equals unity for all periodic orbits. In presence of N-dimensional symmetry Lie group G, further N eigenvalues equal unity. Hence the trace integral $\mathrm{tr}\,\mathcal{L}^t$ requires a separate treatment for the direction along the flow and for the N group transformation directions.

Ole: clarify this — draw a picture.

To evaluate the contribution of a prime cycle p of period T_p, restrict the integration to an infinitesimally thin manifold $\mathcal{M}_p$ enveloping the cycle and all of its rotations by G, pick a point on the cycle, and choose a local coordinate system with a longitudinal coordinate $dx_\parallel$ along the direction of the flow, N coordinates dx_G along the invariant manifold swept by p under the action of the symmetry group G, and $(d-N-1)$ transverse coordinates $x_\perp$

draw figure

$$\mathrm{tr}\,{}_p\mathcal{L}^t_m = d_m \int_G dg\ \chi_m(g) \int_{\mathcal{M}_p} dx_\perp dx_\parallel dx_G\, \delta\big(x - D_m(g) f^t(x)\big)\ . \tag{23.30}$$

The integral along the longitudinal coordinate was computed in (14.18). Eliminating the time dependence by Laplace transform one obtains

$$\int_0^\infty e^{-st} \oint_p dx_\parallel \, \delta\big(x_\parallel - f^t(x_\parallel)\big) \;=\; T_p \sum_{r=1}^{\infty} e^{-sT_p r} \;. \tag{23.31}$$

Example 23.5 Trace group integral for $SO(2)$: *Parametrize rotations on a circle by $\theta \in [0, 2\pi)$. The normalized Haar measure is $dg = d\theta/(2\pi)$, and a trajectory point advanced by time t and shifted by θ can be denoted $x(t, \theta)$. The character is $e^{-im\theta}$. For a circle this is just Fourier analysis, for a general compact semisimple Lie group Weyl's generalization of Fourier analysis. Consider projection on the mth subspace of the integral along the rotational direction*

$$I_G = \int_G dg \; \chi_m(g) \oint dx_G \, \delta(x_G - D_m(g) x_G(t, 0)) \, .$$

Coordinate x_G is the set of points swept by $[0, 2\pi]$ rotation of a point $x_0 = x_G(0, 0)$, so it is natural to parametrize it by the rotation angle θ': $x_G = x(0, \theta')$, and rewrite the circle integral as

$$\oint dx_G \, \delta(x_G - D_m(g) x_G(t, 0)) = \int_0^{2\pi} d\theta' \frac{dx}{d\theta}(0, \theta') \, \delta(x(0, \theta') - x(t, \theta' + \theta)) \, .$$

Inverting the order of integrations,

$$I_G = \int_0^{2\pi} d\theta' \int_0^{2\pi} \frac{d\theta}{2\pi} \, e^{-im\theta} \frac{dx}{d\theta}(0, \theta') \, \delta(x(0, \theta') - x(t, \theta' + \theta)) \, .$$

The integral is novanishing for smallest θ_p for which $x(0, \theta') = x(t, \theta' + \theta_p)$, and for all its repeats. Expand the argument of δ function in each such neighborhood $\theta' = \theta_p + \theta''$.

$$\begin{aligned} x(t, \theta' + \theta_p + \theta'') &= x(t, \theta' + \theta_p) + \theta'' \frac{dx}{d\theta}(t, \theta' + \theta_p) + \cdots \\ &= x(t, \theta') + \theta'' \frac{dx}{d\theta}(t, \theta') + \cdots \, . \end{aligned}$$

substituting back yields

$$\begin{aligned} I_G &= \int_0^{2\pi} \frac{d\theta'}{2\pi} \sum_{r=1}^{\infty} e^{-im\theta_p r} \frac{dx(0, \theta')}{d\theta} \int_{-\epsilon}^{\epsilon} d\theta'' \, e^{-im\theta''} \delta\left(\theta'' \frac{dx}{d\theta}(0, \theta')\right) \\ &= \sum_{r=1}^{\infty} e^{-im\theta_p r} \, . \end{aligned}$$

The m subspace group integral is simple:

$$\int_G dg \; \chi_m(g) \int_{\mathcal{M}_p} dx_G \, \delta\big(x_G - D_m(g) f^{rT_p}(x_G)\big) = \chi_m(g_p^r). \tag{23.32}$$

For the remaining transverse coordinates the fundamental matrix is defined in a $(N{+}1)$-dimensional surface of section $\mathcal{P}$ of constant $(x_\parallel, x_G)$. Linearization of the periodic flow transverse to the orbit yields

trace!formula!symmetry reduced
periodic orbit!relative
relative!periodic orbit

$$\int_{\mathcal{P}} dx_\perp \delta\big(x_\perp - D_m(g_p^r) f^{rT_p}(x_\perp)\big) = \frac{1}{\left|\det\left(\mathbf{1} - \tilde{M}^r_{m,p}\right)\right|}\,, \qquad (23.33)$$

ch from **J** to **M**

where $\tilde{M}_{m,p} = D_m(g_p)M_p$ is the p-cycle $[(d-1-N)\times(d-1-N)]$ symmetry reduced fundamental matrix, computed on the reduced surface of section and rotated by g_p. As in (14.5), we assume hyperbolicity, i.e., that the magnitudes of all transverse eigenvalues are bounded away from unity.

The *classical symmetry reduced trace formula for flows* follows by substituting (23.31) - (23.33) into (23.30):

$$\sum_{\beta=0}^{\infty} \frac{1}{s - s_{m,\beta}} = d_m \sum_p T_p \sum_{r=1}^{\infty} \chi_m(g_p^r) \frac{e^{r(\beta A_p - sT_p)}}{\left|\det\left(\mathbf{1} - \tilde{M}^r_{m,p}\right)\right|}\,. \qquad (23.34)$$

add the determinant

(we can restore $e^{\beta A_p}$ from (23.1) provided that the observable $a(x)$ also commutes with G.) The sum is over all prime *relative periodic orbits* p and their repeats, orbits in state space which satisfy

$$x(t) = D(g_p)x(t+T_p) \qquad (23.35)$$

for a fixed *relative period* T_p and a fixed *shift* g_p.

For example, for the full Navier-Stokes plane Couette flow defined in a box periodic in stream-wise and span-wise directions, a relative periodic solution is a solution that recurs at time T_p with exactly the same disposition of velocity fields over the entire box, but shifted by a 2-dimensional translation g_p.

The $m = (0, 0, \cdots, 0)$ subspace is the one of most relevance to chaotic dynamics, as its leading eigenfunction, with least nodes and the slowest decay rates, corresponds to the natural measure observed in the long time dynamics.

⇓PRIVATE

Example 23.6 Translation group: *The generator $\mathcal{A}$ is reminiscent of the generator of translations. Indeed, for a constant velocity field dynamical evolution is nothing but a translation by (time × velocity):*

9.10 page 148

$$e^{-tv\frac{\partial}{\partial x}} a(x) = a(x - tv)\,. \qquad (23.36)$$

Example 23.7 $SO(2)$ **rotations:** *$SO(2)$ is the group of rotations in a plane, smoothly connected to the unit element (i.e. the inversion $x \to -x$ is excluded). A group element can be parameterized by angle θ', and its action on smooth periodic functions over θ is generated by*

$$g^\theta = e^{-i\theta T}\,, \quad T = \frac{d}{d\theta}$$

g^θ rotates a periodic function $u(\theta + 2\pi) = u(\theta)$ by θ mod 2π:

$$g^\theta u(\theta') = u(\theta' + \theta)$$

The multiplication law is $g^\theta g^{\theta'} = g^{\theta+\theta'}$. If the group G actions consists of N such rotations which commute, for example a N-dimensional box with periodic boundary conditions, the group G is an Abelian group that acts on a torus T^N.

For a compact Lie group the prefactor d_m in (23.23) counts degenerate eigenvalue multiplicities.

For example, the special case for $SO(3)$ where $m = \{j\}$, the degenerate eigenvalue multiplicity $d_m = (2j + 1)$ is familiar from quantum mechanics.

⇑PRIVATE

23.3.1 Trace formula in presence of a discrete symmetry

☞ chapter 22

In the above, we have discussed only the continuous case (23.23), as (23.37) follows by replacing integration over Haar measure by a finite group discrete sum. For a finite group G with $|G|$ group elements, the sum in (23.37) is normalized, $|G|^{-1}\sum_{g\in G} = 1$, so that the group average of an invariant scalar quantity is the quantity itself.

Keep this in mind: Frobenious theory is a generalization of Fourier analysis for $|G|$ points on a circle to all finite groups, For a finite group $G = \{e, g_2, \ldots, g_{|G|}\}$, the fundamental result is Frobenius reduction formula in terms of group characters, with the projection operator onto the irreducible subspace V_m given by

$$P_m = \frac{d_m}{|G|}\sum_{g\in G} \chi_m(g) D_m(g^{-1})\,. \tag{23.37}$$

For a finite group the prefactor d_m in (23.37) reflects the fact that a d_m-dimensional representation occurs d_m times.

In contrast to the case of continuous symmetries, where relative periodic orbits are almost never eventually periodic, i.e., they almost never lie on periodic trajectories in the full state space [23.28], for discrete symmetries all relative periodic orbits are eventually periodic.

We turn to the case of discrete symmetry group: $G = \{e, g_2, \ldots, g_{|G|}\}$, in great detail in chapter 22.

add here parts of PER's Appendix B.

Commentary

Remark 23.1 Semisimple Lie groups: Here we need only basic results, on the level of any standard group theory textbook [23.8, 23.9]. Chapter 2. of ref. [23.10] offers a clear and pedagogical introduction to Lie groups of transformations. For the Group Orthogonality Theorem see, for example, [23.17, 23.19], or google.

find good reference on Peter-Weyl theorem

Remark 23.2 Full reducibility of semisimple Lie groups:

The study of integrals over compact Lie groups with respect to Haar measure is important in many areas of mathematics and physics, see Mehta [23.21]. In 1896-1897 Frobenius introduced notions of "characters" and group "representations," and proved the full reducibility of representations of finite groups. The characters $\chi_m(g)$ for all compact semisimple Lie groups were constructed and the full reducibility proven by Weyl, extending Cartan's local Lie algebra classification to a global theory of group representations. For the history of this period, see the excellent essay by Hawkins [23.16].

Diagrammatic notation for group theory is explained in the `ChaosBook.orgGroupTheory` webbook.

Remark 23.3 A brief history of relativity: The relative equilibria and relative periodic solutions are related to equilibria and periodic solutions respectively of the Hamiltonian reduced by the symmetries. They appear in many physical situations, such as motion of rigid bodies, gravitational N-body problems, molecules and nonlinear waves. Chenciner [23.37, 23.42] says the first attempt to find (relative) periodic solutions of the N-body problem by was the 1896 short note by Poincaré [23.39], in the context of the 3-body problem. Relative equilibria of the N-body problem (known in this context as the Lagrange points, stationary in the co-rotating frame) are circular motions in the inertial frame, and relative periodic orbits correspond to quasiperiodic motions in the inertial frame. In the planar N-body problem the equations are unchanged by rotations. Relative equilibria can exist in a rotating frame, and are called central configurations. For relative periodic orbits in celestial mechanics see also ref. [23.23] .

Lan has some relative equilibria (traveling waves) for KS in his thesis [25.8], and for complex LG in a paper on "MAWs". Viswanath [23.6] found them in the plane Couette problem. Siminos and Davidchack have for box size $L = 22$ some equilibria. Striking application of relative periodic orbits has been the discovery of "choreographies" of N-body problems [23.38, 23.40, 23.41].

In context of semiclassical quantization Creagh and Littlejohn [23.25, 23.26] concentrate on the case when the continuous symmetry family of orbits includes a true periodic orbit (they use infinitesimal variation around true periodic orbit), not the symmetry reduced case considered here (where almost every relative periodic orbit of the symmetry-reduced dynamics is not a periodic orbit in the full space). They emphasize generalized surface of section dynamics. They refer to relative periodic orbits as "generalized periodic orbits", with "generalized period" $\mathbf{T}_p = (T_p, m_p)$. They mention, but do not go to irreducible reps of the symmetry groups, hence no equivariant trace formula in these papers. Instead, the explicitly compute group volumes. In addition to the reduced dynamics $|\det(1 - M_\perp)|$ they get $\partial\theta/\partial J$ which we do not have. The Berkeley group did it right for discrete symmetries [22.14, 22.4].

Probably suffices to refer to Creagh [23.5] only

This is done in ref. [23.5] which we follow here, and in ref. [23.28] for axially-symmetric case. Creagh [23.5] refers to relative periodic orbits as "pseudoperiodic" orbits. Ref. [23.28] refers to relative periodic orbits as "reduced periodic" orbits, and to the corresponding orbits in the full state space as "quasiperiodic". Creagh remarks at the very end of his paper to his formula (6.4) as the "pleasing result that the quantally reduced spectrum is determined by the classically reduced periodic orbits in the usual way". Ref. [23.36] discusses a trace formula in symmetry-reduced space. Muratore-Ginanneschi [23.29] gives an elegant discussion of "zero-modes" in the path integral formulation, but does not go to irreps either.

use word quasiperiodic

but no character weight in (6.4)?

Ref. [23.27] applies the method to the problems of noninteracting identical particles,

give references: [?] [?] [23.33] [23.34] [23.35]

Résumé

One of the goals of nonlinear dynamics is to describe the long time evolution of ensembles of trajectories, when individual trajectories are exponentially unstable. The main tool in this effort have been trace formulas because they make explicit the duality between individual short time trajectories, and long time invariant densities (natural measures, eigenfunctions of evolution operators). So far, the main successes have been in applications to low dimensional flows and iterated mappings, where the compact invariant sets of short-time dynamics are equilibria, periodic points and periodic orbits. Dynamics in higher dimensions requires extension of trace formulas to higher-dimensional compact invariant sets, such as partially hyperbolic invariant tori.

Here we have used a particularly simple direct product structure of a global symmetry that commutes with the flow to reduce the dynamics to a symmetry reduced $(d-N-1)$-dimensional state space $\mathcal{M}/G$. The trace formulas do not require explicit construction (in general difficult) neither of the reduced state space, nor of the Haar measures.

Amusingly, in this extension of "periodic orbit" theory from unstable 1-dimensional closed orbits to unstable $(N+1)$-dimensional compact manifolds invariant under continuous symmetries, there are no or very few periodic orbits. Relative periodic orbits are almost never eventually periodic, i.e., they almost never lie on periodic trajectories in the full state space [23.28], unless forced to do so by a discrete symmetry, so looking for periodic orbits in systems with continuous symmetries is a fool's errand.

The symmetry reduced trace formula for a *finite* symmetry group $G = \{e, g_2, \ldots, g_{|G|}\}$ with $|G|$ group elements, where the integral over Haar measure in projection (23.23) is replaced by a finite group discrete sum $|G|^{-1}\sum_{g\in G} = 1$, was derived in ref. [23.13, 23.14]. In contrast to the continuous symmetry case, for a discrete symmetry all relative periodic orbits are eventually periodic.

Restriction to compact Lie groups in derivation of the trace formula (23.34) was a matter of convenience, as the general case is more transparent

than particular implementations (such as $SO(2)$ and $SO(3)$ rotations, with their explicit Haar measures and characters). This can be relaxed as the need arises - much powerful group theory developed since Cartan-Weyl era is at our disposal. For example, the time evolution is in general non-compact (a generic trajectory is an orbit of infinite length). Nevertheless, the trace formulas have support on compact invariant sets in $\mathcal{M}$, such as periodic orbits and $(N+1)$-dimensional manifolds generated from them by action of the global symmetry groups. Just as existence of a periodic orbit is a consequence of given dynamics, not any global symmetry, higher-dimensional flows beckon us on with nontrivial higher-dimensional compact invariant sets (for example, partially hyperbolic invariant tori) for whom the trace formulas are still to be written.

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⇓PRELIMINARY

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Flotsam

Look for *rpo s* in the pipe flow, relative both in the shift down the pipe and rotation around the pipe axis.

Likelyhood of finding a periodic orbit is *zero*. One expects some only if in addition to a continuous symmetry one has a discrete symmetry which is not a subgroup of the continuous symmetry. THer are none such for the pipe flow. Duct flow would have them.

However, Kuramoto-Sivashinsky and plane Coutte do have discrete symmetries in addition - that is why there are some equilibria (as opposed to relative equilibria) and some periodic orbits in these cases. They belong to discrete symmetry subspaces. Atypical as they are (no turbulent solution will be confined to these subspaces) they are important for periodic orbit theory, as there the shortest orbits dominate.

I believe some but not all of Kawahara's periodic orbits (for the reason explained above).

I have not looked at the isotropic turbulence orbits paper in detail (as I do not think feel that there is interesting physics in such models) and do

not know what discrete symmetries they have; if any periodic orbits are found, they should be in the discrete symmetry reduced subspaces.

Thomas comments

[2] The Liouville operator can only be decomposed into symmetry subspaces of the observable A is G-invariant. You should make a pertinent remark somewhere.

The transformation (5) of the Liouville operator is not clear to me. I transform densities according to $rho(x) -> rho_g(x) = rho(gx)$, and covariance of the Liouville operator requires that $q_g = L[rho_g]$ if $q = L[rho]$, with two densities q, rho. To achieve that, the kernel must satisfy $L(gy, gx) = L(y, x)$ (if the representation matrices have unit determinant). I would therefore define the transformed kernel by $gL(y, x) = L(gy, gx)$.

When deriving the trace formula, you neglect the possibility that the PO might be invariant under a subgroup of G. In this case, the manifold swept out by the PO and its images has dimension less than $N + 1$. This is discussed in Creagh's paper (but I haven't had a look at it for a while).

Thomas: In your example of the flow with periodic boundary conditions, the group of symmetries induced by the boundary conditions is a two-dimensional lattice, which is discrete and non-compact. The derivations of the paper do not apply.

Predrag: No - both plane Couette and KS are compactified on periodic domain and have compact continuous symmetries: S_1 (circle) or $SO(2)$ for Kuramoto-Sivashinsky, T^2 (torus) or $SO(2)^2$ for plane Couette. There is no two-dimensional Z^2 non-compact discrete lattice.

Remember to check this paper: ref. [23.32].

The symmetry is spatial translation by an arbitrary amount, and you are looking for eigenfunctions with the space denpendence exp(i k x), with integer k, which belongs to a representation space of S1. Relative POs are then orbits that return to the initial configuration up to an arbitrary shift. Right?

Right - except shift in not arbitrary, it is as fixed as the time period is. Currently shifts are mod L (system size) as with both Kuramoto-Sivashinsky and plane Couette we turn translational directions into finite boxes with periodic b.c.

Eventual goal is to find these relative periodic orbit for spatially infinite systems (just like we find unsable periodic orbits of fintie period along the infinite time direction). But we have not done it, except for equilibria.

[2] Bartsch: Predrag uses "Liouville" operator only for Hamiltonian flows, "evolution" operator for any dynamical flow

Paolo's comments

Paolo Muratore-Ginanneschi 24 Oct 2006

1)In my opinion difference between your calculation and the one I present in the phys. rep. (which reproduces by other means a result by Littlejohn and Creagh J. Phys A 25 1992) consists only in the fact that localization for classical trace formule is exact.

2) This (obvious) observation has though an important consequence. The result for classical trace formulae must be interpreted as distribution valued as you explained long ago in your paper with Eckhard.

Finite results may be obtained by integrating the trace over smooth test functions as you did in the subspace associated to the zero mode associated with the time derivative of the flow.

Rather than doing this, it seems to me that in the subspace spanned the zero modes generated by the "Jacobi fields" induced by the symmetry group you used the "completeness relation" for the matrix elements of irreducible representations to express the "delta-function" otherwise emerging from the calculation of the trace.

This choice is at the origin of the discrepancy with my calculation. There localization is not exact and the trace of the unitary operator I am computing can be easily regularised for example by defining it as the limit of an euclidean continuation. This latter choice provide in general the simplest way to make mathematical sense of Feynman path integrals. So I do not need to decompose the result in its irreducible components. This difference accounts in my opinion for the difference in the weights. It seems to me that a formal resummation over irreducible representation in your formula provides a positive check of my claim.

3) What the two calculations have in common is the need to separate fluctuations around the periodic orbit into longitudinal and transversal ones. In my report I tried to show as clearly as I could that this change of variable is the core of any degenerate by continuos symmetry stationary point calculation disregarding whether localization is exact or not.

Such technique is in the path-integral context generally referred to as the Faddeev-Popov trick. Over last weekend I became aware of two further references where the above claim is proved for finite dimensional systems and formally for infinite dimensional ones in full generality. It enjoyed the reading of them so I thought to advertise them

Fainberg and Semikhatok, Nuclear Physics B Volume 226, Issue 1 , 26 September 1983, Pages 253-268

this is the pedagogic version of Appendix 2 of

A.S. Schwarts Comm. Math. Phys 64 233

4) Coming to the detail of the calculation, I was assuming that the symmetry group acts freely on the periodic orbit manifold. So the integral over the transversal fluctuation trivially coincided with the group volume time period $T \times G$.}$MaybeIammissingsomethingitbutitseemstomethatasimilarassumptio$ over the group orbit space may not include a non-trivial isotropy subgroup. The weight would be in such a case

$$\mu(G/H)\,, \qquad \mu \text{ is the measure and } H \text{ the stabilizer}$$

The outer integral involving the group characteristic may become then non-trivial. A partial example of this situation would be in my opinion an ODE with SO(3) symmetry. In such a case the x_G variables would be described by two angles rather than three.

Historical notes

(On group characters) presented to the Berlin Academy on July 16 1896 is of fundamental importance. The following year that representations of groups began to enter the picture, and again it was a concept due to Frobenius. Hence 1897 is the year in which the representation theory of groups was born. Only in 1897 did Frobenius learn of Molien's work which he described in a letter to Dedekind as "very beautiful but difficult". He reformulated Molien's work in terms of matrices and then showed that his characters are the traces of the irreducible representations. This work was published in 1897. Frobenius's character theory was used with great effect by Burnside and was beautifully written up in Burnside's 1911 edition of his Theory of Groups of Finite Order.

Consider a smooth continuous time flow

$$x(t) = f^t(x_0) = x_0 + \int_0^t d\tau\, v(x(\tau))\,, \qquad x(0) = x_0\,. \tag{23.38}$$

obtained by integrating a d-dimensional set of ODEs $\dot{x} = v(x)$.

$$\text{Spectrum}\,(\mathcal{L}^t) = \{e^{s_0 t}, e^{s_1 t}, e^{s_2 t}, \cdots\}\,, \qquad \text{Re}\; s_\alpha \geq \text{Re}\, s_{\alpha+1} \tag{23.39}$$

(assumed discrete)

Definition: A Lie group is a topological group G such that (1) G has the structure of a smooth differential manifold. (2) The composition map $G \times G \to G : (g, h) \to gh^{-1}$ is smooth.

By "smooth" we here always mean $\mathbb{C}^\infty$ differentiable.

Definition. Let $\pi_V : G \to GL(V)$ be a representation, then the character of V is $\chi_V = \operatorname{tr} \pi_V$.

Character of the identity group element is $\chi(e) = d$, since $D_{ij}(e) = \delta_{ij}$ for $1 \leq i, j \leq d$.

Essentially, this is a special case of what we have discussed above. The projection operator onto the m irreducible subspace of dimension d_m is given by $P_m = (d_m/|G|) \sum \chi_m(h) h^{-1}$, where $\chi_m(h) = \operatorname{tr} D_m(h)$ are the group characters, and the transfer operator $\mathcal{L}$ splits into a sum of inequivalent irreducible subspace contributions $\sum_m \operatorname{tr} \mathcal{L}_m$,

where is D_m defined

why h and not g?

$$\mathcal{L}_m(y, x) = \frac{d_m}{|G|} \sum_{g \in G} \chi_m(g) \mathcal{L}(h^{-1} y, x) \ . \tag{23.40}$$

The prefactor d_m in the above reflects the fact that a d_m-dimensional representation occurs d_m times.

A dynamical system $(\mathcal{M}, f)$ is invariant under a symmetry group G if the map $f : \mathcal{M} \to \mathcal{M}$ (or the continuous flow f^t) from the d-dimensional manifold $\mathcal{M}$ into itself (with d finite) is invariant

$$f(gx) = g f(x)$$

for any coordinate $x \in \mathcal{M}$ and any finite non-singular linear representation (i.e., a non-singular $[d \times d]$ matrix) g of any element $g \in G$.

in terms the time evolution generator (9.26)

$$\mathcal{A} \ = \ -(\partial \cdot v) - \sum_{i=1}^{d} v_i(x) \partial_i \ . \tag{23.41}$$

We ignore for the time being the question of what function space the eigenfunctions belong to, as we shall compute the eigenvalue spectrum without constructing any explicit eigenfunctions.

The reduced state space is the quotient $\mathcal{M}/G$. This amounts to ignoring "angles" θ and fixing the conjugate variables p_θ to irreducible representation values θ.

Duality between t and s, θ and m endows this extended space with a natural symplectic structure: In deriving trace formulas, we switch from (x, t, g) to $(x_\perp, s, m)$ representation.

The summation is over orbits that start at x with "angular momentum" m and end at y after time t and "rotation" by g.

character!orthonormality need to evaluate Jacobian for transformation

$$dx_{\|}\left|\frac{\partial(\cdots,s,m)}{\partial(\cdots,t,\theta)}\right| = dtd\theta\,.$$

Need to evaluate

$$W_p = \int_p dtdg \;= T_p V_p\,.$$

Or do we? Isn't $V_p \int_G dg \;= 1$ for any relative periodic orbit? Creagh [23.5] gets to that point (modulo discrete symmetry) by the end of his paper.

Consider motion of a test particle of mass $\mu \ll 1$ in the restricted three-body problem [23.22], under the influence of the gravitational force of two heavy bodies with masses 1 and $\mu \ll 1$ fixed at $(-\mu, 0)$ and $(1-\mu, 0)$.

A reduction of the translation symmetry is obtained by fixing the center of mass at the origin or, considering a configuration as an element of the disposition space $\mathbb{R}^D/(1,...,1)\mathbb{R}$ introduced in [2] in the spirit of Jacobi.

Berry and Tabor should be re-derived here in this approach.

Example 23.8 Haar measure for $SU(2)$: *$SU(2)$ acts on vectors in $\mathbb{C}^2$, and preserves their absolute value, hence its action can be parameterized by a 3-sphere S_3, and multiplication can be viewed as an orthogonal transformation of S_3. This is a special case of the formula $N = n^2 - 1$ for the dimension of $SU(n)$. An invariant Haar measure on $SU(2)$ is a normalized surface measure on S_3, such as*

$$\int_{SU(2)} f(g)dg = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{\pi} \sin^2\phi_1 \sin\phi_2 d\theta d\phi_1 d\phi_2 f(\theta,\phi_1,\phi_2)$$

Note the following laws

$$\int_G dg\; D(g) = 0,$$

recheck Furthermore, if we consider all non-equivalent irreducible representations of a group G, then the quantities $D^m_{ij}(g)$ for fixed m, i and j

$$\int_G dg\; \chi_m(hg)\chi_m(g^{-1}k^{-1}) = \chi_m(hk^{-1}) \quad \text{with } h,k \in G \text{ fixed}$$

Characters are orthonormal.

$$P_m = d_m \int_G dg\; \chi^*_m(g) D^m(g)$$

Orthonormality of the set $\{P_m\}$.

Completeness of the set $\{P_m\}$. If operator X commutes with the symmetry, the reduced operator X_m acts on V_m. Such operators:

time evolution $\mathcal{L}^t = e^{t\mathcal{A}}$

resolvent $1/(s-\mathcal{A})$

Study eigenvalues $s_{m,\alpha}$ restricted to subspace V_m.

Symmetries can be used to reduce dynamics to a system with fewer degrees of freedom, elimination of ignorable coordinates.

Consider the dynamics on the extended state space $\mathcal{M} \times \mathbb{R} \times G$ where $\mathbb{R}$ corresponds to time.

$(x', t, g') \to (g f^t(x'), t + t', g g')$

in generalized "time" (t, g).

Let $\mathcal{G}$ be a group, X a set, and $\cdot : G \times X \longrightarrow X$ a group action. For any $x \in X$, the *orbit* of x under the group action is the set

$$\{g \cdot x \mid g \in \mathcal{G}\} \subset X.$$

⇑PRELIMINARY

symmetry!dynamical system
dynamics!symmetry

Exercises

Exercise 23.1 **To be constructed:** *Derive the trace formula (??) for a discrete symmetry*

A dynamical system $(\mathcal{M}, f)$ is invariant under a symmetry group G if the map $f : \mathcal{M} \to \mathcal{M}$ (or the continous flow f^t) from the d-dimensional manifold $\mathcal{M}$ into itself (with d finite) is invariant

$$f(\mathbf{g}x) = \mathbf{g}f(x)$$

for any coordinate $x \in \mathcal{M}$ and any finite non-singular linear representation $\mathbf{g}$ of any element $g \in G$.

Show that the evolution operator *$\mathcal{L}(x, y)$ (23.1) satisfies (23.25):*

$$\mathcal{L}(x, y) = \mathcal{L}(\mathbf{g}x, \mathbf{g}y). ,$$

Solution: (move to solutions!)
Rotate coordinates x'= $\mathbf{g}$x:

$$\mathcal{L}(x', y') = \delta(\mathbf{g}y - f(\mathbf{g}x)). = |\det \mathbf{g}|^{-1} \delta(y - f(x)). = \mathcal{L}(x, y) = |\det \mathbf{g}| \, \mathcal{L}(\mathbf{g}x, \mathbf{g}y). .$$

For a compact semisimple Lie group $|\det \mathbf{g}| = 1$, hence (23.25).

Exercise 23.2 **Relative periodic orbits for circles, bagels and spheres:** *(a) Show that relative periodic orbits for a point scattering specularly in a circular billiard are single scattering arcs. Compute their stability. Compute the spectrum.*

(b) Show that relative periodic orbits for a point scattering specularly in the plane that slices symmetrically upper half of a bagel (floating tire, torus) are single scattering arcs. Compute their stability. Compute the spectrum. Compute the escape rate.

(c) Show that relative periodic orbits for a point scattering specularly within a sphere billiard are single scattering arcs. Compute their stability, spectrum.

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Chapter 24

Deterministic diffusion

> This is a bizzare and discordant situation.
> M.V. Berry

(R. Artuso and P. Cvitanović)

The advances in the theory of dynamical systems have brought a new life to Boltzmann's mechanical formulation of statistical mechanics. Sinai, Ruelle and Bowen (SRB) have generalized Boltzmann's notion of ergodicity for a constant energy surface for a Hamiltonian system in equilibrium to dissipative systems in nonequilibrium stationary states. In this more general setting the attractor plays the role of a constant energy surface, and the SRB measure of sect. 9.1 is a generalization of the Liouville measure. Such measures are purely microscopic and indifferent to whether the system is at equilibrium, close to equilibrium or far from it. "Far for equilibrium" in this context refers to systems with large deviations from Maxwell's equilibrium velocity distribution. Furthermore, the theory of dynamical systems has yielded new sets of microscopic dynamics formulas for macroscopic observables such as diffusion constants and the pressure, to which we turn now.

We shall apply cycle expansions to the analysis of *transport* properties of chaotic systems.

The resulting formulas are exact; no probabilistic assumptions are made, and the all correlations are taken into account by the inclusion of cycles of all periods. The infinite extent systems for which the periodic orbit theory yields formulas for diffusion and other transport coefficients are spatially periodic, the global state space being tiled with copies of a elementary cell. The motivation are physical problems such as beam defocusing in particle accelerators or chaotic behavior of passive tracers in 2-d rotating flows, problems which can be described as deterministic diffusion in periodic arrays.

In sect. 24.1 we derive the formulas for diffusion coefficients in a simple physical setting, the 2-d periodic Lorentz gas. This system, however, is

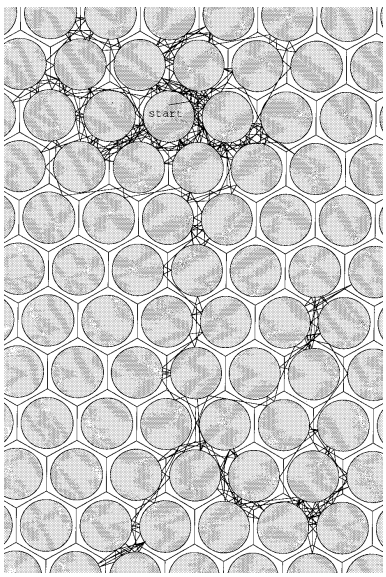

Figure 24.1: Deterministic diffusion in a finite horizon periodic Lorentz gas. (Courtesy of T. Schreiber)

not the best one to exemplify the theory, due to its complicated symbolic dynamics. Therefore we apply the theory first to diffusion induced by a 1-*d* maps in sect. 24.2and return to the Lorentz systems in sect. 24.4, after a brief disscusion of anomalous diffusion in sect. 24.3.

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24.1 Diffusion in periodic arrays

The 2-*d Lorentz gas* is an infinite scatterer array in which diffusion of a light molecule in a gas of heavy scatterers is modelled by the motion of a point particle in a plane bouncing off an array of reflecting disks. The Lorentz gas is called "gas" as one can equivalently think of it as consisting of any number of pointlike fast "light molecules" interacting only with the stationary "heavy molecules" and not among themselves. As the scatterer array is built up from only defocusing concave surfaces, it is a pure hyperbolic system, and one of the simplest nontrivial dynamical systems that exhibits deterministic diffusion, figure 24.1. We shall now show that the *periodic* Lorentz gas is amenable to a purely deterministic treatment. In this class of open dynamical systems quantities characterizing global dynamics, such as the Lyapunov exponent, pressure and diffusion constant, can be computed from the dynamics restricted to the elementary cell. The method applies to any hyperbolic dynamical system that is a periodic tiling $\hat{\mathcal{M}} = \bigcup_{\hat{n}\in T} \mathcal{M}_{\hat{n}}$ of the dynamical state space $\hat{\mathcal{M}}$ by *translates* $\mathcal{M}_{\hat{n}}$ of an *elementary cell* $\mathcal{M}$, with T the Abelian group of lattice translations. If the scattering array has further discrete symmetries, such as reflection symmetry, each elementary cell may be built from a *fundamental domain* $\widetilde{\mathcal{M}}$ by the action of a discrete (not necessarily Abelian) group G. The symbol $\hat{\mathcal{M}}$ refers here to the full state space, i.e.,, both the spatial coordinates and the momenta. The spatial component of $\hat{\mathcal{M}}$ is the complement of the disks in the *whole* space.

We shall now relate the dynamics in $\mathcal{M}$ to diffusive properties of the

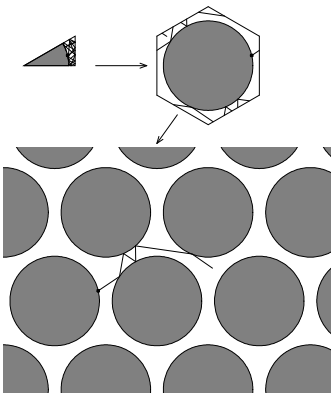

Figure 24.2: Tiling of $\hat{\mathcal{M}}$, a periodic lattice of reflecting disks, by the fundamental domain $\widetilde{\mathcal{M}}$. Indicated is an example of a global trajectory $\hat{x}(t)$ together with the corresponding elementary cell trajectory $x(t)$ and the fundamental domain trajectory $\tilde{x}(t)$. (Courtesy of J.-P. Eckmann)

Lorentz gas in $\hat{\mathcal{M}}$.

These concepts are best illustrated by a specific example, a Lorentz gas based on the hexagonal lattice Sinai billiard of figure 24.2. We distinguish two types of diffusive behavior; the *infinite horizon* case, which allows for infinite length flights, and the *finite horizon* case, where any free particle trajectory must hit a disk in finite time. In this chapter we shall restrict our consideration to the finite horizon case, with disks sufficiently large so that no infinite length free flight is possible. In this case the diffusion is normal, with $\hat{x}(t)^2$ growing like t. We shall return to the anomalous diffusion case in sect. 24.3.

ask J.-P. Eckmann for permission to use his figures

As we will work with three kinds of state spaces, good manners require that we repeat what hats, tildes and nothings atop symbols signify:

$$\begin{array}{ll} \tilde{} & \text{fundamental domain, triangle in figure 24.2} \\ & \text{elementary cell, hexagon in figure 24.2} \\ \hat{} & \text{full state space, lattice in figure 24.2} \end{array} \tag{24.1}$$

It is convenient to define an evolution operator for each of the 3 cases of figure 24.2. $\hat{x}(t) = \hat{f}^t(\hat{x})$ denotes the point in the global space $\hat{\mathcal{M}}$ reached by the flow in time t. $x(t) = f^t(x_0)$ denotes the corresponding flow in the elementary cell; the two are related by

$$\hat{n}_t(x_0) = \hat{f}^t(x_0) - f^t(x_0) \in T\,, \tag{24.2}$$

the translation of the endpoint of the global path into the elementary cell $\mathcal{M}$. The quantity $\tilde{x}(t) = \tilde{f}^t(\tilde{x})$ denotes the flow in the fundamental domain $\widetilde{\mathcal{M}}$; $\tilde{f}^t(\tilde{x})$ is related to $f^t(\tilde{x})$ by a discrete symmetry $g \in G$ which maps $\tilde{x}(t) \in \widetilde{\mathcal{M}}$ to $x(t) \in \mathcal{M}$.

☞ chapter 22

Fix a vector $\beta \in \mathbb{R}^d$, where d is the dimension of the state space. We will compute the diffusive properties of the Lorentz gas from the leading eigenvalue of the evolution operator (10.11)

$$s(\beta) = \lim_{t\to\infty} \frac{1}{t} \log\langle e^{\beta\cdot(\hat{x}(t)-x)}\rangle_{\mathcal{M}}\,, \tag{24.3}$$

where the average is over all initial points in the elementary cell, $x \in \mathcal{M}$.

If all odd derivatives vanish by symmetry, there is no drift and the second derivatives

$$\frac{\partial}{\partial\beta_i}\frac{\partial}{\partial\beta_j}s(\beta)\Big|_{\beta=0} = \lim_{t\to\infty}\frac{1}{t}\langle(\hat{x}(t)-x)_i(\hat{x}(t)-x)_j\rangle_{\mathcal{M}} \ ,$$

yield a (generally anisotropic) diffusion matrix. The spatial diffusion constant is then given by the Einstein relation (10.13)

$$D = \frac{1}{2d}\sum_i \frac{\partial^2}{\partial\beta_i^2}s(\beta)\Big|_{\beta=0} = \lim_{t\to\infty}\frac{1}{2dt}\langle(\hat{q}(t)-q)^2\rangle_{\mathcal{M}} \ ,$$

where the i sum is restricted to the spatial components q_i of the state space vectors $x=(q,p)$, i.e., if the dynamics is Hamiltonian to the number of the degrees of freedom.

reinstate mass, velocity, size to get β, m, σ dependencies right

We now turn to the connection between (24.3) and periodic orbits in the elementary cell. As the full $\hat{\mathcal{M}} \to \widetilde{\mathcal{M}}$ reduction is complicated by the nonabelian nature of G, we shall introduce the main ideas in the abelian $\hat{\mathcal{M}} \to \mathcal{M}$ context.

☞ remark 24.6

24.1.1 Reduction from $\hat{\mathcal{M}}$ to $\mathcal{M}$

The key idea follows from inspection of the relation

$$\left\langle e^{\beta\cdot(\hat{x}(t)-x)}\right\rangle_{\mathcal{M}} = \frac{1}{|\mathcal{M}|}\int_{\substack{x\in\mathcal{M}\\ \hat{y}\in\hat{\mathcal{M}}}} dx d\hat{y}\, e^{\beta\cdot(\hat{y}-x)}\delta(\hat{y}-\hat{f}^t(x)) \ .$$

$|\mathcal{M}| = \int_{\mathcal{M}} dx$ is the volume of the elementary cell $\mathcal{M}$. As in sect. 10.2, we have used the identity $1 = \int_{\mathcal{M}} dy\, \delta(y-\hat{x}(t))$ to motivate the introduction of the evolution operator $\mathcal{L}^t(\hat{y},x)$. There is a unique lattice translation $\hat{n}$ such that $\hat{y} = y - \hat{n}$, with $y \in \mathcal{M}$, and $f^t(x)$ given by (24.2). The difference is a translation by a constant, and the Jacobian for changing integration from $d\hat{y}$ to dy equals unity. Therefore, and this is the main point, translation invariance can be used to reduce this average to the elementary cell:

$$\langle e^{\beta\cdot(\hat{x}(t)-x)}\rangle_{\mathcal{M}} = \frac{1}{|\mathcal{M}|}\int_{x,y\in\mathcal{M}} dx dy\, e^{\beta\cdot(\hat{f}^t(x)-x)}\delta(y-f^t(x)) \ . \qquad (24.4)$$

As this is a translation, the Jacobian is $\delta\hat{y}/\delta y = 1$. In this way the global $\hat{f}^t(x)$ flow averages can be computed by following the flow $f^t(x_0)$ restricted

to the elementary cell $\mathcal{M}$. The equation (24.4) suggests that we study the evolution operator

standing orbit!Lorentz gas
running orbits!Lorentz gas
accelerator mode

$$\mathcal{L}^t(y,x) \;=\; e^{\beta\cdot(\hat{x}(t)-x)}\delta(y-f^t(x)) \;, \tag{24.5}$$

where $\hat{x}(t) = \hat{f}^t(x) \in \hat{\mathcal{M}}$, but x, $f^t(x)$, $y \in \mathcal{M}$. It is straightforward to check that this operator satisfies the semigroup property (10.25), $\int_{\mathcal{M}} dz\, \mathcal{L}^{t_2}(y,z)\mathcal{L}^{t_1}(z,x) = \mathcal{L}^{t_2+t_1}(y,x)$. For $\beta = 0$, the operator (24.5) is the Perron-Frobenius operator (9.10), with the leading eigenvalue $e^{s_0} = 1$ because there is no escape from this system (this will lead to the flow conservation sum rule (19.11) later on).

The rest is old hat. The spectrum of $\mathcal{L}$ is evaluated by taking the trace

☞ sect. 14.3

$$\mathrm{tr}\,\mathcal{L}^t \;=\; \int_{\mathcal{M}} dx\, e^{\beta\cdot\hat{n}_t(x)}\delta(x-x(t)) \;.$$

Here $\hat{n}_t(x)$ is the discrete lattice translation defined in (24.2). Two kinds of orbits periodic in the elementary cell contribute. A periodic orbit is called *standing* if it is also periodic orbit of the infinite state space dynamics, $\hat{f}^{T_p}(x) = x$, and it is called *running* if it corresponds to a lattice translation in the dynamics on the infinite state space, $\hat{f}^{T_p}(x) = x + \hat{n}_p$. In the theory of area–preserving maps such orbits are called *accelerator modes*, as the diffusion takes place along the momentum rather than the position coordinate. The travelled distance $\hat{n}_p = \hat{n}_{T_p}(x_0)$ is independent of the starting point x_0, as can be easily seen by continuing the path periodically in $\hat{\mathcal{M}}$.

recheck usage "standing", "running" with the literature.

explain that "accelerator modes" makes sense for standard map

The final result is the spectral determinant (15.7)

$$\det\left(s(\beta)-\mathcal{A}\right) \;=\; \prod_p \exp\left(-\sum_{r=1}^{\infty}\frac{1}{r}\frac{e^{(\beta\cdot\hat{n}_p-sT_p)r}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|}\right) \;, \tag{24.6}$$

or the corresponding dynamical zeta function (15.16)

$$1/\zeta(\beta,s) \;=\; \prod_p\left(1-\frac{e^{(\beta\cdot\hat{n}_p-sT_p)}}{|\Lambda_p|}\right) \;. \tag{24.7}$$

The dynamical zeta function cycle averaging formula (18.18) for the diffusion constant (10.13), zero mean drift $\langle\hat{x}_i\rangle = 0$, is given by

$$D \;=\; \frac{1}{2d}\frac{\langle\hat{x}^2\rangle_\zeta}{\langle T\rangle_\zeta} \;=\; \frac{1}{2d}\frac{1}{\langle T\rangle_\zeta}\sum{}'\,\frac{(-1)^{k+1}(\hat{n}_{p_1}+\cdots+\hat{n}_{p_k})^2}{|\Lambda_{p_1}\cdots\Lambda_{p_k}|} \;. \tag{24.8}$$

where the sum is over all distinct non-repeating combination of prime cycles. The derivation is standard, still the formula is strange. Diffusion is

unbounded motion accross an infinite lattice; nevertheless, the reduction to the elementary cell enables us to compute relevant quantities in the usual way, in terms of periodic orbits.

A sleepy reader might protest that $x_p = x(T_p) - x(0)$ is manifestly equal to zero for a periodic orbit. That is correct; $\hat{n}_p$ in the above formula refers to a displacement on the *infinite* periodic lattice, while p refers to closed orbit of the dynamics reduced to the elementary cell, with x_p belonging to the closed prime cycle p.

Even so, this is not an obvious formula. Globally periodic orbits have $\hat{x}_p^2 = 0$, and contribute only to the time normalization $\langle T \rangle_\zeta$. The mean square displacement $\langle \hat{x}^2 \rangle_\zeta$ gets contributions only from the periodic runaway trajectories; they are closed in the elementary cell, but on the periodic lattice each one grows like $\hat{x}(t)^2 = (\hat{n}_p/T_p)^2 = v_p^2 t^2$. So the orbits that contribute to the trace formulas and spectral determinants exhibit either ballistic transport or no transport at all: diffusion arises as a balance between the two kinds of motion, weighted by the $1/|\Lambda_p|$ measure. If the system is not hyperbolic such weights may be abnormally large, with $1/|\Lambda_p| \approx 1/{T_p}^\alpha$ rather than $1/|\Lambda_p| \approx e^{-T_p \lambda}$, where λ is the Lyapunov exponent, and they may lead to anomalous diffusion - accelerated or slowed down depending on whether the probabilities of the running or the standing orbits are enhanced.

☞ sect. 24.3

We illustrate the main idea, tracking of a globally diffusing orbit by the associated confined orbit restricted to the elementary cell, with a class of simple 1-d dynamical systems where all transport coefficients can be evaluated analytically. We return to the Lorentz gas in sect. 24.4.

⇓PRELIMINARY
⇑PRELIMINARY

24.2 Diffusion induced by chains of 1-d maps

In a typical deterministic diffusive process, trajectories originating from a given scatterer reach a finite set of neighboring scatterers in one bounce, and then the process is repeated. As was shown in chapter 11, the essential part of this process is the stretching along the unstable directions of the flow, and in the crudest approximation the dynamics can be modelled by 1-d expanding maps. This observation motivates introduction of a class of particularly simple 1-d systems, chains of piecewise linear maps.

We start by defining the map $\hat{f}$ on the unit interval as

$$\hat{f}(\hat{x}) = \begin{cases} \Lambda \hat{x} & \hat{x} \in [0, 1/2) \\ \Lambda \hat{x} + 1 - \Lambda & \hat{x} \in (1/2, 1] \end{cases} , \qquad \Lambda > 2 , \tag{24.9}$$

and then extending the dynamics to the entire real line, by imposing the translation property

$$\hat{f}(\hat{x} + \hat{n}) = \hat{f}(\hat{x}) + \hat{n} \qquad \hat{n} \in \mathbb{Z} . \tag{24.10}$$

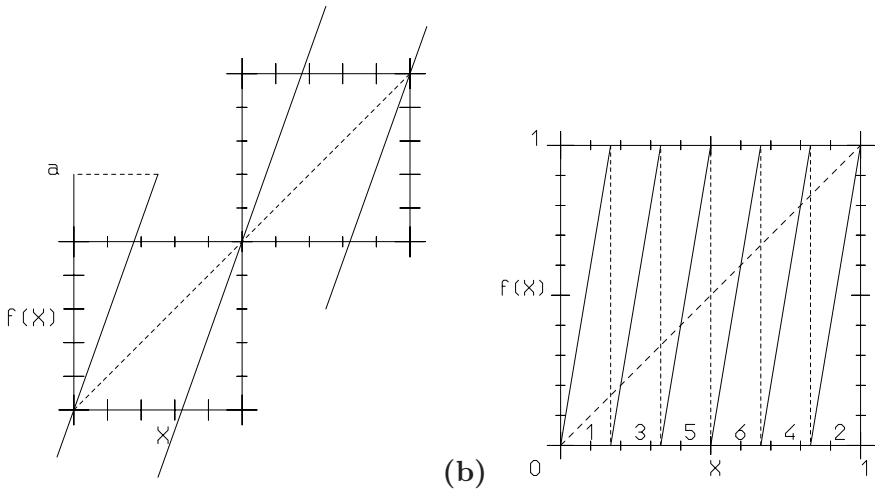

Figure 24.3: (a) $\hat{f}(\hat{x})$, the full space sawtooth map (24.9), $\Lambda > 2$. (b) $f(x)$, the sawtooth map restricted to the unit circle (24.12), $\Lambda = 6$.

As the map is dicontinuous at $\hat{x} = 1/2$, $\hat{f}(1/2)$ is undefined, and the $x = 1/2$ point has to be excluded from the Markov partition. The map is antisymmetric under the $\hat{x}$-coordinate flip

$$\hat{f}(\hat{x}) \;=\; -\hat{f}(-\hat{x})\,, \tag{24.11}$$

so the dynamics will exhibit no mean drift; all odd derivatives of the generating function (10.11) with respect to β, evaluated at $\beta = 0$, will vanish.

The map (24.9) is sketched in figure 24.3 (a). Initial points sufficiently close to either of the fixed points in the initial unit interval remain in the elementary cell for one iteration; depending on the slope Λ, other points jump $\hat{n}$ cells, either to the right or to the left. Repetition of this process generates a random walk for almost every initial condition.

The translational symmetry (24.10) relates the unbounded dynamics on the real line to dynamics restricted to the elementary cell - in the example at hand, the unit interval curled up into a circle. Associated to $\hat{f}(\hat{x})$ we thus also consider the circle map

$$f(x) = \hat{f}(\hat{x}) - \left[\hat{f}(\hat{x})\right]\,, \qquad x = \hat{x} - [\hat{x}] \in [0,1] \tag{24.12}$$

define $\hat{n}$ here as well

figure 24.3 (b), where $[\cdots]$ stands for the integer part (24.2). As noted above, the elementary cell cycles correspond to either standing or running orbits for the map on the full line: we shall refer to $\hat{n}_p \in \mathbb{Z}$ as the *jumping number* of the p cycle, and take as the cycle weight

$$t_p \;=\; z^{n_p} e^{\beta \hat{n}_p} / |\Lambda_p|\,. \tag{24.13}$$

critical!value

For the piecewise linear map of figure 24.3 we can evaluate the dynamical zeta function in closed form. Each branch has the same value of the slope, and the map can be parametrized by a single parameter, for example its critical value $a = \hat{f}(1/2)$, the absolute maximum on the interval $[0, 1]$ related to the slope of the map by $a = \Lambda/2$. The larger Λ is, the stronger is the stretching action of the map.

The diffusion constant formula (24.8) for 1-d maps is

$$D = \frac{1}{2}\frac{\langle \hat{n}^2 \rangle_\zeta}{\langle n \rangle_\zeta} \tag{24.14}$$

where the "mean cycle time" is given by (18.19)

$$\langle n \rangle_\zeta = z\frac{\partial}{\partial z}\frac{1}{\zeta(0,z)}\bigg|_{z=1} = -\sum{}'(-1)^k \frac{n_{p_1}+\cdots+n_{p_k}}{|\Lambda_{p_1}\cdots\Lambda_{p_k}|}\,, \tag{24.15}$$

and the "mean cycle displacement squared" by (20.1)

$$\langle \hat{n}^2 \rangle_\zeta = \frac{\partial^2}{\partial \beta^2}\frac{1}{\zeta(\beta,1)}\bigg|_{\beta=0} = -\sum{}'(-1)^k \frac{(\hat{n}_{p_1}+\cdots+\hat{n}_{p_k})^2}{|\Lambda_{p_1}\cdots\Lambda_{p_k}|}\,, \tag{24.16}$$

the primed sum indicating all distinct non-repeating combinations of prime cycles. The evaluation of these formulas in this simple system will require nothing more than pencil and paper.

24.2.1 Case of unrestricted symbolic dynamics

Whenever Λ is an integer number, the symbolic dynamics is exceedingly simple. For example, for the case $\Lambda = 6$ illustrated in figure 24.3 (b), the elementary cell map consists of 6 full branches, with uniform stretching factor $\Lambda = 6$. The branches have different jumping numbers: for branches 1 and 2 we have $\hat{n} = 0$, for branch 3 we have $\hat{n} = +1$, for branch 4 $\hat{n} = -1$, and finally for branches 5 and 6 we have respectively $\hat{n} = +2$ and $\hat{n} = -2$. The same structure reappears whenever Λ is an even integer $\Lambda = 2a$: all branches are mapped onto the whole unit interval and we have two $\hat{n} = 0$ branches, one branch for which $\hat{n} = +1$ and one for which $\hat{n} = -1$, and so on, up to the maximal jump $|\hat{n}| = a-1$. The symbolic dynamics is thus full, unrestricted shift in $2a$ symbols $\{0_+, 1_+, \ldots, (a-1)_+, (a-1)_-, \ldots, 1_-, 0_-\}$, where the symbol indicates both the length and the direction of the corresponding jump.

For the piecewise linear maps with uniform stretching the weight associated with a given symbol sequence is a product of weights for individual steps,

$t_{sq} = t_s t_q$. For the map of figure 24.3 there are 6 distinct weigths (24.13): kurtosis Helfand moments

$$\begin{aligned} t_1 &= t_2 = z/\Lambda \\ t_3 &= e^{\beta} z/\Lambda\,, \quad t_4 = e^{-\beta} z/\Lambda\,, \quad t_5 = e^{2\beta} z/\Lambda\,, \quad t_6 = e^{-2\beta} z/\Lambda\,. \end{aligned}$$

The piecewise linearity and the simple symbolic dynamics lead to the full cancellation of all curvature corrections in (18.5). The *exact* dynamical zeta function (13.13) is given by the fixed point contributions: why not 1/2 cosh?

$$\begin{aligned} 1/\zeta(\beta,z) &= 1 - t_{0_+} - t_{0_-} - \cdots - t_{(a-1)_+} - t_{(a-1)_-} \\ &= 1 - \frac{z}{a}\left(1 + \sum_{j=1}^{a-1} \cosh(\beta j)\right)\,. \end{aligned} \tag{24.17}$$

The leading (and only) eigenvalue of the evolution operator (24.5) is

$$s(\beta) = \log\left\{\frac{1}{a}\left(1 + \sum_{j=1}^{a-1} \cosh(\beta j)\right)\right\}\,, \qquad \Lambda = 2a,\ a \text{ integer}\,. \tag{24.18}$$

The flow conservation (19.11) sum rule is manifestly satisfied, so $s(0) = 0$. The first derivative $s(0)'$ vanishes as well by the left/right symmetry of the dynamics, implying vanishing mean drift $\langle \hat{x} \rangle = 0$. The second derivative $s(\beta)''$ yields the diffusion constant (24.14):

$$\langle n \rangle_\zeta = 2a\frac{1}{\Lambda} = 1\,, \qquad \left\langle \hat{x}^2 \right\rangle_\zeta = 2\frac{0^2}{\Lambda} + 2\frac{1^2}{\Lambda} + 2\frac{2^2}{\Lambda} + \cdots + 2\frac{(a-1)^2}{\Lambda} \tag{24.19}$$

Using the identity $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ we obtain

$$D = \frac{1}{24}(\Lambda - 1)(\Lambda - 2)\,, \qquad \Lambda \text{ even integer}\,. \tag{24.20}$$

Similar calculation for odd integer $\Lambda = 2k - 1$ yields

24.1 page 503

$$D = \frac{1}{24}(\Lambda^2 - 1)\,, \qquad \Lambda \text{ odd integer}\,. \tag{24.21}$$

24.2.2 Higher order transport coefficients

The same approach yields higher order transport coefficients harmonize with the averages chapter

$$\mathcal{B}_k = \frac{1}{k!}\frac{d^k}{d\beta^k} s(\beta)\Big|_{\beta=0}\,, \qquad \mathcal{B}_2 = D\,, \tag{24.22}$$

Gaussian!integral
Burnett coefficient
Markov!partition

known for $k > 2$ as the Burnett coefficients. The behavior of the higher order coefficients yields information on the relaxation to the asymptotic distribution function generated by the diffusive process. Here $\hat{x}_t$ is the relevant dynamical variable and $\mathcal{B}_k$'s are related to moments $\langle \hat{x}_t^k \rangle$ of arbitrary order.

discuss Kulso formula

recheck

Were the diffusive process purely Gaussian

$$e^{ts(\beta)} = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{+\infty} d\hat{x}\, e^{\beta \hat{x}} e^{-\hat{x}^2/(4Dt)} = e^{\beta^2 D t} \tag{24.23}$$

the only $\mathcal{B}_k$ coefficient different from zero would be $\mathcal{B}_2 = D$. Hence, nonvanishing higher order coefficients signal deviations of deterministic diffusion from a Gaussian stochastic process.

For the map under consideration the first Burnett coefficient coefficient $\mathcal{B}_4$ is easily evaluated. For example, using (24.18) in the case of even integer slope $\Lambda = 2a$ we obtain

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$$\mathcal{B}_4 = -\frac{1}{4! \cdot 60}(a-1)(2a-1)(4a^2-9a+7)\,. \tag{24.24}$$

We see that deterministic diffusion is not a Gaussian stochastic process. Higher order even coefficients may be calculated along the same lines.

24.2.3 Case of finite Markov partitions

For piecewise-linear maps exact results may be obtained whenever the critical points are mapped in finite numbers of iterations onto partition boundary points, or onto unstable periodic orbits. We will work out here an example for which this occurs in two iterations, leaving other cases as exercises.

fix figure 24.4 (c)

credits in acknowledgements

The key idea is to construct a *Markov partition* (11.4), with intervals mapped *onto* unions of intervals. As an example we determine a value of the parameter $4 \leq \Lambda \leq 6$ for which $f(f(1/2)) = 0$. As in the integer Λ case, we partition the unit interval into six intervals, labeled by the jumping number $\hat{n}(x) \in \{\mathcal{M}_{0_+}, \mathcal{M}_{1_+}, \mathcal{M}_{2_+}, \mathcal{M}_{2_-}, \mathcal{M}_{1_-}, \mathcal{M}_{0_-}\}$, ordered by their placement along the unit interval, figure 24.4 (a).

In general the critical value $a = \hat{f}(1/2)$ will not correspond to an interval border, but now we choose a such that the critical point is mapped onto the right border of $\mathcal{M}_{1_+}$. Equating $f(1/2)$ with the right border of $\mathcal{M}_{1_+}$, $x = 1/\Lambda$, we obtain a quadratic equation with the expanding solution $\Lambda = 2(\sqrt{2}+1)$. For this parameter value $f(\mathcal{M}_{1_+}) = \mathcal{M}_{0_+} \bigcup \mathcal{M}_{1_+}$,

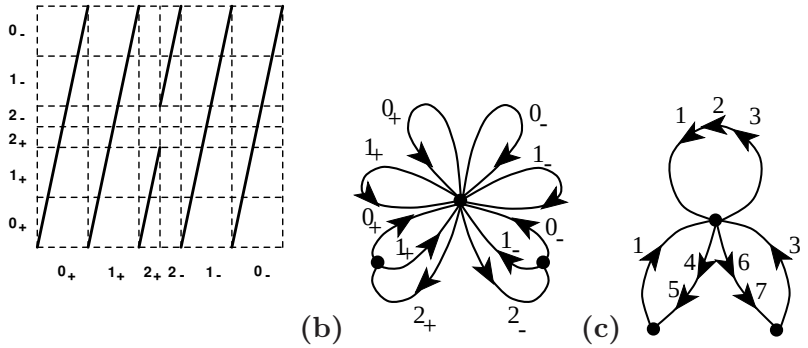
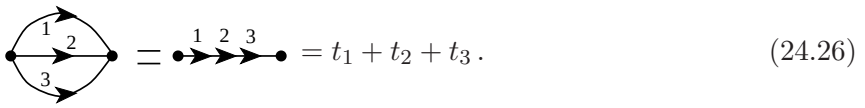


Figure 24.4: (a) A partition of the unit interval into six intervals, labeled by the jumping number $\hat{n}(x)$ $I = \{0_+, 1_+, 2_+, 2_-, 1_-, 0_-\}$. The partition is Markov, as the critical point is mapped onto the right border of $\mathcal{M}_{1_+}$. (b) The Markov graph for this partition. (c) The Markov graph in the compact notation of (24.26) (introduced by Vadim Moroz).

$f(\mathcal{M}_{2_-}) = \mathcal{M}_{0_-} \bigcup \mathcal{M}_{1_-}$, while the remaining intervals map onto the whole unit interval $\mathcal{M}$. The transition matrix (11.2) is given by

$$\phi' = T\phi \;=\; \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \phi_{0_+} \\ \phi_{1_+} \\ \phi_{2_+} \\ \phi_{2_-} \\ \phi_{1_-} \\ \phi_{0_-} \end{pmatrix} . \tag{24.25}$$

One could diagonalize (24.25) on a computer, but, as we saw in sect. 11.5, the Markov graph figure 24.4 (b) corresponding to figure 24.4 (a) offers more insight into the dynamics. The graph figure 24.4 (b) can be redrawn more compactly as Markov graph figure 24.4 (c) by replacing parallel lines in a graph by their sum



$$= t_1 + t_2 + t_3 \,. \tag{24.26}$$

The dynamics is unrestricted in the alphabet

$$\mathcal{A} = \{0_+,\, 1_+,\, 2_+0_+,\, 2_+1_+,\, 2_-1_-,\, 2_-0_-,\, 1_-,\, 0_-\}\,.$$

Applying the loop expansion (13.13) of sect. 13.3, we are led to the dynamical zeta function

$$\begin{aligned} 1/\zeta(\beta, z) &= 1 - t_{0_+} - t_{1_+} - t_{2_+0_+} - t_{2_+1_+} - t_{2_-1_-} - t_{2_-0_-} - t_{1_-} - t_{0_-} \\ &= 1 - \frac{2z}{\Lambda}\,(1 + \cosh(\beta)) - \frac{2z^2}{\Lambda^2}\,(\cosh(2\beta) + \cosh(3\beta)) \end{aligned} \tag{24.27}$$

For grammar as simple as this one, the dynamical zeta function is the sum over fixed points of the unrestricted alphabet. As the first check of this expression for the dynamical zeta function we verify that

$$1/\zeta(0,1) = 1 - \frac{4}{\Lambda} - \frac{4}{\Lambda^2} = 0\,,$$

as required by the flow conservation (19.11). Conversely, we could have started by picking the desired Markov partition, writing down the corresponding dynamical zeta function, and then fixing Λ by the $1/\zeta(0,1) = 0$ condition. For more complicated Markov graphs this approach, together with the factorization (24.36), is helpful in reducing the order of the polynomial condition that fixes Λ.

24.3
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The diffusion constant follows from (24.14)

$$\begin{aligned} \langle n\rangle_\zeta &= 4\frac{1}{\Lambda} + 4\frac{2}{\Lambda^2}\,, \quad \langle \hat{n}^2\rangle_\zeta = 2\frac{1^2}{\Lambda} + 2\frac{2^2}{\Lambda^2} + 2\frac{3^2}{\Lambda^2} \\ D &= \frac{15+2\sqrt{2}}{16+8\sqrt{2}}\,. \end{aligned} \tag{24.28}$$

It is by now clear how to build an infinite hierarchy of finite Markov partitions: tune the slope in such a way that the critical value $f(1/2)$ is mapped into the fixed point at the origin in a finite number of iterations p $f^P(1/2) = 0$. By taking higher and higher values of p one constructs a dense set of Markov parameter values, organized into a hierarchy that resembles the way in which rationals are densely embedded in the unit interval. For example, each of the 6 primary intervals can be subdivided into 6 intervals obtained by the 2-nd iterate of the map, and for the critical point mapping into any of those in 2 steps the grammar (and the corresponding cycle expansion) is finite. So, if we can prove continuity of $D = D(\Lambda)$, we can apply the periodic orbit theory to the sawtooth map (24.9) for a random "generic" value of the parameter Λ, for example $\Lambda = 4.5$. The idea is to bracket this value of Λ by a sequence of nearby Markov values, compute the exact diffusion constant for each such Markov partition, and study their convergence toward the value of D for $\Lambda = 4.5$. Some details of how this is accomplished are given in appendix F.2 for a related problem, the pruned Bernulli shift. Judging how difficult such problem is already for a tent map (see sect. 13.6 and appendix F.1), this is not likely to take only a week of work.

⇓PRELIMINARY

⇑PRELIMINARY

Expressions like (24.20) may lead to an expectation that the diffusion coefficient (and thus transport properties) are smooth functions of parameters controling the chaoticity of the system. For example, one might expect that the diffusion coefficient increases smoothly and monotonically as the slope Λ of the map (24.9) is increased, or, perhaps more physically, that the diffusion coefficient is a smooth function of the Lyapunov exponent λ. This turns out not to be true: D as a function of Λ is a fractal, nowhere differentiable curve illustrated in figure 24.5. The dependence of D on

the map parameter Λ is rather unexpected - even though for larger Λ more points are mapped outside the unit cell in one iteration, the diffusion constant does not necessarily grow.

structural stability
stability!structural
spectral stability
stability!spectral
anomalous diffusion
diffusion!anomalous
Jonqui"ere function

This is a consequence of the lack of structural stability, even of purely hyperbolic systems such as the Lozi map and the 1-d diffusion map (24.9). The trouble arises due to non-smooth dependence of the topological entropy on system parameters - any parameter change, no mater how small, leads to creation and destruction of ininitely many periodic orbits. As far as diffusion is concerned this means that even though local expansion rate is a smooth function of Λ, the number of ways in which the trajectory can re-enter the the initial cell is an irregular function of Λ.

The lesson is that lack of structural stabily implies lack of spectral stability, and no global observable is expected to depend smoothly on the system parameters. If you want to master the material, working through the project S.1 and/or project S.2 is strongly recommended.

24.3 Marginal stability and anomalous diffusion

What effect does the intermittency of chapter 21 have on transport properties of 1-d maps? Consider a $1-d$ map of the real line on itself with the same properties as in sect. 24.2, except for a marginal fixed point at $x = 0$.

A marginal fixed point unbalances the role of running and standing orbits, thus generating a mechanism that may result in anomalous diffusion. Our model example is the map shown in figure 24.6 (a), with the corresponding circle map shown in figure 24.6 (b). As in sect. 21.2.1, a branch with support in $\mathcal{M}_i$, $i = 1, 2, 3, 4$ has constant slope Λ_i, while $f|_{\mathcal{M}_0}$ is of intermittent form. To keep you nimble, this time we take a slightly different choice of slopes. The toy example of sect. 21.2.1 was cooked up so that the $1/s$ branch cut in dynamical zeta function was the whole answer. Here we shall take a slightly different route, and pick piecewise constant slopes such that the dynamical zeta function for intermittent system can be expressed in terms of the Jonquière function

☞ remark 24.8

$$J(z,s) = \sum_{k=1}^{\infty} z^k/k^s \tag{24.29}$$

Once the $\overline{0}$ fixed point is pruned away, the symbolic dynamics is given by the infinite alphabet $\{1, 2, 3, 4, 0^i1, 0^j2, 0^k3, 0^l4\}$, $i, j, k, l = 1, 2, \ldots$ (compare with table 21.1). The partitioning of the subinterval $\mathcal{M}_0$ is induced by $\mathcal{M}_{0^k(right)} = \phi^k_{(right)}(\mathcal{M}_3 \bigcup \mathcal{M}_4)$ (where $\phi_{(right)}$ denotes the inverse of the right branch of $\hat{f}|_{\mathcal{M}_0}$) and the same reasoning applies to the leftmost branch. These are regions over which the slope of $\hat{f}|_{\mathcal{M}_0}$ is constant. Thus

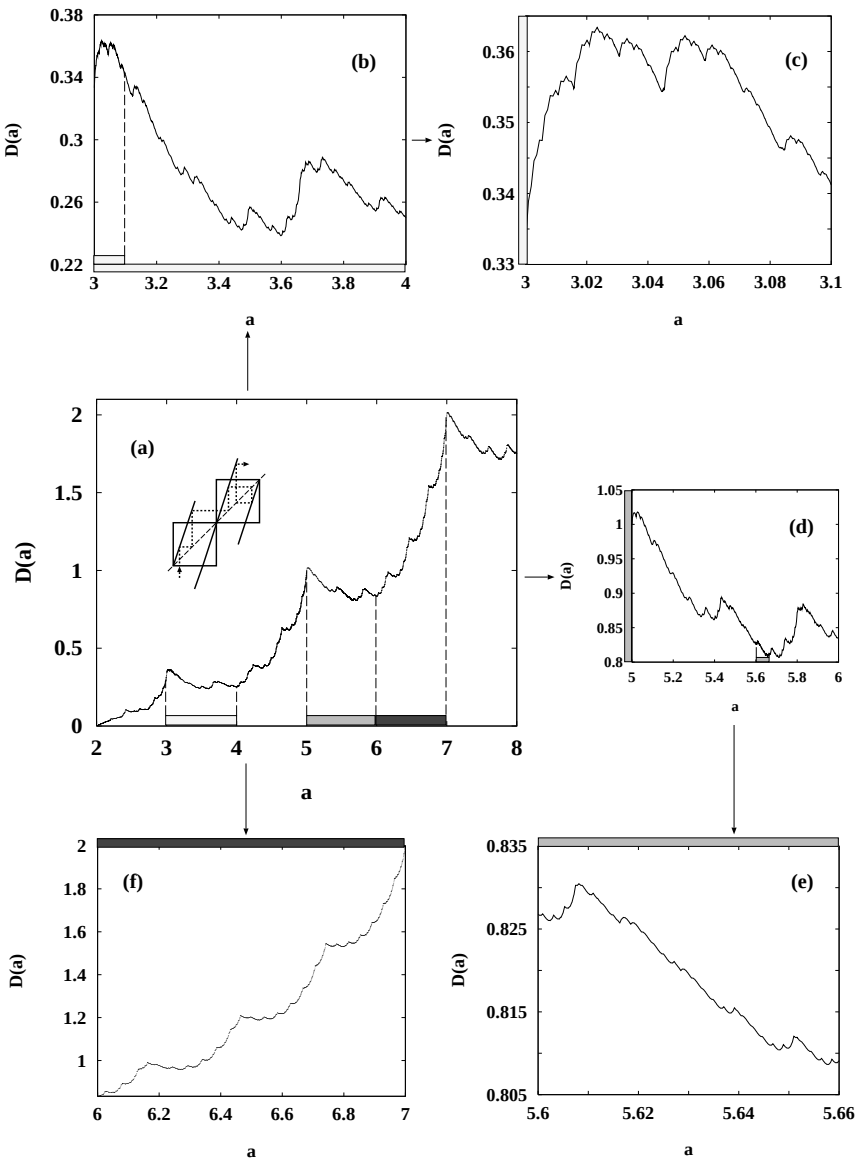

Figure 24.5: The dependence of D on the map parameter a is continuous, but not monotone. (From ref. [24.7]). Here a stands for the slope Λ in (24.9).

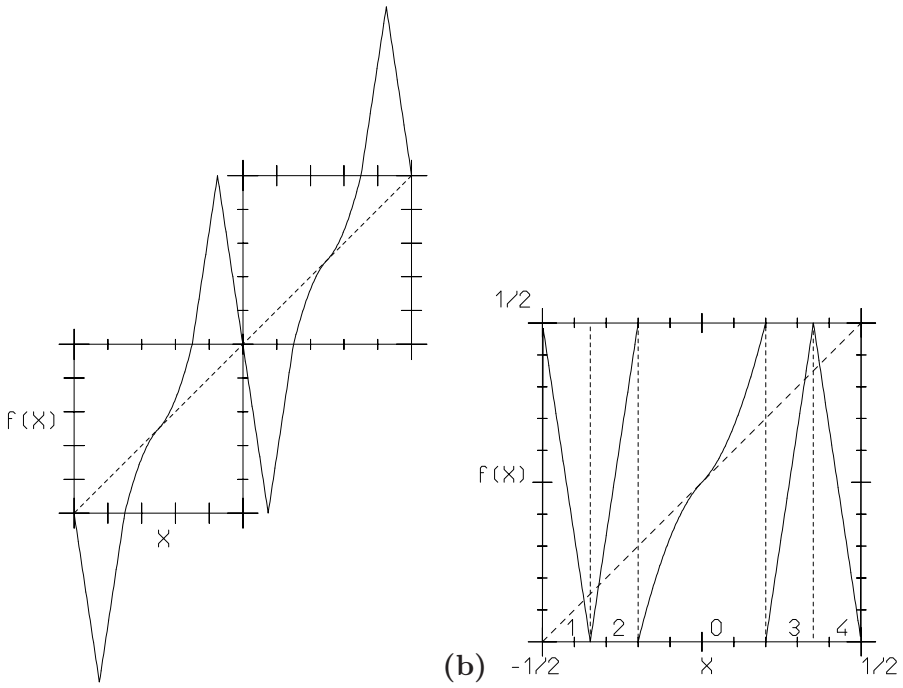

Figure 24.6: (a) A map with marginal fixed point. (b) The map restricted to the unit circle.

we have the following stabilities and jumping numbers associated to letters:

$$\begin{array}{rlll} 0^k3,\, 0^k4 & \Lambda_p = \frac{k^{1+\alpha}}{q/2} & \hat{n}_p = 1 \\ 0^l1,\, 0^l2 & \Lambda_p = \frac{l^{1+\alpha}}{q/2} & \hat{n}_p = -1 \\ 3,\, 4 & \Lambda_p = \pm\Lambda & \hat{n}_p = 1 \\ 2,\, 1 & \Lambda_p = \pm\Lambda & \hat{n}_p = -1\,, \end{array} \tag{24.30}$$

where $\alpha = 1/s$ is determined by the intermittency exponent (21.1), while q is to be determined by the flow conservation (19.11) for $\hat{f}$: —PCdefine R dropped R from Roberto's ζ_R

$$\frac{4}{\Lambda} + 2q\zeta(\alpha+1) = 1$$

so that $q = (\Lambda-4)/2\Lambda\zeta(\alpha+1)$. The dynamical zeta function picks up contributions just by the alphabet's letters, as we have imposed piecewise linearity, and can be expressed in terms of a Jonguiere function (24.29): dropped R from ζ_R

$$1/\zeta_0(z,\beta) \;=\; 1 - \frac{4}{\Lambda} z\cosh\beta - \frac{\Lambda-4}{\Lambda\zeta(1+\alpha)} z\cosh\beta \cdot J(z,\alpha+1)\,. \tag{24.31}$$

Its first zero $z(\beta)$ is determined by

$$\frac{4}{\Lambda} z + \frac{\Lambda-4}{\Lambda\zeta(1+\alpha)} z\cdot J(z,\alpha+1) \;=\; \frac{1}{\cosh\beta}\,.$$

By using implicit function derivation we see that D vanishes (i.e., $z''(\beta)|_{\beta=1} = 0$) when $\alpha \leq 1$. The physical interpretation is that a typical orbit will stick for long times near the $\overline{0}$ marginal fixed point, and the 'trapping time' will be larger for higher values of the intermittency parameter s (recall $\alpha = s^{-1}$). Hence, we need to look more closely at the behavior of traces of high powers of the transfer operator.

The evaluation of transport coefficient requires one more derivative with respect to expectation values of state space observables (see sect. 24.1): if we use the diffusion dynamical zeta function (24.7), we may write the diffusion coefficient as an inverse Laplace transform,in such a way that any distinction between maps and flows has vanished. In the case of 1-d diffusion we thus have

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$$D = \lim_{t\to\infty} \frac{d^2}{d\beta^2} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} ds\, e^{st} \frac{\zeta'(\beta,s)}{\zeta(\beta,s)}\bigg|_{\beta=0} \tag{24.32}$$

where the ζ' refers to the derivative with respect to s.

The evaluation of inverse Laplace transforms for high values of the argument is most conveniently performed using Tauberian theorems. We shall take

$$\omega(\lambda) = \int_0^\infty dx\, e^{-\lambda x} u(x)\,,$$

with $u(x)$ monotone as $x \to \infty$; then, as $\lambda \mapsto 0$ and $x \mapsto \infty$ respectively (and $\rho \in (0,\infty)$),

$$\omega(\lambda) \sim \frac{1}{\lambda^\rho} L\left(\frac{1}{\lambda}\right)$$

if and only if

$$u(x) \sim \frac{1}{\Gamma(\rho)} x^{\rho-1} L(x)\,,$$

where L denotes any slowly varying function with $\lim_{t\to\infty} L(ty)/L(t) = 1$. Now

$$\frac{1/\zeta_0{}'(e^{-s},\beta)}{1/\zeta_0(e^{-s},\beta)} = \frac{\left(\frac{4}{\Lambda} + \frac{\Lambda-4}{\Lambda\zeta(1+\alpha)}\left(J(e^{-s},\alpha+1) + J(e^{-s},\alpha)\right)\right)\cosh\beta}{1 - \frac{4}{\Lambda}e^{-s}\cosh\beta - \frac{\Lambda-4}{\Lambda\zeta(1+\alpha)}e^{-s}(e^{-s},\alpha+1)\cosh\beta J}\,.$$

why 0?

We then take the double derivative with respect to β and obtain

$$\frac{d^2}{d\beta^2}\left(1/\zeta_0{}'(e^{-s},\beta)/\zeta^{-1}(e^{-s},\beta)\right)_{\beta=0}$$

$$= \frac{\frac{4}{\Lambda} + \frac{\Lambda-4}{\Lambda\zeta(1+\alpha)}\left(J(e^{-s},\alpha+1) + J(e^{-s},\alpha)\right)}{\left(1 - \frac{4}{\Lambda}e^{-s} - \frac{\Lambda-4}{\Lambda\zeta(1+\alpha)}e^{-s}J(e^{-s},\alpha+1)\right)^2} = g_\alpha(s) \qquad (24.33)$$

The asymptotic behavior of the inverse Laplace transform (24.32) may then be evaluated via Tauberian theorems, once we use our estimate for the behavior of Jonquière functions near $z = 1$. The deviations from normal behavior correspond to an explicit dependence of D on time. Omitting prefactors (which can be calculated by the same procedure) we have

$$g_\alpha(s) \sim \begin{cases} s^{-2} & \text{for } \alpha > 1 \\ s^{-(\alpha+1)} & \text{for } \alpha \in (0,1) \\ 1/(s^2 \ln s) & \text{for } \alpha = 1\,. \end{cases}$$

The anomalous diffusion exponents follow:

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$$\langle (x - x_0)^2 \rangle_t \sim \begin{cases} t & \text{for } \alpha > 1 \\ t^\alpha & \text{for } \alpha \in (0,1) \\ t/\ln t & \text{for } \alpha = 1\,. \end{cases} \qquad (24.34)$$

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24.4 Lorentz gas

> In economics, reflexivity runs counter to the notion of equilibrium which an free market system is supposed to tend towards. It is, indeed, the reason why open systems do not reach equilibrium but are prone to instability and self-increasing trends, cycles of boom and bust. Development is not linear but feeds on itself. Instead of balancing towards the point of equilibrium self-reinforcing trends create unstable far-from-equilibrium conditions.
> George Soros, "The Crisis of Global Capitalism"

(L. Rondoni, R. Artuso and P. Cvitanović)

We now return to the problem of computing the diffusion coefficient for a "finite horizon" Lorentz gas on a hexagonal lattice introduced in sect. 24.1.

LR: Comment by Lamberto

We are not going to describe in full detail all the calculations that have performed for such a system, our main goal is to provide a physically relevant example where the symbolic dynamics is enormously complicated, and application of formulas like (24.8) is problematic. The complexity of symbolic dynamics for this system has a very clear and profound mathematical justification: Markov partitions for disperse billiards are made of an infinite countable collection of pieces, very unevenly sized. The source of troubles are the grazing singularities, corresponding to the orbits that are tangent to

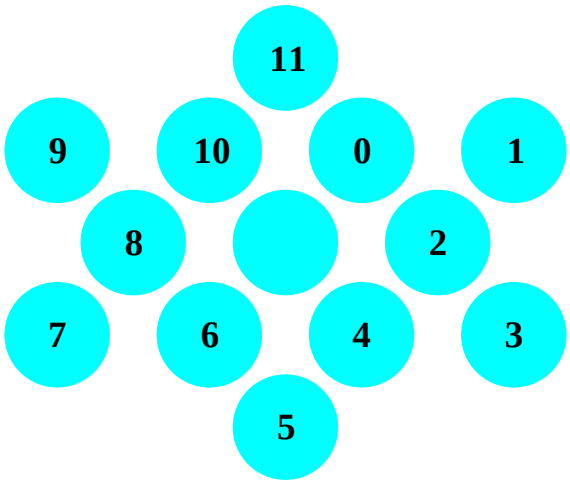

Figure 24.7: Symbolic dynamics for closely packed Lorentz gas.

a disk: a cone of trajectories close to the tangent one is split into two pieces at the tangency point, and the process repeats at every tangency. This mechanism causes the global unstable and stable manifolds to consist of a countable number of smooth components. Thus the presence of singularities in a sense weakens the strong chaotic properties induced by complete hyperbolicity (due to the fact that any single collision leads to defocusing, and that the fime between two successive collisions if bounded from above by a finite quantity).

Let us first define the parameters of the model: we set the disk radius $R = 1$ and denote by w the disk to disk distance, so that the centers of two neighboring disks are separated by a distance $2+w$. For a hexagonal lattice we require $0 \leq w \leq 4/\sqrt{3} - 2$ in order that the finite horizon condition be satisfied.

24.4.1 Lorentz gas: symbolic dynamics

If we start a trajectory from any scatterer in the lattice, under the condition of finite horizon, there are two possibilities for the next hit: either it takes place on one of the six neighboring disks, or on one of the six next-to-nearest disks. Hence we may introduce an alphabet, by labeling these disks clockwise from 0 to 11: if we assign the label 0 to one of the closest obstacles, even labels will then correspond to short flights, see figure 24.7.

Then, any trajectory (without grazing collisions) may be uniquely identified by assigning one of these twelve symbols to each free flight. The way this is done is to let the symbol for a free flight depend on the vector separation between initial and final point of such a segment of trajectory. Then, one should imagine that the center of figure 24.7 is always translated to the scatterer where the free flight starts, and that the symbol for the flight is the one of the scatterer where it ends. So, in practice, for a spacing $w = 0.3$, which is just below the finite horizon condition, one has only

Orbit	Degeneracy	$\tau\lambda$
(0 6)	3	1.51286399
(1 7)	3	3.51394510
(1 5)	6	4.65944481
(0 5)	12	3.15831208
(0 4 8)	4	3.27128792
(0 2 7)	12	6.23520184
(0 2 6)	24	6.31880379
(0 2 5)	24	8.30080318

Table 24.1: Dynamical properties of length-2 and 3 cycles, at $w = 0.3$.

four types of cycles with two collisions, i.e., orbits which indefinitely repeat the same symbol sequence of two symbols: the are of course short orbits between two neighboring scatterers, like (0 6) (next neighbors) or (1 7) (next to nearest neighbors), as well as (1 5) and (0 5) orbits. All these orbits belong to *groups*, characterized by the same layout, and differing just in the orientation: this is of course due to the symmetry propreties of the system: as the symmetry group is discrete there is a maximum number of different orbits in each group (which is 24 in the case under consideration). Obviously, the fact that the different orbits of one kind are oriented in different directions does not affect their length τ and their Lyapunov exponent λ, which are exactly the same for all orbits within one group. The number of distinct orbits within one group, and the corresponding product $\tau\lambda$, which appears in the definition of the stability eigenvalues are listed in table 24.1, for cycles of length 2 and 3.

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It is clear from these considerations that not all possible symbol strings actually give rise to cycles. This is because the symbolic dynamics for the Lorentz gas is marred by a large amount of pruning. For instance, there is an obvious pruning rule that the same symbol cannot be repeated twice, as this would correspond to a trajectory crossing a disk. Many more pruning rules are present: for instance, in the dense gas we consider here not only a letter cannot repeat, but the next symbol has to vary at least by two units after a short flight, and by three units after a long flight. The immense amount of pruning present in the problem may be appreciated by looking at the second column of table 24.2, in which the number of cycles is reported as a function of cycle length: order by order we get roughly an increase by 3, instead of the factor 11 which would come from an unrestricted grammar (keeping into account only the non–crossing condition, which effectively reduces by one the possible symbols that can follow any given one).

24.4.2 Diffusion

We now apply the formula (24.8) for a particular value of the spacing $w = 0.3$.

Together with estimates of the diffusion coefficient we may also provide estimates for the Lyapunov exponent (see sect. **??**). Moreover, since no trajectory escapes, the escape rate is zero, and the spectral determinant

length	# cycles	$\zeta(0,0)$	λ	D
1	0	-	-	-
2	24	-0.31697	1.330	0.375
3	64	-0.54152	1.435	0.338
4	156	-0.09718	1.902	0.282
5	492	0.02383	2.324	0.212
6	1484	0.02812	1.931	0.129
7	5244	0.02044	1.836	0.185
8	19008	-0.00036	1.754	0.256
ref. [24.1], numerical experiment				0.25(1)
ref. [24.2], numerical experiment				0.2492(3)

Table 24.2: Elementary cell, w=0.3. The cycles are sorted with the method of remark T.9. The last column is taken from ref. [24.2].

Figure 24.8: Finite order estimates for the diffusione coefficient from cycle expansions and trace formulas.

and the dynamical zeta function must satisfy the material flow conservation rule (19.11): the results for all these quantities are reported in table 24.2, where cells have been sampled with the variational method described in remark T.9.

LR: column 5 fixed by Lamberto

As mentioned above a lot of symbol sequences are pruned when the required finite horizon is achieved by making the spacing between disks small enough. So very poor convergence of cycle expansions has to be expected. With this in mind the numbers given in table 24.2 are in a reasonable accord with the probability conservation and offer a rather poor, but not unreasonable, estimate of the Lyapunov exponent. Nevertheless, the estimates of the diffusion constant up to the number of cycles employed so far appear to converge very slowly. They seem to be more sensitive to the bad shadowing than the Lyapunov exponent and the probability conservation. Of course, one should remember that the diffusion coefficient is a higher order moment of the generating function, hence it should be expected to be harder to get.

A way to circumvent the poorness of shadowing, due to severe and badly controllable pruning, is just going back to trace formulas (**??**)

$$D_{tr} = \frac{1}{4} \lim_{n\to\infty} \frac{\sum_{x\in Fix(n)} \hat{n}_x^2/|\Lambda_x|}{\sum_{x\in Fix(n)} T_x/|\Lambda_x|} \tag{24.35}$$

For cycle length 9 ref. [24.2] reports $D_{tr} \simeq 0.240$, and for cycle length 10 $D_{tr} \simeq 0.2501$, which is not too far from the result of direct simulations (see figure 24.8). The apparent better performance of trace formulas over cycle expansions in this context is rather unexpected, and no strong theoretical clue is offered to justify this observation, but the very simple fact that trace formulas automatically include probability conservation, as the average is always computed by using a normalized measure. This fact suggests that the performance of cycle expansions for systems with strong pruning, might

figure 24.8 tex fil is using too much memory, cannot latex book

be improved by imposing by hand the condition of probability conservation at all orders of approximation.

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Commentary

Remark 24.1 Lorentz gas. The original pinball model proposed by Lorentz [24.3] consisted of randomly, rather than regularly placed scatterers.

Remark 24.2 Who's dun it? Cycle expansions for the diffusion constant of a particle moving in a periodic array have been introduced independently by R. Artuso [24.4] (exact dynamical zeta function for 1-d chains of maps (24.8)), by W.N. Vance [24.5](the trace formula (24.35) for the Lorentz gas), and by P. Cvitanović, J.-P. Eckmann, and P. Gaspard [24.6] (the dynamical zeta function cycle expansion (24.8) applied to the Lorentz gas).

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add Schreiber ref.

Remark 24.3 Lack of structural stability for D. Expressions like (24.20) may lead to an expectation that the diffusion coefficient (and thus transport properties) are smooth functions of the chaoticity of the system (parametrized, for example, by the Lyapunov exponent $\lambda = \ln \Lambda$). This turns out not to be true: D as a function of Λ is a fractal, nowhere differentiable curve shown in figure 24.5. The dependence of D on the map parameter Λ is rather unexpected - even though for larger Λ more points are mapped outside the unit cell in one iteration, the diffusion constant does not necessarily grow. The fractal dependence of diffusion constant on the map parameter is discussed in refs. [24.7, 24.8, 24.9]. Statistical mechanicians tend to believe that such complicated behavior is not to be expected in systems with very many degrees of freedom, as the addition to a large integer dimension of a number smaller than 1 should be as unnoticeable as a microscopic perturbation of a macroscopic quantity. No fractal-like behavior of the conductivity for the Lorentz gas has been detected so far [24.10].

Remark 24.4 Diffusion induced by 1-d maps. We refer the reader to refs. [24.11, 24.12] for early work on the deterministic diffusion induced by 1-dimenional maps. The sawtooth map (24.9) was introduced by Grossmann and Fujisaka [24.13] who derived the integer slope formulas (24.20) for the diffusion constant. The sawtooth map is also discussed in refs. [24.14].

Remark 24.5 Symmetry factorization in one dimension. In the $\beta = 0$ limit the dynamics (24.11) is symmetric under $x \to -x$, and the zeta functions factorize into products of zeta functions for the symmetric and antisymmetric subspaces, as described in sect. 22.1.2:

$$\frac{1}{\zeta(0,z)} = \frac{1}{\zeta_s(0,z)}\,\frac{1}{\zeta_a(0,z)}\,, \qquad \frac{\partial}{\partial z}\frac{1}{\zeta} = \frac{1}{\zeta_s}\frac{\partial}{\partial z}\frac{1}{\zeta_a} + \frac{1}{\zeta_a}\frac{\partial}{\partial z}\frac{1}{\zeta_s}\,. \tag{24.36}$$

Jonqui"'ere function

length	# cycles	$\zeta(0,0)$	λ
1	5	-1.216975	-
2	10	-0.024823	1.745407
3	32	-0.021694	1.719617
4	104	0.000329	1.743494
5	351	0.002527	1.760581
6	1243	0.000034	1.756546

Table 24.3: Fundamental domain, w=0.3 .

The leading (material flow conserving) eigenvalue $z = 1$ belongs to the symmetric subspace $1/\zeta_s(0,1) = 0$, so the derivatives (24.15) also depend only on the symmetric subspace:

$$\langle n \rangle_\zeta = z\frac{\partial}{\partial z}\frac{1}{\zeta(0,z)}\bigg|_{z=1} = \frac{1}{\zeta_a(0,z)}\, z\frac{\partial}{\partial z}\,\frac{1}{\zeta_s(0,z)}\bigg|_{z=1} . \tag{24.37}$$

Implementing the symmetry factorization is convenient, but not essential, at this level of computation.

Remark 24.6 Lorentz gas in the fundamental domain. The vector valued nature of the generating function (24.3) in the case under consideration makes it difficult to perform a calculation of the diffusion constant within the fundamental domain. Yet we point out that, at least as regards scalar quantities, the full reduction to $\tilde{\mathcal{M}}$ leads to better estimates. A proper symbolic dynamics in the fundamental domain has been introduced in ref. [24.15], numerical estimates for scalar quantities are reported in table 24.3, taken from ref. [24.16].

In order to perform the full reduction for diffusion one should express the dynamical zeta function (24.7) in terms of the prime cycles of the fundamental domain $\tilde{\mathcal{M}}$ of the lattice (see figure 24.2) rather than those of the elementary (Wigner-Seitz) cell $\mathcal{M}$. This problem is complicated by the breaking of the rotational symmetry by the auxilliary vector β, or, in other words, the non-commutativity of translations and rotations: see ref. [24.6].

Remark 24.7 Anomalous diffusion. Anomalous diffusion for 1-d intermittent maps was studied in the continuous time random walk approach in refs. [21.10, 21.11]. The first approach within the framework of cycle expansions (based on truncated dynamical zeta functions) was proposed in ref. [21.12]. Our treatment follows methods introduced in ref. [21.13], applied there to investigate the behavior of the Lorentz gas with unbounded horizon.

Remark 24.8 Jonquière functions. In statistical mechanics Jonquière functions

$$J(z,s) = \sum_{k=1}^{\infty} z^k/k^s \tag{24.38}$$

appear in the theory of free Bose-Einstein gas, see refs. [21.21, 21.22].

Résumé

stosszahlansatz
Boltzmann!equation

The classical Boltzmann equation for evolution of 1-particle density is based on *stosszahlansatz*, neglect of particle correlations prior to, or after a 2-particle collision. It is a very good approximate description of dilute gas dynamics, but a difficult starting point for inclusion of systematic corrections. In the theory developed here, no correlations are neglected - they are all included in the cycle averaging formula such as the cycle expansion for the diffusion constant

Boltzmann refs.

$$D = \frac{1}{2d}\frac{1}{\langle T\rangle_\zeta}\sum{}'(-1)^{k+1}\frac{(\hat{n}_p+\cdots)}{|\Lambda_p\cdots|}\frac{(\hat{n}_{p_1}+\cdots+\hat{n}_{p_k})^2}{|\Lambda_{p_1}\cdots\Lambda_{p_k}|}\,.$$

Such formulas are *exact*; the issue in their applications is what are the most effective schemes of estimating the infinite cycle sums required for their evaluation. Unlike most statistical mechanics, here there are no phenomenological macroscopic parameters; quantities such as transport coefficients are calculable to any desired accuracy from the microscopic dynamics.

Though superficially indistinguishable from the probabilistic random walk diffusion, deterministic diffusion is quite recognizable, at least in low dimensional settings, through fractal dependence of the diffusion constant on the system parameters, and through non-Gaussion relaxation to equilibrium (non-vanishing Burnett coefficients).

For systems of a few degrees of freedom these results are on rigorous footing, but there are indications that they capture the essential dynamics of systems of many degrees of freedom as well.

Actual evaluation of transport coefficients is a test of the techniques developped above in physical settings. In cases of severe pruning the trace formulas and ergodic sampling of dominant cycles might be more effective strategy than the cycle expansions of dynamical zeta functions and systematic enumeration of all cycles.

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Exercises

Exercise 24.1 Diffusion for odd integer Λ. *Show that when the slope $\Lambda = 2k-1$ in (24.9) is an odd integer, the diffusion constant is given by $D = (\Lambda^2-1)/24$, as stated in (24.21).*

Exercise 24.2 Fourth-order transport coefficient. *Verify (24.24). You will need the identity*

$$\sum_{k=1}^{n} k^4 \;=\; \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1)\,.$$

Exercise 24.3 Finite Markov partitions. *Verify (24.28).*

Exercise 24.4 Maps with variable peak shape:
Consider the following piecewise linear map

$$f_\delta(x) \;=\; \begin{cases} \frac{3x}{1-\delta} & \text{for } x \in \left[0, \frac{1}{3}(1-\delta)\right] \\ \frac{3}{2} - \left(\frac{2}{\delta}\left|\frac{4-\delta}{12} - x\right|\right) & \text{for } x \in \left[\frac{1}{3}(1-\delta), \frac{1}{6}(2+\delta)\right] \\ 1 - \frac{3}{1-\delta}\left(x - \frac{1}{6}(2+\delta)\right) & \text{for } x \in \left[\frac{1}{6}(2+\delta), \frac{1}{2}\right] \end{cases} \qquad (24.39)$$

and the map in $[1/2, 1]$ is obtained by antisymmetry with respect to $x = 1/2$, $y = 1/2$. Write the corresponding dynamical zeta function relevant to diffusion and then show that

$$D \;=\; \frac{\delta(2+\delta)}{4(1-\delta)}$$

See refs. [24.17, 24.18] for further details. rescue

Exercise 24.5 Two-symbol cycles for the Lorentz gas. *Write down all cycles labeled by two symbols, such as $(0\ 6)$, $(1\ 7)$, $(1\ 5)$ and $(0\ 5)$.*

Appendix S contains several project-length deterministic diffusion exercises.

Exercise 24.6 Accelerated diffusion. *Consider a map h, such that $\hat{h} = \hat{f}$ of figure 24.6 (b), but now running branches are turner into standing branches and vice versa, so that $1, 2, 3, 4$ are standing while 0 leads to both positive and negative jumps. Build the corresponding dynamical zeta function and show that*

$$\sigma^2(t) \;\sim\; \begin{cases} t & \text{for } \alpha > 2 \\ t \ln t & \text{for } \alpha = 2 \\ t^{3-\alpha} & \text{for } \alpha \in (1,2) \\ t^2/\ln t & \text{for } \alpha = 1 \\ t^2 & \text{for } \alpha \in (0,1) \end{cases}$$

Exercise 24.7 Recurrence times for Lorentz gas with infinite horizon. *Consider the Lorentz gas with unbounded horizon with a square lattice geometry, with disk radius R and unit lattice spacing. Label disks according to the (integer) coordinates of their center: the sequence of recurrence times $\{t_j\}$ is given by the set of collision times. Consider orbits that leave the disk sitting at the origin and hit a disk far away after a free flight (along the horizontal corridor). Initial conditions are characterized by coordinates (ϕ, α) (ϕ determines the initial position along the disk, while α gives the angle of the initial velocity with respect to the outward normal: the appropriate measure is then $d\phi \cos\alpha \, d\alpha$ ($\phi \in [0, 2\pi)$, $\alpha \in [-\pi/2, \pi/2]$. Find how $\psi(T)$ scales for large values of T: this is equivalent to investigating the scaling of portions of the state space that lead to a first collision with disk $(n, 1)$, for large values of n (as $n \mapsto \infty$ $n \simeq T$).* move this solutions

Suggested steps

(a) *Show that the condition assuring that a trajectory indexed by (ϕ, α) hits the (m, n) disk (all other disks being transparent) is written as*

$$\left| \frac{d_{m,n}}{R} \sin(\phi - \alpha - \theta_{m,n}) + \sin\alpha \right| \leq 1 \tag{24.40}$$

where $d_{m,n} = \sqrt{m^2 + n^2}$ and $\theta_{m,n} = \arctan(n/m)$. You can then use a small R expansion of (24.40).

(b) *Now call j_n the portion of the state space leading to a first collision with disk $(n, 1)$ (take into account screening by disks $(1, 0)$ or $(n - 1, 1)$). Denote by $J_n = \bigcup_{k=n+1}^{\infty} j_k$ and show that $J_n \sim 1/n^2$, from which the result for the distribution function follows.*

infinite-dimensional flows—(
flow!infinite-dimensional—(
Navier-Stokes equation
turbulence!problem of
partial differential equations@"pdes
PDEs
ordinary differential equations@"pdes
ODEs

Chapter 25

Turbulence?

> I am an old man now, and when I die and go to Heaven there are two matters on which I hope enlightenment. One is quantum electro-dynamics and the other is turbulence of fluids. About the former, I am rather optimistic.
>
> Sir Horace Lamb

There is only one honorable cause that would justify sweating through so much formalism - this is but the sharpening of a pencil in order that we may attack the Navier-Stokes equation,

ref to Poul Martin: En dansk students eventyr [1.41]

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{f}\,, \tag{25.1}$$

and solve the problem of turbulence.

Flows described by partial differential equations [PDEs] are said to be 'infinite dimensional' because if one writes them down as a set of ordinary differential equations [ODEs], one needs infinitely many of them to represent the dynamics of one partial differential equation. Even though the state space is infinite-dimensional, the long-time dynamics of many systems of physical interest is finite-dimensional, contained within an *inertial manifold*.

Being realistic, we are not so foolhardy to immediately plunge into *the* problem – there are too many dimensions and indices. Instead, we start small, in one spatial dimension, $\mathbf{u} \to u$, $\mathbf{u}\cdot\nabla\mathbf{u} \to u\partial_x$, assume constant ρ, forget about the pressure p, and so on. This line of reasoning, as well as many other equally sensible threads of thought, such as the amplitude equations obtained via weakly nonlinear stability analysis of steady flows, leads to a small set of frequently studied nonlinear PDEs, like the one that we turn to now.

spatiotemporal chaos
chaos!spatiotemporal

25.1 Fluttering flame front

> Romeo: 'Misshapen chaos of well seeming forms!'
> W. Shakespeare, *Romeo and Julliet*, Act I, Scene I

The Kuramoto-Sivashinsky [KS] system describes the flame front flutter of gas burning on your kitchen stove, figure 25.1 (a), and many other problems of greater import, is one of the simplest nonlinear systems that exhibit 'turbulence' (in this context often referred to more modestly as 'spatiotemporally chaotic behavior'). The time evolution of the 'flame front velocity' $u = u(x,t)$ on a periodic domain $u(x,t) = u(x+L,t)$ is given by

$$u_t + \tfrac{1}{2}(u^2)_x + u_{xx} + u_{xxxx} = 0\,, \qquad x \in [0, L]\,. \tag{25.2}$$

In this equation t is the time and x is the spatial coordinate. The subscripts x and t denote partial derivatives with respect to x and t: $u_t = \partial u/d\partial$, u_{xxxx} stands for the 4th spatial derivative of $u = u(x,t)$ at position x and time t. In what follows we use interchangeably the "dimensionless system size" $\tilde{L}$, or the periodic domain size $L = 2\pi\tilde{L}$, as the system parameter. We take note, as in the Navier-Stokes equation (25.1), of the "inertial" term $u\partial_x u$, the "anti-diffusive" term $\partial_x^2 u$ (with a "wrong" sign), etc..

The term $(u^2)_x$ makes this a *nonlinear system*. This is one of the simplest conceivable nonlinear PDE, playing the role in the theory of spatially extended systems a bit like the role that the x^2 nonlinearity plays in the dynamics of iterated mappings. The time evolution of a typical solution of the Kuramoto-Sivashinsky system is illustrated by figure 25.1 (b).

☞ sect. 3.4
☞ remark 25.1

Spatial periodicity $u(x,t) = u(x+L,t)$ makes it convenient to work in the Fourier space,

$$u(x,t) = \sum_{k=-\infty}^{+\infty} a_k(t) e^{ikx/\tilde{L}}\,, \tag{25.3}$$

with the 1-dimensional PDE (25.2) replaced by an infinite set of ODEs for the complex Fourier coefficients $a_k(t)$:

$$\dot{a}_k = v_k(a) = ((k/\tilde{L})^2 - (k/\tilde{L})^4)\, a_k - i\frac{k}{2\tilde{L}} \sum_{m=-\infty}^{+\infty} a_m a_{k-m}\,. \tag{25.4}$$

Since $u(x,t)$ is real, $a_k = a_{-k}^*$, and we can replace the sum in (25.10) by a sum over $k > 0$.

⇓PRIVATE
⇑PRIVATE

Due to the hyperviscous damping u_{xxxx}, long time solutions of Kuramoto-Sivashinsky equation are smooth, a_k drop off fast with k, and truncations

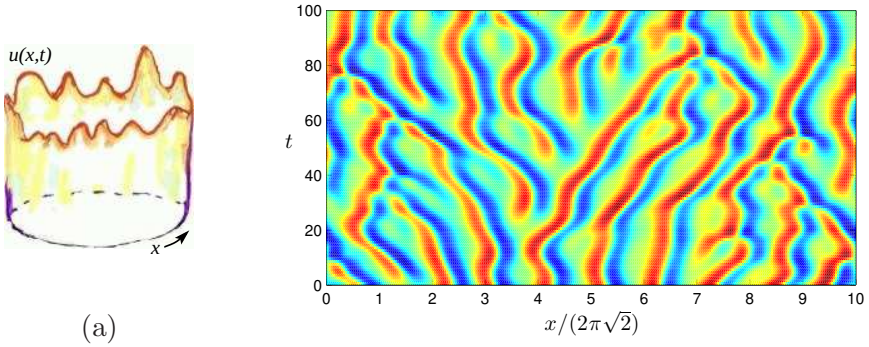

Figure 25.1: (a) Kuramoto-Sivashinsky dynamics visualized as the Bunsen burner flame flutter, with $u = u(x,t)$ the "velocity of the flame front" at position x and time t. (b) A typical "turbulent" solution of the Kuramoto-Sivashinsky equation, system size $L = 88.86$. The color (gray scale) indicates the value of u at a given position and instant in time. The x coordinate is scaled with the most unstable wavelength $2\pi\sqrt{2}$, which is approximately also the mean wavelength of the turbulent flow. The dynamics is typical of a large system, in this case approximately 10 mean wavelengths wide, see refeq??. (from ref. [25.10])

of (25.10) to N terms, $16 \leq N \leq 128$, yield highly accurate solutions for system sizes considered here. Robustness of the Fourier representation of KS as a function of the number of modes kept in truncations of (25.10) is, however, a subtle issue. Adding an extra mode to a truncation of the system introduces a small perturbation. However, this can (and often will) throw the dynamics into a different asymptotic state. A chaotic attractor for $N = 15$ can collapse into an attractive period-3 cycle for $N = 16$, and so on. If we compute, for example, the Lyapunov exponent $\lambda(\tilde{L}, N)$ for a strange attractor of the system (25.10), there is no reason to expect $\lambda(\tilde{L}, N)$ to smoothly converge to a limit value $\lambda(\tilde{L}, \infty)$ as $N \to \infty$, because of the lack of structural stability both as a function of truncation N, and the system size $\tilde{L}$. The topology is more robust for $\tilde{L}$ windows of transient turbulence, where the system can be structurally stable, and it makes sense to compute Lyapunov exponents, escape rates, etc., for the repeller, i.e., the closure of the set of all unstable periodic orbits.

Spatial representations of PDEs (such as the $3d$ snapshots of velocity and vorticity fields in Navier-Stokes) offer little insight into detailed dynamics of low-Re flows. Much more illuminating are the state space representations.

this text to be moved to a more appropriate place

The objects explored in this paper: equilibria and short periodic orbits, are robust both under mode truncations and small system parameter $\tilde{L}$ changes.

25.1.1 Scaling and symmetries

The Kuramoto-Sivashinsky equation (25.2) is space translationally invariant, time translationally invariant, and invariant under reflection $x \to -x$,

Galilean invariance
invariance!Galilean
Fourier!mode!truncation
truncations!Fourier

$u \to -u$.

Comparing u_t and $(u^2)_x$ terms we note that u has dimensions of $[x]/[t]$, hence u is the "velocity," rather than the "height" of the flame front. Indeed, the Kuramoto-Sivashinsky equation is Galilean invariant: if $u(x,t)$ is a solution, then $v + u(x + 2vt, t)$, with v an arbitrary constant velocity, is also a solution. Without loss of generality, in our calculations we shall work in the mean zero velocity frame

$$\int dx\, u = 0\,. \tag{25.5}$$

In terms of the system size L, the only length scale available, the dimensions of terms in (25.2) are $[x] = L$, $[t] = L^2$, $[u] = L^{-1}$, $[\nu] = L^2$. Scaling out the "viscosity" ν

$$x \to x\nu^{\frac{1}{2}}\,, \quad t \to t\nu\,, \quad u \to u\nu^{-\frac{1}{2}}\,,$$

brings the Kuramoto-Sivashinsky equation (25.2) to a non-dimensional form

$$u_t = (u^2)_x - u_{xx} - u_{xxxx}\,, \qquad x \in [0, L\nu^{-\frac{1}{2}}] = [0, 2\pi\tilde{L}]\,. \tag{25.6}$$

In this way we trade in the "viscosity" ν and the system size L for a single dimensionless system size parameter

$$\tilde{L} = L/(2\pi\sqrt{\nu}) \tag{25.7}$$

which plays the role of a "Reynolds number" for the Kuramoto-Sivashinsky system.

In the literature sometimes L is used as the system parameter, with ν fixed to 1, and at other times ν is varied with L fixed to either 1 or 2π. To minimize confusion, in what follows we shall state results of all calculations in units of dimensionless system size $\tilde{L}$. Note that the time units also have to be rescaled; for example, if T_p^* is a period of a periodic solution of (25.2) with a given ν and $L = 2\pi$, then the corresponding solution of the non-dimensionalized (25.6) has period

how fast?

$$T_p = T_p^*/\nu\,. \tag{25.8}$$

expand this into a visualization subsection: how we use d-dimensional vectors (stability eigenvectors, etc) to project from d-dimensions to 2 or 3 dimensions. Not Fourier modes as coordinates!

25.1.2 Fourier space representation

Spatial periodic boundary condition $u(x,t) = u(x+2\pi\tilde{L},t)$ makes it convenient to work in the Fourier space,

infinite-dimensional flows
flow!infinite-dimensional
inertial manifold

$$u(x,t) = \sum_{k=-\infty}^{+\infty} b_k(t) e^{ikx/\tilde{L}} \,. \tag{25.9}$$

with (25.6) replaced by an infinite tower of ODEs for the Fourier coefficients:

$$\dot{b}_k = (k/\tilde{L})^2 \left(1 - (k/\tilde{L})^2\right) b_k + i(k/\tilde{L}) \sum_{m=-\infty}^{+\infty} b_m b_{k-m} \,. \tag{25.10}$$

This is the infinite set of ordinary differential equations promised in this chapter's introduction.

Since $u(x,t)$ is real, $b_k = b^*_{-k}$, so we can replace the sum over m in (25.10) by a sum over $m > 0$. As $\dot{b}_0 = 0$, b_0 is a conserved quantity, in our calculations fixed to $b_0 = 0$ by the vanishing mean $\langle u \rangle$ condition (25.5) for the front velocity.

motivate 2π factor by the mean wavelength, refer to the equation number

MAKE SURE that all periods in tables of computed cycles are stated for that case, and not for $L = 2\pi$.

Example 25.1 Kuramoto-Sivashinsky antisymmetric subspace: *The Fourier coefficients b_k are in general complex numbers. We can isolate the antisymmetric subspace $u(x,t) = -u(-x,t)$ by considering the case of b_k pure imaginary, $b_k = ia_k$, where $a_k = -a_{-k}$ are real, with the evolution equations*

$$\dot{a}_k = (k/\tilde{L})^2 \left(1 - (k/\tilde{L})^2\right) a_k - (k/\tilde{L}) \sum_{m=-\infty}^{+\infty} a_m a_{k-m} \,. \tag{25.11}$$

By picking this subspace we eliminate the continuous translational symmetry from our considerations; that is not an option for an experimentalist, but will do for our purposes. In the antisymmetric subspace the translational invariance of the full system reduces to the invariance under discrete translation by half a spatial period L. In the Fourier representation (25.11) this corresponds to invariance under

$$a_{2m} \to a_{2m} \,, a_{2m+1} \to -a_{2m+1} \,. \tag{25.12}$$

The antisymmetric condition amounts to imposing $u(0,t) = 0$ boundary condition.

25.2 Infinite-dimensional flows: Numerics

The trivial solution $u(x,t) = 0$ is an equilibrium point of (25.2), but that is basically all we know as far as useful analytical solutions are concerned. To develop some intuition about the dynamics we turn to numerical simulations.

How are solutions such as figure 25.1 (b) computed? The salient feature of such partial differential equations is a theorem saying that for state space

Galerkin@G“”alerkin truncation
truncation!G“”alerkin

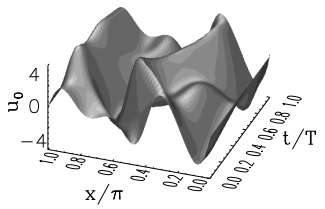

Figure 25.2: Spatiotemporally periodic solution $u_0(x,t)$, with period $T_0 = 30.0118$. The antisymmetric subspace, $u(x,t) = -u(-x,t)$, so we plot $x \in [0, L/2]$. System size $\tilde{L} = 2.89109$, $N = 16$ Fourier modes truncation. (From ref. [25.4])

contracting flows, the asymptotic dynamics is describable by a *finite* set of "inertial manifold" ordinary differential equations. How you solve the equation (25.2) numerically is up to you. Here are some options: be explicit

Discrete mesh: You can divide the x interval into a sufficiently fine discrete grid of N points, replace space derivatives in (25.2) by approximate discrete derivatives, and integrate a finite set of first order differential equations for the discretized spatial components $u_j(t) = u(jL/N, t)$, by any integration routine you trust.

Fourier modes: You can integrate numerically the Fourier modes (25.10), truncating the ladder of equations to a finite number of modes N, i.e., set $a_k = 0$ for $k > N$. In the applied mathematics literature more sophisticated variants of such truncations are called *Gälerkin truncations*, or *Gälerkin projections*. You need to worry about "stiffness" of the equations and the stability of your integrator. For the parameter values explored in this chapter, truncations N in range 16 to 64 yields sufficient accuracy.

make antisymmetric space into an exercise

Pseudo-spectral methods: You can mix the two methods, exploiting the speed of Fast Fourier Transforms.

define inertial manifold
2.7
page 46
link Trefethen

Example 25.2 Kuramoto-Sivashinsky simulation, antisymmetric subspace: *To get started, we set $\nu = 0.029910$, $L = 2\pi$ in the Kuramoto-Sivashinsky equation (25.2), or, equivalently, $\nu = 1$, $L = 36.33052$ in the non-dimensionalized (25.6). Consider the antisymmetric subspace (25.11), so the non-dimensionalized system size is $\tilde{L} = L/4\pi = 2.89109$. Truncate (25.11) to $0 \le k \le 16$, and integrate an arbitrary initial condition. Let the transient behavior settle down.*

add Ruslan's appendix

Why this $\tilde{L}$? For this system size $\tilde{L}$ the dynamics appears to be chaotic, as far as can be determined numerically. Why $N = 16$? In practice one repeats the same calculation at different truncation cutoffs N, and makes sure that the inclusion of additional modes has no effect within the desired accuracy. For this system size $N = 16$ suffices.

Once a trajectory is computed in Fourier space, we can recover and plot the corresponding spatiotemporal pattern $u(x,t)$ over the configuration space using (25.9), as in figure 25.1 (b) and figure 25.2. Such patterns give us a qualitative picture of the flow, but no detailed dynamical information; for that, tracking the evolution in a high-dimensional state space, such as the space of Fourier modes, is much more informative.

Kuramoto-Sivashinsky!system
obscure!topology

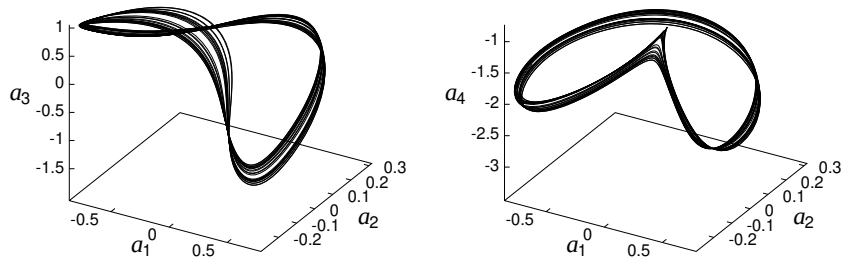

Figure 25.3: Projections of a typical 16-dimensional trajectory onto different 3-dimensional subspaces, coordinates (a) $\{a_1, a_2, a_3\}$, (b) $\{a_1, a_2, a_4\}$. System size $\tilde{L} = 2.89109$, $N = 16$ Fourier modes truncation. (From ref. [25.4].)

25.3 Visualization

period-doubling

The problem with high-dimensional representations, such as truncations of the infinite tower of equations (25.10), is that the dynamics is difficult to visualize. The best we can do without much programming is to examine the trajectory's projections onto any three axes a_i, a_j, a_k, as in figure 25.3.

distinction between spacetime dependent velocity field $u(x,t) \in \mathbb{R}^3$ and a point x in the ∞-dimensional state space $\mathcal{M}$.

The question is: how is one to look at such a flow? It is not clear that restricting the dynamics to a Poincaré section necessarily helps - after all, a section reduces a $(d+1)$-dimensional flow to a d-dimensional map, and how much is gained by replacing a continuous flow in 16 dimensions by a set of points in 15 dimensions? The next example illustrates the utility of visualization of dynamics by means of Poincaré sections.

Example 25.3 Kuramoto-Sivashinsky Poincaré return maps: *Consider the Kuramoto-Sivashinsky equation in the N Fourier modes representation. We pick (arbitrarily) the hyperplane $a_1 = 0$ as the Poincaré section, and integrate (25.10) with $a_1 = 0$, and an arbitrary initial point $(a_2, \ldots, a_N)$. When the flow crosses the $a_1 = 0$ hyperplane in the same direction as initially, the initial point is mapped into $(a'_2, \ldots a'_N) = P(a_2, \ldots, a_N)$. This defines P, the Poincaré return map (3.1) of the $(N-1)$-dimensional $a_1 = 0$ hyperplane into itself.*

Figure 25.4 is a typical result. We have picked - again arbitrarily - a subspace such as $a_6(n+1)$ vs. $a_6(n)$ in order to visualize the dynamics. While the topology of the attractor is still obscure, one thing is clear: even though the flow state space is infinite dimensional, the attractor is finite and thin, barely thicker than a line. [1]

☞ sect. 25.3

The above example illustrates why a Poincaré section gives a more informative snapshot of the flow than the full flow portrait. While no fine structure is discernible in the full state space flow portraits of the Kuramoto-Sivashinsky dynamics, figure 25.3, the Poincaré return map figure 25.4 reveals the fractal structure in the asymptotic attractor.

In order to find a better representation of the dynamics, we now turn to its topological invariants.

[1] Mason: duplicate this example for CGL

25.4 Equilibria of equilibria

(Y. Lan and P. Cvitanović)

The set of equilibria and their stable / unstable manifolds form the coarsest topological framework for organizing state space orbits.

The equilibrium condition $u_t = 0$ for the Kuramoto-Sivashinsky equation PDE (25.6) is the ODE

$$(u^2)_x - u_{xx} - u_{xxxx} = 0$$

which can be analyzed as a dynamical system in its own right. Integrating once we get

$$u^2 - u_x - u_{xxx} = c\,, \tag{25.13}$$

where c is an integration constant whose value strongly influences the nature of the solutions. Written as a 3–d dynamical system with spatial coordinate x playing the role of "time", this is a volume preserving flow

$$u_x = v\,, \qquad v_x = w\,, \qquad w_x = u^2 - v - c\,, \tag{25.14}$$

with the "time" reversal symmetry,

$$x \to -x, \quad u \to -u, \quad v \to v, \quad w \to -w\,.$$

From (25.14) we see that

$$(u + w)_x = u^2 - c\,.$$

If $c < 0$, $u + w$ increases without bound with $x \to \infty$, and every solution escapes to infinity. If $c = 0$, the origin $(0, 0, 0)$ is the only bounded solution.

For $c > 0$ there is much c-dependent interesting dynamics, with complicated fractal sets of bounded solutions. The sets of the solutions of the equilibrium condition (25.14) are themselves in turn organized by the equilibria of the equilibrium condition, and the connections between them. For $c > 0$ the equilibrium points of (25.14) are $c_+ = (\sqrt{c}, 0, 0)$ and $c_- = (-\sqrt{c}, 0, 0)$. Linearization of the flow around c_+ yields stability eigenvalues $[2\lambda\,, -\lambda \pm i\theta]$ with

$$\lambda = \frac{1}{\sqrt{3}} \sinh\phi\,, \qquad \theta = \cosh\phi\,,$$

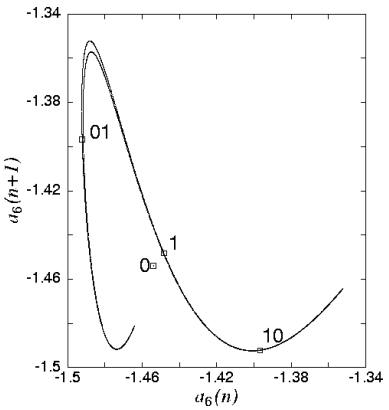

Figure 25.4: The attractor of the Kuramoto-Sivashinsky system (25.10), plotted as the a_6 component of the $a_1 = 0$ Poincaré section return map. Here 10,000 Poincaré section returns of a typical trajectory are plotted. Also indicated are the periodic points 0, 1, 01 and 10. System size $\tilde{L} = 2.89109$, $N = 16$ Fourier modes truncation. (From ref. [25.4].)

and ϕ fixed by $\sinh 3\phi = 3\sqrt{3c}$. Hence c_+ has a 1–d unstable manifold and a 2–d stable manifold along which solutions spiral in. By the $x \to -x$ "time reversal" symmetry, the invariant manifolds of c_- have reversed stability properties.

The non–wandering set fo this dynamical system is quite pretty, and surprisingly hard to analyze. However, we do not need to explore the fractal set of the Kuramoto-Sivashinsky equilibria for infinite size system here; for a fixed system size L with periodic boundary condition, the only surviving equilibria are those with periodicity L. They satisfy the equilibrium condition for (25.10)

make figure 25.2, figure 25.3, figure 25.4 into a problem set

$$(k/\tilde{L})^2 \left(1 - (k/\tilde{L})^2\right) b_k + i(k/\tilde{L}) \sum_{m=-\infty}^{+\infty} b_m b_{k-m} = 0\,. \tag{25.15}$$

Periods of spatially periodic equilibria are multiples of L. Every time $\tilde{L}$ crosses an integer value $\tilde{L} = n$, n-cell states are generated through pitchfork bifurcations. In the full state space they form an invariant circle due to the translational invariance of (25.6). In the antisymmetric subspace considered here, they corresponds to two points, half-period translates of each other of the form

$$u(x,t) = -2 \sum_k b_{kn} \sin(knx)\,,$$

where $b_{kn} \in \mathbb{R}$.

For any fixed period L the number of spatially periodic solutions is finite up to a spatial translation. This observation can be heuristically motivated as follows. Finite dimensionality of the inertial manifold bounds the size of Fourier components of all solutions. On a finite-dimensional compact manifold, an analytic function can only have a finite number of zeros. So, the equilibria, i.e., the zeros of a smooth velocity field on the inertial manifold, are finitely many.

Kuramoto-Sivashinsky!equilibria
equilibria!Kuramoto-Sivashinsky
local!stability
linear!stability
[?], to remarks
stability!linear
explain the theory; say that in practice it is
equilibrium!point
Kuramoto-Sivashinsky!system
Lyapunov exponent!equilibrium

For a sufficiently small L the number of equilibria is small, mostly concentrated on the low wave number end of the Fourier spectrum. These solutions may be obtained by solving the truncated versions of (25.15).

Example 25.4 Some Kuramoto-Sivashinsky equilibria:

25.5 Why does a flame front flutter?

We start by considering the case where a_* is an equilibrium point (2.7). Expanding around the equilibrium point a_*, and using the fact that the matrix $\mathbf{A} = \mathbf{A}(a_*)$ in (4.2) is constant, we can apply the simple formula (4.27) also to the fundamental matrix of an equilibrium point of a PDE,

$$J^t(a_*) = e^{\mathbf{A}t} \qquad \mathbf{A} = \mathbf{A}(a_*)\,.$$

Example 25.5 Stability matrix, antisymmetric subspace: *The Kuramoto-Sivashinsky flat flame front $u(x,t) = 0$ is an equilibrium point of (25.2). The stability matrix (4.3) follows from (25.10)*

$$A_{kj}(a) = \frac{\partial v_k(a)}{\partial a_j} = ((k/\tilde{L})^2 - (k/\tilde{L})^4)\delta_{kj} - 2(k/\tilde{L})a_{k-j}\,. \tag{25.16}$$

For the $u(x,t) = 0$ equilibrium solution the stability matrix is diagonal, and – as in (4.15) – so is the fundamental matrix $J^t_{kj}(0) = \delta_{kj} e^{((k/\tilde{L})^2-(k/\tilde{L})^4)t}$.

For $\tilde{L} < 1$, $u(x,t) = 0$ is the globally attractive stable equilibrium. As the system size $\tilde{L}$ is increased, the "flame front" becomes increasingly unstable and turbulent, the dynamics goes through a rich sequence of bifurcations on which we shall not dwell here.

The $|k| <$?? long wavelength perturbations of the flat-front equilibrium are linearly unstable, while all $|k| >$?? short wavelength perturbations are strongly contractive. The high k eigenvalues, corresponding to rapid variations of the flame front, decay so fast that the corresponding eigendirections are physically irrelevant. To illustrate the rapid contraction in the non-leading eigendirections we plot in figure 25.6 the eigenvalues of the equilibrium in the unstable regime, for relatively small system size, and compare them with the stability eigenvalues of the least unstable cycle for the same system size. The equilibrium solution is very unstable, in 5 eigendirections, the least unstable cycle only in one. Note that for $k > 7$ the rate of contraction is so strong that higher eigendirections are numerically meaningless for either solution; even though the flow is infinite-dimensional, the attracting set must be rather thin.

While in general for $\tilde{L}$ sufficiently large one expects many coexisting attractors in the state space,in numerical studies most random initial conditions settle converge to the same chaotic attractor.

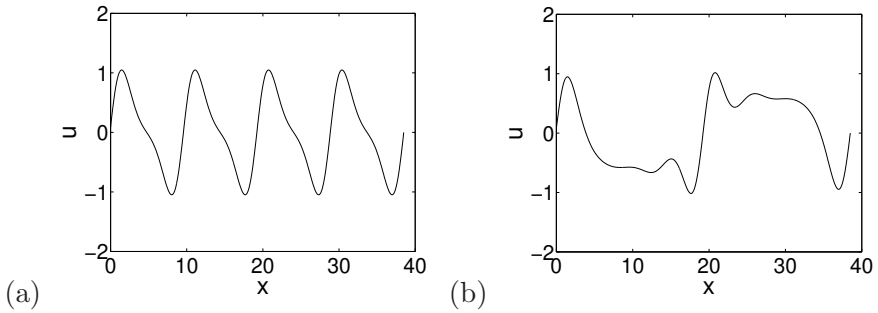

Figure 25.5: The non–wandering set under study appears to consist of three patches: the left part (S_L), the center part (S_C) and the right part (S_R), each centered around an unstable equilibrium: (a) central C_1 equilibrium, (b) side R_1 equilibrium on the interval $[0, L]$.

From (25.10) we see that the origin $u(x,t) = 0$ has Fourier modes as the linear stability eigenvectors. When $|k| \in (0, \tilde{L})$, the corresponding Fourier modes are unstable. The most unstable modes has $|k| = \tilde{L}/\sqrt{2}$ and defines the scale of basic building blocks of the spatiotemporal dynamics of the Kuramoto-Sivashinsky equation in large system size limit, as shown in sect. **??**.

Consider now the case of initial a_k sufficiently small that the bilinear $a_m a_{k-m}$ terms in (25.10) can be neglected. Then we have a set of decoupled linear equations for a_k whose solutions are exponentials, at most a finite number for which $k^2 > \nu k^4$ is growing with time, and infinitely many with $\nu k^4 > k^2$ decaying in time. The growth of the unstable long wavelengths (low $|k|$) excites the short wavelengths through the $a_m a_{k-m}$ nonlinear term in (25.10). The excitations thus transferred are dissipated by the strongly damped short wavelengths, and a "chaotic equilibrium" can emerge. The very short wavelengths $|k| \gg 1/\sqrt{\nu}$ remain small for all times, but the intermediate wavelengths of order $|k| \sim 1/\sqrt{\nu}$ play an important role in maintaining the dynamical equilibrium. As the damping parameter decreases, the solutions increasingly take on shock front character poorly represented by the Fourier basis, and many higher harmonics may need to be kept in truncations of (25.10).

say somewhere: " The task of the theory is to describe this spatio-temporal turbulence and yield quantitative predictions for its measurable consequences. "

Hence, while one may truncate the high modes in the expansion (25.10), care has to be exercised to ensure that no modes essential to the dynamics are chopped away.

In other words, even though our starting point (25.2) is an infinite-dimensional dynamical system, the asymptotic dynamics unfolds on a finite-dimensional attracting manifold, and so we are back on the familiar territory of sect. 2.2: the theory of a finite number of ODEs applies to this infinite-dimensional PDE as well. [2]

[2] Mason: I volunteer to write a section on amplitude equations, *a la* my thesis

obscure!topology
Kuramoto-Sivashinsky!equilibria

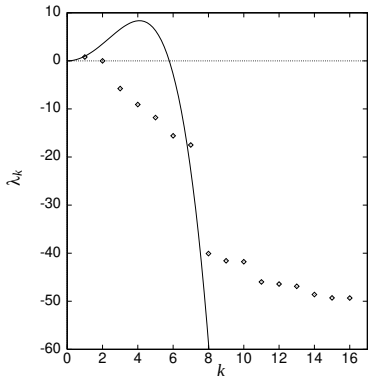

Figure 25.6: Lyapunov exponents $\lambda_{\overline{1},k}$ versus k for the least unstable spatio-temporally periodic orbit $\overline{1}$ of the Kuramoto-Sivashinsky system, compared with the stability exponents of the $u(x,t) = 0$ stationary solution, $\lambda_k = k^2 - \nu k^4$. The eigenvalue $\lambda_{\overline{1},k}$ for $k \geq 8$ falls below the numerical accuracy of integration and are not meaningful. The cycle $\overline{1}$ was computed using methods of chapter 17. System size $\tilde{L} = 2.89109$, $N = 16$ Fourier modes truncation. (From ref. [25.4])

We can now start to understand the remark on page 38 that for infinite dimensional systems time reversibility is not an option: evolution forward in time strongly damps the higher Fourier modes. There is no turning back: if we reverse the time, the infinity of high modes that contract strongly forward in time now explodes, instantly rendering evolution backward in time meaningless. As so much you are told about dynamics, this claim is also wrong, in a subtle way: if the initial $u(x,0)$ is *in* the non–wandering set (2.2), the trajectory is well defined both forward and backward in time. For practical purposes, this subtlety is not of much use, as any time-reversed numerical trajectory in a finite-mode truncation will explode very quickly, unless special precautions are taken.

When is an equilibrium important? There are two kinds of roles equilibria play:

"Hole" in the natural measure. The more unstable eigendirections it has (for example, the $u = 0$ solution), the more unlikely it is that an orbit will recur in its neighborhood.

unstable manifold of a "least unstable" equilibrium. Asymptotic dynamics spends a large fraction of time in neighborhoods of a few equilibria with only a few unstable eigendirections.

Example 25.6 Stability of Kuramoto-Sivashinsky equilibria:

spiraling out in a plane, *all other directions contracting*

Stability of "center" equilibrium

linearized stability exponents:

$$(\mu_1 \pm i\,\nu_1, \mu_2 \pm i\,\nu_2, \cdots) = (0.044 \pm i\,0.262\,,\, -0.255 \pm i\,0.431\,,\, \cdots)$$

The plane spanned by $\mu_1 \pm i\,\nu_1$ *eigenvectors rotates with angular period* $T \approx 2\pi/\nu_1 = 24.02$.

a trajectory that starts near the C_1 *equilibrium point spirals away per one rotation with multiplier* $\Lambda_{radial} \approx \exp(\mu_1 T) = 2.9$.

each Poincaré section return, contracted into the stable manifold by factor of $\Lambda_2 \approx \exp(\mu_2 T) = 0.002$

The local Poincaré return map is in practice $1 - dimensional$

Kuramoto-Sivashinsky!system

Table 25.1: Important Kuramoto-Sivashinsky equilibria: the first few stability exponents

S	$\mu_1 \pm i\,\nu_1$	$\mu_2 \pm i\,\nu_2$	$\mu_3 \pm i\,\nu_3$
C_1	$0.04422 \pm i\,0.26160$	$-0.255 \pm i\,0.431$	$-0.347 \pm i\,0.463$
R_1	$0.01135 \pm i\,0.79651$	$-0.215 \pm i\,0.549$	$-0.358 \pm i\,0.262$
T	0.25480	$-0.07 \pm i\,0.645$	-0.264

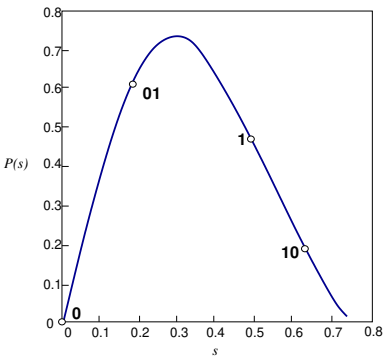

Figure 25.7: The Poincaré return map of the Kuramoto-Sivashinsky system (25.10) figure 25.4, from the unstable manifold of the $\overline{1}$ fixed point to the (neighborhood of) the unstable manifold. Also indicated are the periodic points $\overline{0}$ and $\overline{01}$.

25.6 Periodic orbits

expanding eigenvalue of the least unstable spatio-temporally periodic orbit $\overline{1}$: $\Lambda_1 = -2.0\ldots$

very thin Poincaré section
thickness $\propto$ least contracting eigenvalue $\Lambda_2 = 0.007\ldots$

15-$d \to$ 15-d Poincaré return map projection on the $[a_6 \to a_6]$ Fourier component is not even $1 \to 1$.

25.7 Intrinsic parametrization

probably drop?

Both in the Rössler flow of example 3.3, and in the Kuramoto-Sivashinsky system of example 25.3 we have learned that the attractor is very thin, but otherwise the return maps that we found were disquieting – neither figure 3.3 nor figure 25.4 appeared to be one-to-one maps. This apparent loss of invertibility is an artifact of projection of higher-dimensional return maps onto lower-dimensional subspaces. As the choice of lower-dimensional subspace is arbitrary, the resulting snapshots of return maps look rather arbitrary, too. Other projections might look even less suggestive.

Such observations beg a question: Does there exist a "natural", intrinsically optimal coordinate system in which we should plot of a return map?

insert into figure 25.7: This unimodal map is an approximation to the repeller given in figure 25.8.

As we shall now argue (see also sect. 17.1), the answer is yes: The intrinsic coordinates are given by the stable/unstable manifolds, and a

center manifold
Kuramoto-Sivashinsky!system

return map should be plotted as a map from the unstable manifold back onto the immediate neighborhood of the unstable manifold.

Examination of numerical plots such as figure 25.3 suggests that a more thoughtful approach would be to find a coordinate transformation $y = h(x)$ to a "center manifold", such that in the new, curvilinear coordinates large-scale dynamics takes place in (y_1, y_2) coordinates, with exponentially small dynamics in $y_3, y_4 \cdots$. But - thinking is extra price - we do not know how to actually accomplish this.

Both in the example of the Rössler flow and of the Kuramoto-Sivashinsky system we sketched the attractors by running a long chaotic trajectory, and noted that the attractors are very thin, but otherwise the return maps that we plotted were disquieting – neither figure 3.3 nor figure 25.4 appeared to be 1-to-1 maps. In this section we show how to use such information to approximately locate cycles.

needs expanding!

Example 25.7 Kuramoto-Sivashinsky system. *In sect. 20.1 we have developed basic understanding of*[3] *The full return map is* $(N-1)$*-dimensional* $\mathbf{a} \to \mathbf{P}(a_2, \ldots, a_N) = \mathbf{a}'$ *and single-valued, and for the examples considered the attractor is essentially 1-dimensional, but its projection into the* $\{a_k, P_k(a_2, \ldots, a_N)\}$ *plane, a return map for a given coordinate* $a_k \to a'_k = P_k(a_2, \ldots, a_N)$ *can appear multi-valued and self-intersecting. The multiple-valuedness in figure 25.4 arises from the fact that the return map is a 2-dimensional projection of a convoluted 1-dimensional curve embedded into a high-dimensional space. One can imagine a situation where no "good" projection is possible, that is, any projection onto any 2-dimensional plane is a multiple-valued function. The question is: does there exist an intrinsically optimal representation of a return map? and the answer is, yes, the intrinsic coordinates are given by the stable/unstable manifolds, and a return map should be plotted as a map from the unstable manifold back onto the unstable manifold.*

explain return map here

We now show how this works for the Kuramoto-Sivashinsky example. We shall show that it is possible to find an intrinsic *parametrization* s *along the unstable manifold, such that the map* $s \to f(s)$ *induced by the full* d*-dimensional flow is approximately* 1-dimensional. *Strictly speaking, the attractor on figure 25.4 has a certain thickness transverse to it, but the contraction in the transverse directions is so strong that the invariant set is effectively* 1*-dimensional.*

Suppose we already have determined some of the shorter cycles for our system, i.e., the fixed points of the Poincaré map and its iterates. This is accomplished relatively easily by checking a trajectory of a random initial point for close returns and then using these as initial guesses for a cycle search algorithm. We now assume that the invariant set can be approximated by a curve passing close to all periodic points, and determine the order of periodic points along such curve. This is done in the following way: there exists a fixed point which is not connected to the attractor (the point $\overline{0}$ *in figure 25.4) - we choose this fixed point as the starting point and assign it number* 1. *Point number* 2 *is the periodic point in the sample which is closest (in the full space) to this fixed point, and the* n*th point is determined as the point which has the minimum distance from the point number* $n-1$ *among all the periodic points which have not yet been enumerated. Proceeding this way, we order all the periodic points that we have found so far.*

Since all periodic points belong to cycles, their images are known and are simply the successive periodic points along the cycle. We use this fact to recursively construct a

[3] Dorte: ulaeseligt

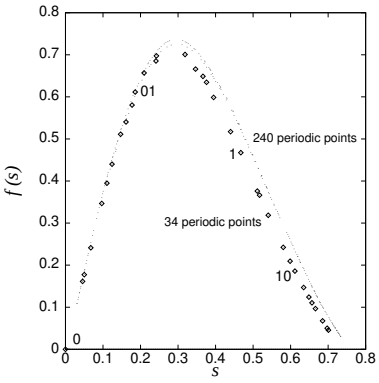

Figure 25.8: The return map $s_{n+1} = f(s_n)$ constructed from the images of periodic points. The diamonds were obtained by using 34 periodic points, the dots were obtained by using 240 periodic points. We have indicated the periodic points $\overline{0}$, $\overline{1}$ and $\overline{01}$. Note that the transverse fractal structure of the map shows when the number of points is increased. System size $\tilde{L} = 2.89109$, $N = 16$ Fourier modes truncation. (From ref. [25.4])

1-dimensional mapping $s_i \to f(s_i)$. We approximate parametrization length s along the invariant set by computing the Euclidean inter-distances between the successive periodic points in the full dynamical space, $s_1 = 0, s_2 = \|\mathbf{a}_2 - \mathbf{a}_1\|, s_i - s_{i-1} = \|\mathbf{a}_i - \mathbf{a}_{i-1}\|$. The ith cycle point s_i is mapped onto its image $s_{\sigma i} = f(s_i)$, where σ_i denotes the label of the next periodic point in the cycle. We can now find longer cycles of the 1-dimensional map f by standard methods such as inverse iteration, and guess the location of the corresponding points in the full N-dimensional space by interpolating between the nearest known periodic points. These will not be exact cycles of the full system, but are very useful as good starting guesses in a search for the exact cycles. Iteratively, more and more cycles can be computed. While it only pays to refine the 1-dimensional parametrization until the density of the periodic points become so high that the width of the attractor becomes noticeable, the 1-dimensional map continues to provide good initial guesses to longer cycles. More sophisticated methods are needed only if high accuracy around the folding region of $f(s)$ is required in order to distinguish between long cycles.

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For the values of ν we are working with, the approximate return map $s \to f(s)$ is unimodal.

We use the unimodal map $s \to f(s)$ to construct binary symbolic dynamics for the system in the usual way: assign the symbol '0' to points to the left of the maximum, '1' to the points to the right. In the period-3 window with the stable cycle $\overline{001}$, the pruning rules are very easy: except for the stable $\overline{001}$ cycle and the $\overline{0}$ fixed point (both disjoint from the invariant set) two 0's in a row are forbidden. In this case it is convenient to redefine the alphabet by denoting the symbol pair 01 by a and the symbol 1 by b. This renders the symbolic dynamics of the points on the repeller complete binary: all sequences of the letters a and b are admissible.

reminder should go to the fixed points chapter

25.8 Energy budget

The space average of a function $a = a(x,t)$ on the interval L,

$$\langle a \rangle = \frac{1}{L}\int_0^L dx\, a(x,t)\,, \tag{25.17}$$

is in general time dependent. Its mean value is given by the time average

$$\overline{a} = \lim_{t\to\infty}\frac{1}{t}\int_0^t d\tau\ \langle a \rangle = \lim_{t\to\infty}\frac{1}{tL}\int_0^t\int_0^L d\tau dx\, a(x,\tau)\,. \tag{25.18}$$

The mean value $\overline{a}$, $a = a(u)$ evaluated on an equilibrium or relative equilibrium $u(x,t) = u_q(x - ct)$ is

$$a_q = \langle a \rangle_q \,. \tag{25.19}$$

Evaluation of the infinite time average (25.18) on a function of a period T_p periodic orbit or relative periodic orbit $u_p(x,t)$ requires only a single traversal of the periodic solution,

$$a_p = \frac{1}{T_p} \int_0^{T_p} d\tau \, \langle a \rangle \,. \tag{25.20}$$

Equation (25.2) can be written as

$$u_t = -V_x \,, \qquad V(x,t) = \tfrac{1}{2} u^2 + u_x + u_{xxx} \,. \tag{25.21}$$

u is related to the "flame-front height" $h(x,t)$ by $u = h_x$, so E can be interpreted as the mean energy density (25.22). So, even though KS is a phenomenological small-amplitude equation, the time-dependent quantity

$$E = \frac{1}{L} \int_0^L dx \, V(x,t) = \frac{1}{L} \int_0^L dx \, \frac{u^2}{2} \tag{25.22}$$

has a physical interpretation [**?**] as the average "energy" density of the flame front. This analogy to the corresponding definition of the mean kinetic energy density for the Navier-Stokes will be useful in what follows.

⇑PRELIMINARY

The energy (25.22) is also the quadratic norm in the Fourier space,

$$E = \sum_{k=1}^{\infty} E_k \,, \qquad E_k = \tfrac{1}{2} |a_k|^2 \,. \tag{25.23}$$

Take time derivative of the energy density (25.22), substitute (25.2) and integrate by parts. Total derivatives vanish by the spatial periodicity on the L domain:

$$\begin{aligned} \dot{E} &= \langle u_t \, u \rangle = - \left\langle \left(\frac{u^2}{2} + u \, u_x + u \, u_{xxx} \right)_x u \right\rangle \\ &= \left\langle + u_x \, \frac{u^2}{2} + (u_x)^2 + u_x \, u_{xxx} \right\rangle \,. \end{aligned} \tag{25.24}$$

Substitution by (**??**) verifies that for an equilibrium E is constant:

$$\dot{E} = \left\langle \left(\frac{u^2}{2} + u_x + u_{xxx} \right) u_x \right\rangle = E \, \langle u_x \rangle = 0 \,.$$

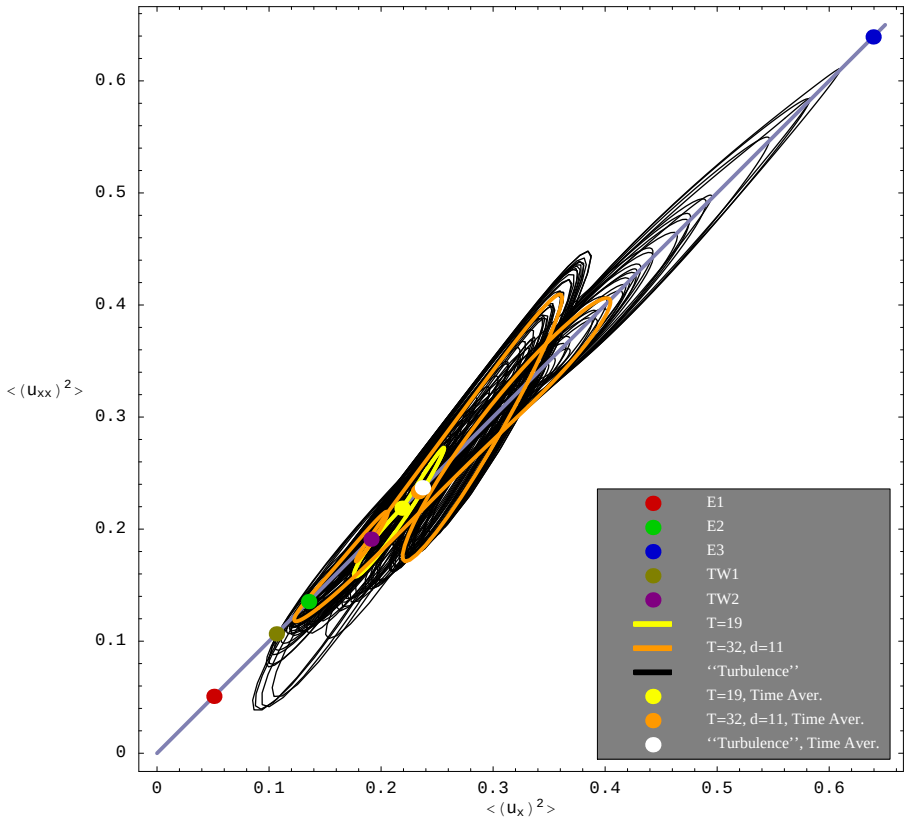


Figure 25.9: Power input $\langle (u_x)^2 \rangle$ vs. dissipation $\langle (u_{xx})^2 \rangle$ for $L = 22$ equilibria and relative equilibria, for several periodic orbits and relative periodic orbits, and for a typical "turbulent" state. Note that $\overline{(u_{p,x})^2}$ of the $(T_p, d_p) = (32.8, 10.96)$ relative periodic orbit, figure **??**(c), which appears well embedded within the turbulent state, is close to the turbulent expectation $\overline{(u_x)^2}$ [temporary Japanese heresy].

The first term in (25.24) vanishes by integration by parts, $\langle (u^3)_x \rangle = 3 \langle u_x\, u^2 \rangle = 0$, and integrating the third term by parts yet again we get that the energy variation

$$\dot{E} = \langle (u_x)^2 \rangle - \langle (u_{xx})^2 \rangle \tag{25.25}$$

balances the KS equation (25.2) power pumped in by the anti-diffusion u_{xx} against energy dissipated by the hypervicosity u_{xxxx} [**?**].

In figure 25.9 we plot the power input $\langle (u_x)^2 \rangle$ vs. dissipation $\langle (u_{xx})^2 \rangle$ for all $L = 22$ equilibria and relative equilibria determined so far, several periodic orbits and relative periodic orbits, and for a typical "turbulent" evolution. The time averaged energy density $\overline{E}$ computed on a typical orbit goes to a constant, so the expectation values (25.26) of drive and

from bounds on energy, we might be able to bound the number of equilibria as function of systems size L, and thus be sure we have them all.

bit weird: can use Galilean invariance to set $\overline{E} = 0$ for any given $u(x,t)$?

Implementing Ruslan: all axes in figure 25.10 changed by a factor of 4?

ks22TurbConn2_xfig.eps for figure 25.10 not

inertial manifold
Kuramoto, Y.
Sivashinsky, G.I.
Holmes, P.

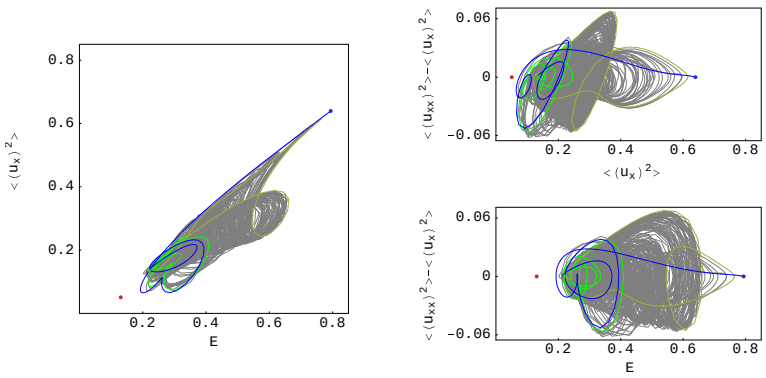


Figure 25.10: E_1 (red), E_2 (green), E_3 (blue), connections from E_1 to $A(L/4)E_1$ (green), from $A(L/4)E_1$ to E_1 (yellow-green) and from E_3 to $A(L/4)E_1$ (blue), along with a generic long-time "turbulent" evolution (grey) for $L = 22$. Three different projections of the $(E, \langle (u_x)^2 \rangle, \langle (u_{xx})^2 \rangle) - \langle (u_x)^2 \rangle)$ representation are shown.

dissipation exactly balance each out:

$$\overline{\dot{E}} = \lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau\, \dot{E} = \overline{(u_x)^2} - \overline{(u_{xx})^2} = 0\,. \tag{25.26}$$

In particular, the equilibria and relative equilibria sit on the diagonal in figure 25.9, and so do time averages computed on periodic orbits and relative periodic orbits:

$$\begin{aligned} \overline{E}_p &= \frac{1}{T_p} \int_0^{T_p} d\tau\, E(\tau) \\ \overline{(u_x)^2}_p &= \frac{1}{T_p} \int_0^{T_p} d\tau\, \left\langle (u_x)^2 \right\rangle = \overline{(u_{xx})^2}_p\,. \end{aligned} \tag{25.27}$$

In the Fourier basis (25.23) the conservation of energy on average takes form

$$0 = \sum_{k=1}^{+\infty} ((k/\tilde{L})^2 - (k/\tilde{L})^4)\, \overline{E_k}\,, \qquad E_k(t) = |a_k(t)|^2\,. \tag{25.28}$$

The large k convergence of this series is insensitive to the system size L; $\overline{E_k}$ have to decrease much faster than $1/(k/\tilde{L})^4$. Deviation of E_k from this bound for small k determines the active modes. This may be useful to bound the number of equilibria, with the upper bound given by zeros of a small number of long wavelength modes.

Re figure 25.9: (b) current type figure, with chaotic trajectory, all equilibria, $(32.\cdots, 11)$ as example of well embedded, and a typical periodic orbit *not* embedded into the "turbulent" attractor. (c) a separate figure with more periodic orbits, no "turbulent" attractor. (e) replace gray window with a white background, black font. (f) in the publication version replace colored dots with symbols of varying shapes, fine black border, can be filled in with colors

Commentary

Remark 25.1 Model PDE systems. The theorem on finite dimensionality of inertial manifolds of state space contracting PDE flows is proven in ref. [25.1].

The Kuramoto-Sivashinsky equation was introduced in refs. [25.2, 25.3]. Holmes, Lumley and Berkooz [25.5] offer a delightful discussion of why this system deserves study as a staging ground for studying turbulence in full-fledged Navier-Stokes equation. How good a description of a flame front this equation is not a concern here; suffice it to say that such model amplitude equations for interfacial instabilities arise in a variety of contexts - see e.g. ref. [25.6] - and this one is perhaps the simplest physically interesting spatially extended nonlinear system. [4]

Kuramoto-Sivashinsky!system

For equilibria the L-independent bound on E is given by Michaelson [**?**]. The best current bound[**?**, **?**] on the long-time limit of E as a function of the system size L scales as $E \propto L^{3/2}$.

believe it or not, we are now set to compute $\overline{u^2}$ and $\overline{(u_x)^2}$ using cycle expansions

determine the decay rate, presumably exponential in k

The work described in this chapter was initiated by Putkaradze's 1996 term project (see `ChaosBook.org/extras`), and continued by Christiansen Cvitanović, Davidchack, Gibson, Halcrow, Lan, and Siminos [25.4, 25.7, 25.8, 37.16, 37.15, 25.10, 25.11, 25.9].

Comment om MAWs, BECS and CGLe

refer to Trefethen's program for fast integration [25.12]

Remark 25.2 Kuramoto-Sivashinsky system, truncations. We describe here our criterion for reliable truncations of the infinite ladder of ordinary differential equations (25.10).

Adding an extra dimension to a truncation of the system (25.10) introduces a small perturbation, and this can (and often will) throw the system into a totally different asymptotic state. A chaotic attractor for $N = 15$ can become a period three window for $N = 16$, and so on. If we compute, for example, the Lyapunov exponent $\lambda(\nu, N)$ for the strange attractor of the system (25.10), there is no reason to expect $\lambda(\nu, N)$ to smoothly converge to the limit value $\lambda(\nu, \infty)$ as $N \to \infty$. The situation is different in the periodic windows, where the system is structurally stable, and it makes sense to compute Lyapunov exponents, escape rates, etc. for the *repeller*, i.e., the closure of the set of all *unstable* periodic orbits. Here the power of cycle expansions comes in: to compute quantities on the repeller by direct averaging methods is generally more difficult, because the asymptotic motion collapses to the stable cycle.

Constantin says that the answer is in refs. [**?**]. Eckmann says: the best bound is by Otto[**?**]; the only bound close to k=0, better in essential way. See also ref. [**?**]. Eckmann had $L^{8/5}$, but conceptually Otto is the best. Recheck whether it is $|u|$ or $E \propto L^{3/2}$. When the solution is big, how long can it stay big? They found it cannot stay big for long.

Next for fluid guys: read Lieb and Ruelle to learn how to bound E for plane Couette by Sobolev bounds

Remark 25.3 A brief history of dynamicist's vision of turbulence. Dynamical approach to study of turbulence are - surprisingly - still a cutting-edge research area. You might find Appendix A.5 amusing.

⇓PRELIMINARY

⇑⇑PRELIMINARY

⇓⇓PRELIMINARY

link to Dahlqvist-Russberg paper

Résumé

rescue Spiegel Hopf story

Turbulence is the graveyard of theories
Hans W Liepmann

reference Kawahara-Kida

We have learned that an instanton is an analytic solution of Yang-Mills equations of motion, but shouldn't a strongly nonlinear field theory

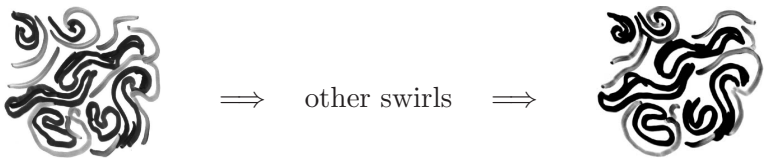
[4] Mason: I disagree - CGL is more important

turbulence
dimension!intrisic
degree of freedom
infinite-dimensional flows—)
flow!infinite-dimensional—)

dynamics be dominated by turbulent solutions? How are we to think about systems where every spatiotemporal solution is unstable?

Here we think of turbulence in spatially extended systems in terms of recurrent spatiotemporal patterns. Pictorially, dynamics drives a given spatially extended system through a repertoire of unstable patterns; as we watch a turbulent system evolve, every so often we catch a glimpse of a familiar pattern:

⇑PRELIMINARY



For any finite spatial resolution, the system follows approximately for a finite time a pattern belonging to a finite alphabet of admissible patterns, and the long term dynamics can be thought of as a walk through the space of such patterns. Recasting this image into mathematics is the subject of this book.

The problem one faces with high-dimensional flows is that their topology is hard to visualize, and that even with a decent starting guess for a point on a periodic orbit, methods like the Newton-Raphson method are likely to fail. Methods that start with initial guesses for a number of points along the cycle, such as the multipoint shooting method of sect. 17.3, are more robust. The relaxation (or variational) methods take this strategy to its logical extreme, and start by a guess of not a few points along a periodic orbit, but a guess of the entire orbit. As these methods are intimately related to variational principles and path integrals, we postpone their introduction to chapter 37.

Google only returns Doyle as a source, and that I do not find reliable. Amazon.com finds it in *Chance and Chaos* by D. Ruelle, p. 52 as "Turbulence is a [not "the"] graveyard of theories", without attribution. Then *The Self-Made Tapestry: Pattern Formation in Nature* by P. Ball ascribes it to Ruelle (!).

At present the theory is in practice applicable only to systems with a low intrinsic *dimension* – the minimum number of coordinates necessary to capture its essential dynamics. If the system is very turbulent (a description of its long time dynamics requires a space of very high intrinsic dimension) we are out of luck.

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Galilean invariance
invariance!Galilean
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Exercises

Exercise 25.1 Galilean invariance of the Kuramoto-Sivashinsky equation.

(a) *Verify that the Kuramoto-Sivashinsky equation is Galilean invariant: if $u(x,t)$ is a solution, then $v+u(x+2vt,t)$, with v an arbitrary constant velocity, i s also a solution.*

(b) *Verify that mean*

$$\langle u\rangle = \frac{1}{L}\int_L dx\, u$$

is conserved by the flow.

(c) *Argue that the choice (25.5) of the vanishing mean velocity, $\langle u\rangle = 0$ leads to no loss of generality in calculations that follow.*

(d) *[thinking is extra cost] Inspection of various "turbulent" solutions of Kuramoto-Sivashinsky equation reveals subregions of "traveling waves" with locally nonzero $\langle u\rangle$. Is there a way to use Galilean invariance locally, even though we eliminated it by the $\langle u\rangle = 0$ condition?*

Exercise 25.2 Infinite dimensional dynamical systems are not smooth.
Many of the operations we consider natural for finite dimensional systems do not have smooth behavior in infinite dimensional vector spaces. Consider, as an example, a concentration ϕ diffusing on $\mathbb{R}$ according to the diffusion equation

$$\partial_t\phi = \frac{1}{2}\nabla^2\phi\,.$$

(a) *Interpret the partial differential equation as an infinite dimensional dynamical system. That is, write it as $\dot{x} = F(x)$ and find the velocity field.*

(b) *Show by examining the norm*

$$\|\phi\|^2 = \int_{\mathbb{R}} dx\, \phi^2(x)$$

that the vector field F is not continuous.

(c) *Try the norm*

$$\|\phi\| = \sup_{x\in\mathbb{R}} |\phi(x)|\,.$$

Is F continuous?

(d) *Argue that the semi-flow nature of the problem is not the cause of our difficulties.*

(e) *Do you see a way of generalizing these results?*

⇓PRIVATE

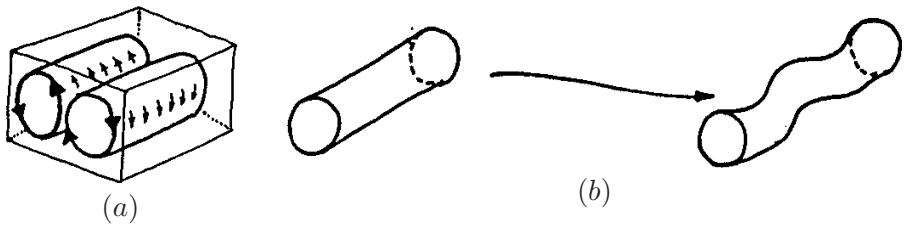
Chapter 26

Universality in transitions to chaos

The developments that we shall describe next are one of those pleasing demonstrations of the unity of physics. The key discovery was made by a physicist not trained to work on problems of turbulence. In the fall of 1975 Mitchell J. Feigenbaum, an elementary particle theorist, discovered a universal transition to chaos in one-dimensional unimodal map dynamics. At the time the physical implications of the discovery were nil. During the next few years, however, numerical and mathematical studies established this universality in a number of realistic models in various physical settings, and in 1980 the universality theory passed its first experimental test.

The discovery was that large classes of nonlinear systems exhibit transitions to chaos which are *universal* and *quantitatively* measurable. This advance was akin to (and inspired by) earlier advances in the theory of phase transitions; for the first time one could, predict and measure "critical exponents" for turbulence. But the breakthrough consisted not so much in discovering a new set of universal numbers, as in developing a new way to solve strongly nonlinear physical problems. Traditionally, we use regular motions (harmonic oscillators, plane waves, free particles, etc.) as zeroth-order approximations to physical systems, and account for weak nonlinearities perturbational. We think of a dynamical system as a smooth system whose evolution we can follow by integrating a set of differential equations. The universality theory tells us that the zeroth-order approximations to strongly nonlinear systems should be quite different. They show an amazingly rich structure which is not at all apparent in their formulation in terms of differential equations; instead, they exhibit self-similar structures which can be encoded by universality equations of a type which we will describe here. To put it more provocatively: junk your old equations and look for guidance in clouds' repeating patterns.

In this chapter we reverse the chronology, describing first a turbulence experiment, then a numerical experiment, and finally explain the observations using the universality theory. We will try to be intuitive and concentrate



(a) (b)

Figure 26.1:

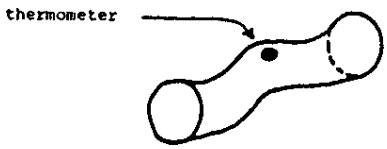


Figure 26.2:

on a few key ideas. Even though we illustrate it by onset of turbulence, the universality theory is by no means restricted to the problems of fluid dynamics.

26.1 Onset of turbulence

We start by describing schematically the 1980 experiment of Libchaber and Maurer. In the experiment a liquid is contained in a small box heated from the bottom. The salient points are:

1. There is a *controllable parameter*, the Rayleigh number, which is proportional to the temperature difference between the bottom and the top of the cell.

2. The system is *dissipative*. Whenever the Rayleigh number is increased, one waits for the *transients to die out*.

3. The container, figure 26.1 (a), has a small "aspect ratio"; its width is a small integer multiple of its height, approximately.

For small temperature gradients there is a heat flow across the cell, but the liquid is static. At a critical temperature a convective flow sets in. The hot liquid rises in the middle, the cool liquid flows down at the sides, and two convective rolls appear. So far everything is as expected from standard bifurcation scenarios. As the temperature difference is increased further, the rolls become unstable in a very specific way - a wave starts running along the roll, figure 26.1 (b).

As the warm liquid is rising on one side of the roll, while cool liquid is descending down the other side, the position and the sideways velocity of the ridge can be measured with a thermometer, figure 26.2. One observes a sinusoid, figure 26.3. The periodicity of this instability suggests two other ways of displaying the measurement, figure 26.4.

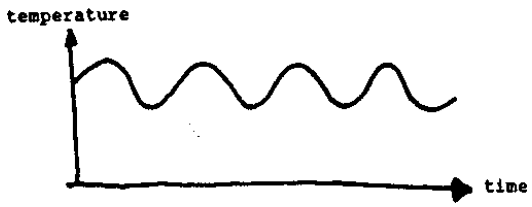


Figure 26.3:

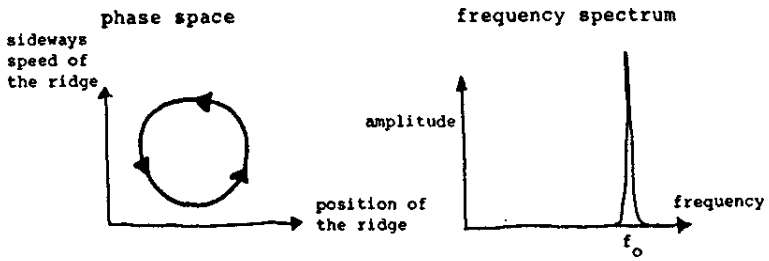


Figure 26.4:

Now the temperature difference is increased further. After the stabilization of the state space trajectory, a new wave is observed superimposed on the original sinusoidal instability. The three ways of looking at it (real time, state space, frequency spectrum) are sketched in figure 26.5. A coarse measurement would make us believe that T_0 is the periodicity. However, a closer look reveals that the state space trajectory misses the starting point at T_0, and closes on itself only after $2T_0$. If we look at the frequency spectrum, a new wave band has appeared at half the original frequency. Its amplitude is small, because the state space trajectory is still approximately a circle with periodicity T_0.

As one increases the temperature very slightly, a fascinating thing happens: the state space trajectory undergoes a very fine splitting, see figure 26.6. We see that there are three scales involved here. Looking casually, we see a circle with period T_0; looking a little closer, we see a pretzel of period $2T_0$; and looking very closely, we see that the trajectory closes on itself only after $4T_0$. The same information can be read off the frequency spectrum; the

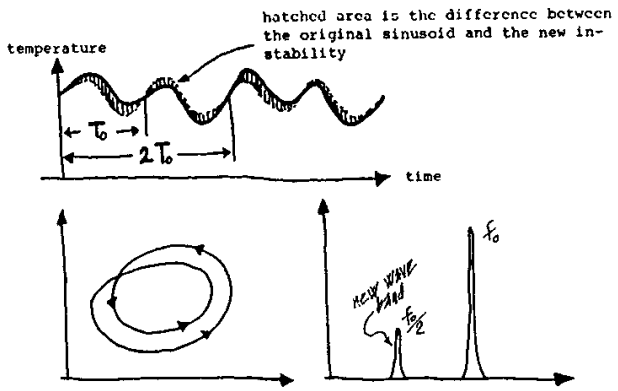


Figure 26.5:

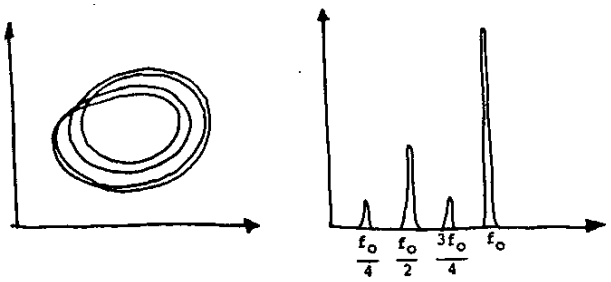


Figure 26.6:

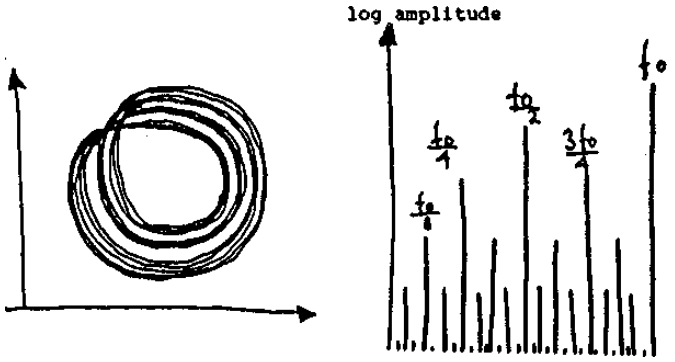
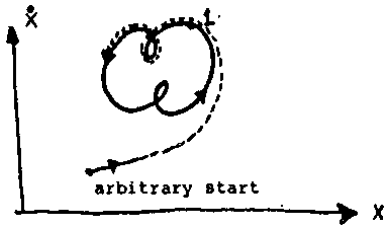


Figure 26.7:

dominant frequency is f_0 (the circle), then $f_0/2$ (the pretzel), and finally, much weaker $f_0/4$ and $3f_0/4$.

The experiment now becomes very difficult. A minute increase in the temperature gradient causes the state space trajectory to split on an even finer scale, with the periodicity 2^3T_0. If the noise were not killing us, we would expect these splittings to continue, yielding a trajectory with finer and finer detail, and a frequency spectrum of figure 26.7, with families of ever weaker frequency components. For a critical value of the Rayleigh number, the periodicity of the system is $2^\infty T_0$, and the convective rolls have become turbulent. This *weak turbulence* is today usually referred to as the "onset of chaos". Globally, the rolls persist but are wiggling irregularly. The ripples which are running along them show no periodicity, and the spectrum of an idealized, noise-free experiment contains infinitely many subharmonics, figure 26.8. If one increases the temperature gradient

Figure 26.8:



Duffing!damped

Figure 26.9:

beyond this critical value, there are further surprises (see, for example, figure 26.16) which we will not discuss here.

We now turn to a numerical simulation of a simple nonlinear oscillator in order to start understanding why the state space trajectory splits in this peculiar fashion.

26.2 Onset of chaos in a numerical experiment

In the experiment that we have just described, limited experimental resolution makes it impossible to observe more than a few bifurcations. Much longer sequences can be measured in numerical experiments. A typical example is the nonlinear oscillator

cite Arecchi and Lisi (1982)

26.2
page 551

$$\ddot{x} + \gamma\dot{x} - x + 4x^3 = A\cos(\omega t)\,. \tag{26.1}$$

The oscillator is driven by an external force of frequency ω, with amplitude A period $T_0 = 2\pi/\omega$. The dissipation is controlled by the friction coefficient γ. (See (2.6) and example 5.1.) Given the initial displacement and velocity one can easily follow numerically the state state space trajectory of the system. Due to the dissipation it does not matter where one starts; for a wide range of initial points the state space trajectory converges to an attracting *limit cycle* (trajectory loops onto itself) which for some $\gamma = \gamma_0$ looks something like figure 26.9. If it were not for the external driving force, the oscillator would have simply come to a stop. As it is, executing a motion forced on it externally, independent of the initial displacement and velocity. Starting at the point marked 1, the pendulum returns to it after the unit period T_0.

However, as one decreases, the same phenomenon is observed as in the turbulence experiment; the limit cycle undergoes a series of period-doublings, figure 26.10. The trajectory keeps on nearly missing the starting point, until it hits after exactly $2^n T_0$. The state space trajectory is getting increasingly hard to draw; however, the sequence of points 1, 2, ..., 2^n, which corresponds to the state of the oscillator at times T_0, $2T_0$, ..., $2^n T_0$, sits in a small region of the state space, so in figure 26.11 we enlarge it for a closer look. Globally the trajectories of the turbulence experiment

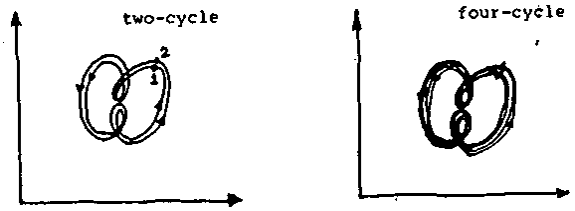


Figure 26.10:

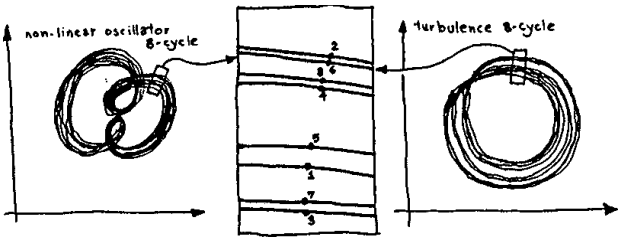


Figure 26.11:

and of the non-linear oscillator numerical experiment look very different. However, the above sequence of near misses is *local*, and looks roughly the same for both systems. This sequence of points lies approximately on a straight line, figure 26.12. Let us concentrate on this line, reducing the dimensionality of the state space by a Poincaré map. The Poincaré map contains all the information we need; from it we can read off when an instability occurs, and how large it is. One varies continuously the non-linearity parameter (friction, Rayleigh number, etc.) and plots the location of the intersection points; in the present case, the Poincaré surface is - for all practical purposes - a smooth 1-dimensional curve, and the result is a *bifurcation tree* of figure 26.13. We already have some *qualitative* understanding of this plot. The state space trajectories we have drawn are localized (the energy of the oscillator is bounded) so the tree has a finite span. Bifurcations occur simultaneously because we are cutting a single trajectory; when it splits, it does so everywhere along its length. Finer and finer scales characterize both the branch separations and the branch lengths.

Feigenbaum's discovery consists of the following *quantitative* observations:

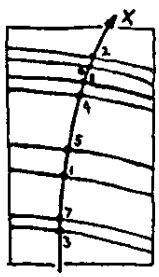


Figure 26.12:

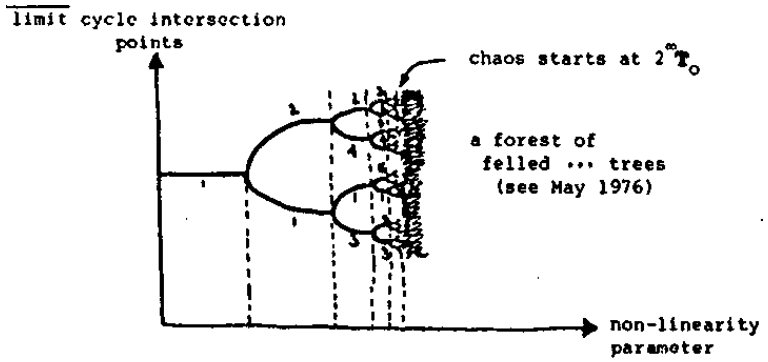


Figure 26.13:

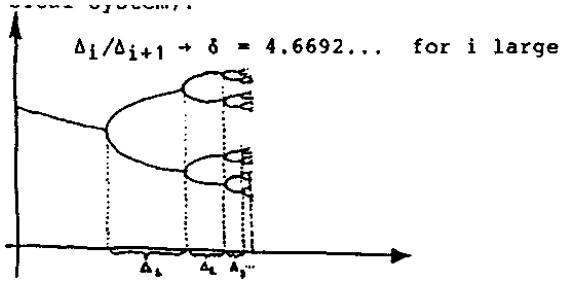


Figure 26.14:

1. The *parameter convergence* is universal (i.e., independent of the particular physical system), $\Delta_i/\Delta_{i+1} \to 4.6692\ldots$ for i large, see figure 26.14.

2. The relative scale of successive *branch splittings* is universal: $\epsilon_i/\epsilon_{i+1} \to 2.5029\ldots$ for i large, see figure 26.15.

The beauty of this discovery is that if turbulence (chaos) is arrived at through an infinite sequence of bifurcations, we have two quantitative predictions:

1. The convergence of the critical Rayleigh numbers corresponding to the cycles of length 2, 4, 8, 16, ... is controlled by the universal convergence parameter $\delta = 4.6692016\ldots$.

2. The splitting of the state space trajectory is controlled by the universal scaling parameter $\alpha = 2.50290787\ldots$. As we have indicated in

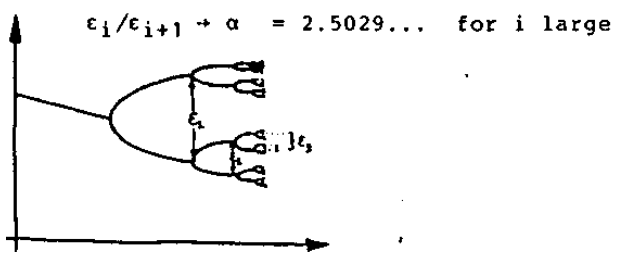


Figure 26.15:

period-doubling!tree

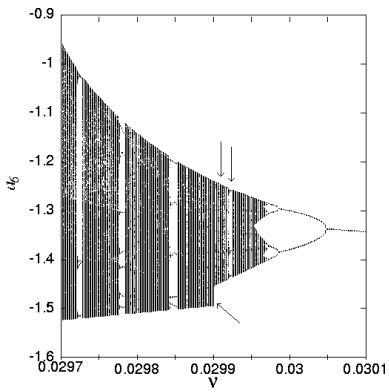


Figure 26.16: A period-doubling tree observed in a small size Kuramoto-Sivashinsky system, generated under adiabatic change of the damping parameter (system size). The choice of projection down to the coordinate a_6 is arbitrary; projected down to any coordinate, the tree is qualitatively the same. The two upper arrows indicate typical values: for $\nu = 0.029910$ dynamics appears chaotic, and $\nu = 0.029924$ there is a "golden-mean" repelling set coexisting with attractive period-3 window. The lower arrow indicates the value at which upper invariant set with this merges with its $u(x) \to -u(-x)$ symmetry partner. $N = 16$ Fourier modes truncation of (25.10). Truncation to $N = 17$ modes yields a similar figure, with values for specific bifurcation points shifted by $\sim 10^{-5}$ with respect to the $N = 16$ values. (from ref. [25.4])

our discussion of the turbulence experiment, the relative heights of successive subharmonics measure this splitting and hence α.

⇓PRELIMINARY

These universal numbers are measured in a variety of experiments: a sample of early experiments is given in table ?.

⇑PRELIMINARY

While this universality was derived through study of simple, few-dimensional systems (pendulum, oscillations along a convective roll), it also applies to high- or even infinite-dimensional systems, such as. discretizations of the Navier-Stokes equations, and in the literature there are innumerable other examples of period-doublings in many-dimensional systems. A wonderful thing about this universality is that it does not matter much how close our equations are to the ones chosen by nature; as long as the model is in the same universality class (in practice this means that it can be modeled by a mapping of form (26.2)) as the real system, both will undergo a period-doubling sequence. That means that we can get the right physics out of very simple models, and this is precisely what we will do next.

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Example 26.1 Period doubling tree in a flame flutter. *For $\nu > 1$, $u(x,t) = 0$ is the globally attractive stable equilibrium; starting with $\nu = 1$ the solutions go through a rich sequence of bifurcations.*

Figure 26.16 is a representative plot of the period-doubling tree for the Poincaré map P. To obtain this figure, we took a random initial point, iterated it for a some time to let it settle on the attractor and then plotted the a_6 coordinate of the next 1000 intersections with the Poincaré section. Repeating this for different values of the damping parameter ν, one can obtain a picture of the attractor as a function of ν; the dynamics exhibits a rich variety of behaviors, such as strange attractors, stable limit cycles, and so on.

The reason why multidimensional *dissipative* systems become effectively one-dimensional is that: for a dissipative system state space volumes shrink.

Figure 26.17:

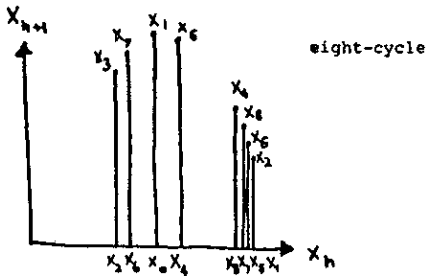

Figure 26.18:

They shrink at different rates in different directions, as in figure 26.17. The direction of the slowest convergence defines a one-dimensional line which will contain the attractor (the region of the state space to which the trajectory is confined at asymptotic times):

What we have presented so far are a few experimental facts; we now have to convince you that they are universal.

26.3 What does all this have to do with fishing?

Looking at the state space trajectories shown earlier, we observe that the trajectory bounces within a restricted region of the state space. How does this happen? One way to describe this bouncing is to plot the $(n+1)$th intersection of the trajectory with the Poincaré surface as a function of the preceding intersection. Referring to figure 26.12 we find the map of figure 26.18. This is a Poincaré return map for the limit cycle. If we start at various points in the state space (keeping the non-linearity parameter fixed) and mark all passes as the trajectory converges to the limit cycle, we trace an approximately continuous curve $f(x)$ of figure 26.19. which gives

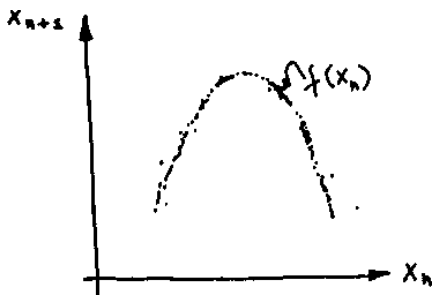

Figure 26.19:

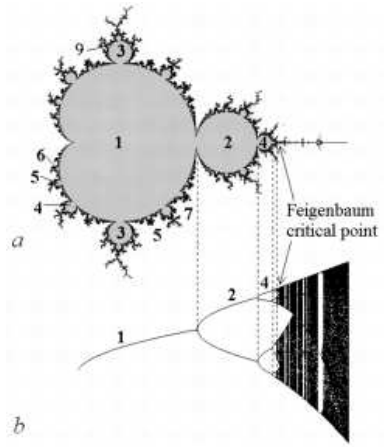

Figure 26.20: Correspondence between (a) the Mandelbrot set, shown in plane $(\mathrm{Re}\lambda, \mathrm{Im}\lambda)$ for the map $z_{k+1} = \lambda - z_k^2$, and (b) the period-doubling bifurcation tree, plane (λ, x), $x, \lambda \in \mathbb{R}$. (from ref. [26.14])

the location of the trajectory at time $t + T_0$ as a function of its location at time t:

$$x_{n+1} = f(x_n), \tag{26.2}$$

The trajectory bounces within a trough in the state space, and $f(x)$ gives a local description of the way the trajectories converge to the limit cycle. In principle we know $f(x)$, as we can measure it (see Simoyi, Wolf and Swinney (1982) for a construction of a return map in a chemical turbulence experiment , or compute it from the equations of motion. The form of $f(x)$ depends on the choice of Poincaré map, and an analytic expression for $f(x)$ is in general not available (see Gonzales and Piro (1983) for an example of an explicit return map), but we know what $f(x)$ should look like; it has to fall on both sides (to confine the trajectory), so it has a maximum. Around the maximum it looks like a parabola

⇓PRELIMINARY

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$$f(x) = a_o + a_2(x - x_c)^2 + \dots \tag{26.3}$$

ask for permission to use figure 26.20

like any sensible polynomial approximation to a function with a hump.

This brings us to the problem of a rational approach to fishery. By means of a Poincaré map we have reduced a continuous trajectory in state space to one-dimensional iteration. This one-dimensional iteration is studied in population biology, where $f(x)$ is interpreted as a population curve (the number of fish x_{n+1} in the given year as a function of the number of fish x_n the preceding year), and the bifurcation tree figure 26.13 has been studied in considerable detail.

The first thing we need to understand is the way in which a trajectory converges to a limit cycle. A numerical experiment will give us something like figure 26.21. In the Poincaré map the limit trajectory maps onto itself, $x^* = f(x^*)$. Hence a limit trajectory corresponds to a *fixed point* of $f(x)$. Take a programmable pocket calculator and try to determine x^*. Type in a simple approximation to $f(x)$, such as

$$f(x) = \lambda - x^2 \,. \tag{26.4}$$

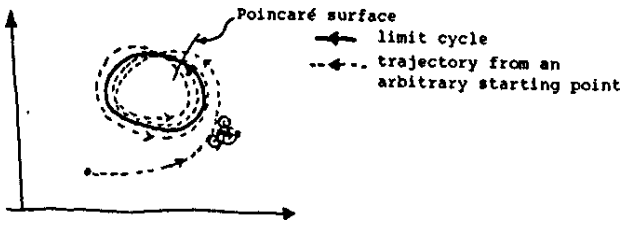

Figure 26.21:

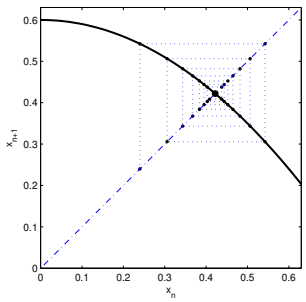

Figure 26.22:

Here λ is the non-linear parameter. Enter your guess x_0 and press the button. The number x_1 appears on the display. Is it a fixed point? Press the button again, and again, until $x_{n+1} = x_n$ to desired accuracy. Diagrammatically, this is illustrated by the web traced out be the trajectory in figure 26.22. Note the tremendous simplification gained by the use of the Poincaré map. Instead of computing the entire state space trajectory by a numerical integration of the equations of motion, we are merely pressing a button on a pocket calculator.

This little calculation confirms one's intuition about fishery. Given a fishpond, and sufficient time, one expects the number of fish to stabilize. However, no such luck - a rational fishery manager soon discovers that anything can happen from year to year. The reason is that the fixed point x^* need not be attractive, and our pocket calculator computation need not converge.

26.4 A universal equation

universal function → universal fixed-point function?

Why is the naive fishery manager wrong in concluding that the number of fish will eventually stabilize? He is right when he says that $x^* = f(x^*)$ corresponds to the same number of fish every year. However, this is not necessarily a stable situation. Reconsider how we got to the fixed point in figure 26.22. Starting with a sufficiently good guess, the iterates converge to the fixed point. Now start increasing gently the non-linearity parameter (Rayleigh number, the nutritional value of the pond, etc.). $f(x)$ will slowly change shape, getting steeper and steeper at the fixed point, until the fixed point becomes unstable and gives birth to a cycle of two points. This is

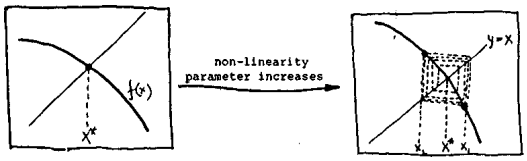

Figure 26.23:

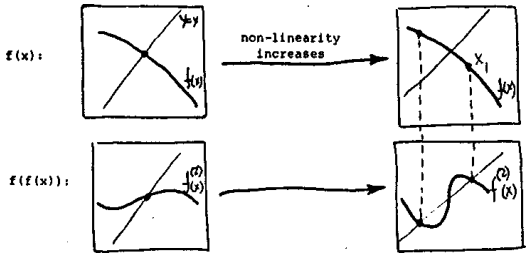

Figure 26.24:

precisely the first bifurcation observed in our experiments.

Example 26.2 Fixed point stability.

fix *The fixed point condition for map (26.4) $x^2 + x - \lambda = 0$ yields 2 fixed points.*

$$x_p m = \frac{-1 \pm \sqrt{1+4\lambda}}{what?}$$

The fixed point x_+ loses stability at $\lambda = -1$. Inserted into $\lambda = f'(x) = -2x$, this yields

$$\lambda = 3/4\,, \qquad x_+ = 1/2$$

as the value at which fixed point x_+ loses stability.

This is the only gentle way in which our trajectory can become unstable (cycles of other lengths can be created, but that requires delicate fiddling with parameters; such bifurcations are not generic). Now we return to the same point after every second iteration

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$$x_i = f(f(x_i))\,, \qquad i = 1, 2\,.$$

so the cycle points of $f(x)$ are the fixed points of $f(f(x))$.

To study their stability, we plot $f(f(x))$ alongside $f(x)$ in figure 26.24. What happens as we continue to increase the "Rayleigh number"? $f(x)$ becomes steeper at its fixed point, and so does $f(f(x))$. Eventually the magnitude of the slope at the fixed points of $f(f(x))$ exceeds one, and they bifurcate. Now the cycle is of length four, and we can study the stability of the fixed points of the fourth iterate. They too will bifurcate, and so forth. This is why the state space trajectories keep on splitting $2 \to 4 \to 8 \to 16 \to$

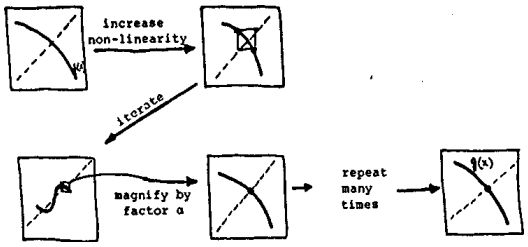

Figure 26.25:

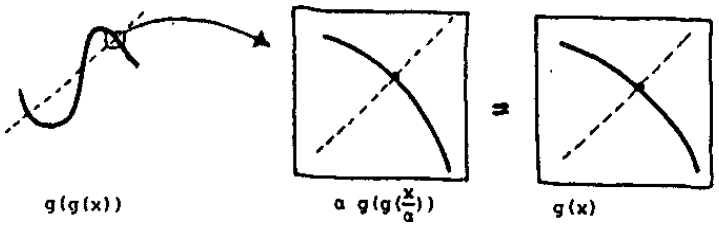

Figure 26.26:

32 $\cdots$ in our experiments The argument does not depend on the precise form of $f(x)$, and therefore the phenomenon of successive period-doublings is *universal*.

More amazingly, this universality is not only qualitative. In our analysis of the stability of fixed points we kept on magnifying the neighborhood of the fixed point, figure 26.25. The neighborhoods of successive fixed points look very much the same after iteration and rescaling. After we have magnified the neighborhoods of fixed points many times, practically all information about the global shape of the starting function $f(x)$ is lost, and we are left with a *universal function* $g(x)$. Denote by T the operation indicated in figure 26.25 iterate twice and rescale by (without changing the non-linearity parameter),

$$Tf(x) = \alpha f(f(x/\alpha)), \tag{26.5}$$

$g(x)$ is self-replicating under rescaling and iteration, figure 26.26. More precisely, this can be stated as the *universal equation*

$$g(x) = \alpha g(g(-x/\alpha)), \tag{26.6}$$

which determines both the universal function $g(x)$ and $\alpha = -1/g(1) = 2.50290787\ldots$, with normalization convention $g(0) = 1$.

Example 26.3 An approximate period doubling renormalization. remark, contributor credits to IsaKuz05c; also Hellemann

As the simplest examples of period-doubling cascades. Consider the map

$$x_{n+1} = f_\lambda(x_n) = \lambda - x_n^2\,, \qquad \lambda, x \epsilon \mathbb{R}\,. \tag{26.7}$$

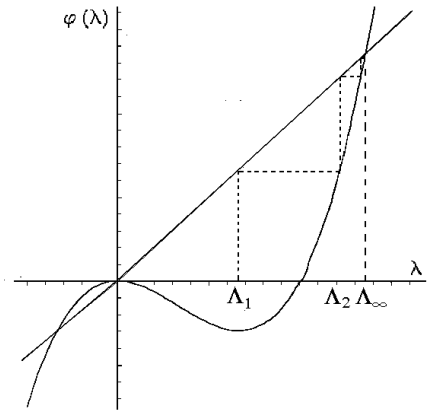

Figure 26.27: Iteration of the approximate renormalization transformation (26.10). Dashed line designates the backward iterations starting at the first period doubling bifurcation point, $\lambda_1 = 3/4$, and mapping to the further bifurcation points λ_m.

cite Hellemann, ask about Landau

The two fixed points of f, $x_\pm = \frac{1\pm\sqrt{1+4\lambda}}{2}$, are the roots of $x_* = \lambda - x_*^2$. At $\lambda = 3/4$ the stability multiplier $\Lambda = f'_\lambda(x_*)$ of the fixed point $x_* = \frac{1+\sqrt{1+4\lambda}}{2}$ is marginal, $\Lambda = -2x_* = -1$. For $\lambda > 3/4$, the fixed point loses its stability and undergoes a period-doubling bifurcation. Values λ for subsequent bifurcations can be found by means of the following approximate renormalization method. Apply the map (26.7) two times:

$$x_{n+2} = \lambda - \lambda^2 + 2\lambda x_n^2 - x_n^4\,, \tag{26.8}$$

and drop the quartic term x_n^4. By the scale transformation

$$x_n \to x_n/\alpha_0, \qquad \alpha_0 = -2\lambda\,, \tag{26.9}$$

this can be rewritten in the original form $x_{n+2} = \lambda_1 - x_n^2$, which differs from (26.7) only by renormalization of λ

$$\lambda_1 = \varphi(\lambda) = -2\lambda(\lambda - \lambda^2)\,. \tag{26.10}$$

The map parametrized by λ, approximates two applications of the original map. Repeating the renormalization transformation (26.10) with scale factors $\alpha_m = -2\lambda_m$, one obtains a sequence of the form

$$x_{n+2^m} = \lambda_m - x_n^2\,, \qquad \lambda_m = \varphi(\lambda_{m-1})\,. \tag{26.11}$$

Fixed points of these maps correspond to the 2^m-cycles of the original map. All these cycles, as well as the fixed point of the map (26.7), become unstable at $\lambda_m = \Lambda_1 = 3/4$. Solving the chain of equations

$$\Lambda_1 = \varphi(\Lambda_2)\,\Lambda_2 = \varphi(\Lambda_3)\,...\,\Lambda_{m-1} = \varphi(\Lambda_m)\,, \tag{26.12}$$

we get the corresponding sequence of bifurcation values of parameter λ (with $\lambda \approx \Lambda_m$ the 2^m-cycle of (26.7)). From iteration diagram of figure 26.27 it is evident, that this sequence converges with $m \to \infty$ to a definite limit Λ_∞, the fixed point of the renormalization transformation. It satisfies the equation $\Lambda_\infty = \varphi(\Lambda_\infty)$, thus $\Lambda_\infty = (1+\sqrt{3})/2 \approx 1.37$. The scaling factors also converge to the limit: $\alpha_m \to \alpha$, where

$\alpha = -2\Lambda_\infty \approx 2.74$. *The multipliers (Floquet eigenvalues of the 2^m-cycles) converge to* $\mu_m \to \mu = \sqrt{1-4\Lambda_\infty} \approx -1.54$.

From transformation (26.11) on can also describe the convergence of the bifurcation sequence:

$$\Lambda_m = \varphi(\Lambda_\infty) + \varphi'(\Lambda_\infty)(\Lambda_{m+1} - \Lambda_\infty) = \\ = \Lambda_\infty + \delta(\Lambda_{m+1} - \Lambda_\infty) \quad , \tag{26.13}$$

where the Feigenbaum $\delta = \varphi'(\Lambda_\infty) = 4 + \sqrt{3} \approx 5.73$ *characterizes parameter rescaling for each successive period doubling.*

The approximate values of Feigenbaum's universal space and parameter scaling constants are reasonably close to the exact values,

		exact	approximate
α	=	-2.502$\cdots$	-2.74
δ	=	4.669$\cdots$	5.73 ,

considering the crudeness of the approximation: the universal fixed-point function $g(x)$ *is here truncated to a quadratic polynomial.*

(O.B. Isaeva and S.P. Kuznetsov)

If you arrive at $g(x)$ the way we have, by successive bifurcations and rescaling, you can hardly doubt its existence. However, if you start with (26.6) as an equation to solve, it is not obvious what its solutions should look like. The simplest thing to do is to approximate $g(x)$ by a finite polynomial and solve the universal equation numerically, by Newton's method. This way you can compute α and δ to much higher accuracy than you can ever hope to measure them to experimentally.

There is much pretty mathematics in universality theory. Despite its simplicity, nobody seems to have written down the universal equation before 1976, so the subject is still young. We do not have a series expansion for α, or an analytic expression for $g(x)$; the numbers that we have are obtained by boring numerical methods. So far, all we know is that $g(x)$ exists. What has been proved is that the Newton iteration converges, so we are no wiser for the result. In some situations the universal equation (26.6) has analytic solutions; we shall return to this in the discussion of intermittency (SECTION 10). The universality theory has also been extended to iterations of complex polynomials (SECTION 12).

comment about universal equation

⇓PRELIMINARY

make up a example

⇑PRELIMINARY

To see why the universal function must be a rather crazy function, consider high iterates of $f(x)$ for parameter values corresponding to 2-, 4- and 8-cycles, figure 26.28. If you start anywhere in the unit interval and iterate a very large number of times, you end up in one of the cycle points. For the 2-cycle there are two possible limit values, so $f(f(\dots f(x)))$ resembles a castle battlement. Note the infinitely many intervals accumulating at the unstable $x = 0$ fixed point. In a bifurcation of the 2-cycle into the 4-cycle each of these intervals gets replaced by a smaller battlement. After infinitely many bifurcations this becomes a fractal (i.e., looks the same under any enlargement), with battlements within battlements on every

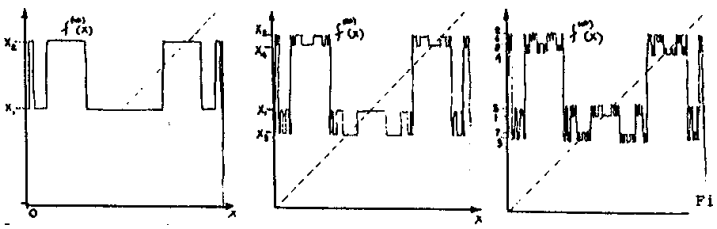

Figure 26.28:

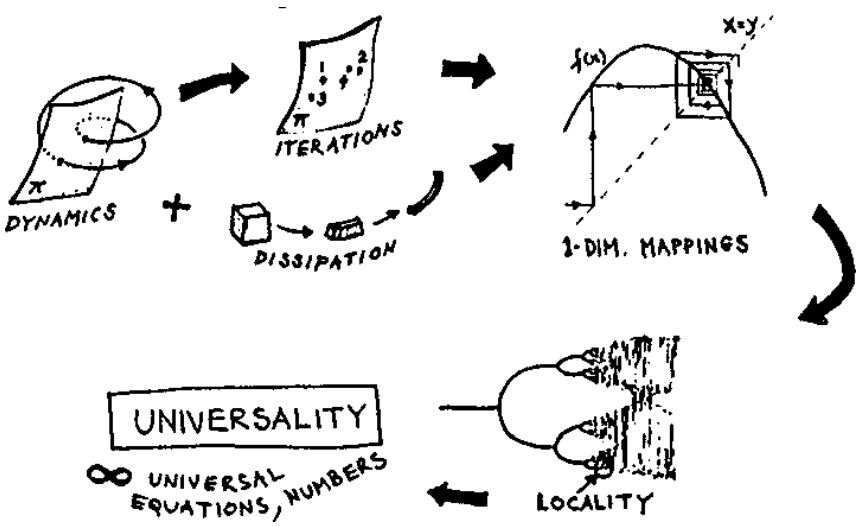

Figure 26.29:

scale. Our universal function $g(x)$ does not look like that close to the origin, because we have enlarged that region by the factor $\alpha = 2.5029\ldots$ after each period-doubling, but all the wiggles are still there; you can see them in Feigenbaum's (1978) plot of $g(x)$. For example, (26.6) implies that if x^* is a fixed point of $g(x)$, so is $\alpha(x^*)$. Hence $g(x)$ must cross the lines $y = x$ and $y = -x$ infinitely many times. It is clear that while around the origin $g(x)$ is roughly a parabola and well approximated by a finite polynomial, something more clever is needed to describe the infinity of $g(x)$'s wiggles further along the real axis and in the complex plane.

remark 1978

All this is fun, but not essential for understanding the physics of the onset of chaos. The main thing is that we now understand where the universality comes from. We start with a complicated many-dimensional dynamical system. A Poincaré map reduces the problem from a study of differential equations to a study of discrete iterations, and dissipation reduces this further to a study of one-dimensional iterations (now we finally understand why the state space trajectory in the turbulence experiment undergoes a series of bifurcations as we turn the heat up!). The successive bifurcations take place in smaller and smaller regions of the state space. After n bifurcations the trajectory splittings are of order $\alpha^{-n} = (0.399\ldots)-n$ and practically all memory of the global structure of the original dynamical system is lost (see figure 26.29). The asymptotic self-similarities can be encoded by universal equations. The physically interesting scaling numbers

can be quickly estimated by simple truncations of the universal equations, such as example 26.3 (May and Oster 1980, Derrida and Pomeau 1980, Helleman 1980a, Hu 1981). The full universal equations are designed for accurate determinations of universal numbers; as they have built-in rescaling, the round-off errors do not accumulate, and the only limit on the precision of the calculation is the machine precision of the computer.

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Anything that can be extracted from the asymptotic period-doubling regime is universal; the trick is to identify those universal features that have a chance of being experimentally measurable. We will discuss several such extensions of the universality theory in the remainder of this introduction.

26.5 The unstable manifold

Feigenbaum delta

$$\delta = \lim_{n\to\infty} = \frac{r_{n-1} - r_n}{r_n - r_{n+1}} \qquad = 4.6692016\ldots \tag{26.14}$$

is the universal number of the most immediate experimental import - it tells us that in order to reach the next bifurcation we should increase the Rayleigh number (or friction, or whatever the controllable parameter is in the given experiment) by about one fifth of the preceding increment. Which particular parameter is being varied is largely a question of experimental expedience; if r is replaced by another parameter $R = R(r)$, then the Taylor expansion

$$R(r) = R(r_\infty) + (r - r_\infty)R'(r_\infty) + (r - r_\infty)^2 R''(r_\infty)/2 + \cdots$$

yields the same asymptotic delta

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$$\delta \simeq \frac{R(r_{n-1}) - R(r_n)}{R(r_n) - R(r_{n+1})} = \frac{r_{n+1} - r_n}{r_n - r_{n+1}} + O(\delta^n) \tag{26.15}$$

providing, of course, that $R'(r_\infty)$ is non-vanishing (the chance that a physical system just happens to be parametrized in such a way that $R'(r_\infty) = 0$ is nil).

In deriving the universal equation (26.6) we were intentionally sloppy, because we wanted to introduce the notion of encoding self-similarity by universal equations without getting entangled in too much detail. We obtained a universal equation which describes the self-similarity in the x-space, under iteration and rescaling by α. However, the bifurcation tree figure 26.13 is self-similar both in the x-space and the parameter space: each branch looks like the entire bifurcation tree. We will exploit this fact to construct a universal equation which determines both α and δ.

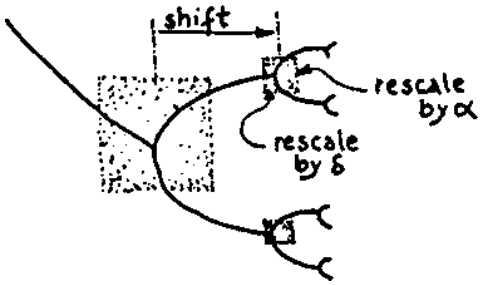


Figure 26.30:

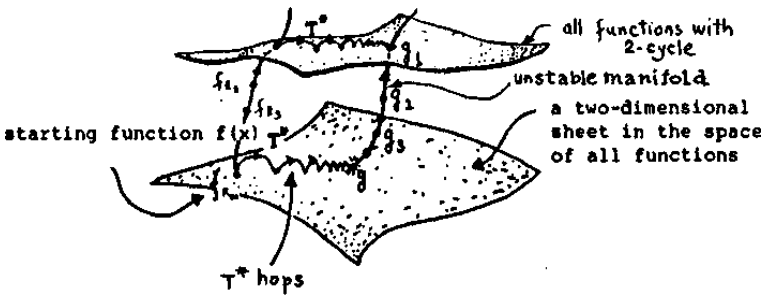


Figure 26.31:

Let T^* denote the operation of iterating twice, rescaling x by α, *shifting* the non-linearity parameter to the corresponding value at the next bifurcation (more precisely, the value of the nonlinearity parameter with the same *stability*, i.e., the same slope at the cycle points), and *rescaling* it by δ:

$$T^* f_{R_n+\Delta_n p}(x) = \alpha_n f^{(2)}_{R_n+\Delta_n(1+p/\delta_n)}(x/\alpha_n) \tag{26.16}$$

Here R_n is a value of the non-linearity parameter for which the limit cycle is of length 2^n, Δ_n is the distance to R_{n+1}, $\delta_n = \Delta_n/\Delta_{n+1}$, p provides a continuous parametrization, and we apologize that there are so many subscripts. T^* operation encodes the self-similarity of the bifurcation tree figure 26.13, see figure 26.30:

For example, if we take the fish population curve $f(x)$ (26.4) with R value corresponding to a cycle of length 2^n, and act with T^*, the result will be a similar cycle of length 2^n, but on a scale α times smaller. If we apply T^* infinitely many times, the result will be a universal function with a cycle of length 2^n:

$$g_p(x) = (T^*)^\infty f_{R+p_\Delta}(x) \tag{26.17}$$

If you can visualize a space of all functions with quadratic maximum, you will find figure 26.31 helpful. Each transverse sheet is a manifold consisting of functions with 2^n-cycle of given stability. T^* moves us across this transverse manifold toward g_p.

$g_p(x)$ is invariant under the self-similarity operation T^*, so it satisfies a universal equation

$$g_p(x) = \alpha g_{1+p/\delta}(g_{1+p/\delta}(x/\alpha)) \tag{26.18}$$

p parameterizes a one-dimensional continuum family of universal functions. Our first universal equation (26.6) is the fixed point of the above equation:

$$p^* = 1 + p^*/\delta \tag{26.19}$$

and corresponds to the asymptotic 2^∞-cycle.

The family of universal functions parametrized by p is called the *unstable manifold* because T-operation (26.5) drives p away from the fixed point value $g(x) = g_{p^*}(x)$.

You have probably forgotten by now, but we started this section promising a computation of δ. This we do by linearizing the equations (26.18) around the fixed point p^*. Close to the fixed point $g_p(x)$ does not differ much from $g(x)$, so one can treat it as a small deviation from $g(x)$:

$$g_p(x) = g(x) + (p - p^*)h(x)$$

Substitute this into (26.18), keep the leading term in $p - p^*$, and use the universal equation (26.6). This yields a universal equation for δ:

$$g'(g(x))h(x) + h(g(x)) = (\delta/\alpha)h(x) \tag{26.20}$$

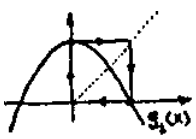
We already know $g(x)$ and α, so this can be solved numerically by polynomial approximations, yielding $\delta = 4.6692016\ldots$, together with a part of the spectrum of eigenvalues and eigenvectors $h(x)$.

Actually, one can do better with less work; T^*-operation treats the coordinate x and the parameter p on the same footing, which suggests that we should approximate the entire unstable manifold by a double power series

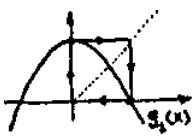
$$g_p(x) = \sum_{j=0}^{N}\sum_{k=0}^{N} c_{jk} x^{2j} p^k \tag{26.21}$$

The scale of x and p is arbitrary. We will fix it by the normalization conditions

$$g_0(0) = 0\,, \tag{26.22}$$



$$\begin{aligned} g_1(0) &= 1 \\ g_1(1) &= 0\,, \end{aligned} \tag{26.23}$$



The first condition means that the origin of p corresponds to the superstable fixed point. The second condition sets the scale of x and p by the superstable 2-cycle. (Superstable cycles are the cycles which have the maximum of the map as one of the cycle points.) Start with any simple approximation to $g_p(x)$ which satisfies the above conditions (for example, $g(x) = p - x^2$). Apply the T^*-operation (26.16) to it. This involves polynomial expansions in which terms of order higher than M and N in (26.21) are dropped. Now find by Newton's method the value of δ which satisfies normalization (26.23). This is the only numerical calculation you have to do; the condition (26.23) automatically yields the value of α. The result is a new approximation to g_p. Keep applying T^* until the coefficients in (26.21) repeat; this has moved the approximate g_p toward the unstable manifold along the transverse sheets indicated in figure 26.31. Computationally this is straightforward, the accuracy of the computation is limited only by computer precision, and at the end you will have α, δ and a polynomial approximation to the unstable manifold $g_p(x)$.

As δ controls the convergence of the high iterates of the initial mapping toward their universal limit $g(x)$, it also controls the convergence of most other numbers toward their universal limits, such as the scaling number $\alpha_n = \alpha + O(\delta^{-n})$, or even δ itself, $\delta_n = \delta + O(\delta^{-n})$. As $1/\delta = 0.2141\ldots$, the convergence is very rapid, and already after a few bifurcations the universality theory is good to a few per cent. This rapid convergence is both a blessing and a curse. It is a theorist's blessing because the asymptotic theory applies already after a few bifurcations; but it is an experimentalist's curse because a measurement of every successive bifurcation requires a fivefold increase in the experimental accuracy of the determination of the non-linearity parameter r.

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26.6 Cookie-cutter's universality

Refs. [18.2, 18.6]: cycle expansion for δ.

Ref. [26.11]: method for computation of the generalized dimensions of fractal attractors at the period-doubling transition to chaos, via an eigenvalue problem formulated in terms of functional equations, with a coefficient expressed in terms of Feigenbaum's universal fixed-point function.

⇑PRELIMINARY

Commentary

Remark 26.1 Sources. This chapter is based on a Nordita lecture prepared together with Mogens Høgh Jensen (Cvitanović and Høgh Jensen 1982). Ulla Selmer prepared the drawings, Oblivia Kaypro stood for initial execution.

The universal equation (26.6) was formulated by P. Cvitanović in March 1976, in collaboration with M.J. Feigenbaum [26.2]. Equation (26.19) is not necessarily the only way to formulate universality. Coullet and Tresser [26.5] have

proposed similar equations, after Feigenbaum's work became known, but before Joel Lebowitz rescued him from the referees and arranged for its publication. Daido (1981a) has introduced a different set of universal equations. find Hoppensteadt (1978)

We recommend review by May [26.6]

If $f(x)$ is not quadratic around the maximum, the universal numbers will be different - see Villela Mendés (1981) and Hu and Mao (1982b) for their values. According to Kuramoto and Koga (1982) such mappings can arise in chemical turbulence.

The elegant unstable manifold formulation of universality (26.18) is due to Vul and Khanin (1982) and Goldberg, Sinai and Khanin (1983).

Derrida, Gervois and Pomeau (1979) have extracted a great many metric universalities from the asymptotic regime. Grassberger (1981) has computed the Hausdorff dimension of the asymptotic attractor. Lorenz (1980) and Daido (1981b) have found a universal ratio relating bifurcations and reverse bifurcations.

A nice description of initial experiments has been given by Libchaber and Maurer(1981). The most thorough exposition available is the Collet and Eckmann [26.4] monograph. We also recommend Hu (1982), Crutchfield, Farmer and Huberman (1982), Eckmann (1981) and Ott (1981).

Collet and Eckmann (1980a) and Collet, Eckmann and Koch (1980) give a detailed description of how a dissipative system becomes one-dimensional.

The period route to turbulence that we describe is by no means the only one; see Eckmann (1981) discussion of other routes to chaos.

The geometric parameter convergence was first noted by Myrberg (1958), and independently of Feigenbaum, by Grossmann and Thomae (1977). These authors have not emphasized the universality of δ.

Refs. [26.9, S.5, 26.10] are interesting; they compute solutions of the period-doubling fixed point equation using methods of Schöder and Abel, yielding what are so far the most accurate δ and α.

The theory of period-doubling universal equations and scaling functions is developed in Kenway's notes of Feigenbaum 1984 Edinburgh lectures [26.3] (trifle hard to track down).

A nice discussion of circle maps and their physical applications is given in refs. [28.3] The universality theory for golden mean scalings is developed in refs. [**?**, 28.7, 28.14] The scaling functions for circle maps are discussed in ref. [28.4].

Nonlinear oscillator; quadratic potenital with damping and harmonic driving force exhibit cascades of period-doubling bifurcations [27.49, 27.50]

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Remark 26.2 Universality theory for conservative systems. The details of the universality theory are different for dissipative and conservative systems; however, the spirit is the same. Almost all that we shall say applies to dissipative systems we will not discuss conservative systems, but refer the reader to ???.

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Résumé

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copied from Isaeva [27.3]

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⇑PRELIMINARY

Exercises

Exercise 26.1 **Period doubling in your pocket:** *Take a programmable pocket calculator or Matlab or whatever makes you feel good and program the function*

$$f(x) = \lambda - x^2 \,.$$

The game consists in staring at the display, and looking for regularities in the sequences of iterates.

(a) *(no thinking) Try to determine fixed point $x_* = f(x_*)$ by blind iteration. Chose some value of λ a bit bigger than 0, and initial x between -1 and 1. Enter the initial x_0 and read off the next x_1. Start again, with x_1 as input. The number x_2 appears on the display. Is it a fixed point? Press the button again, and again, until $x_n = x_*$ to desired accuracy.*

(f) *(no thinking) Increase λ in small steps, as long as the trajectory does not blow up, let transients die, and then plot few hundred consecutive x_n. Generate a figure to replace the hand drawn figure 26.14.*

(c) *(thinking) Determine the smallest positive λ for which almost any initial x_0 iterates to $-\infty$.*

(b) *(no thinking) Try $\lambda = 3/4$. How's the convergence now?*

(c) *(thinking) Determine λ for which the fixed point x_* goes unstable.*

(d) *(no thinking) Try also λ: 1, 1.31070274134, 1.38154748443, 1.3979453597.*

(e) *(thinking) Compute the next number in this series. Estimate Feigenbaum δ.*

(g) *(thinking) Determine numerically scaling factors α_m which overlay (approximately) neighborhood of $x = 0$ for superstable $f^{2(m-1)}(x)$ over the neighborhood for $\alpha_m f^{2^m}(f^{2^m}(x/\alpha_m))$ for 4, 8, 16, $\cdots$ superstable cycles. Draw a figure to replace the hand drawn figure 26.26.*

check Sethna, Vivaldi

⇓PRELIMINARY

F. Vivaldi: period doubling problem set

⇑PRELIMINARY

(P. Cvitanović)

Exercise 26.2 **Period doubling in a 3-d flow:** *This is a more time consuming problem, but it gives you a feel for how numerical experiments in nonlinear dynamics are* really *done, and sets you up for doing real-life problems later on.*

Consider the nonlinear oscillator

$$\frac{d^2}{dt^2}x + k\frac{d}{dt}x - x + 4x^3 = A\cos(\omega t)\,. \tag{26.24}$$

The oscillator is driven by an external force of frequency ω, with amplitude A and the natural time unit $T_0 = 2\pi/\omega$. The dissipation is controlled by the friction coefficient k.

Given the initial displacement and velocity one can easily follow numerically (by the Runge-Kutta method, for example) the state space trajectories of the system. Due to the dissipation it does not matter where one starts in the state space. For a strong friction or a weak forcing a wide range of initial points converge to a fixed point or a limit cycle (trajectory loops onto itself). However, as one decreases the friction, bifurcations and chaos are observed.

(a) *Rewrite as a nonautonomous system of 2 first order ODEs.*

(b) *Rewrite as an autonomous system of 3 first order ODEs (add an equation whose solution is $\cos(\omega t)$).*

(c) *Observe long-time trajectories for $k = 0.154$, $\omega = 1.2199778$ and A in some range like $A = 0.05$ to 0.2. Anything interessting happening? [Helpful tips: let the system run for sufficiently long time that the transients have had time to die out. Whenever changing the parameter (in this case, parameter A), increase parameter in small steps (adiabatically), and then restart the trajectory from the last point of the preceeding simulation. This minimizes the transients.]*

(d) *Construct a stroboscopic Poincaré section by recording values of $(x_n, \dot{x}_n)$ at times separated T_0, $2T_0, \cdots$, $2^n T_n, \cdots$*

(e) *Play a bit with plotting various combinations of x_n, x_{n+1}, $\dot{x}_n$, and $\dot{x}_{n+1}$. Often a Poincaré return map - such as (x_n, x_{n+1}) - is more informative than a Poincaré section, such as $(x_n, \dot{x}_n)$.*

(f) *Increase A in small steps, as long as the trajectory does not blow up, let transients die, and then plot a few hundred consecutive x_n (or whatever variable you like best). Plot a (A, x_n) period-doubling tree to replace the hand drawn figure 26.14.*

(g) *Observe long-time trajectories for $(k, \omega) = (0.154, 1.2199778)$ and $A = 0.1$, 0.11, 0.114, 0.11437, ... Do you observe bifurcations?*

There is nothing special about these parameter values; we give them just to help you with finding your first bifurcation sequence in a realistic nonlinear flow.

(P. Cvitanović)

Exercise 26.3 Renormalization in a 3-d flow: *Consider the nonlinear oscillator (26.24).*

(a) *For $(k, \omega) = (0.154, 1.2199778)$ determine accurately amplitude values $A_1, A_2, \cdots, A_n, \cdots$, for which 2^n-cycle bifurcates into 2^{n+1}-cycle. [Tip: use the Newton method on the Poincaré section. You'll need to also integrate the linearized stability matrix.]*

(b) *Estimate Feigenbaum δ.*

(g) *Estimate Feigenbaum α.*

(P. Cvitanović)

border of order
Holomorphic dynamics

Chapter 27

Complex universality

uncomment ref in preface
find a sillier quote

> In its original form, the existence of δ seems a technical analytic result. Now it proves to be an aspect of a broader property of fractal scaling.
>
> Benoit B. Mandelbrot

The easy part of dynamical systems theory is the regular, "ordered" motions; attracting fixed points and limit cycles, integrable "elliptic" regions. As we now know, fully chaotic dynamics is often not much harder: it can be described by Smale horseshoes, finite grammars and unstable periodic orbits. The hard part lies in between, the dynamics to which we shall refer to here as the "border of order" - trajectories that are neither stable nor unstable.

The closer to the border of order, the harder it is to decide whether a given trajectory is stable or unstable. We need to inspect the trajectory with a higher and higher resolution, for longer and longer times, as in figure 27.1. In general this is a hopeless undertaking. However, there are situations where the border of order is a self-similar fractal. By a magnification of spatial scales and a replacement of time by logarithmic time, the dynamics close to the border of order can be turned into a clean, uniformly hyperbolic dynamics, and described to exponential accuracy by techniques developed for nice chaotic sets. Not only that, but the fine structure of the border turns out to be topologically and quantitatively universal, i.e., common to large classes of dynamical systems.

In this chapter we illustrate the key concepts underlying the renormalization theory of transitions to chaos by studying sequences of period n-tuplings for complex maps. We have chosen this example for its beauty and simplicity; here you should be able to visualize the renormalization transformations and the universal scalings as encodings of the self-similar patterns generated by deterministic dynamics.

27.1 Holomorphic dynamics

define Holomorphic

Fatou and Julia maps

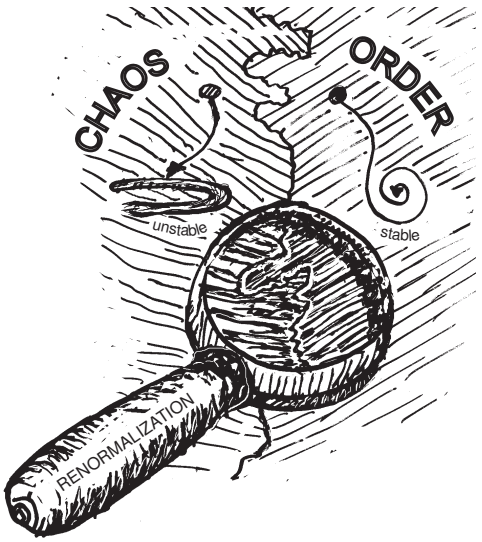


Figure 27.1: Renormalization is the process of magnifying a neighborhood of the border of order, and inspecting closer and closer returns for longer and longer times.

We shall study properties of the asymptotic iterates of

$$z_{n+1} = f(z_n)\,, \tag{27.1}$$

where $f(z)$ is a polynomial in the complex variable z with a quadratic critical point, i.e., with Taylor series expansion of the form

$$f(z) = a_0 + a_2(z - z_c)^2 + \dots\,.$$

Typical model mappings of this type are the Fatou and Julia maps, respectively:

$$f(z) = \lambda - z^2 \tag{27.2}$$
$$f(z) = \lambda\, z(1 - z)\,. \tag{27.3}$$

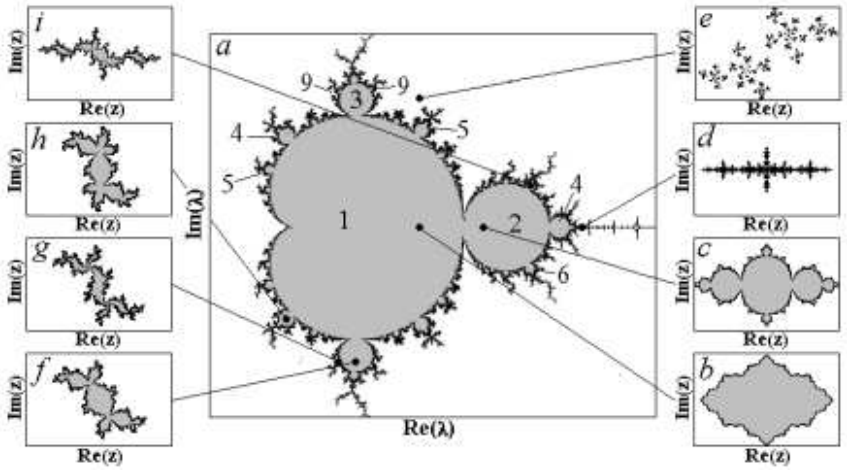
change (27.3) parameters

When such mappings are used to model dynamical systems with z a real variable and the "nonlinearity" parameter λ real, the asymptotic attractor is conveniently represented by a "bifurcation tree" of figure 26.13, i.e., by a two-dimensional plot with λ as one axis and values of the asymptotic iterates for given λ plotted along the other axis.

It is not possible to visualize asymptotics of complex iterations in this way, as their iteration space has two (real) dimensions, and period n-tuplings are induced by tuning a pair of (real) parameters. To describe the asymptotic iterates of complex maps we proceed in two steps.

ask for permission to use figure 27.2

First, we describe the parameter space by its *Mandelbrot set* M. The Mandelbrot set is the set of all values of the mapping parameter (parameter



"em critical point $z˙c$
Mandelbrot set M
basin of attraction
attractor
Julia set J
superstable

Figure 27.2: (a) The Mandelbrot set. Gray: critical point trajectory converges to an attractive periodic orbit of period $1, 2, \ldots, 6$. White: the critical point trajectory escapes to infinity. Julia sets for the quadratic map $z_{k+1} = \lambda - z^2$ parameter values (b) $\lambda = 0.5$, (c) $\lambda = 0.8$, (d) $\lambda = 1.42$, (e) $\lambda = 0.5 + 0.7i$, (f) $\lambda = 0.123 - 0.745i$, (g) $\lambda = 0.0315 - 0.7908i$, (h) $\lambda = -0.282 + 0.530i$, (i) $\lambda = 1.16 + 0.25i$. (from ref. [27.3])

p in the model mapping (27.2)) for which iterates of the critical point do not escape to infinity. A *critical point* z_c is a value of z for which the mapping $f(z)$ has vanishing derivative, $f'(z_c) = 0$. For example $z = 0$ in (27.2) is the critical point. The Mandelbrot set for the mapping (27.2) is plotted in figure 27.2.

Second, we characterize the *asymptotic iterates* for a given value of the parameter either by their *basin of attraction*, or by their *attractor*. The basin of attraction K is the set of all values of z which are attracted toward the attractor under iteration by $f(z)$. A typical basin of attraction is plotted in figure 27.3.

The boundary of K, or the *Julia set* J, is the closure of all unstable fixed points of all iterates of $f(z)$.

Theorem. For parameter values within the Mandelbrot set M, the Julia set J is connected. If all critical points iterate to infinity, J is a Cantor set.

If the nth iterate of $f(z)$ equals z, the set of points $z_k = f^k(z_0)$, $k = 0, 1, 2, ..., n-1$ form a periodic orbit (or cycle) of length n. If

$$|df^n(z_k)/dz| < 1 \tag{27.4}$$

the cycle is *attractive*, with the *attractor* L the periodic orbit $z_0, z_1, ..., z_{n-1}$. If the derivative (27.4) is vanishing, the orbit is *superstable*, and (by the chain rule) a critical point is one of the cycle points. For polynomial mappings $z = \infty$ plays a special role; it is always a superstable fixed point. The following theorem eases attractor searches:

"em Mandelbrot cactus

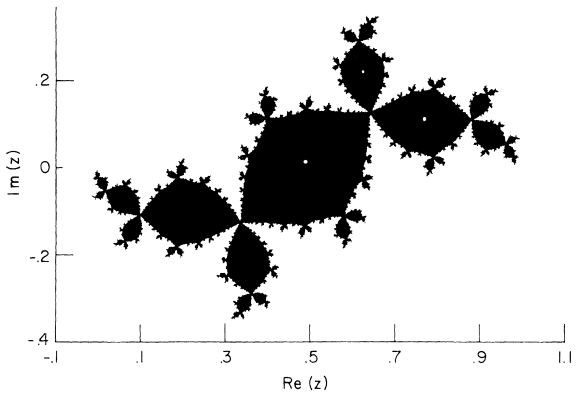


Figure 27.3: The basin of attraction for the superstable 3-cycle of mapping (27.3), $\lambda = 0.123 - 0.745i$. Any initial z from the black region converges toward the superstable 3-cycle, denoted by the three white dots. The basin of attraction for mapping (27.2) superstable 3-cycle is the same, up to a coordinate shift and rescaling.

Theorem. The basin of attraction K contains at least one critical point.

The precise shape of the Mandelbrot set M depends on $f(z)$, but it always resembles a cactus, see figure 27.4. Here we are not so much interested in the entire M, as in the *Mandelbrot cactus*, figure 27.4, the set of components of M generated from a single fixed point by all possible sequences of all possible period n-tuplings.

n-tuplings not defined

To summarize, the parameter dependence of asymptotic iterates of mapping $f(z)$ is described by the Mandelbrot set M. For each point inside M, the asymptotic iterates are characterized by their basin of attraction K, the Julia set J and the attractor L.

27.2 Mandelbrot cactus

Now that the general setting is established, we can turn to a detailed study of the way in which a fixed point of the complex mapping (27.1) branches into an n-cycle. The fact that the same analysis applies to period n-tupling of any k-cycle into an nk-cycle will be seen to be the origin of the self-similarity of the Mandelbrot cactus.

Denote the stability of a fixed point by

$$\Lambda = \frac{d}{dz} f(z)\,. \tag{27.5}$$

Example 27.1 Border of fixed-print stability: *Fixed point condition for map (27.2), $z^2 + z - \lambda = 0$ yields fixed points. The one of interest here is*

Smooth conjugacies

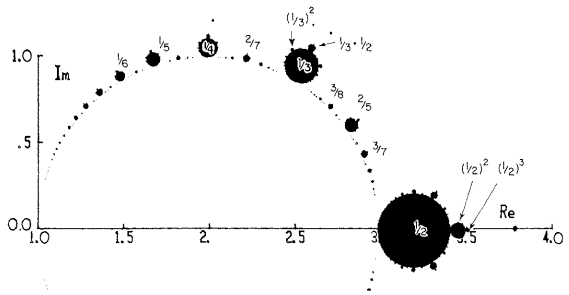


Figure 27.4: The Mandelbrot cactus for the Julia mapping (27.3). Inside the big circle (left open for clarity) iterations converge to a fixed point. The full region has two symmetry axes, $\mathrm{Re}\,\lambda = 1$ and $\mathrm{Im}\,\lambda = 0$, so only one quarter is shown. The period-doubling sequence is on the real axis. The winding numbers of the periodic orbits corresponding to larger leafs of M are indicated. See ref. [27.10] for detailed scans of the set.

with stability (27.5)

$$\Lambda = f'(z*) = -2z* = 1 - \sqrt{1+4\lambda}$$

The fixed point loses its stability on the marginal stability curve $\Lambda = e^{i\varphi}$. In the complex λ parameter plane, $\varphi = [0, 2\pi]$ traces cut a cardioid

$$\lambda = \frac{1}{4}(e^{i2\phi} - ze^{id})\,, \tag{27.6}$$

the border of the central component of the Mandelbrot set in figure 27.2.

We take, without loss of generality, the fixed point to be at $z = 0$, and $f(z)$ a power series expansion

$$f(z) = \Lambda z + \sum_{j=2}^{\infty} a_j z^j\,. \tag{27.7}$$

explain normal form

To bring the map into a normal form, we change the variable

$$w = z + \sum_{j=2}^{\infty} h_j z^j\,, \tag{27.8}$$

and "flatten" out the mapping close to the fixed point by choosing successively h_2, $h_3, \ldots$ in such a way that as many leading nonlinear terms as possible vanish in (27.7).

Example 27.2 Smooth conjugacies of a fixed point: *The idea is to perform a smooth nonlinear change of coordinates that flattens out the vicinity of a fixed point and makes the map* linear *in an open neighborhood. This can be implemented only for an isolated nondegenerate fixed point (as we shall see here, higher terms will contribute*

invariant period
doubling operator
spectrum

to the normal form expansion around the point), and only in a finite neigborhood of a point, as the conjugating function in general has a finite radius of convergence. For example, a quadratic map has two zeros, and there is no global linear map that can capture more one zero.

remark that h^- entire if f is?

explain "conjugation"

Let the fixed point of analytic function $f(z)$ be $z = 0$ and the stability of that point be $\Lambda = f'(0)$. If $|\Lambda| \neq 1$, there exists a smooth conjugation $h(x)$ satisfying $h(0) = 0$ such that:

$$f(z) = h(\Lambda h^{-1}(z))\,. \tag{27.9}$$

In several dimensions, Λ is replaced by the Jacobian matrix, and one has to check that its eigenvalues are non-resonant, that is, there is no integer linear relation between their logarithms. If $h(z)$ is a conjugation, so is any scaling $h(\alpha z)$ of the function for a case number α. Hence the value of $h'(0)$ is not determined by the functional equation (27.9); we shall set $h'(0) = 1$.

explain word "resonsnt"

To compute the conjugation h we use the functional equation $h(\Lambda z) = f(h(z))$ and the expansions

$$\begin{aligned} f(z) &= \Lambda z + z^2 f_2 + z^3 f_3 + \dots \\ h(z) &= z + z^2 h_2 + z^3 h_3 + \dots\ . \end{aligned} \tag{27.10}$$

In the present context absorbing the factorials into the definition of expansion coefficients turns out to be more convenient than the usual Taylor expansion. Equating recursively coefficients in expansions in terms of z=h(u)

$$\begin{aligned} h(\Lambda u) - \Lambda h(u) &= \sum_{m=2}^{\infty} f_m \left(h(u)\right)^m \\ \sum_{n=2}^{\infty} (\Lambda^n - \Lambda) h_n u^n &= \sum_{m=2}^{\infty} f_m u^m \left(1 + \sum_{k=2}^{\infty} h_k u^{k-1}\right)^m \end{aligned} \tag{27.11}$$

yields

$$h_2 = \frac{f_2}{\Lambda(\Lambda-1)}\,, \qquad h_3 = \frac{2f_2^2 + \Lambda(\Lambda-1)f_3}{\Lambda^2(\Lambda-1)(\Lambda^2-1)}\,, \qquad \dots \tag{27.12}$$

As long as $|\Lambda| \neq 1$, all is well. But if Λ is n-th root of unity $\Lambda = e^{2\pi}$ prefactor in the conjugation (27.11) vanishes, u^n can not be eliminated, justify the local normal for (27.14).

mention Euler formula?

We find it convenient to factorize h_n as

$$h_n = \frac{b_n}{D_n}\,, \qquad D_n = \left(1 - \frac{1}{\Lambda}\right)\left(1 - \frac{1}{\Lambda^2}\right)\cdots\left(1 - \frac{1}{\Lambda^{n-1}}\right)\Lambda^{\frac{(n+2)(n-1)}{2}}\,.$$

Computer algebra then yields

$$\begin{aligned} b_2 &= f_2 \\ b_3 &= 2f_2^2 + \Lambda\,(\Lambda-1)\,f_3 \\ b_4 &= (5+\Lambda)f_2^3 - \Lambda(5 - 2\Lambda - 3\Lambda^2)f_2 f_3 + \Lambda^2(\Lambda-1)(\Lambda^2-1)f_4\,. \end{aligned} \tag{27.13}$$

If Λ is sufficiently close to nth root of unity, $\omega = \exp(i2\pi m/n)$, and z is close to 0, the typical behavior of the new iteration function is the same as

$$f(z) = \Lambda z + z^{n+1} \,. \tag{27.14}$$

The normal function (27.14) has an n-cycle

$$z_j = \omega^j z_0 \,, \qquad z_0^n = \omega - \Lambda \,. \tag{27.15}$$

For $\Lambda = \omega$ this n-cycle coincides with the fixed point $z = 0$. In the neighborhood of $\Lambda = \omega$ we have

$$\begin{aligned} \frac{dz_n}{dz_0} &= f'(z_0)f'(z_1)...f'(z_{n-1}) = (\Lambda + (n+1)z_0^n)^n \\ &= 1 - (\Lambda - \omega)n^2/\omega + \dots \,. \end{aligned} \tag{27.16}$$

For $\Lambda = (1+\epsilon)\omega$ the n-cycle (27.15) of the mapping (27.14) is stable if

$$|1 - n^2\epsilon| < 1 \,, \tag{27.17}$$

while the fixed point is stable if

$$|1 + \epsilon| < 1 \,. \tag{27.18}$$

The mapping (27.14) is equivalent to (27.7) only for small z, so the above analysis of how a fixed point of (27.7) becomes unstable and branches into the n-cycle is valid only for infinitesimal $n\epsilon$.

In conclusion, whenever a fixed point becomes unstable at $\Lambda = n$th root of unity, it branches into an n-cycle which immediately becomes stable. As any stable cycle becomes unstable in the same fashion, branching into a new stable cycle with a multiple of the original cycle length, and as any such cycle is stable inside a disklike region in the complex parameter plane, the union of all these stability regions form a self-similar *Mandelbrot cactus* of figure 27.4.

Next we turn to a study of infinite sequences of period n-tuplings, each branching characterized by the same ratio m/n.

As discussed above, a stable n^k-cycle becomes unstable and branches into an n^{k+1}-cycle when the parameter λ passes through a value such that the stability $\Lambda_k(\lambda)$ (as defined in (27.5)) is the nth root of unity, explain f^n fix points

$$\Lambda(\lambda) = \omega = e^{i2\pi m/n} \,.$$

Renormalization
universality
Feigenbaum
$"delta$

For λ sufficiently close to this value the system is modeled by (27.14). From (27.16) it follows that near the transition from an n^k-cycle to an n^{k+1}-cycle

$$\Lambda_{k+1} = 1 - (\Lambda_k - \omega)n^2/\omega + \cdots , \tag{27.19}$$

hence

$$\left.\frac{d\Lambda_{k+1}}{d\lambda}\right|_{\Lambda_{k+1}=1} = -\frac{n^2}{\omega}\left.\frac{d\Lambda_k}{d\lambda}\right|_{\Lambda_k=\omega} , \tag{27.20}$$

and at the transition there is a scale change by the complex factor $-n^2/\omega$ which is independent of k.

27.3 Renormalization and universality

Each leaf of the Mandelbrot cactus figure 27.4 corresponds to an m/n cycle, and the parameter value for the superstable m/n cycle corresponds to the center of the leaf. The above argument suggests that the leaf is approximately n^2 times smaller than the cactus, and that it is rotated by a phase factor $-1/\omega$. The very geometry of the Mandelbrot cactus, figure 27.4, suggests such scaling. This scaling is not exact, because the above analysis applies only to the infinitesimal neighborhood of the junction of a leaf to the cactus; however, the evaluation of the exact scaling numbers shows that this is a rather good approximation to the exact scaling. Numerical evaluation of $\delta_{m/n}$'s supports the conjecture that $\delta_{m/n} \to -n^2/\omega$ as $m/n \to 0$, exactly.

The exact scaling is obtained by comparing values of the parameter λ corresponding to successive $(m/n)^k$ superstable cycles, i.e., λ values such that $\Lambda_k(\lambda_k) = 0$. As each cactus leaf is similar to the entire cactus, the ratios of the sizes of the successive stability regions corresponding to successive $(m/n)^k$-cycles tend to a limit as $k \to \infty$:

$$\delta_{m/n} = \lim_{k\to\infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_{k+1} - \lambda_k} . \tag{27.21}$$

The scaling number δ tells us by how much we have to change the parameter λ in order to cause the next m/n period n-tupling. $\delta_{1/2} = 4.669...$ is the Feigenbaum δ for the period doublings in the real one-dimensional mappings, discussed in chapter 26

Scaling in the parameter space (generalized Feigenbaum δ) is a consequence of the self-similarity of the Mandelbrot cacti. In the same way the self-similarity of the Julia sets (or the asymptotic attractors) suggests a scaling law in the iteration space z, which we discuss next. This law will characterize the scales of successive trajectory splittings (generalized Feigenbaum α).

Feigenbaum "alpha

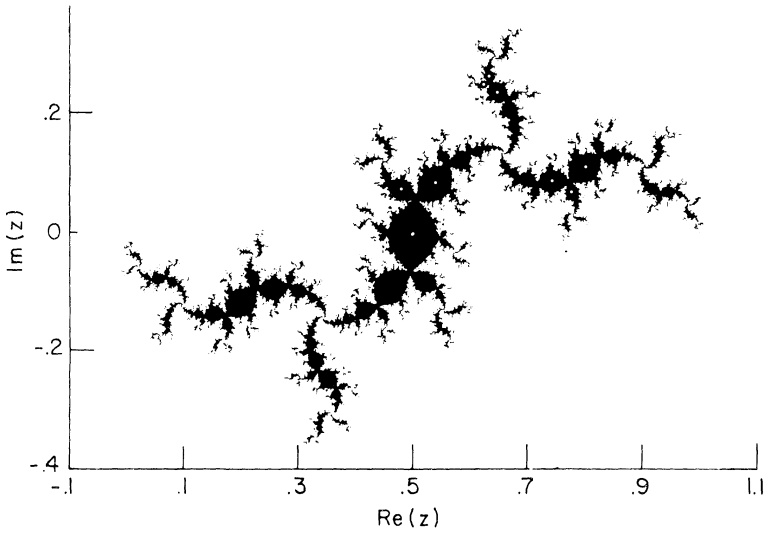


Figure 27.5: The basin of attraction for the superstable 9-cycle for iterates of the model mapping (27.3). The scaled down version of the 3-cycle basin of attraction, figure 27.3, is visible in the center.

The self-similarity we are alluding to can be seen by comparing the basin of attraction for the superstable 3-cycle, figure 27.3, and for the superstable 9-cycle, figure 27.5. In the latter figure the 3-cycle basin of attraction is visible in the center, rotated and scaled down by a factor whose asymptotic limit is the generalization of Feigenbaum α to period triplings.

This scaling number α can be computed by comparing the successive superstable cycles, at successive parameter values λ_k, λ_{k+1}. As $k \to \infty$, the sequence of λ's converges to λ_∞, and the superstable n^k-cycles converge to an n^∞-cycle. The attractor is self-similar: the orbits on succeeding levels are related by rescaling and rotation by a complex number which asymptotically approaches

27.1
page 568

$$\alpha_{m/n} = \lim_{k\to\infty} \frac{z_{n^k} - z_0}{z_{n^{k+1}} - z_0} \,. \tag{27.22}$$

α characterizes the scale of trajectory splitting at each period n-tupling. For $m/n = 1/2$ this is the Feigenbaum $\alpha = -2.5029\ldots$,

So period n-tuplings are self-similar both in the iteration space and in the parameter space: not only does the asymptotic orbit resembles itself under rescaling and rotation by α, but also each leaf of the Mandelbrot cactus resembles the entire cactus under rescaling and rotation by δ.

These self-similarities can be described by means of the following three operations:

The first operation is a rescaling of the parameter and iteration spaces:

$$[Rf]_p(z) = a f_{p/d}(z/a) \,. \tag{27.23}$$

With the appropriate choice of complex numbers d and a, a leaf of the Mandelbrot cactus (a part of the attractor) can be rescaled and rotated to approximately overlap the entire cactus (entire attractor, respectively).

We fix the origin of p and z by requiring that $z = 0$ be a critical point of the mapping $f_p(z)$, and, for the parameter value $p = 0$, a superstable fixed point as well. Mapping (27.2) is an example. We fix the scale of p and z by requiring that the superstable m/n cycle occurs for the parameter value $p = 1$ and that

$$f_1(0) = 1\,. \tag{27.24}$$

The second operation shifts the origin of the parameter space to the center of the m/n-leaf of the Mandelbrot cactus (p corresponding to the superstable m/n cycle):

$$[Sf]_p(z) = f_{1+p}(z)\,. \tag{27.25}$$

The third operation iterates $f_p(z)$ n times:

$$[Nf]_p(z) = f_p^n(z)\,. \tag{27.26}$$

By definition, $[Sf]_0(z) = f_1(z)$ has a superstable m/n cycle, so its nth iterate has a superstable fixed point, $[NSf]_0(0) = 0$.

The parameter shift S overlays the Mandelbrot cactus over its m/n leaf, and the Julia set for $[Nf]_1(z)$ resembles the Julia set for the superstable fixed point $f_0(z)$ (compare figure 27.3 with figure 27.5, for example). Finally we adjust the scale of the new M, J sets by requiring that the scale factors a, d in (27.23) are such that $[RNSf]_p(z)$ satisfies the same normalization condition (27.24) as the initial function $f_0(z)$. This shifting and rescaling is illustrated in figure 27.6.

The combined effect of the rescaling, parameter shift and iteration is summarized by the operator $T^* = RNS$

$$[T^*f]_p(z) = a f_{1+p/d}^n(z/a)\,. \tag{27.27}$$

If we take a polynomial $f_p(z)$ and act on it with T^*, the result will be a longer polynomial with similar M and J sets. For a finite number of T^* operations the scaling numbers d and a depend on the choice of the initial mapping $f_p(z)$. If we apply T^* infinitely many times, a and d converge to the universal numbers α and δ, and $T^*f_p(z)$ converges to a one-parameter family of universal functions which is the fixed line of the operator T^*:

$$g_p(z) = [T^*g]_p(z) = \alpha g_{1+p/\delta}^n(z/\alpha)\,. \tag{27.28}$$

universal
unstable manifold

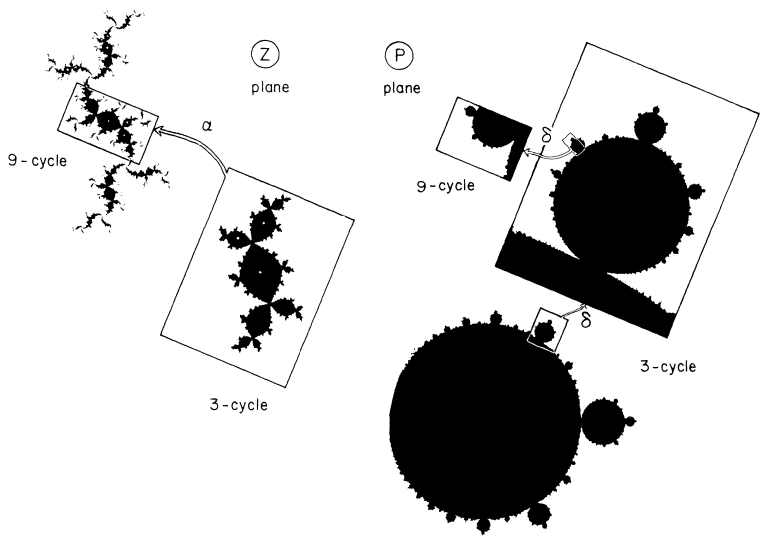


Figure 27.6: The unstable manifold method illustrated by period triplings. The parameter is shifted from the center of a cactus leaf to its 1/3 leaf, the 1/3 leaf is rescaled and rotated by δ, and the basin of attraction of third iterates is rescaled and rotated by α. The Mandelbrot cactus and the basin of attraction for the unstable manifold $g_p(z)$ is self-similar under such shifting and rescaling.

This universal equation determines both $g_p(z)$ and the universal numbers α and δ. The 1-*d* family of universal functions $g_p(z)$ parametrized by p is called the *unstable manifold*.

ain "unstable
ifold"

To summarize, the T^* operation encodes simultaneously the self-similarity of the parameter space (Mandelbrot cacti) and of the iteration space (Julia sets). Being no more than a redefinition of variables, it is an explicit implementation of the above self-similarities; T^* magnifies the nth iterate of the $(m/n)^{k+1}$-cycle and overlays it onto the $(m/n)^k$-cycle (see figure 27.6). Asymptotically the self-similarities are exact, and the procedure converges to the unstable manifold, a one-dimensional line of universal functions g_p.

Not only are the N, S, R operations the natural encoding of the complex universality, but they are also useful computational tools.

The universal equation (27.28) can be solved numerically by approximating the unstable manifold by a truncation of the double power series expansion

$$g_p(z) = \sum_{j,k\geq 0}^{N,M} c_{jk} z^{2j} p^k \,. \tag{27.29}$$

We start with (27.2) as a two term approximation to $g_p(z)$. Repeated applications of the T^* operation (27.27) generate a longer and longer double polynomial in z and p; this procedure converges asymptotically to the unstable manifold $g_p(z)$. We implement the shifting and iteration operations S and N as numerical polynomial substitution routines, truncating all

Farey numbers
Mandelbrot Set
Douady's rabbits
Mandelbrot's My Book

polynomials as in (27.29). The T^* operation is completed by the rescaling operation R, equation (27.23). The scaling numbers d and a are fixed by the normalization conditions (27.24). We use the Newton's method to find the parameter value corresponding to the superstable m/n-cycle. This determines d, and a then follows directly from the condition (27.24). The result is a new approximation to $g_p(z)$. Asymptotically d's converge to δ and a's converge to α. We keep applying the truncated T^* operation until the coefficients in (27.29) stabilize to desired accuracy.

The self-similar structure of the Mandelbrot cactus, figure 27.4, suggests a systematic way of presenting the universal numbers that we have computed in the previous section. Observe that roughly halfway between any two large leafs on the periphery of a Mandelbrot cactus (such as 1/2 and 1/3) there is the next largest leaf (such as 2/5). Furthermore, we know from (27.20) that the size of the "cactus leaf" corresponding to period n-tupling is of order n^{-2}. Hence the natural hierarchy is provided by an interpolation scheme which organizes rational numbers m/n into self-similar levels of increasing period lengths n. Such scheme is provided by Farey numbers.

Implicit in the Farey numbers are scaling laws that relate the universal numbers. It turns out that the same Farey structure is a very useful tool for the study of mode-locking intervals for circle maps. We shall discuss this at length in chapter 28.

⇓PRELIMINARY

Commentary

Remark 27.1 Complex universality. This chapter is based on a collaboration with Jan Myrheim. The universality theory for complex period n-tuplings was developed in ref. [28.27, 27.7]. As we were so rash as to submit the 2nd paper to the *Communications in Mathematical Physics*, it took 6 years before it appeared in print. The referee report, when it arrived, asked authors to change "universality" to "universal" throughout. We shall never make that error again. Unbeknownst to us, accumulation points for period-tripling and period-quadrupling bifurcation cascades were studied a bit earlier by Golberg, Sinai and Khanin [27.8].

move to appeHistory

move to appeHistory

A computer picture of what B.B. Mandelbrot calls My Set was first published by. Today the credit tends to be given to 1980 Mandelbrot's paper [27.1]. Unaware of existence of such pictures, Jan Myrhaim constructed during the summer of 1982 the "cactus" part of this set by thinking (he developed the bifurcation analysis of sect. 27.2), assisted by pencil and paper. We then plotted this on Jan's own proto-home computer, using Jan's machine-language coded graphics for a dot-printer, wholly innocent of Mandelbrot's dragons and Douady's rabbits. Very pretty early depictions of the Mandelbrot cactus, figure 27.4, are given on pp. 180-192 of Mandelbrot's My Book [27.2]. Today any mathematically inclined 12-year old can produce prettier pictures than best of IBM printers of 1980's.

find the reference

keep here, add Demidov website

REST is clipped from Isaeva paper, reedit in entirety

Siegel disks [27.31, 27.32, 27.33].

Violation of the analyticity leads to drastic changes of the dynamics [27.44, 27.34, 27.35, 27.36, 27.37].

The physical applications of complex dynamics: renormalization group in phase transitions theory of a percolation [27.38, 27.39, 27.40, 27.41].

possibility of the construction of the physical system, in which the Mandelbrot set would arise, Beck [27.42]

accumulation points for period-tripling and period-quadrupling bifurcation cascades Golberg, Sinai and Khanin [27.30].

first experimental observation of the Mandelbrot set [27.43]

The special structure of the Fourier spectrum of signal generated by experimental system at the period-tripling accumulation point is presented in [27.45].

period multiplication cascades survive in complex Hénon map [27.48]

⇑PRELIMINARY
⇓PRELIMINARY

Résumé

⇑PRELIMINARY

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COMPLETE
ref. [27.49]

⇑PRELIMINARY

Exercises

Exercise 27.1 Approximate period tripling renormalization:

Implement an approximate renormalization for period-tripling sequence of figure 27.6, using the the approximate period-doubling renormalization scheme of example 26.3, applied to the complex polynomial

$$z_{n+1} = \lambda - z_n^2 \,.$$

The idea is to drop from $f_\lambda \circ f_\lambda \circ f_\lambda(z)$ all terms higher than z^2 in each period-tripling step. Evaluate numerically the complex coordinate and parameter rescaling universal constants $\alpha_{1/3}$ and $\delta_{1/3}$.

(O.B. Isaeva and S.P. Kuznetsov)

⇑PRIVATE

Hopf!bifurcation
bifurcation!Hopf
Hopf, Ebehardt

Chapter 28

Irrationally winding

I don't care for islands, especially very small ones.
D.H. Lawrence

(R. Artuso and P. Cvitanović)

out of the main line of argument, move to an appendix?

This chapter is concerned with the mode locking problems for circle maps: besides its physical relevance it nicely illustrates the use of cycle expansions away from the dynamical setting, in the realm of renormalization theory at the transition to chaos.

The physical significance of circle maps is connected with their ability to model the two–frequencies mode–locking route to chaos for dissipative systems. In the context of *dissipative* dynamical systems one of the most common and experimentally well explored routes to chaos is the two-frequency mode-locking route. Interaction of pairs of frequencies is of deep theoretical interest due to the generality of this phenomenon; as the energy input into a dissipative dynamical system (for example, a Couette flow) is increased, typically first one and then two of intrinsic modes of the system are excited. After two Hopf bifurcations (a fixed point with inward spiralling stability has become unstable and outward spirals to a limit cycle) a system lives on a two-torus. Such systems tend to mode-lock: the system adjusts its internal frequencies slightly so that they fall in step and minimize the internal dissipation. In such case the ratio of the two frequencies is a rational number. An irrational frequency ratio corresponds to a quasiperiodic motion - a curve that never quite repeats itself. If the mode-locked states overlap, chaos sets in. The likelihood that a mode-locking occurs depends on the strength of the coupling of the two frequencies.

Our main concern in this chapter is to illustrate the "global" theory of circle maps, connected with universality properties of the whole irrational winding set. We shall see that critical global properties may be expressed via cycle expansions involving "local" renormalization critical exponents. The renormalization theory of critical circle maps demands rather tedious numerical computations, and our intuition is much facilitated by approximating circle maps by number-theoretic models. The models that arise in this way

inflection point
winding number
shift!map

are by no means mathematically trivial, they turn out to be related to number-theoretic abysses such as the Riemann conjecture, already in the context of the "trivial" models.

28.1 Mode locking

The simplest way of modeling a nonlinearly perturbed rotation on a circle is by 1-dimensional circle maps $x \to x' = f(x)$, restricted to the one dimensional torus, such as the *sine map*

$$x_{n+1} = f(x_n) = x_n + \Omega - \frac{k}{2\pi}\sin(2\pi x_n) \qquad \mod 1 \ . \tag{28.1}$$

$f(x)$ is assumed to be continuous, have a continuous first derivative, and a continuous second derivative at the inflection point (where the second derivative vanishes). For the generic, physically relevant case (the only one considered here) the inflection is cubic. Here k parametrizes the strength of the nonlinear interaction, and Ω is the *bare* frequency.

The state space of this map, the unit interval, can be thought of as the elementary cell of the map

$$\hat{x}_{n+1} = \hat{f}(\hat{x}_n) = \hat{x}_n + \Omega - \frac{k}{2\pi}\sin(2\pi\hat{x}_n) \ . \tag{28.2}$$

where $\hat{}$ is used in the same sense as in chapter 24.

The winding number is defined as

$$W(k,\Omega) = \lim_{n\to\infty} (\hat{x}_n - \hat{x}_0)/n. \tag{28.3}$$

and can be shown to be independent of the initial value $\hat{x}_0$.

For $k = 0$, the map is a simple rotation (the *shift map*) see figure 28.1 [1]

$$x_{n+1} = x_n + \Omega \qquad \mod 1 \ , \tag{28.4}$$

and the rotation number is given by the parameter Ω.

$$W(k=0,\Omega) = \Omega \ .$$

For given values of Ω and k the winding number can be either rational or irrational. For invertible maps and rational winding numbers $W = P/Q$

[1] Dahlqvist: Exciting figure

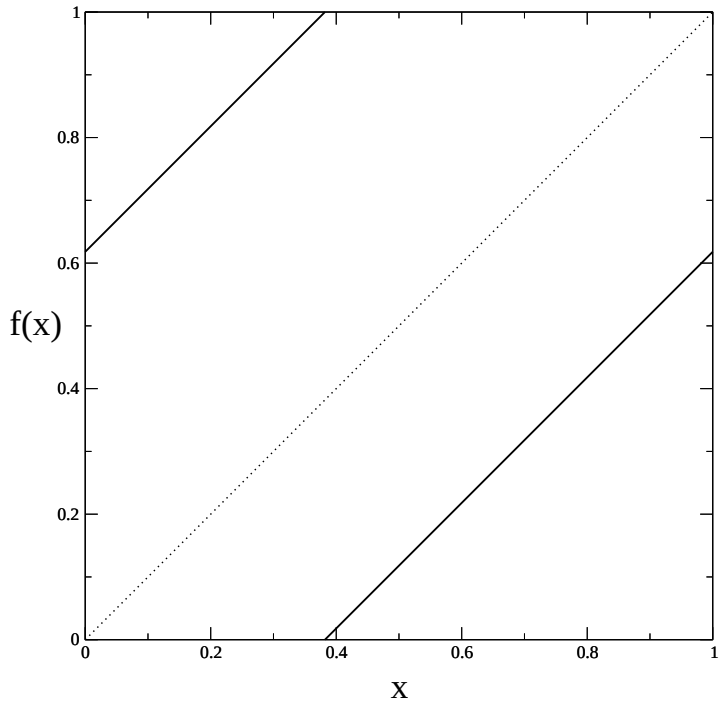

Figure 28.1: Unperturbed circle map ($k = 0$ in (28.1)) with golden mean rotation number.

the asymptotic iterates of the map converge to a unique attractor, a stable periodic orbit of period Q

$$\hat{f}^Q(\hat{x}_i) = \hat{x}_i + P, \quad i = 0, 1, 2, \cdots, Q-1 \ .$$

This is a consequence of the independence of $\hat{x}_0$ previously mentioned. There is also an unstable cycle, repelling the trajectory. For any rational winding number, there is a finite interval of values of Ω values for which the iterates of the circle map are attracted to the P/Q cycle. This interval is called the P/Q *mode-locked* (or *stability*) interval, and its width is given by

28.1
page 592
define right, left

$$\Delta_{P/Q} = Q^{-2\mu_{P/Q}} = \Omega^{right}_{P/Q} - \Omega^{left}_{P/Q} \ . \tag{28.5}$$

where $\Omega^{right}_{P/Q}$ ($\Omega^{left}_{P/Q}$) denote the biggest (smallest) value of Ω for which $W(k, \Omega) = P/Q$. Parametrizing mode lockings by the exponent μ rather than the width Δ will be convenient for description of the distribution of the mode-locking widths, as the exponents μ turn out to be of bounded variation. The stability of the P/Q cycle is

$$\Lambda_{P/Q} = \frac{\partial x_Q}{\partial x_0} = f'(x_0) f'(x_1) \cdots f'(x_{Q-1})$$

For a stable cycle $|\Lambda_{P/Q}|$ lies between 0 (the superstable value, the "center" of the stability interval) and 1 (the $\Omega^{right}_{P/Q}$, $\Omega^{left}_{P/Q}$ endpoints of (28.5)). For

winding number
circle map!critical

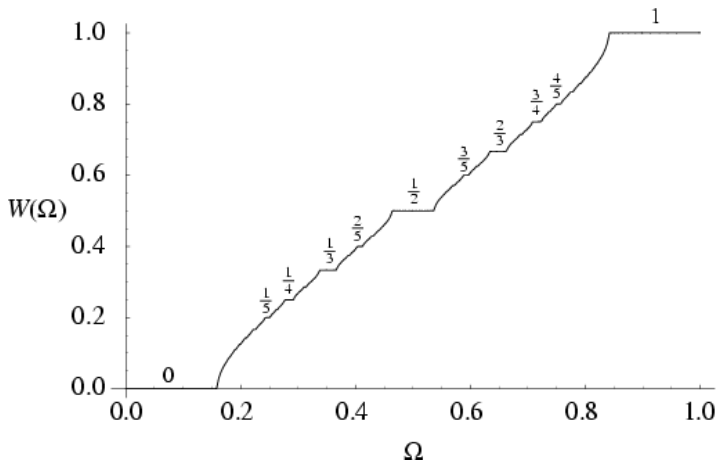

Figure 28.2: The critical circle map ($k = 1$ in (28.1)) devil's staircase [28.3]; the winding number W as function of the parameter Ω.

the shift map (28.4), the stability intervals are shrunk to points. As Ω is varied from 0 to 1, the iterates of a circle map either mode-lock, with the winding number given by a rational number $P/Q \in (0, 1)$, or do not mode-lock, in which case the winding number is irrational. A plot of the winding number W as a function of the shift parameter Ω is a convenient visualization of the mode-locking structure of circle maps. It yields a monotonic "devil's staircase" of figure 28.2 whose self-similar structure we are to unravel. Circle maps with zero slope at the inflection point x_c (see figure 28.3)

$$f'(x_c) = 0\,, \qquad f''(x_c) = 0$$

($k = 1$, $x_c = 0$ in (28.1)) are called *critical*: they delineate the borderline of chaos in this scenario.

As the nonlinearity parameter k increases, the mode-locked intervals become wider, and for the critical circle maps ($k = 1$) they fill out the whole interval. A critical map has a superstable P/Q cycle for any rational P/Q, as the stability of any cycle that includes the inflection point equals zero. If the map is non-invertible ($k > 1$), it is called supercritical; the bifurcation structure of this regime is extremely rich and beyond the scope of this exposition.

The physically relevant transition to chaos is connected with the critical case, however the apparently simple "free" shift map limit is quite instructive: in essence it involves the problem of ordering rationals embedded in the unit interval on a hierarchical structure. From a physical point of view, the main problem is to identify a (number-theoretically) consistent hierarchy susceptible of experimental verification. We will now describe a few ways of organizing rationals along the unit interval: each has its own advantages as well as its drawbacks, when analyzed from both mathematical and physical perspective.

Farey!series

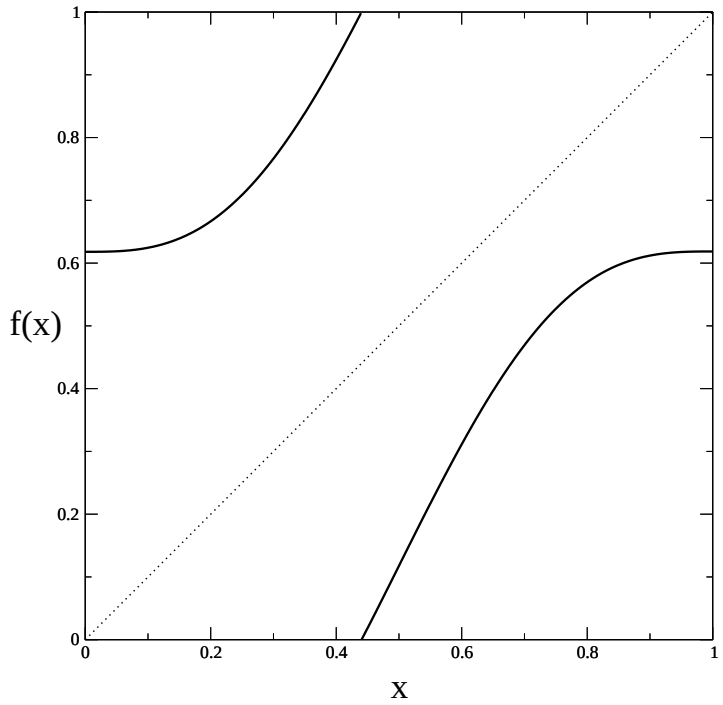

Figure 28.3: Critical circle map ($k = 1$ in (28.1)) with golden mean bare rotation number.

28.1.1 Hierarchical partitions of the rationals

Intuitively, the longer the cycle, the finer the tuning of the parameter Ω required to attain it; given finite time and resolution, we expect to be able to resolve cycles up to some maximal length Q. This is the physical motivation for partitioning mode lockings into sets of cycle length up to Q. In number theory such sets of rationals are called *Farey series*. They are denoted by $\mathcal{F}_Q$ and defined as follows. The Farey series of order Q is the monotonically increasing sequence of all irreducible rationals between 0 and 1 whose denominators do not exceed Q. Thus P_i/Q_i belongs to $\mathcal{F}_Q$ if $0 < P_i \leq Q_i \leq Q$ and $(P_i|Q_i) = 1$. For example

$$\mathcal{F}_5 = \left\{ \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}$$

A Farey series is characterized by the property that if P_{i-1}/Q_{i-1} and P_i/Q_i are consecutive terms of $\mathcal{F}_Q$, then

$$P_i Q_{i-1} - P_{i-1} Q_i = 1.$$

The number of terms in the Farey series F_Q is given by

$$\Phi(Q) = \sum_{n=1}^{Q} \phi(Q) = \frac{3Q^2}{\pi^2} + O(Q \ln Q). \tag{28.6}$$

Euler!totient function
totient function
Gauss map
Gauss map

Here the Euler function $\phi(Q)$ is the number of integers not exceeding and relatively prime to Q. For example, $\phi(1) = 1$, $\phi(2) = 1$, $\phi(3) = 2$, $\ldots$, $\phi(12) = 4, \phi(13) = 12, \ldots$

From a number-theorist's point of view, the *continued fraction partitioning* of the unit interval is the most venerable organization of rationals, preferred already by Gauss. The continued fraction partitioning is obtained by ordering rationals corresponding to continued fractions of increasing length. If we turn this ordering into a way of covering the complementary set to mode-lockings in a circle map, then the first level is obtained by deleting $\Delta_{[1]}$, $\Delta_{[2]}, \cdots, \Delta_{[a_1]}, \cdots$ mode-lockings; their complement are the *covering* intervals $\ell_1, \ell_2, \ldots, \ell_{a_1}, \ldots$ which contain all windings, rational and irrational, whose continued fraction expansion starts with $[a_1, \ldots]$ and is of length at least 2. The second level is obtained by deleting $\Delta_{[1,2]}$, $\Delta_{[1,3]}$, $\cdots, \Delta_{[2,2]}$, $\Delta_{[2,3]}$, $\cdots, \Delta_{[n,m]}, \cdots$ and so on.

The nth level continued fraction partition $\mathcal{S}_n = \{a_1 a_2 \cdots a_n\}$ is defined as the monotonically increasing sequence of all rationals P_i/Q_i between 0 and 1 whose continued fraction expansion is of length n:

$$\frac{P_i}{Q_i} = [a_1, a_2, \cdots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots \cfrac{1}{a_n}}}$$

The object of interest, the set of the irrational winding numbers, is in this partitioning labeled by $\mathcal{S}_\infty = \{a_1 a_2 a_3 \cdots\}$, $a_k \in Z^+$, i.e., the set of winding numbers with infinite continued fraction expansions. The continued fraction labeling is particularly appealing in the present context because of the close connection of the Gauss shift to the renormalization transformation R, discussed below. The Gauss map

$$\begin{aligned} T(x) &= \frac{1}{x} - \left[\frac{1}{x}\right] \quad x \neq 0 \\ &\quad 0 \ , \qquad\quad x = 0 \end{aligned} \tag{28.7}$$

($[\cdots]$ denotes the integer part) acts as a shift on the continued fraction representation of numbers on the unit interval

$$x = [a_1, a_2, a_3, \ldots] \to T(x) = [a_2, a_3, \ldots] \ . \tag{28.8}$$

into the "mother" interval $\ell_{a_2 a_3 \ldots}$.

However natural the continued fractions partitioning might seem to a number theorist, it is problematic in practice, as it requires measuring infinity of mode-lockings even at the first step of the partitioning. Thus numerical and experimental use of continued fraction partitioning requires

at least some understanding of the asymptotics of mode–lockings with large continued fraction entries.

Farey!tree
Farey!mediant

The *Farey tree partitioning* is a systematic bisection of rationals: it is based on the observation that roughly halfways between any two large stability intervals (such as 1/2 and 1/3) in the devil's staircase of figure 28.2 there is the next largest stability interval (such as 2/5). The winding number of this interval is given by the Farey mediant $(P+P')/(Q+Q')$ of the parent mode-lockings P/Q and P'/Q'. This kind of cycle "gluing" is rather general and by no means restricted to circle maps; it can be attained whenever it is possible to arrange that the Qth iterate deviation caused by shifting a parameter from the correct value for the Q-cycle is exactly compensated by the Q'th iterate deviation from closing the Q'-cycle; in this way the two near cycles can be glued together into an exact cycle of length $Q+Q'$. The Farey tree is obtained by starting with the ends of the unit interval written as 0/1 and 1/1, and then recursively bisecting intervals by means of Farey mediants.

We define the nth *Farey tree level* T_n *as the monotonically increasing sequence of those continued fractions* $[a_1, a_2, \ldots, a_k]$ *whose entries* $a_i \geq 1$, $i = 1, 2, \ldots, k-1, \quad a_k \geq 2$, *add up to* $\sum_{i=1}^{k} a_i = n+2$. For example

$$T_2 = \{[4], [2,2], [1,1,2], [1,3]\} = \left(\frac{1}{4}, \frac{1}{5}, \frac{3}{5}, \frac{3}{4}\right). \tag{28.9}$$

The number of terms in T_n is 2^n. Each rational in T_{n-1} has two "daughters" in T_n, given by

draw in the arrows

$$\begin{array}{ccc} & [\cdots, a] & \\ [\cdots, a-1, 2] & & [\cdots, a+1] \end{array}$$

Iteration of this rule places all rationals on a binary tree, labeling each by a unique binary label, figure 28.4.

The smallest and the largest denominator in T_n are respectively given by

$$[n-2] = \frac{1}{n-2}, \qquad [1,1,\ldots,1,2] = \frac{F_{n+1}}{F_{n+2}} \propto \rho^n , \tag{28.10}$$

where the Fibonacci numbers F_n are defined by $F_{n+1} = F_n + F_{n-1}$; $\quad F_0 = 0$, $F_1 = 1$, and ρ is the golden mean ratio

$$\rho = \frac{1+\sqrt{5}}{2} = 1.61803\ldots \tag{28.11}$$

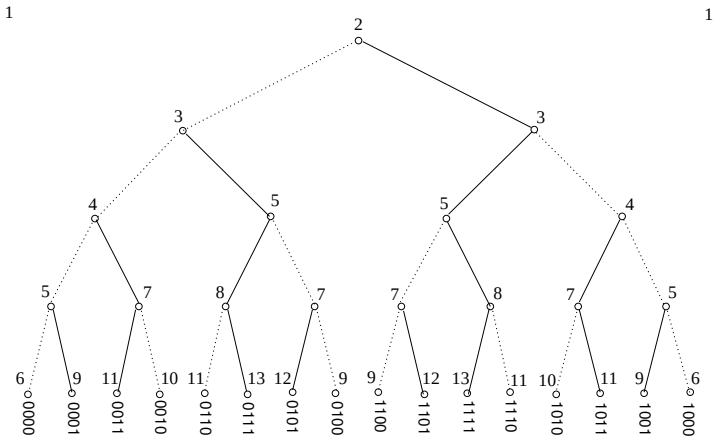

Figure 28.4: Farey tree: alternating binary ordered labeling of all Farey denominators on the nth Farey tree level.

Note the enormous spread in the cycle lengths on the same level of the Farey tree: $n \leq Q \leq \rho^n$. The cycles whose length grows only as a power of the Farey tree level will cause strong non-hyperbolic effects in the evaluation of various averages.

Having defined the partitioning schemes of interest here, we now briefly summarize the results of the circle-map renormalization theory.

28.2 Local theory: "Golden mean" renormalization

The way to pinpoint a point on the border of order is to recursively adjust the parameters so that at the recurrence times $t = n_1, n_2, n_3, \cdots$ the trajectory passes through a region of contraction sufficiently strong to compensate for the accumulated expansion of the preceding n_i steps, but not so strong as to force the trajectory into a stable attracting orbit. The *renormalization operation* R implements this procedure by recursively magnifying the neighborhood of a point on the border in the dynamical space (by rescaling by a factor α), in the parameter space (by shifting the parameter origin onto the border and rescaling by a factor δ), and by replacing the initial map f by the nth iterate f^n restricted to the magnified neighborhood

$$f_p(x) \to Rf_p(x) = \alpha f^n_{p/\delta}(x/\alpha)$$

There are by now many examples of such renormalizations in which the new function, framed in a smaller box, is a rescaling of the original function, i.e., the fix-point function of the renormalization operator R. The best known is the period doubling renormalization, with the recurrence times $n_i = 2^i$. The simplest circle map example is the golden mean renormalization, with recurrence times $n_i = F_i$ given by the Fibonacci numbers (28.10).

Intuitively, in this context a metric self-similarity arises because iterates of critical maps are themselves critical, i.e., they also have cubic inflection points with vanishing derivatives.

The renormalization operator appropriate to circle maps acts as a generalization of the Gauss shift (28.38); it maps a circle map (represented as a pair of functions (g, f), of winding number $[a, b, c, \ldots]$ into a rescaled map of winding number $[b, c, \ldots]$:

$$R_a \begin{pmatrix} g \\ f \end{pmatrix} = \begin{pmatrix} \alpha g^{a-1} \circ f \circ \alpha^{-1} \\ \alpha g^{a-1} \circ f \circ g \circ \alpha^{-1} \end{pmatrix}, \tag{28.12}$$

Acting on a map with winding number $[a, a, a, \ldots]$, R_a returns a map with the same winding number $[a, a, \ldots]$, so the fixed point of R_a has a quadratic irrational winding number $W = [a, a, a, \ldots]$. This fixed point has a single expanding eigenvalue δ_a. Similarly, the renormalization transformation $R_{a_p} \ldots R_{a_2} R_{a_1} \equiv R_{a_1 a_2 \ldots a_p}$ has a fixed point of winding number $W_p = [a_1, a_2, \ldots, a_{n_p}, a_1, a_2, \ldots]$, with a single expanding eigenvalue δ_p.

For short repeating blocks, δ can be estimated numerically by comparing successive continued fraction approximants to W. Consider the P_r/Q_r rational approximation to a quadratic irrational winding number W_p whose continued fraction expansion consists of r repeats of a block p. Let Ω_r be the parameter for which the map (28.1) has a superstable cycle of rotation number $P_r/Q_r = [p, p, \ldots, p]$. The δ_p can then be estimated by extrapolating from

$$\Omega_r - \Omega_{r+1} \propto \delta_p^{-r}. \tag{28.13}$$

What this means is that the "devil's staircase" of figure 28.2 is self-similar under magnification by factor δ_p around any quadratic irrational W_p.

The fundamental result of the renormalization theory (and the reason why all this is so interesting) is that the ratios of successive P_r/Q_r mode-locked intervals converge to *universal* limits. The simplest example of (28.13) is the sequence of Fibonacci number continued fraction approximants to the golden mean winding number $W = [1, 1, 1, ...] = (\sqrt{5} - 1)/2$.

When global problems are considered, it is useful to have at least and idea on extemal scaling laws for mode–lockings. This is achieved, in a first analysis, by fixing the cycle length Q and describing the range of possible asymptotics.

For a given cycle length Q, it is found that the *narrowest* interval shrinks with a power law

$$\Delta_{1/Q} \propto Q^{-3} \tag{28.14}$$

For fixed Q the *widest* interval is bounded by $P/Q = F_{n-1}/F_n$, the nth continued fraction approximant to the *golden mean*. The intuitive reason is that the golden mean winding sits as far as possible from any short cycle mode-locking.

The golden mean interval shrinks with a universal exponent

$$\Delta_{P/Q} \propto Q^{-2\mu_1} \tag{28.15}$$

where $P = F_{n-1}$, $Q = F_n$ and μ_1 is related to the universal Shenker number δ_1 (28.13) and the golden mean (28.11) by

$$\mu_1 = \frac{\ln|\delta_1|}{2\ln\rho} = 1.08218\ldots \tag{28.16}$$

The closeness of μ_1 to 1 indicates that the golden mean approximant mode-lockings barely feel the fact that the map is critical (in the k=0 limit this exponent is $\mu = 1$).

To summarize: for critical maps the spectrum of exponents arising from the circle maps renormalization theory is bounded from above by the harmonic scaling, and from below by the geometric golden-mean scaling:

$$3/2 > \mu_{m/n} \geq 1.08218\cdots. \tag{28.17}$$

28.3 Global theory: Thermodynamic averaging

Consider the following average over mode-locking intervals (28.5):

$$\Omega(\tau) = \sum_{Q=1}^{\infty} \sum_{(P|Q)=1} \Delta_{P/Q}^{-\tau}. \tag{28.18}$$

The sum is over all irreducible rationals P/Q, $P < Q$, and $\Delta_{P/Q}$ is the width of the parameter interval for which the iterates of a critical circle map lock onto a cycle of length Q, with winding number P/Q.

The qualitative behavior of (28.18) is easy to pin down. For sufficiently negative τ, the sum is convergent; in particular, for $\tau = -1$, $\Omega(-1) = 1$, as for the critical circle maps the mode-lockings fill the entire Ω range [28.11]. However, as τ increases, the contributions of the narrow (large Q) mode-locked intervals $\Delta_{P/Q}$ get blown up to $1/\Delta_{P/Q}^{\tau}$, and at some critical value of τ the sum diverges. This occurs for $\tau < 0$, as $\Omega(0)$ equals the number of all rationals and is clearly divergent.

The sum (28.18) is infinite, but in practice the experimental or numerical mode-locked intervals are available only for small finite Q. Hence it is necessary to split up the sum into subsets $\mathcal{S}_n = \{i\}$ of rational winding numbers P_i/Q_i on the "level" n, and present the set of mode-lockings hierarchically, with resolution increasing with the level:

$$\bar{Z}_n(\tau) = \sum_{i\in\mathcal{S}_n} \Delta_i^{-\tau}. \tag{28.19}$$

The original sum (28.18) can now be recovered as the $z = 1$ value of a "generating" function $\Omega(z,\tau) = \sum_n z^n \bar{Z}_n(\tau)$. As z is anyway a formal parameter, and n is a rather arbitrary "level" in some *ad hoc* partitioning of rational numbers, we bravely introduce a still more general, P/Q weighted generating function for (28.18):

$$\Omega(q,\tau) = \sum_{Q=1}^{\infty} \sum_{(P|Q)=1} e^{-q\nu_{P/Q}} Q^{2\tau\mu_{P/Q}} \ . \tag{28.20}$$

The sum (28.18) corresponds to $q = 0$. Exponents $\nu_{P/Q}$ will reflect the importance we assign to the P/Q mode-locking, i.e., the *measure* used in the averaging over all mode-lockings. Three choices of of the $\nu_{P/Q}$ hierarchy that we consider here correspond respectively to the Farey series partitioning

$$\Omega(q,\tau) = \sum_{Q=1}^{\infty} \Phi(Q)^{-q} \sum_{(P|Q)=1} Q^{2\tau\mu_{P/Q}} \ , \tag{28.21}$$

the continued fraction partitioning

$$\Omega(q,\tau) = \sum_{n=1}^{\infty} e^{-qn} \sum_{[a_1,\ldots,a_n]} Q^{2\tau\mu_{[a_1,\ldots,a_n]}} \ , \tag{28.22}$$

and the Farey tree partitioning

$$\Omega(q,\tau) = \sum_{k=n}^{\infty} 2^{-qn} \sum_{i=1}^{2^n} Q_i^{2\tau\mu_i} \ , \ \ Q_i/P_i \in T_n \ . \tag{28.23}$$

We remark that we are investigating a set arising in the analysis of the parameter space of a dynamical system: there is no "natural measure" dictated by dynamics, and the choice of weights reflects only the choice of hierarchical presentation.

28.4 Hausdorff dimension of irrational windings

A finite cover of the set irrational windings at the "nth level of resolution" is obtained by deleting the parameter values corresponding to the mode-lockings in the subset $\mathcal{S}_n$; left behind is the set of complement *covering* intervals of widths

$$\ell_i = \Omega^{min}_{P_r/Q_r} - \Omega^{max}_{P_l/Q_l} \ . \tag{28.24}$$

Here $\Omega^{min}_{P_r/Q_r}$ ($\Omega^{max}_{P_l/Q_l}$) are respectively the lower (upper) edges of the mode-locking intervals Δ_{P_r/Q_r} (Δ_{P_l/Q_l}) bounding ℓ_i and i is a symbolic dynamics label, for example the entries of the continued fraction representation $P/Q = [a_1, a_2, ..., a_n]$ of one of the boundary mode-lockings, $i = a_1 a_2 \cdots a_n$. ℓ_i provide a finite cover for the irrational winding set, so one may consider the sum

$$Z_n(\tau) \ = \ \sum_{i \in \mathcal{S}_n} \ell_i^{-\tau} \tag{28.25}$$

The value of $-\tau$ for which the $n \to \infty$ limit of the sum (28.25) is finite is the *Hausdorff dimension* D_H of the irrational winding set. Strictly speaking, this is the Hausdorff dimension only if the choice of covering intervals ℓ_i is optimal; otherwise it provides an upper bound to D_H. As by construction the ℓ_i intervals cover the set of irrational winding with no slack, we expect that this limit yields the Hausdorff dimension. This is supported by all numerical evidence, but a proof that would satisfy mathematicians is lacking.

The physically relevant statement is that for critical circle maps $D_H = 0.870\ldots$ is a (global) universal number.

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28.4.1 The Hausdorff dimension in terms of cycles

Estimating the $n \to \infty$ limit of (28.25) from finite numbers of covering intervals ℓ_i is a rather unilluminating chore. Fortunately, there exist considerably more elegant ways of extracting D_H. We have noted that in the case of the "trivial" mode-locking problem (28.4), the covering intervals are generated by iterations of the Farey map (28.37) or the Gauss shift (28.38). The nth level sum (28.25) can be approximated by $\mathcal{L}^n_\tau$, where

$$\mathcal{L}_\tau(y,x) = \delta(x - f^{-1}(y))|f'(y)|^\tau$$

This amounts to approximating each cover width ℓ_i by $|df^n/dx|$ evaluated on the ith interval. We are thus led to the following determinant

$$\det(1 - z\mathcal{L}_\tau) \ = \ \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{z^{rn_p}}{r} \frac{|\Lambda_p^r|^\tau}{1 - 1/\Lambda_p^r}\right)$$

$$= \prod_p \prod_{k=0}^{\infty} \left(1 - z^{n_p} |\Lambda_p|^\tau / \Lambda_p^k\right) \ . \tag{28.26}$$

Gauss map!cycle stability
cycle!stability!Gauss map

The sum (28.25) is dominated by the leading eigenvalue of $\mathcal{L}_\tau$; the Hausdorff dimension condition $Z_n(-D_H) = O(1)$ means that $\tau = -D_H$ should be such that the leading eigenvalue is $z = 1$. The leading eigenvalue is determined by the $k = 0$ part of (28.26); putting all these pieces together, we obtain a pretty formula relating the Hausdorff dimension to the prime cycles of the map $f(x)$:

$$0 = \prod_p \left(1 - 1/|\Lambda_p|^{D_H}\right) \ . \tag{28.27}$$

For the Gauss shift (28.38) the stabilities of periodic cycles are available analytically, as roots of quadratic equations: For example, the x_a fixed points (quadratic irrationals with $x_a = [a, a, a \ldots]$ infinitely repeating continued fraction expansion) are given by

$$x_a = \frac{-a + \sqrt{a^2+4}}{2}, \qquad \Lambda_a = -\left(\frac{a + \sqrt{a^2+4}}{2}\right)^2 \tag{28.28}$$

and the $x_{ab} = [a, b, a, b, a, b, \ldots]$ 2–cycles are given by

$$\begin{aligned} x_{ab} &= \frac{-ab + \sqrt{(ab)^2 + 4ab}}{2b} \\ \Lambda_{ab} &= (x_{ab} x_{ba})^{-2} = \left(\frac{ab + 2 + \sqrt{ab(ab+4)}}{2}\right)^2 \end{aligned} \tag{28.29}$$

We happen to know beforehand that $D_H = 1$ (the irrationals take the full measure on the unit interval, or, from another point of view the Gauss map is not a repeller), so is the infinite product (28.27) merely a very convoluted way to compute the number 1? Possibly so, but once the meaning of (28.27) has been grasped, the corresponding formula for the *critical* circle maps follows immediately:

$$0 = \prod_p \left(1 - 1/|\delta_p|^{D_H}\right) \ . \tag{28.30}$$

The importance of this formula relies on the fact that it expresses D_H in terms of *universal* quantities, thus providing a nice connection from local universal exponents to global scaling quantities: actual computations using (28.30) are rather involved, as they require a heavy computational effort to extract Shenker's scaling δ_p for periodic continued fractions, and moreover dealing with an infinite alphabet requires control over tail summation if an accurate estimate is to be sought. In table 28.1 we give a small selection of computed Shenker's scalings.

p	δ_p
[1 1 1 1 ...]	-2.833612
[2 2 2 2 ...]	-6.7992410
[3 3 3 3 ...]	-13.760499
[4 4 4 4 ...]	-24.62160
[5 5 5 5 ...]	-40.38625
[6 6 6 6 ...]	-62.140
[1 2 1 2 ...]	17.66549
[1 3 1 3 ...]	31.62973
[1 4 1 4 ...]	50.80988
[1 5 1 5 ...]	76.01299
[2 3 2 3 ...]	91.29055

Table 28.1: Shenker's δ_p for a few periodic continued fractions, from ref. [28.1].

28.5 Thermodynamics of Farey tree: Farey model

We end this chapter by giving an example of a number theoretical model motivated by the mode-locking phenomenology. We will consider it by means of the thermodynamic formalism of chapter 20, by looking at the free energy.

Consider the Farey tree partition sum (28.23): the narrowest mode-locked interval (28.15) at the nth level of the Farey tree partition sum (28.23) is the golden mean interval

$$\Delta_{F_{n-1}/F_n} \propto |\delta_1|^{-n}. \tag{28.31}$$

It shrinks exponentially, and for τ positive and large it dominates $q(\tau)$ and bounds $dq(\tau)/d\tau$:

$$q'_{max} = \frac{\ln|\delta_1|}{\ln 2} = 1.502642\ldots \tag{28.32}$$

However, for τ large and negative, $q(\tau)$ is dominated by the interval (28.14) which shrinks only harmonically, and $q(\tau)$ approaches 0 as

$$\frac{q(\tau)}{\tau} = \frac{3\ln n}{n\ln 2} \to 0. \tag{28.33}$$

So for finite n, $q_n(\tau)$ crosses the τ axis at $-\tau = D_n$, but in the $n \to \infty$ limit, the $q(\tau)$ function exhibits a phase transition; $q(\tau) = 0$ for $\tau < -D_H$, but is a non-trivial function of τ for $-D_H \leq \tau$. This non-analyticity is rather severe - to get a clearer picture, we illustrate it by a few number-theoretic models (the critical circle maps case is qualitatively the same).

An approximation to the "trivial" Farey level thermodynamics is given by the "Farey model", in which the intervals $\ell_{P/Q}$ are replaced by Q^{-2}:

$$Z_n(\tau) = \sum_{i=1}^{2^n} Q_i^{2\tau}. \tag{28.34}$$

Here Q_i is the denominator of the ith Farey rational P_i/Q_i. For example (see figure 28.4),

$$Z_2(1/2) \;=\; 4 \,+\, 5 \,+\, 5 \,+\, 4.$$

By the annihilation property (28.38) of the Gauss shift on rationals, the nth Farey level sum $Z_n(-1)$ can be written as the integral

$$Z_n(-1) = \int dx \delta(f^n(x)) = \sum 1/|f'_{a_1 \ldots a_k}(0)| \,,$$

and in general

$$Z_n(\tau) = \int dx \mathcal{L}^n_\tau(0,x) \,,$$

with the sum restricted to the Farey level $a_1 + \ldots + a_k = n+2$. It is easily checked that $f'_{a_1 \ldots a_k}(0) = (-1)^k Q^2_{[a_1,\ldots,a_k]}$, so the Farey model sum is a partition generated by the Gauss map preimages of $x = 0$, i.e., by rationals, rather than by the quadratic irrationals as in (28.26). The sums are generated by the same transfer operator, so the eigenvalue spectrum should be the same as for the periodic orbit expansion, but in this variant of the finite level sums we can can evaluate $q(\tau)$ *exactly* for $\tau = k/2$, k a nonnegative integer. First one observes that $Z_n(0) = 2^n$. It is also easy to check that $Z_n(1/2) = \sum_i Q_i = 2 \cdot 3^n$. More surprisingly, $Z_n(3/2) = \sum_i Q^3 = 54 \cdot 7^{n-1}$. A few of these "sum rules" are listed in the table 28.2, they are consequence of the fact that the denominators on a given level are Farey sums of denominators on preceding levels.

add the plot of $q(\tau)$

28.3
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A bound on D_H can be obtained by approximating (28.34) by

$$Z_n(\tau) \;=\; n^{2\tau} \,+\, 2^n \rho^{2n\tau}. \qquad (28.35)$$

In this approximation we have replaced all $\ell_{P/Q}$, except the widest interval $\ell_{1/n}$, by the narrowest interval ℓ_{F_{n-1}/F_n} (see (28.15)). The crossover from the harmonic dominated to the golden mean dominated behavior occurs at the τ value for which the two terms in (28.35) contribute equally:

$$D_n \;=\; \hat{D} \,+\, O\left(\frac{\ln n}{n}\right), \quad \hat{D} \;=\frac{\ln 2}{2 \ln \rho} = .72\ldots \qquad (28.36)$$

For negative τ the sum (28.35) is the lower bound on the sum (28.25) , so $\hat{D}$ is a lower bound on D_H.

From a general perspective the analysis of circle maps thermodynamics has revealed the fact that physically interesting dynamical systems often

$\tau/2$	$Z_n(\tau/2)/Z_{n-1}(\tau/2)$
0	2
1	3
2	$(5+\sqrt{17})/2$
3	7
4	$(5+\sqrt{17})/2$
5	$7+4\sqrt{6}$
6	26.20249...

Table 28.2: Partition function sum rules for the Farey model.

exhibit mixtures of hyperbolic and marginal stabilities. In such systems there are orbits that stay 'glued' arbitrarily close to stable regions for arbitrarily long times. This is a generic phenomenon for Hamiltonian systems, where elliptic islands of stability coexist with hyperbolic homoclinic webs. Thus the considerations of chapter 21 are important also in the analysis of renomarmalization at the onset of chaos.

Commentary

Remark 28.1 The physics of circle maps. Mode–locking phenomenology is reviewed in ref. [28.5], a more theoretically oriented discussion is contained in ref. [28.3]. While representative of dissipative systems we may also consider circle mapsas a crude approximation to Hamiltonian local dynamics: a typical island of stability in a Hamiltonian 2-d map is an infinite sequence of concentric KAM tori and chaotic regions. In the crudest approximation, the radius can here be treated as an external parameter Ω, and the angular motion can be modelled by a map periodic in the angular variable [28.8, 28.9]. By losing all of the "island-within-island" structure of real systems, circle map models skirt the problems of determining the symbolic dynamics for a realistic Hamiltonian system, but they do retain some of the essential features of such systems, such as the golden mean renormalization [17.4, 28.8] and non-hyperbolicity in form of sequences of cycles accumulating toward the borders of stability. In particular, in such systems there are orbits that stay "glued" arbitrarily close to stable regions for arbitrarily long times. As this is a generic phenomenon in physically interesting dynamical systems, such as the Hamiltonian systems with coexisting elliptic islands of stability and hyperbolic homoclinic webs, development of good computational techniques is here of utmost practical importance.

Remark 28.2 Critical mode–locking set The fact that mode-lockings completely fill the unit interval at the critical point has been proposed in refs. [**?**, 28.10]. The proof that the set of irrational windings is of zero Lebesgue measure in given in ref. [28.11].

Remark 28.3 Counting noise for Farey series. The number of rationals in the Farey series of order Q is $\phi(Q)$, which is a highly irregular function of Q: incrementing Q by 1 increases $\Phi(Q)$ by anything from 2 to Q terms. We refer to this fact as the "Euler noise".

The Euler noise poses a serious obstacle for numerical calculations with the Farey series partitionings; it blocks smooth extrapolations to $Q \to \infty$ limits from

finite Q data. While this in practice renders inaccurate most Farey-sequence partitioned averages, the finite Q Hausdorff dimension estimates exhibit (for reasons that we do not understand) surprising numerical stability, and the Farey series partitioning actually yields the *best* numerical value of the Hausdorff dimension (28.25) of any methods used so far; for example the computation in ref. [28.12] for critical sine map (28.1), based on $240 \leq Q \leq 250$ Farey series partitions, yields $D_H = .87012 \pm .00001$. The quoted error refers to the variation of D_H over this range of Q; as the computation is not asymptotic, such numerical stability can underestimate the actual error by a large factor. Gauss map

Remark 28.4 Farey tree presentation function. The Farey tree rationals can be generated by backward iterates of 1/2 by the Farey presentation function [28.13]:

$$\begin{array}{rcll} f_0(x) & = & x/(1-x) & 0 \leq x < 1/2 \\ f_1(x) & = & (1-x)/x & 1/2 < x \leq 1 \ . \end{array} \tag{28.37}$$

The Gauss shift (28.7) corresponds to replacing the binary Farey presentation function branch f_0 in (28.37) by an infinity of branches

$$\begin{array}{rcll} f_a(x) & = & f_1 \circ f_0^{(a-1)}(x) = \dfrac{1}{x} - a, & \dfrac{1}{a-1} < x \leq \dfrac{1}{a}, \\ f_{ab\cdots c}(x) & = & f_c \circ \cdot \circ f_b \circ f_a(x) \ . & \end{array} \tag{28.38}$$

A rational $x = [a_1, a_2, \ldots, a_k]$ is annihilated by the kth iterate of the Gauss shift, $f_{a_1 a_2 \cdots a_k}(x) = 0$. The above maps look innocent enough, but note that what is being partitioned is not the dynamical space, but the parameter space. The flow described by (28.37) and by its non-trivial circle-map generalizations will turn out to be a *renormalization group* flow in the function space of dynamical systems, not an ordinary flow in the state space of a particular dynamical system.

The Farey tree has a variety of interesting symmetries (such as "flipping heads and tails" relations obtained by reversing the order of the continued-fraction entries) with as yet unexploited implications for the renormalization theory: some of these are discussed in ref. [28.4].

An alternative labeling of Farey denominators has been introduced by Knauf [28.6] in context of number-theoretical modeling of ferromagnetic spin chains: it allows for a number of elegant manipulations in thermodynamic averages connected to the Farey tree hierarchy.

Remark 28.5 Circle map renormalization The idea underlying golden mean renormalization goes back to Shenker [28.9]. A renormalization group procedure was formulated in refs. [28.7, 28.14], where moreover the uniqueness of the relevant eigenvalue is claimed. This statement has been confirmed by a computer–assisted proof [28.15], and in the following we will always assume it. There are a number of experimental evidences for local universality, see refs. [28.16, 28.17].

On the other side of the scaling tale, the power law scaling for harmonic fractions (discussed in refs. [28.2, **?**, 28.4]) is derived by methods akin to those used in describing intermittency [28.21]: $1/Q$ cycles accumulate toward the edge of 0/1 mode-locked interval, and as the successive mode-locked intervals $1/Q$, $1/(Q-1)$ lie on a parabola, their differences are of order Q^{-3}.

Remark 28.6 Farey series and the Riemann hypothesis The Farey series thermodynamics is of a number theoretical interest, because the Farey series provide uniform coverings of the unit interval with rationals, and because they are closely related to the deepest problems in number theory, such as the Riemann hypothesis [28.22, 28.23] . The distribution of the Farey series rationals across the unit interval is surprisingly uniform - indeed, so uniform that in the pre-computer days it has motivated a compilation of an entire handbook of Farey series [28.24]. A quantitive measure of the non-uniformity of the distribution of Farey rationals is given by displacements of Farey rationals for $P_i/Q_i \in \mathcal{F}_Q$ from uniform spacing:

$$\delta_i = \frac{i}{\Phi(Q)} - \frac{P_i}{Q_i}, \quad i = 1, 2, \cdots, \Phi(Q)$$

The Riemann hypothesis states that the zeros of the Riemann zeta function lie on the $s = 1/2 + i\tau$ line in the complex s plane, and would seem to have nothing to do with physicists' real mode-locking widths that we are interested in here. However, there is a real-line version of the Riemann hypothesis that lies very close to the mode-locking problem. According to the theorem of Franel and Landau [28.25, 28.22, 28.23], the Riemann hypothesis is equivalent to the statement that

$$\sum_{Q_i \le Q} |\delta_i| = o(Q^{\frac{1}{2}+\epsilon})$$

for all ϵ as $Q \to \infty$. The mode-lockings $\Delta_{P/Q}$ contain the necessary information for constructing the partition of the unit interval into the ℓ_i covers, and therefore implicitly contain the δ_i information. The implications of this for the circle-map scaling theory have not been worked out, and is not known whether some conjecture about the thermodynamics of irrational windings is equivalent to (or harder than) the Riemann hypothesis, but the danger lurks.

use text from ref. [28.26] here

Remark 28.7 Farey tree partitioning. The Farey tree partitioning was introduced in refs. [28.26, 28.27, 28.4] and its thermodynamics is discussed in detail in refs. [28.12, 28.13]. The Farey tree hierarchy of rationals is rather new, and, as far as we are aware, not previously studied by number theorists. It is appealing both from the experimental and from the the golden-mean renormalization point of view, but it has a serious drawback of lumping together mode-locking intervals of wildly different sizes on the same level of the Farey tree.

Remark 28.8 Local and global universality. Numerical evidences for global universal behavior have been presented in ref. [28.3]. The question was reexamined in ref. [28.12], where it was pointed out how a high-precision numerical estimate is in practice very hard to obtain. It is not at all clear whether this is the optimal global quantity to test but at least the Hausdorff dimension has the virtue of being independent of how one partitions mode-lockings and should thus be the same for the variety of thermodynamic averages in the literature.

The formula (28.30), linking local to global behavior, was proposed in ref. [28.1].

The derivation of (28.30) relies only on the following aspects of the "hyperbolicity conjecture" of refs. [28.4, 28.18, 28.19, 28.20]:

1. *limits* for Shenker δ's *exist* and are universal. This should follow from the renormalization theory developed in refs. [28.7, 28.14, 28.15], though a general proof is still lacking.
2. δ_p grow *exponentially* with n_p, the length of the continued fraction block p.
3. δ_p for $p = a_1 a_2 \dots n$ with a large continued fraction entry n grows as a *power* of n. According to (28.14), $\lim_{n\to\infty} \delta_p \propto n^3$. In the calculation of ref. [28.1] the explicit values of the asymptotic exponents and prefactors were not used, only the assumption that the growth of δ_p with n is not slower than a power of n.

find Friedland, U of Illinois, Chicago, who has studied Farey trees and Riemann hypothesis

Remark 28.9 Farey model. The Farey model (28.33) has been proposed in ref. [28.12]; though it might seem to have been pulled out of a hat, the Farey model is as sensible description of the distribution of rationals as the periodic orbit expansion (28.26).

Remark 28.10 Symbolic dynamics for Hamiltonian rotational orbits. The rotational codes of ref. [5.4] are closely related to those for maps with a natural angle variable, for example for circle maps [28.34, 28.36] and cat maps [28.37]. Ref. [5.4] also offers a systematic rule for obtaining the symbolic codes of "islands around islands" rotational orbits [28.39]. These correspond, for example, to orbits that rotate around orbits that rotate around the elliptic fixed point; thus they are defined by a sequence of rotation numbers.

extracted verbatim from [5.4]

A different method for constructing symbolic codes for "islands around islands" was given in refs. [28.42, 28.40]; however in these cases the entire set of orbits in an island was assigned the same sequence and the motivation was to study the transport implications for chaotic orbits outside the islands [28.39, 28.41].

Résumé

The mode locking problem, and the quasiperiodic transition to chaos offer an opportunity to use cycle expansions on hierarchical structures in parameter space: this is not just an application of the conventional thermodynamic formalism, but offers a clue on how to extend universality theory from local scalings to global quantities.

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Gauss map!metric entropy
metric entropy!Gauss map
entropy!Gauss map

Exercises

Exercise 28.1 Mode-locked intervals. *Check that when $k \neq 0$ the interval $\Delta_{P/Q}$ have a non-zero width (look for instance at simple fractions, and consider k small). Show that for small k the width of $\Delta_{0/1}$ is an increasing function of k.*

Exercise 28.2 Bounds on Hausdorff dimension. *By making use of the bounds (28.17) show that the Hausdorff dimension for critical mode lockings may be bounded by*

$$2/3 \leq D_H \leq .9240\ldots$$

Exercise 28.3 Farey model sum rules. *Verify the sum rules reported in table 28.2. An elegant way to get a number of sum rules for the Farey model is by taking into account an lexical ordering introduced by Contucci and Knauf, see ref. [28.28].*

Exercise 28.4 Metric entropy of the Gauss shift. *Check that the Lyapunov exponent of the Gauss map (28.7) is given by $\pi^2/6\ln 2$. This result has been claimed to be relevant in the discussion of "mixmaster" cosmologies, see ref. [28.30].*

Exercise 28.5 Refined expansions. *Show that the above estimates can be refined as follows:*

$$F(z,2) \sim \zeta(2) + (1-z)\log(1-z) - (1-z)$$

and

$$F(z,s) \sim \zeta(s) + \Gamma(1-s)(1-z)^{s-1} - S(s)(1-z)$$

for $s \in (1,2)$ ($S(s)$ being expressed by a converging sum). You may use either more detailed estimate for $\zeta(s,a)$ (via Euler summation formula) or keep on subtracting leading contributions [28.31].

Exercise 28.6 Hitting condition. *Prove (24.40). Hint: together with the real trajectory consider the line passing through the starting point, with polar angle $\theta_{m,n}$: then draw the perpendiculars to the actual trajectory, passing through the center of the $(0,0)$ and (m,n) disks.*

Exercise 28.7 j_n and α_{cr}. *Look at the integration region and how it scales by plotting it for increasing values of n.*

Exercise 28.8 Estimates of the Riemann zeta function. *Try to approximate numerically the Riemann zeta function for $s = 2,4,6$ using different acceleration algorithms: check your results with refs. [28.32, 28.33].*

Exercise 28.9 Farey tree and continued fractions I. *Consider the Farey tree presentation function $f : [0,1] \mapsto [0,1]$, such that if $I = [0,1/2)$ and $J = [1/2,1]$, $f|_I = x/(1-x)$ and $f|_J = (1-x)/x$. Show that the corresponding induced map is the Gauss map $g(x) = 1/x - [1/x]$.*

Exercise 28.10 Farey tree and continued fraction II. (Lethal weapon II). *Build the simplest piecewise linear approximation to the Farey tree presentation function (hint: substitute first the righmost, hyperbolic branch with a linear one): consider then the spectral determinant of the induced map $\hat{g}$, and calculate the first two eigenvalues besides the probability conservation one. Compare the results with the rigorous bound deduced in ref. [21.17].*

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Chapter 29

Periodic orbit sum rules

(P. Cvitanović, Kim Hansen and G. Vattay)

A family of exact sum rules for stabilities of periodic points of d-dimensional polynomial mappings is derived. Associated spectral determinants are of particularly simple polynomial form. The sum rules are applied to the 3-disk repeller and the Hénon map periodic orbits, and other possible applications to periodic orbit calculations for continuous time flows are discussed.

this chapter to be still written, check the "Beyond periodic orbit theory" paper on my the homepage

Chapter 30

Limbo

30.1 Phase transitions

RM: When stability goes to one the scaling function falls off slower than $1/n^2$ and there is a phase transition.

The "thermodynamic" functions extracted from the partition sum (?!) exhibit a phenomenon that might have gone unnoticed were it not for the thermodynamic formalism. They can undergo "phase transitions". These phase transitions can be visualized in the following way: the exponent τ acts as a "magnifying lens" which blows up some of the covering intervals, and (relatively) shrinks the others, figure C.2. For negative τ, the fat intervals are expanded, and the thin ones are made even thinner. For positive τ, the thin intervals are (relatively) blown up. Below the phase transition, the sum is dominated by a few fat intervals. Visually, the cover consists of a few black blocks and vanishing amount of fine "dust". Above the phase transition, the large number of thin intervals overwhelms the few fat ones; visually, the set looks gray. The phase transition can be of first order, or infinite order.

30.2 Scaling, presentation functions

RM: How to interpret the dynamics in terms of statistical mechanics by thinking of the scaling function as the potential.

Commentary

Résumé

remember to incorporate book/chapter/scaling.tex

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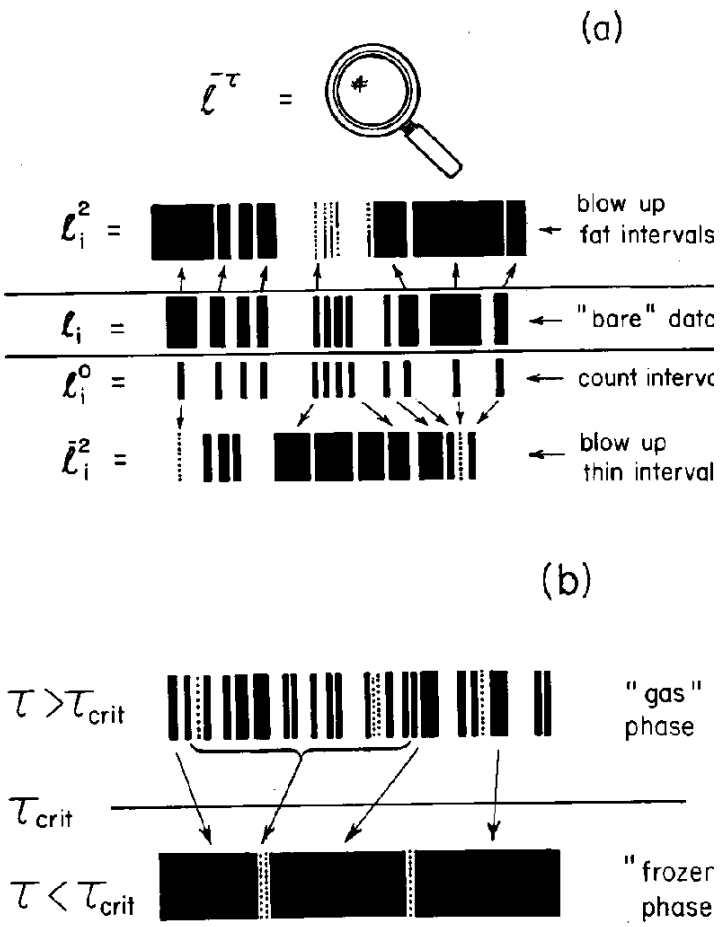

Figure 30.1: A phase transition visualized.