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FIELD THEORY

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(lecture notes prepared by Ejnar Gyldenkerne)

NORDITA LECTURE NOTES
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TO MY GIRLS

CONTENTS:

PREFACE

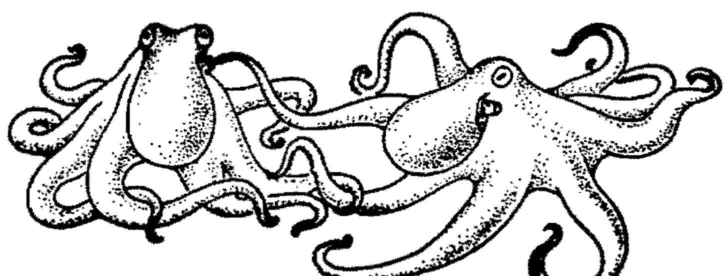
In the fall of 1979, Benny Lautrup and I set out to write the ultimate Quantum Chromodynamics review. The report was going to consist of four parts, one for each line of

From Ghoulies and Ghosties
and Long-leggety Beasties
and Things that go bump in the Night
Good Lord, deliver us!

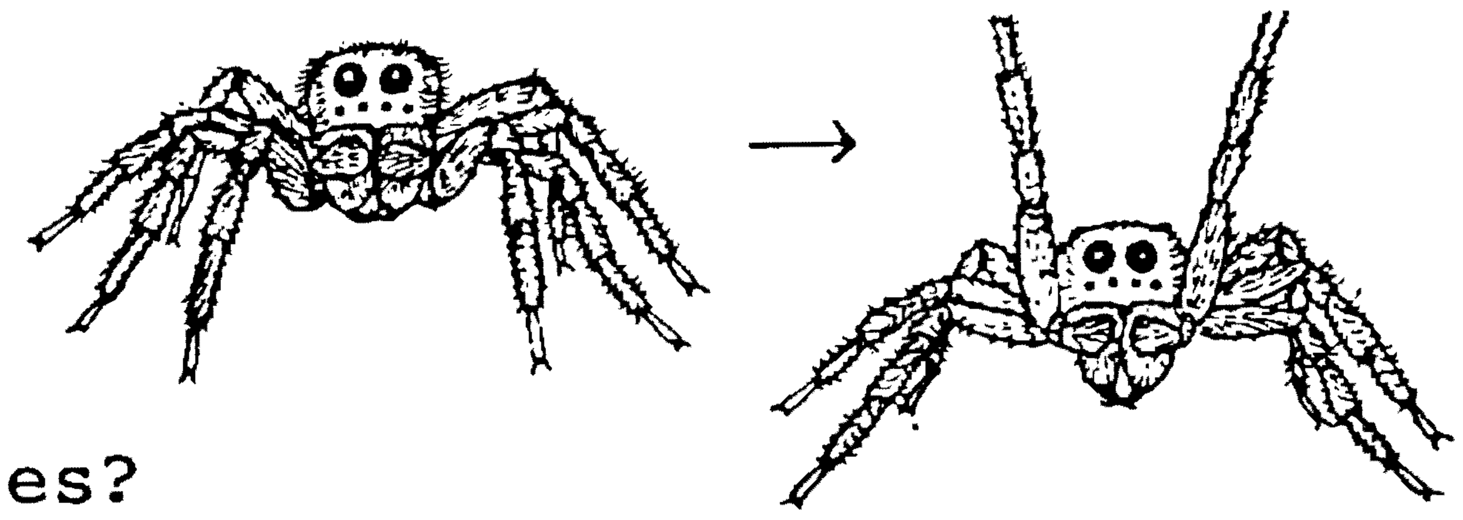
Ghoulies are body-snatchers and grave robbers; they are those revel in that which is revolting.

Benny had previously described ghosties in a very nice set of QCD lectures† which we were going to use as the first part. Green functions resemble long-leggety beasties; they, and the general formalism of field theory, were to be developed in the second part. The things that go bump in the night are clearly the many unpleasant surprises of field theory; divergences, together with the regularization and computation techniques, were to be covered in the third part. Finally, good Lord deliver us, we were going to actually calculate a few basic QCD integrals.

Well, while I was lecturing about the long-leggety beasties, Benny deserted me for lattice, and the ultimate QCD review was never written. That these lectures appear at all is largely due to tireless work by Ejnar Gyldenkerne and to the criticisms of the QCD study group at the Niels Bohr Institute. In writing these lectures I have profited much from discussions with Benny Lautrup, to whom I direct my thanks.



† B. Lautrup, "Of ghoulies and ghosties - an introduction to QCD", Basko Polje 1976 lectures, available as Niels Bohr Institute preprint NBI-HE-76-14.



1. INTRODUCTION

What are long-leggety beasties?

Long-leggety beasties are to be seen in any field theory or statistical mechanics textbook; they are Feynman diagrams, Green functions, S-matrix elements, correlation functions, and so on. They represent sums of probabilities (statistical mechanics) or probability amplitudes (quantum mechanics).

There are two ways of visualizing long-leggety beasties†.

In the first picture the transition probabilty (amplitude) is the sum of all ways in which particles can propagate, disintegrate and recombine before reaching a detector. Each possibility is represented by a Feynman diagram, and the penalty associated with each choice is given by a Feynman integral.

In the second picture the transition probability (amplitude) is a sum over all "paths" which the system can take between the initial and the final state. The penalty to be paid for a particular path is assessed by a Boltzmann factor (phase factor). A process is dominated by the classical paths, and the fluctuation (quantum) effects arise from the heavily penalized deviations away from the beaten path.

The two pictures are equivalent. The second (path integrals) is a "Fourier" transform of the first (generating functionals). In some contexts, such as in perturbative calculations, generating functionals are the practical choice. In others, such as in identifying the dominant classical configurations, or in exploiting symmetries of a theory, the path integral formulation might be more suggestive.

In these notes we put the usual logic of field theory textbooks on its head; we start with the Feynman rules and end with Lagrangians. We find it easier to understand field theory this way: for many particle physicists, diagrams are an important tool for developing field-theoretic intuition.

Our attitude will be eclectic. We shall start by building up generating functionals using vertices and propagators as

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† R. Herrick has in his poem "On Julia's Legs" suggested a third way: "Fain would I kiss my Julia's dainty leg, which is as white and hairless as an egg".

simple building blocks. Then we shall rewrite the results in terms of path integrals, and from then on use either formalism, whichever may be more expedient. Each particular physical theory brings in its own set of ailments (ultraviolet divergences, ill-defined path integrals, etc.), but the general formalism should be good enough to describe anything under the sun, from statistical mechanics to lattice gauge theories to continuum theories to gravity and cosmology. The general formalism is straightforward and intuitive. The real work starts only with specialization to a particular theory; the dominant classical configurations have to be identified, divergent sums (integrals) regularized, etc.

We will apply the general formalism to QCD. Chapter 6 is a rehash of Benny Lautrup's "Ghoulies and Ghosties". This construction yields QCD Feynman rules and bare Ward identities. In chapter 7 we feed these into the general formalism to obtain the Ward identities for full Green functions. At this point our patience runs out, and the proof of renormalizability of QCD and the evaluation of the running coupling constants, scaling violations and hadron masses are left as exercises for the reader.

I have included much graphic gore in these notes. The reason is that I fear that the perturbation theory is here to stay; it will not go away even if the gauge theories do. At least, if I ever have to do a perturbative calculation again, I will know where to look up the diagrams. The reader is advised to skip over lengthy perturbative expansions - most particle physicists reach tenure without doing anything more strenuous than one-loop Feynman integrals. The exercises are another matter - we have relegated much of the conceptually dull but technically important material to the exercises. They are of three kinds: trivial, undoable, and wrong.

There is nothing in these lectures that is not well-known and has not been published many other places. The only excuse for writing them up is that they seem to resemble no other field theory text on the market. It cannot be precluded that that might be considered a virtue.

A. Land of Quefithe

Once (and it was not yesterday) there lived a very young mole and a very young crow who, having heard of the fabulous land called Quefithe, decided to visit it. Before starting out, they went to the wise owl and asked what Quefithe was like.

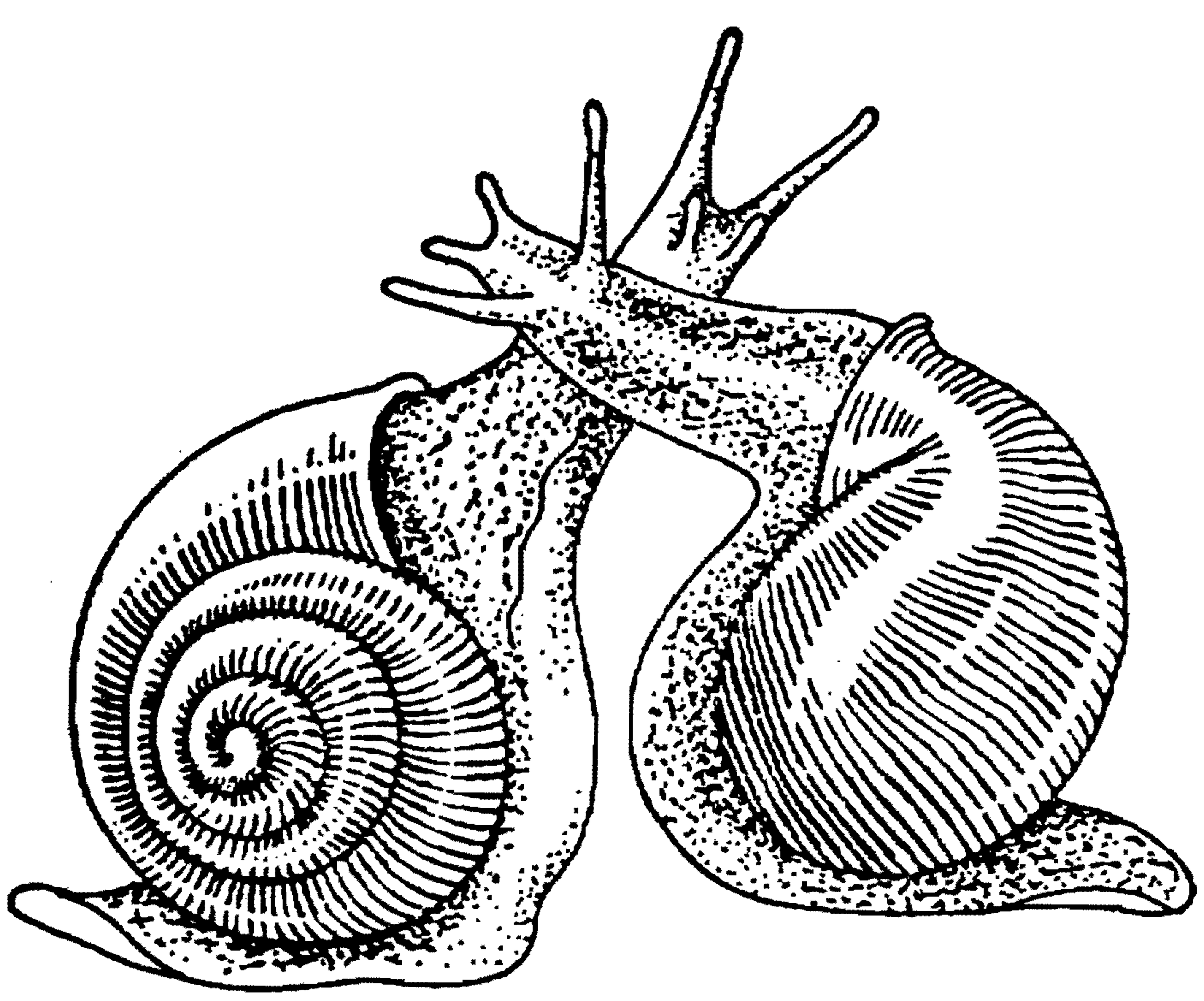
Owl's description of Quefithe was quite confusing. He said that in Quefithe everything was both up and down. If you knew where you were, there was no way of knowing where you were going, and conversely, if you knew where you were going, there was no way of knowing where you were. The young mole and the young crow did not understand much, so they went instead to the old eagle and asked him what Quefithe was like. The eagle shook his white-feathered head, sized them up with his fierce eyes, and said: "Action gives automatically invariant description of Quefithe. You must study the unitary representations of the Lorentz group". The mole and the crow waited for more, but the eagle remained silent, his gaze fixed on an unfathomable string in the sky.

Clearly, if they were ever going to learn anything about Quefithe, they had to see it for themselves. And that is what they did.

After a few years had passed, the mole came back. He said that Quefithe consisted of lots of tunnels. One entered a hole and wandered through a maze, tunnels splitting and rejoining, until one found the next hole and got out. Quefithe sounded like a place only a mole would like, and nobody wanted to hear more about it.

Not much later the crow landed, flapping its wings and crowing excitedly. Quefithe was amazing, it said. The most beautiful landscape with high mountains, perilous passes and deep valleys. The valley floors were teeming with little moles who were scurrying down rutted paths. The crow sounded like he had taken too many bubble baths, and many who heard him shook their heads. The frogs kept on croaking "it is not rigorous, it is not rigorous!" The eagle said: "It is frightful nonsense. One must study the unitary representations of the Lorentz group". But there was something about crow's enthusiasm that was infectious.

The most puzzling thing about it all was that the mole's description of Quefithe sounded nothing like the crow's description. Some even doubted that the mole and the crow had ever gotten to the mythical land. Only the fox, who was by nature very curious, kept running back and forth between the mole and the crow and asking questions, until he was sure that he understood them both. Nowadays, anybody can get to Quefithe - even snails.



two hermaphroditic snails.

2. GENERATING FUNCTIONALS

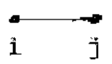
A. Propagators and vertices

A particle (an elementary excitation of a theory) is specified by a list of attributes; its name, its state (spin up, ingoing, ...), its spacetime location, etc. To develop the formalism of field theory, one does not need any specific part of this information, so we hide it in a single collective index:

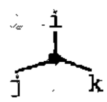
$$i = \{q, a, \alpha, \mu, x_\mu, \;\ldots .\}$$

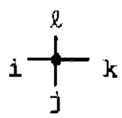
q : particle type
a : colour
α : spin
μ : Minkowski indices
x_μ : spacetime coordinates (2.1)

A particle is an interesting particle only if it can do something. The simplest thing it can do is to change its position, its spin or some other attribute. The probability (amplitude) that this happens is described by the (bare) <u>propagators</u>:

$\Delta_{ij} =$  . (2.2)

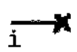
Beyond this, many things can happen; a particle can split into two, or three, or many other particles. The probability (amplitude) that this happens is described by (bare) <u>vertices</u>:

$\gamma_{ijk} =$ 

$\gamma_{ijk\ell} =$ 

$\gamma_{ijk\ell m} =$ $\vdots$ (2.3)

A particle can also be created (or removed from the system). This is described by a <u>source</u> (or a <u>sink</u>):

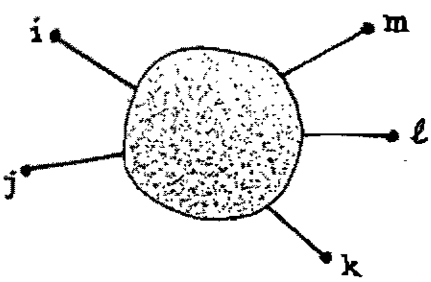
$J_i =$  . (2.4)

The concept of a particle makes sense only if its persistence probability (2.2) is appreciable, i.e. if (2.3), the probability of its disintegration, is relatively small. In that case the interactions (2.3) may be treated as small corrections, and the perturbation theory applies. If the "particle" described by attributes (2.1) has a negligible persistence probability, the theory should be reformulated in terms of another set of "elementary excitations" which are a better approximation to the physical spectrum of the theory (an easy thing to say).

How many identical particles (particles with all the same labels) can coexist? We shall consider two extremes: infinity (bosons) or at most one (fermions). Other more perverse possibilities cannot be excluded. Assumption of additivity of probabilities/amplitudes then implies that the bosonic propagators and vertices must be symmetric under interchange of indices $\Delta_{ij} = \Delta_{ji}$, $\gamma_{ijk} = \gamma_{jik} = \gamma_{ikj} = \ldots.$ (The argument is similar to the one we shall use for fermions in chapter 4). For the time being, we assume that the vertices (2.3) are symmetric.

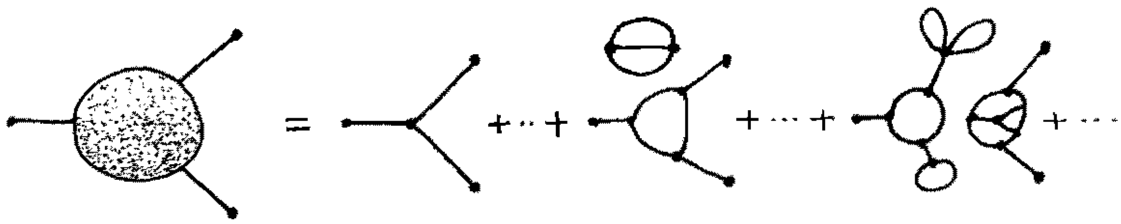
B. Green functions

A typical experiment consists of a setup of the initial particle configuration, followed by a measurement of the final configuration. The theoretical prediction is expressed in terms of the Green functions. For example, if we are considering an experiment in which particles i and j interact, and the outcome is particles k, ℓ, and m, we draw the corresponding Green functions

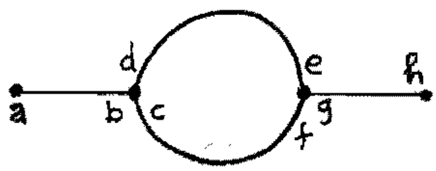
$G_{ijk\ell m} =$  (2.5)

(remember that labels i, j, ... stand for all variables and indices which specify a particle.)

A Green function is a sum of the probabilities (amplitudes) associated with all possible ways in which the final state can be reached. This is represented by an infinite sum of Feynman diagrams:



Each Feynman diagram corresponds to a sum (or an integral). For example, diagram



represents the probability that 1) a particle whose type, location, etc. is described by the collective index a reached any state labeled b; 2) that b splits into any two particles labeled c and d, and so forth. The intermediate states are summed over the entire range of possible index values

$$= \sum_{b,c,d,e,f,g} \Delta_{ab}\gamma_{bcd}\Delta_{cf}\Delta_{de}\gamma_{efg}\Delta_{gh} \; .$$

Here the summation signs imply sums over discrete indices (such as spin) and integrals over continuous indices (such as position). In the future we shall drop the explicit summation signs, and use instead Einstein's repeated index convention; if an index appears twice in a term, it is summed (integrated) over.

Exercise 2.B.1 Continuous indices. For QCD the collective index i stands for:

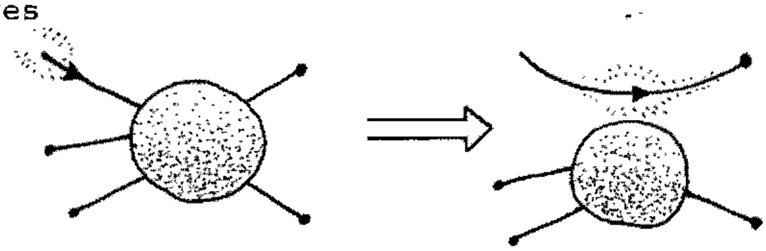
x^μ	spacetime coordinates,
$\mu = 1,2,\ldots,d$	Minkowski indices,
$j = 1,2,\ldots,N$	gluon colours.

If the propagator is denoted by $D^{ij}_{\mu\nu}(x,y)$ and the three-gluon vertex by $\gamma^{ijk}_{\mu\nu\sigma}(x,y,z)$, write down the complete expression for the above self-energy diagram.

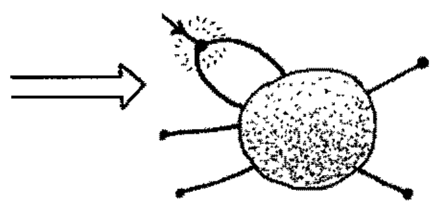
C. Dyson-Schwinger equations

A Green function consists of an infinity of Feynman diagrams. For a theory to be manageable, it is essential that these diagrams can be generated systematically, in order of their relative importance.

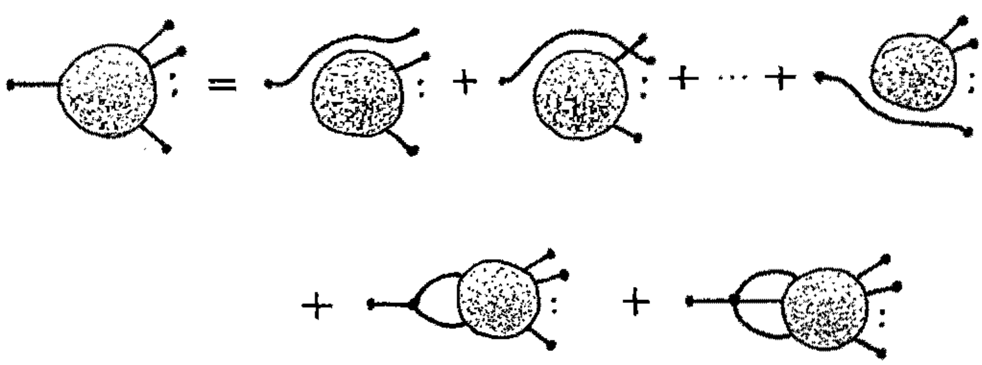
Consider (for simplicity) a theory with only cubic and quartic vertices†. Take a Green function and follow a particle into the blob. Two things can happen; either the particle survives



or it interacts at least once:



More precisely, entering the diagram via leg 1, we either reach leg 2, or leg 3, ... , or hit a three-vertex, or a four-vertex, etc. Adding up all the possibilities, we end up with the Dyson-Schwinger equations:



(2.6)

†Remember that the different particle types are covered by a single collective index, so QCD is also this type.

Iteration of the Dyson-Schwinger (DS) equations yields all Feynman diagrams contributing to a given process, ordered by the number of vertices (the order in perturbation theory).

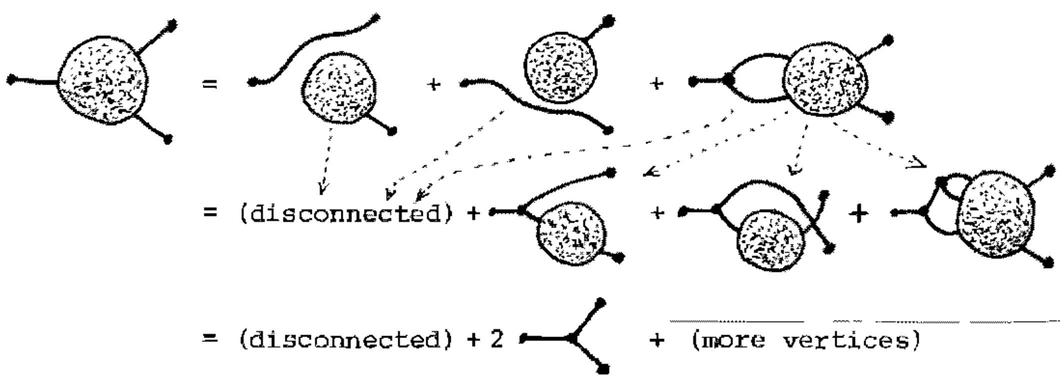
A few words about the diagrammatic notation; a diagrammatic equation like (2.6) contains precisely the same information as its algebraic transcription

$$G_{ij..k\ell} = \Delta_{i\ell}G_{j..k} + \Delta_{ik}G_{j..\ell} + \dots + \Delta_{ij}G_{..k\ell}$$
$$+ \Delta_{ir}\gamma_{rst}G_{tsj..k\ell} + \Delta_{ir}\gamma_{rstu}G_{utsj..k\ell} \,.$$

Indices can always be omitted. An internal line implies a summation/integration over the corresponding indices, and for external lines the equivalent points on each diagram represent the same index in all terms of a diagrammatic equation. The advantages of the diagrammatic notation are obvious to all those who prefer the comic strip editions of "The greatest story ever told" to the unwieldy, fully indexed version[†]. Two of the principal benefits are that it eliminates "dummy indices" and that it does not force Feynman integrals into one-dimensional format (both being means whereby identical integrals can be made to look totally different).

D. Combinatoric factors

For a three-leg Green function the DS equations yield



It is rather unnatural that an expansion of a three-leg Green function does not start with the bare three-vertex, but twice

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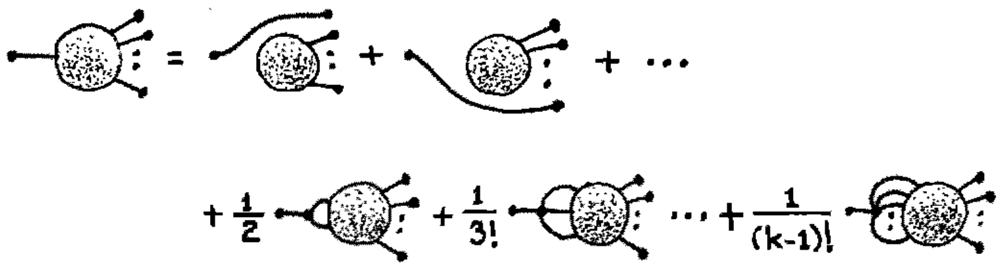
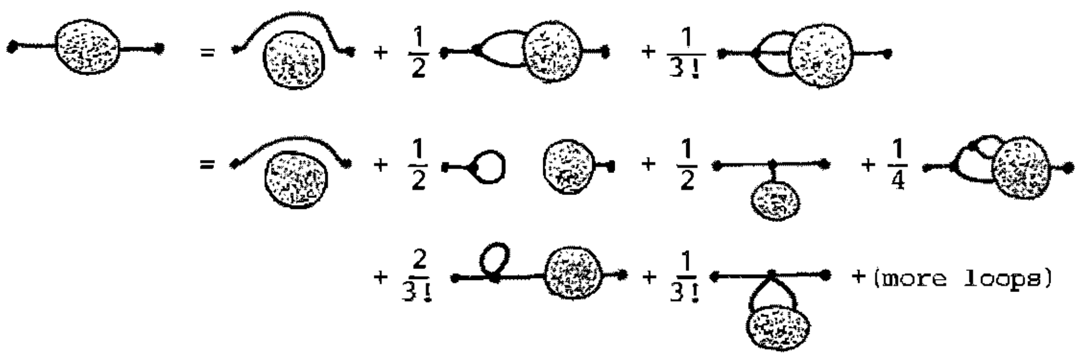
† C. Itzykson and J.-B. Zuber, Quantum Field Theory (McGraw-Hill, N.Y., 1980).

the bare three-vertex. This is easily fixed-up by including compensating combinatorial factors into DS equations:

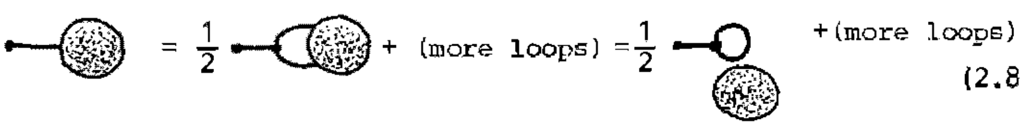


(2.7)

To illustrate how the DS equations generate the perturbation expansion, we expand a two-leg Green function up to one loop:

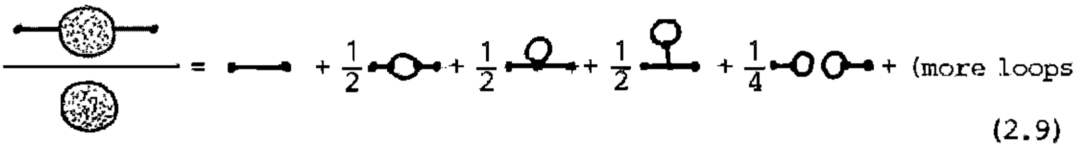


The one-loop tadpole is given by



(2.8)

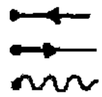
Substituting the tadpole into the above, we finally obtain the self-energy expansion up to two vertices with all the correct combinatoric factors:



(2.9)

This expansion looks like the usual $\phi^3 + \phi^4$ theory, but it is not only that: the combinatoric factors are correct for any theory with cubic and quartic vertices, such as QCD with its full particle content.

Exercise 2.D.1 Feynman diagrams in the collective index notation look like diagrams for scalar field theories. Nevertheless, they do contain the perturbative expansion for theories with arbitrary particle content. As an example, consider a QED-type theory with an "in" particle (electron), and "out" particle (positron) and a scalar particle (photon). The collective index (2.1) now ranges over an array of three sub-collective indices

$$i = \begin{bmatrix} a, \text{ in} \\ a, \text{ out} \\ \mu \end{bmatrix} =$$

electron
positron
photon

Index a stands for the charged particle's position and spin, and index μ stands for all labels characterizing the neutral particle. The "in" - "out" labels can be eliminated by taking a to be an upper index for "in" particles, and a lower index for "out" particles. Diagrammatically they are distinguished by drawing arrows pointing away from upper indices and down into lower indices:

$$\Delta^a_b = a$$  b

$$\gamma_{\mu a}{}^{b} =$$ 

Show that if the sources and fields are replaced by $J = (\eta^a, \eta_b, J_\mu)$, $\phi = (\psi_a, \psi^b, A^\mu)$, the combinatoric factors in (2.9) cancel, and the vertices such as the electron-positron-photon vertex have no combinatoric weight:

$$\frac{1}{2}$$  $$= \frac{1}{2} J_i \Delta_{ij} J_j =$$  $$+ \frac{1}{2}$$ 

$$\frac{1}{3!}$$  $$= \frac{1}{3!} \gamma_{ijk} \phi_i \phi_j \phi_k = \gamma_{\mu b}{}^{a} \psi^b A_\mu \psi_a \ .$$

Exercise 2.D.2 Write the Dyson-Schwinger equations for QED-like theories. (We say "QED-like" because electrons are fermions. We shall return to the fermion DS equations later.)

Exercise 2.D.3 Determine the one-loop self-energy diagrams (2.9) for QED-like theories.

E. Generating functionals

The structure of the DS equations is very general; still, at present we have to write them separately for two-leg Green function, three-leg Green function, and so on. To state relations between Green functions in a more compact way we introduce generating functionals. A generating functional is the vacuum (legless) Green function for a theory with sources (2.4):

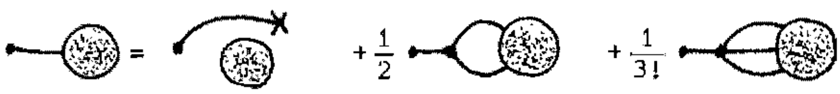
$$Z[J] = \sum_{m=0}^{\infty} \frac{1}{m!} G_{i_1 i_2 \ldots i_m} J_{i_1} J_{i_2} \ldots J_{i_m}$$

 $= 1 +$  $+ \frac{1}{2}$  $+ \frac{1}{3!}$  $+ \ \ldots \ ,$ (2.10)

(as J_i is a function which depends on both discrete and continuous indices, $Z[J]$ is a functional). The coefficients in this expansion are the usual Green functions. They can be retrieved from the generating functional by differentiation:

$$G_{ijk} = \frac{d}{dJ_i}\frac{d}{dJ_j}\frac{d}{dJ_k} Z[J]\bigg|_{J=0} \text{, etc.} \tag{2.11}$$

The DS equations (2.7) can be written as



$$\frac{d}{dJ_i} Z[J] = \Delta_{ij}\left\{J_j + \frac{1}{2}\gamma_{jk\ell}\frac{d}{dJ_\ell}\frac{d}{dJ_k} + \frac{1}{3!}\gamma_{jk\ell m}\frac{d}{dJ_m}\frac{d}{dJ_\ell}\frac{d}{dJ_k}\right\} Z[J] \ . \tag{2.12}$$

The bare propagators and vertices can themselves be collected in a functional called the <u>action</u>:

$$S[\phi] = -\frac{1}{2}\phi_i \Delta^{-1}_{ij}\phi_j + S_I[\phi] \ , \tag{2.13}$$

$$S_I[\phi] = \sum_m \underbrace{\gamma_{ijk..\ell}}_{m \text{ legs}} \frac{\phi_i\phi_j\ldots\phi_\ell}{m!} \ . \tag{2.14}$$

Now the Dyson-Schwinger equations can be stated in an even more elegant way:

$$0 = \left(\frac{dS}{d\phi_i}\left[\frac{d}{dJ}\right] + J_i\right) Z[J] \ , \tag{2.15}$$

where

$$\frac{dS}{d\phi_i}\left[\frac{d}{dJ}\right] \equiv \frac{dS[\phi]}{d\phi_i}\bigg|_{\phi=\frac{d}{dJ}} \ .$$

The action (or the Lagrangian) is just another way of defining the propagators and vertices for a given theory. Giving the Lagrangian or listing the Feynman rules is one and the same thing.

<u>Exercise 2.E.1</u> <u>Functional derivatives.</u> For continuous indices the Kronecker deltas are replaced by Dirac deltas. For example, check that in d-dimensions

$$\frac{dJ(x)}{dJ(y)} = \delta^{d}(x-y) \quad ,$$

is the correct definition of the derivative in (2.11).

<u>Exercise 2.E.2</u> <u>Feynman rules.</u> Consider ϕ^3 theory given by the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}\,\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}\,m^2\phi(x)^2 - \frac{\mu}{3!}\,\phi(x)^3$$

$$S = \int d^d x \mathcal{L}(x) \,.$$

Read off the bare propagators and vertices (the Feynman rules) from the Lagrangian.

Hint: $$\gamma_{ij..k} = \frac{d}{d\phi_i}\frac{d}{d\phi_j}\cdots\frac{d}{d\phi_k}\,S[\phi]\Big|_{\phi=0} \quad ,$$

and the derivatives are in this case functional derivatives.

<u>Exercise 2.E.3.</u> <u>Zero-dimensional field theory.</u> Consider a ϕ^3 theory defined by trivial Feynman rules

 $= 1$,  $= g$.

The value of a graph with k vertices is g^k, and k-th order contribution to Green function is basically the number of contributing diagrams. More precisely, if

$$Z[J] = \sum_{k,m} G_k^{(m)} g^k \frac{J^m}{m!}$$

the Green function

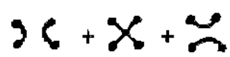
$$G_k^{(m)} = \sum_G C_G$$

is the sum of combinatoric factors of all diagrams with m legs and k vertices. Use the Dyson-Schwinger equation (2.7) to show that for a free field theory

$$G_0^{(m)} = (m-1)!! \qquad m \text{ even}$$
$$= 0 \qquad m \text{ odd} \quad .$$

Diagrammatically

$G^{(2)} =$ $= 1$

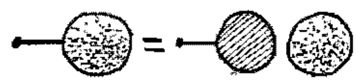
$G^{(4)} =$  $= 3$, etc.

The zero-dimensional field theory is about the only field theory which is easily computable to all orders. We shall use it often to illustrate in a concrete way various field-theoretic relations.

F. Connected Green functions

Generating functionals are a powerful tool for stating relations between Green functions. For example, we can use them to derive relations between the full and the connected Green functions:

Pick out a leg and follow it into a full Green function. This separates all associated Feynman diagrams into two parts - the part that is connected to the initial leg, and the remainder:



$$\frac{d}{dJ_i} Z[J] = \frac{dW[J]}{dJ_i} Z[J] \tag{2.16}$$

The generating functional for the connected Green functions is defined in the same way as (2.10), the generating functional for the full Green functions:

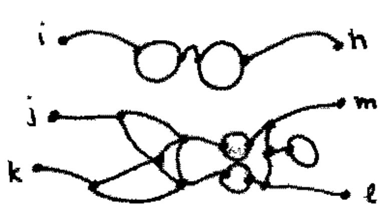
$$W[J] = \sum_{m=1}^{\infty} \frac{1}{m!} G^{(c)}_{i_1 i_2 \ldots i_m} J_{i_1} J_{i_2} \ldots J_{i_m} \quad ,$$

(2.17)

The differential equation (2.16) is easily solved

$$Z[J] = e^{W[J]} \tag{2.18}$$

A disconnected Feynman diagram such as



represents a product of two independent processes; one could take place on the moon, and the other in Aarhus. The physically interesting processes are described by the connected Green functions. To obtain a systematic perturbation series which includes only the connected Feynman diagrams, we use the identity[+]

$$\frac{1}{Z[J]}\frac{d}{dJ_i}Z[J] = \frac{dW[J]}{dJ_i} + \frac{d}{dJ_i} \tag{2.19}$$

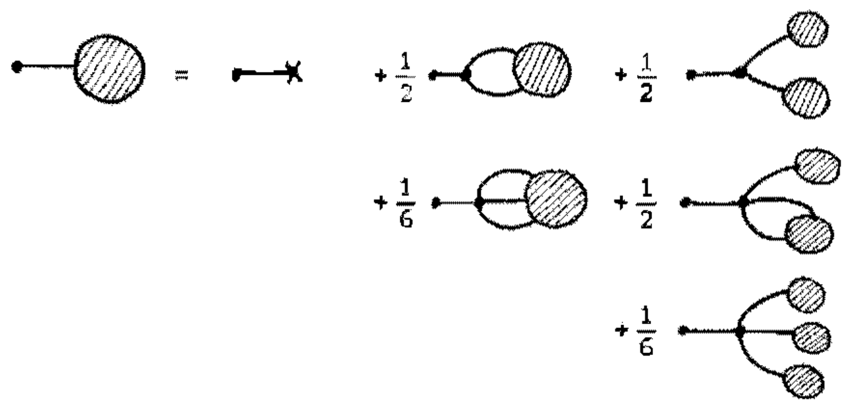
to rewrite the DS equations (2.15) in terms of the connected Green functions:

$$0 = \frac{dS}{d\phi_i}\left[\frac{dW[J]}{dJ} + \frac{d}{dJ}\right] + J_i \ . \tag{2.20}$$

This is very elegant, but possibly not too transparent. To get a feeling for these equations, take the $\phi^3 + \phi^4$ DS equations (2.12) and substitute $Z[J] = \exp(W[J])$. The result is, in the functional notation

$$\frac{dW[J]}{dJ_i} = \Delta_{ij}\left\{ J_j + \frac{1}{2}\gamma_{jk\ell}\left(\frac{d^2W[J]}{dJ_\ell dJ_k} + \frac{dW[J]}{dJ_k}\frac{dW[J]}{dJ_\ell}\right) + \frac{1}{6}\gamma_{jk\ell m}\left(\frac{d^3W[J]}{dJ_m dJ_\ell dJ_k} + 3\frac{dW[J]}{dJ_k}\frac{d^2W[J]}{dJ_m dJ_\ell} + \frac{dW[J]}{dJ_k}\frac{dW[J]}{dJ_\ell}\frac{dW[J]}{dJ_m}\right)\right\}, \tag{2.21}$$

and in the longleggedy notation



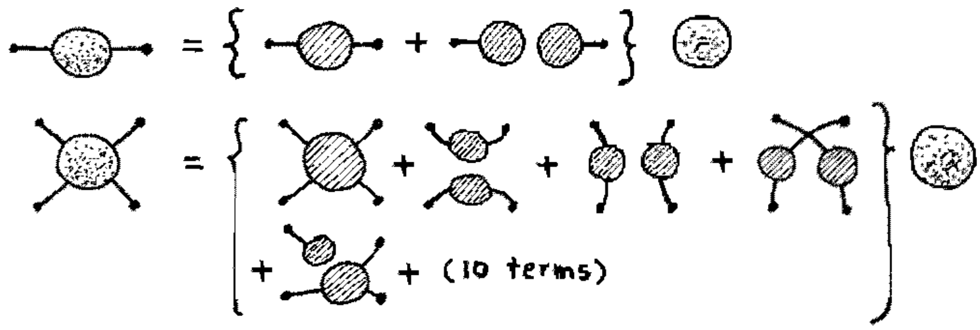
After reaching a vertex, one continues into diagrams that are either mutually disconnected, or connected - that is the reason that there are extra terms in the connected DS equations, compared with the full Green functions equations (2.12).

[+] more explicitly

$$\frac{1}{Z[J]}\frac{d}{dJ}(Z[J]f[J]) = \left(\frac{dW[J]}{dJ} + \frac{d}{dJ}\right)f[J] .$$

Exercise 2.F.1 Use DS equations (2.21) to compute self-energy to one loop. How does the result differ from (2.9)?

Exercise 2.F.2 Expand some full Green functions in terms of the connected ones:

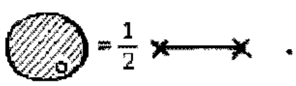


Hint: iterating (2.19) is probably the fastest way.

G. Free field theory

The connected generating functional for a free field theory is trivial: there are no interactions, so the only connected Feynman diagram is the propagator:

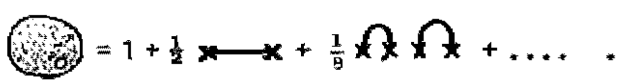
$$W_0[J] = \frac{1}{2} J_i \Delta_{ij} J_j$$



(2.22)

For the free field theory (2.18) gives an explicit expression for the generating functional:

$$Z_0[J] = e^{\frac{1}{2} J_i \Delta_{ij} J_j}$$

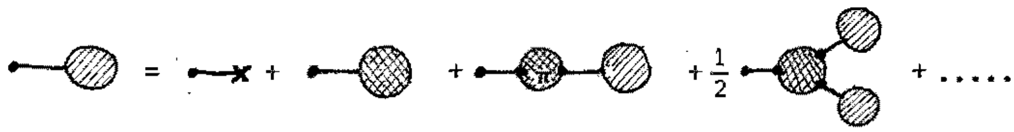


(2.23)

H. One-particle irreducible Green functions

A one-particle irreducible (1PI) diagram cannot be cut into two disconnected parts by cutting a single internal line. An arbitrary connected diagram has in general a number of such lines. The connected and the 1PI Green functions can be related by our usual diagrammatic trick:

Pick out a leg of a connected diagram. This pulls out a 1PI piece, which ends in 0, 1, 2, ... lines whose cutting would disconnect the diagram. Those lines continue into further connected pieces:



$$\phi_i = \Delta_{ij}(J_j + \Gamma_j + \pi_{jk}\phi_k + \tfrac{1}{2}\,\Gamma_{jk\ell}\phi_k\phi_\ell + \ldots) \,. \qquad (2.24)$$

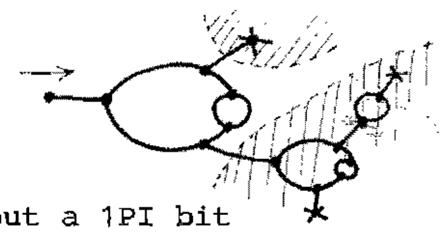
Here the "field" ϕ is defined by

$$= \phi_i = \frac{dW[J]}{dJ_i} \,. \qquad (2.25)$$

We draw the 1PI Green functions as cross-hatched blobs

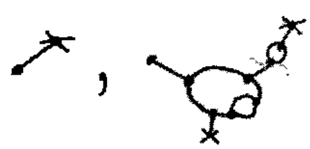
$$\Gamma_{ij..k} =$$

Unlike the full and the connected Green functions, the 1PI ones do not have propagators on external legs - the external indices always belong to a vertex of an 1PI diagram. This is indicated by drawing dots on the edges of 1PI Green functions. Any connected diagram belongs to one and only one term in the expansion (2.24). For example, going into connected diagram



we pull out a 1PI bit

followed by connected bits



Multiplying both sides of (2.24) by the inverse of the bare propagator we obtain

$$0 = J_i + \Gamma_i + (-\Delta^{-1} + \pi)_{ij}\phi_j + \tfrac{1}{2}\,\Gamma_{ijk}\phi_k\phi_j + \ldots \,.$$

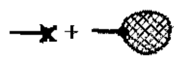
(For reasons which should soon be clear, it is convenient to define the two-leg Γ as $\Gamma_{ij} = -\Lambda^{-1}_{ij} + \pi_{ij}$, where π_{ij} is the 1PI two-leg Green function, or the proper self-energy.)

Collecting all 1PI Green functions into the _effective action_ functional

$$\Gamma[\phi] = \sum_{m=1} \Gamma_{ij..k} \frac{\phi_k \cdot\cdot \phi_j \phi_i}{m!} \quad , \tag{2.26}$$

we can write (2.24), the relation between the connected and the 1PI Green functions, as:

$$0 = J_i + \frac{d\Gamma[\phi]}{d\phi_i} \; ,$$

0 =  .   (2.27)

This, together with (2.25), can be summarized by a _Legendre transformation_

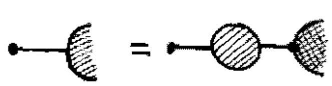
$$W[J] = \Gamma[\phi] + \phi_i J_i \; . \tag{2.28}$$

(2.27) guarantees that W is independent of ϕ, and (2.25) guarantees that Γ is independent of J:

$$\frac{dW[J]}{d\phi} = 0 , \qquad \frac{d\Gamma[\phi]}{dJ} = 0 \; .$$

This is elegant, but how does it help us to get 1PI Green functions? The point is that we are not interested in extracting 1PI Green functions from the connected ones; what we need are the 1PI Dyson-Schwinger equations, i.e. the systematics of generating 1PI diagrams (and only 1PI diagrams). To achieve this, we must first eliminate J-derivatives in favour of ϕ-derivatives (cf. (2.25)):

$$\frac{d}{dJ_i} = \frac{d\phi_j}{dJ_i} \frac{d}{d\phi_j} = \frac{d^2W[J]}{dJ_i dJ_j} \frac{d}{d\phi_j}$$

   (2.29)

This accounts for all self-energy insertions. The right-hand side can be expressed in terms of 1PI Green functions by taking a derivative of (2.27):

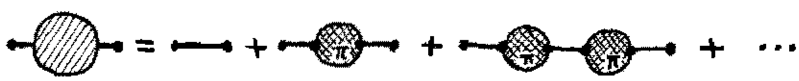
$$0 = \delta_{ij} + \frac{d}{dJ_j}\,\frac{d\Gamma[\phi]}{d\phi_i} = \delta_{ij} + \frac{d^2W[J]}{dJ_j dJ_k}\,\frac{d^2\Gamma[\phi]}{d\phi_k d\phi_i}\ . \qquad (2.30)$$

In order to understand this relation diagrammatically, we separate out the bare propagator in (2.26) by defining the "interaction" part of Γ:

$$\Gamma[\phi] = -\tfrac{1}{2}\phi_i \Delta^{-1}_{ij}\phi_j + \Gamma_I[\phi]\ . \qquad (2.31)$$

Now (2.30) can be written as

$$\frac{d^2W[J]}{dJ_i dJ_j} = \Delta_{ij} + \Delta_{ik}\,\frac{d^2\Gamma_I[\phi]}{d\phi_k d\phi_\ell}\,\Delta_{\ell j} + \ldots.$$



$$W[J]'' = \frac{1}{\Delta^{-1} - \Gamma_I[\phi]''}\ . \qquad (2.32)$$

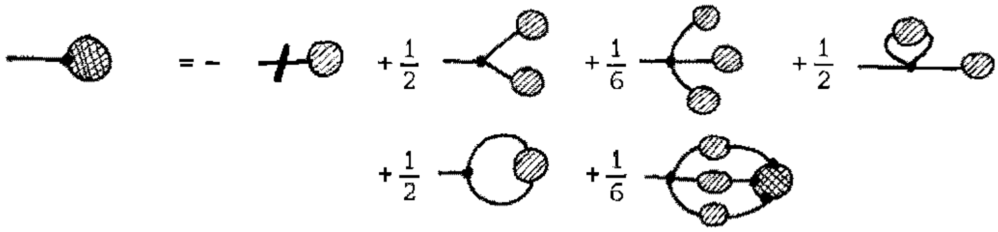
Diagrammatically W'' is a complete propagator which sums up all proper self-energies.

We can use (2.25) and (2.27) to eliminate source-dependent functionals in favour of field-dependent functionals, and (2.29) to replace J-derivatives by ϕ-derivatives, in order to rewrite (2.20) as the <u>1PI Dyson-Schwinger equation</u>:

$$\frac{d\Gamma[\phi]}{d\phi_i} = \frac{dS}{d\phi_i}\left[\phi + W''[J]\frac{d}{d\phi}\right]\ . \qquad (2.33)$$

The form of this equation is one of the reasons why the generating functional for 1PI Green functions is called the <u>effective action</u>. If the derivatives are dropped, the effective action reduces to the classical action. The role of the derivatives is to generate loops, i.e. quantum corrections (or statistical fluctuations). We shall return to this in our discussion of path integrals.

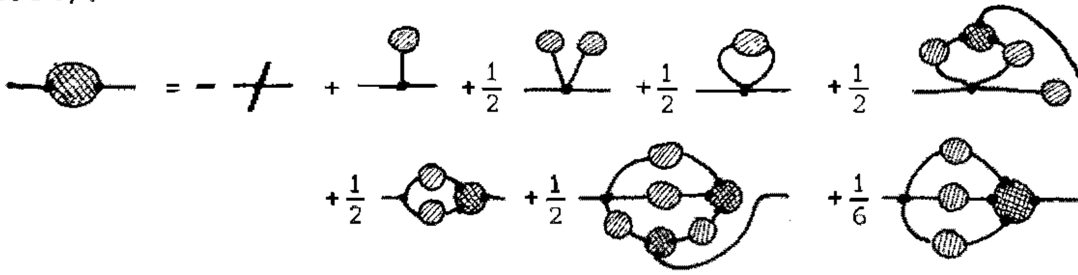
DS equations (2.33) are again so elegant that one is probably at a loss as to what to do with them. To get a feeling for their utility, we write them out for the $\phi^3 + \phi^4$ example (2.21):



$$\frac{d\Gamma[\phi]}{d\phi_i} = -\Delta^{-1}_{ij}\phi_j + \tfrac{1}{2}\gamma_{ijk}\phi_k\phi_j + \tfrac{1}{6}\gamma_{ijk\ell}\phi_\ell\phi_k\phi_j + \tfrac{1}{2}\gamma_{ijk\ell}\phi_\ell \frac{d^2W[J]}{dJ_k dJ_j}$$

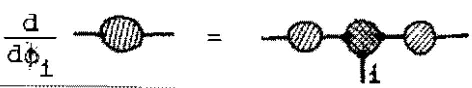
$$+ \tfrac{1}{2}\gamma_{ijk}\frac{d^2W[J]}{dJ_k dJ_j} + \tfrac{1}{6}\gamma_{ijk\ell}\frac{d^2W[J]}{dJ_j dJ_m}\frac{d^2W[J]}{dJ_k dJ_n}\frac{d^2W[J]}{dJ_\ell dJ_\sigma}\frac{d^3\Gamma[\phi]}{d\phi_m d\phi_n d\phi_\sigma} \quad . \tag{2.34}$$

Such equations are used iteratively. For example, to obtain the DS equation for the proper self-energy†, take a derivative of (2.34):



(2.35)

Exercise 2.H.1 Use (2.32) to show that

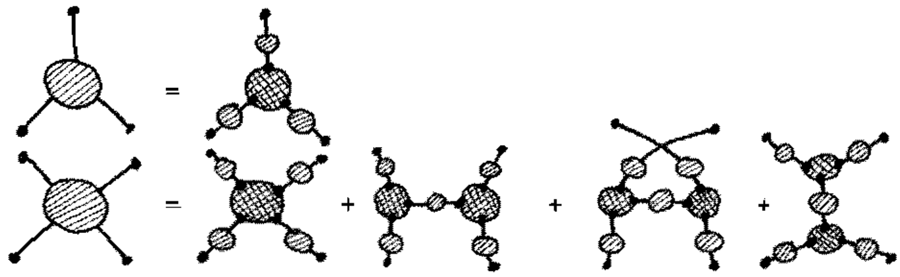


(2.36)

This is a useful identity for deriving relations such as (2.34) and (2.35).

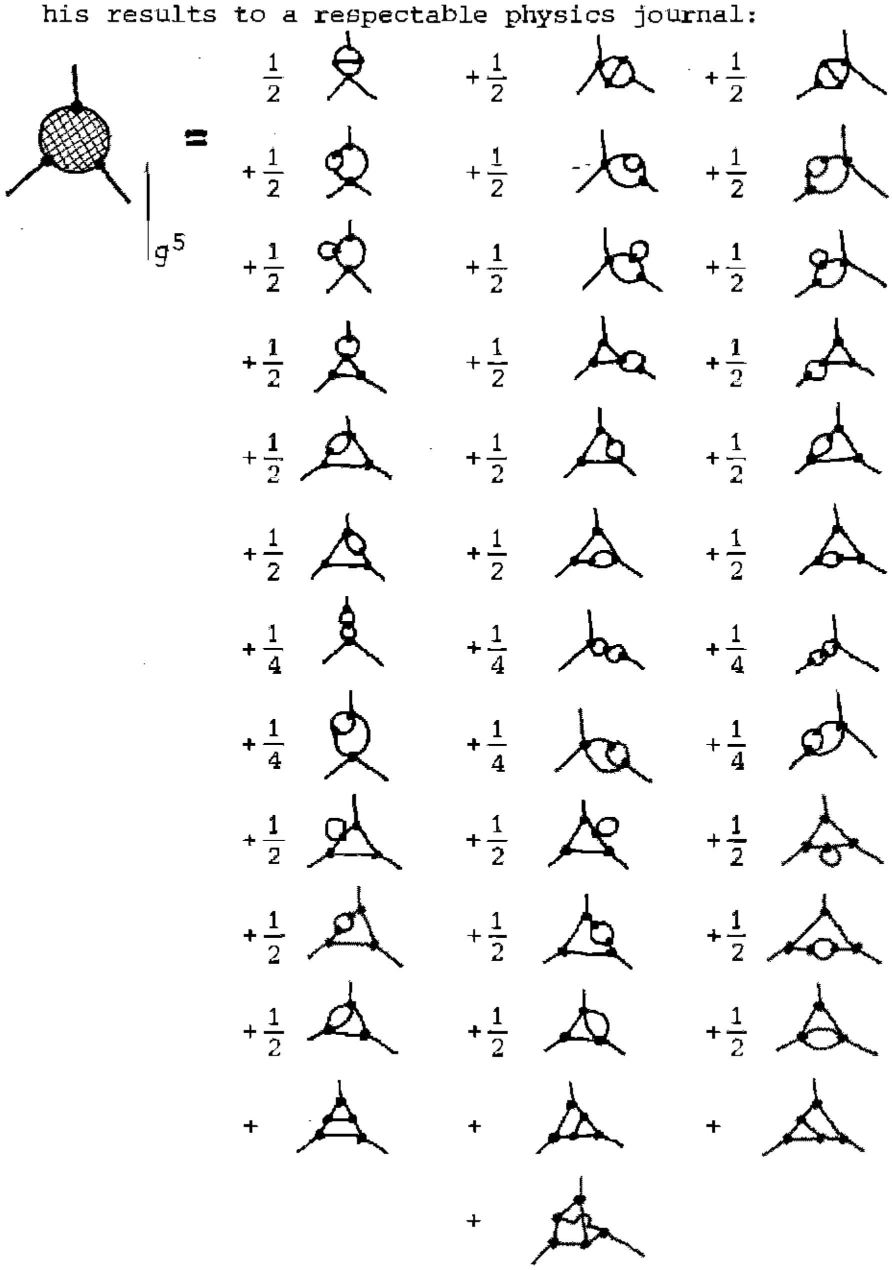
Exercise 2.H.2 Take successive derivatives of (2.30) to show that the connected Green functions can be expanded in terms of 1PI Green functions as

† Here the slash stands for inverse propagator; diagrammatically it is a two-leg vertex. Other vertices are denoted by dots, and a line connecting two vertices is always a propagator, so that $\Delta^{-1}_{ij}\Delta_{jk} = i$—✚—$k = i$——$k = \delta_{ik}$.



(2.37)

Exercise 2.H.3 Jens J. Jensen, a serious young student of field theory, is getting set to compute the two-loop QCD beta-function. He has drawn up a list of gluon corrections to the three-gluon vertex. Use the 1PI Dyson-Schwinger equations to check this list and make Jens aware of 7 (seven) errors before he rushes his results to a respectable physics journal:



I. Vacuum bubbles

The Green function formalism we have developed so far is tailored to scattering problems; all the Green functions we have considered had external legs. Processes without external particles (the corresponding legless diagrams are called vacuum bubbles) are also physically interesting. For example, if a particle is propagating through a hot, dense soup†, a particle-particle scattering experiment would be a hopeless and messy undertaking. Such systems are probed by varying bulk parameters, such as temperature. Indeed, the generating functionals do not depend only on the single-particle sources J_i, but on all interaction parameters

$$Z[J] \equiv Z[J,\gamma_{ij},\gamma_{ijk},\gamma_{ijk\ell},\ldots] \quad . \tag{2.38}$$

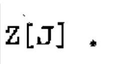
Any of these, or any combination of these, can be varied. Diagrammatically we view an n-vertex as an n-particle source. For example, if we rescale $\gamma_{ij..k} \rightarrow g\gamma_{ij..k}$ and vary infinitesimally the coupling constant g, we "touch" each $\gamma_{ij..k}$ vertex in a Green function:

$$g\frac{d}{dg}Z[J] = \frac{1}{k!}$$  $$= \frac{g}{k!}\gamma_{ij..k}\frac{d}{dJ_k}\cdots\frac{d}{dJ_j}\frac{d}{dJ_i}Z[J] \quad . \tag{2.39}$$

We can use such generalizations of the Dyson-Schwinger equations (from varying single-particle sources J_i to varying many-particle sources $\gamma_{ijk..\ell}$) to compute hosts of physically significant quantities. One such quantity is the expectation value of the action. We rescale the entire action (2.13)

$$\frac{1}{\hbar}S[\phi] = -\frac{1}{2\hbar}\phi_i\Delta^{-1}_{ij}\phi_j + \frac{1}{3!\hbar}\gamma_{ijk}\phi_k\phi_j\phi_k + \cdots , \tag{2.40}$$

and vary $\hbar$ (depending on the context, $\hbar$ could be the Planck constant, coupling constant, inverse temperature or something else):

$$\hbar\frac{d}{d\hbar}Z[J] = -\frac{1}{\hbar}\Big(-\frac{1}{2}$$  $$+\frac{1}{3!}$$ $$+\frac{1}{4!}$$ $$+\ldots\Big)$$

$$= -\frac{1}{\hbar}S\left[\frac{d}{dJ}\right]Z[J] \quad . \tag{2.41}$$

---

†minestrone, to be specific.

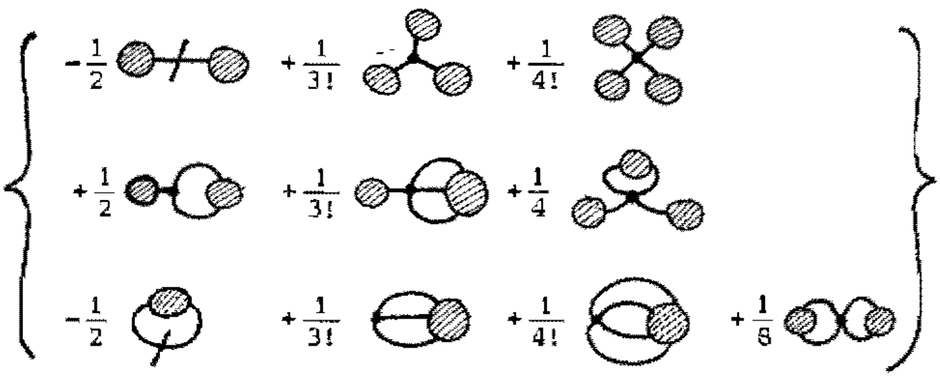
To normalize the expectation value properly, we divide by $Z[J]$:

$$\langle S[\phi]\rangle = \frac{1}{Z[J]}\, S\left[\frac{d}{dJ}\right] Z[J] \quad . \tag{2.42}$$

That this is really an expectation value will perhaps be easier to grasp in the path-integral formalism, cf. (3.11) in the next chapter. Anyway, we can use (2.19) to rewrite the above in terms of connected Green functions:

$$\langle \frac{1}{\hbar} S[\phi]\rangle = -\hbar \frac{dW[J]}{d\hbar} = \frac{1}{\hbar}\, S\left[\frac{dW[J]}{dJ_i} + \frac{d}{dJ_i}\right]$$

$$= \frac{1}{\hbar}$$



(2.43)

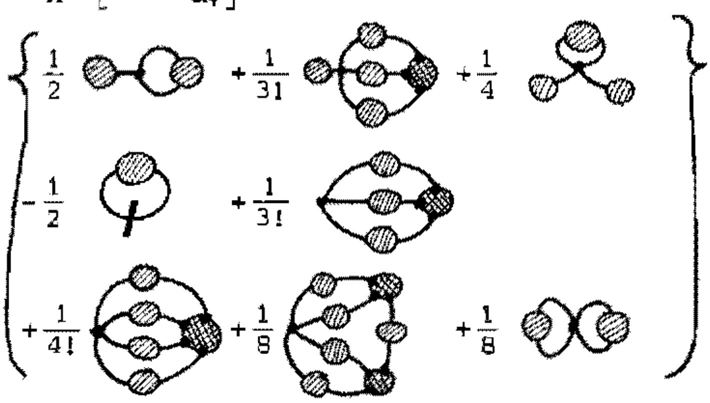
(the diagrammatic expansion is for the $\phi^3 + \phi^4$ theories). Even better, we can use (2.25) and (2.29) together with the identity (follows from (2.28))

$$\frac{dW[J]}{d\hbar} = \frac{d\Gamma[\phi]}{d\hbar} \tag{2.44}$$

to relate the $\langle S[\phi]\rangle$ to the effective action:

$$\frac{1}{\hbar}\langle S[\phi]\rangle = -\hbar \frac{d\Gamma[\phi]}{d\hbar} = \frac{1}{\hbar}\, S\left[\phi + W''\frac{d}{d\phi}\right]$$

$$= \frac{1}{\hbar} S[\phi] + \frac{1}{\hbar}$$



(2.45)

The above expansions can be used to compute the perturbative expansions for the connected and 1PI vacuum bubbles (see exercises). Their physical significance will become clearer in the next chapter.

Exercise 2.I.1 Loop expansion. Show that with action (2.40) the expansion in powers of $\hbar$ is the loop expansion, i.e. that each loop in a Feynman diagram carries a factor $\hbar$. Hence the loop expansion offers a systematic way of computing quantum corrections (or thermal fluctuations in statistical mechanics). Hint: each propagator carries a factor $\hbar$, while each vertex carries $\hbar^{-1}$.

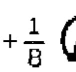
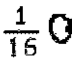
Exercise 2.I.2 Free energy $W[0]$. Compute

$$\frac{1}{\hbar} W[0] = \frac{\delta_{ii}}{2} \frac{\ln \hbar}{\hbar} + \frac{1}{12}$$  $+\frac{1}{8}$  $+\frac{1}{8}$  $+ \ldots$

for $\phi^3 + \phi^4$ theory. Hint: use (2.43) and the DS equations (2.21).

Exercise 2.I.3 Gibbs free energy $\Gamma[0]$. Compute

$$\frac{1}{\hbar} \Gamma[0] = \frac{\delta_{ii}}{2} \frac{\ln \hbar}{\hbar} + \{\frac{1}{12}$$  $+\frac{1}{8}$  $\}$

$+\{\frac{1}{24}$  $+\frac{1}{16}$  $+\frac{1}{8}$  $+\frac{1}{8}$  $+\frac{1}{16}$  $+\frac{1}{48}$  $\}\hbar$

$+ \ldots$ (2.46)

for $\phi^3 + \phi^4$ theory. Hint: use (2.45) and the DS equations (2.34). Note that the one-particle reducible diagrams from $W[0]$ are indeed missing. The vacuum-bubble combinatoric weights are not always obvious - equation (2.45) provides the fastest way of computing them, as far as I know.

Exercise 2.I.4 Show that for the zero-dimensional ϕ^3 theory (continuation of exercise 2.E.3)

$$G^{(m)} = (m - 1 + 3g \frac{d}{dg}) G^{(m-2)} \quad .$$

Hint: use (2.39) together with the Dyson-Schwinger equations (2.12).

Show also that

$$G^{(1)} = \frac{g}{2} G^{(2)} \quad .$$

Hence all Green functions can be computed from $Z \equiv G^{(0)}$, the vacuum bubbles. Show that these satisfy

$$g\frac{d}{dg} Z = g^2 \left(\frac{5}{12} + \frac{9}{4} g \frac{d}{dg} + \frac{3}{4} g^2 \frac{d^2}{dg^2} \right) Z \quad .$$

Compute the first few terms of the expansion in powers of g. The complete solution is given in exercise 3.C.1.

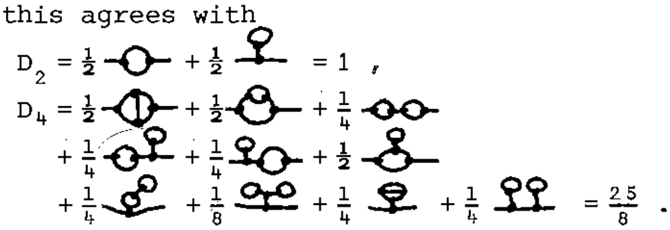
Exercise 2.I.5 Zero-dimensional field theory. Show that the connected vacuum bubbles $W \equiv W[0]$ satisfy

$$g\frac{d}{dg}W = g^2\left[\frac{5}{12} + \frac{9}{4}\, g\frac{dW}{dg} + \frac{3}{4}\, g^2\left(\frac{d^2W}{dg^2} + \left(\frac{dW}{dg}\right)^2\right)\right] .$$

Use this equation to derive recursion relations for connected m-leg Green functions. Compute the exact propagator $D = G^{c(2)}$

$$D = 1 + g^2 + \frac{25}{8}\, g^4 + \frac{390}{32}\, g^6 + \ldots\ldots$$

and check that this agrees with



Hint: establish first that

$$\frac{d}{dJ}W[J] = \frac{g}{2} + J + \frac{g}{2}\left(J\frac{d}{dJ} + 3g\frac{d}{dg}\right)W[J] .$$

That relates $G^{c(m)}$ to the vacuum bubbles W.

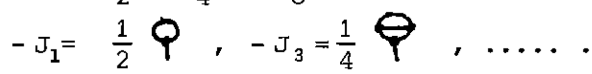
Exercise 2.I.6 Zero-dimensional ϕ^3 theory. Combine the DS equation (2.34) and the previous results to relate 1PI Green functions with different numbers of legs:

$$\frac{d}{d\phi}\Gamma[\phi] = \frac{g}{2} - \phi + \frac{g}{2}\left(-\phi\frac{d}{d\phi} + 3g\frac{d}{dg}\right)\Gamma[\phi] ,$$

and show that the proper tadpoles $J = -\Gamma^{(1)}$ satisfy

$$J = -\frac{g}{2} + \frac{g}{2}\left(1 - \frac{3}{2}g\frac{d}{dg}\right)J^2$$

$$= -\frac{1}{2}\, g - \frac{1}{4}\, g^3 - \frac{5}{8}\, g^5 - \ldots .$$

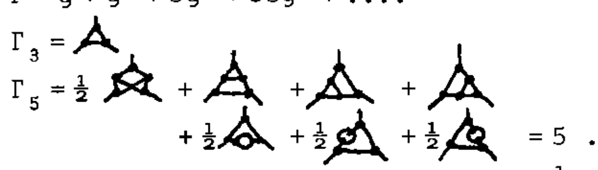


Compute the proper self-energy

$$\pi = \frac{1}{2}\, g^2 + g^4 + \frac{35}{8}\, g^6 + \ldots\ldots$$

and the proper three-vertex $\Gamma \equiv \Gamma^{(3)}$

$$\Gamma = g + g^3 + 5g^5 + 35g^7 + \ldots .$$



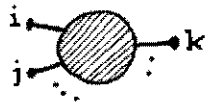
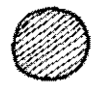
Compare π with the preceeding exercise, $D = (1 - \pi)^{-1}$.

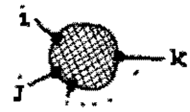
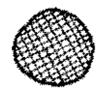
Exercise 2.I.7 Check (2.44).

J. Summary of the generating functional formalism

generating functionals

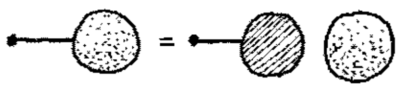
Full Green functions: $G_{ij\ldots k} =$  $Z[J] =$ 

Connected Green functions: $G^{(c)}_{ij\ldots k} =$  $W[J] =$ 

1PI Green functions: $\Gamma_{ij\ldots k} =$  $\Gamma[\phi] =$ 

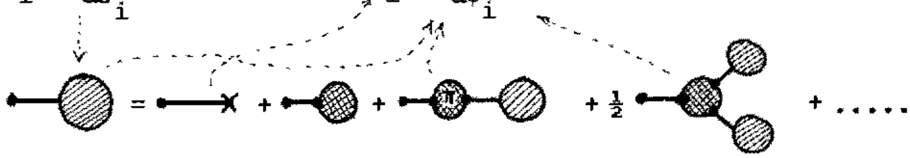
Full $\leftrightarrow$ connected relation:

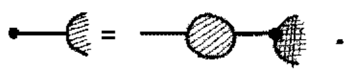
$$\frac{1}{Z[J]}\frac{d}{dJ_i}Z[J] = \frac{dW[J]}{dJ_i} + \frac{d}{dJ_i}$$



Connected $\leftrightarrow$ 1PI relations:

$$\phi_i = \frac{dW[J]}{dJ_i} \qquad\qquad 0 = J_i + \frac{d\Gamma[\phi]}{d\phi_i}$$



$$\frac{d}{dJ_i} = \frac{d^2W[J]}{dJ_i dJ_k}\frac{d}{d\phi_k} \quad \leftrightarrow$$ 

Dyson-Schwinger equations:

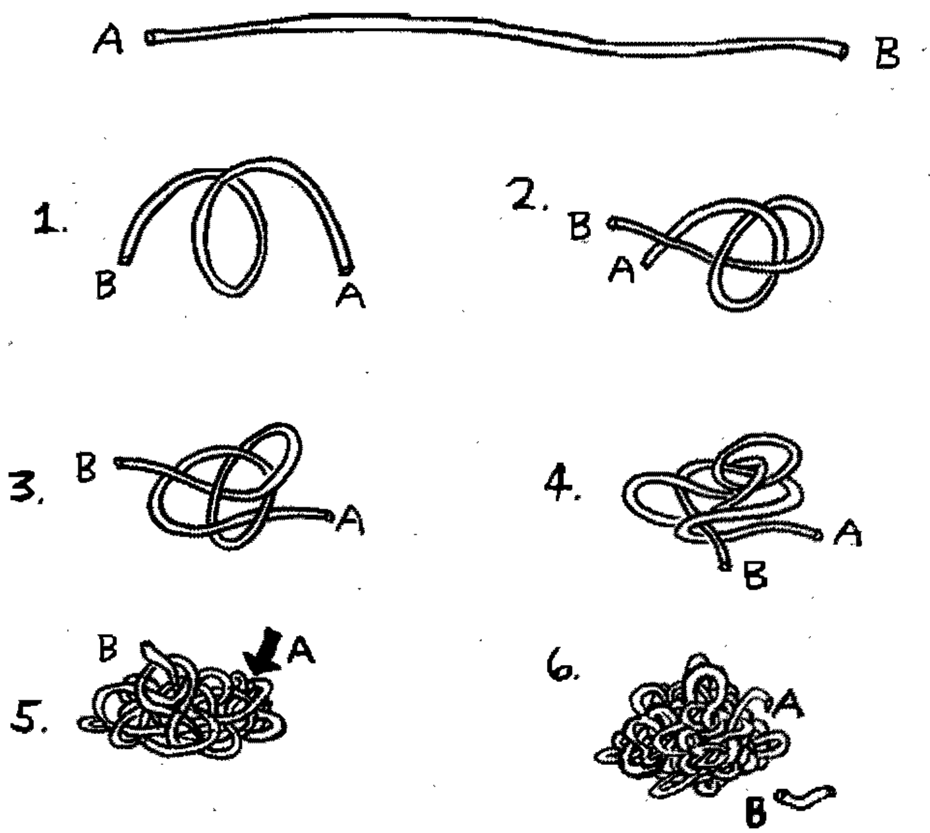
full $$\left(\frac{dS}{d\phi_i}\left[\frac{d}{dJ}\right] + J_i\right)Z[J] = 0 \quad ,$$

connected $$\frac{dS}{d\phi_i}\left[\frac{dW[J]}{dJ} + \frac{d}{dJ}\right] + J_i = 0 \quad ,$$

1PI $$\frac{d\Gamma[\phi]}{d\phi_i} = \frac{dS}{d\phi_i}\left[\phi + W''[J]\frac{d}{d\phi}\right] .$$

By now we are thoroughly fed up with longleggedy beasties, and the diagrammatic manipulations:

TYING THE NUDO DEL DIABLO OR DEVIL'S KNOT



(CONTINUED NEXT WEEK)

Let us now see whether the crow's vision of Quefithe is any more fun than the mole's version.

3. PATH INTEGRALS

An inconvenient aspect of the generating functional formalism is the proliferation of derivatives. Green function legs are pulled out by taking derivatives with respect to sources, equation (2.11), so that the Dyson-Schwinger equations are differential equations. This is a familiar problem. It is usually resolved by finding a transformation (such as Fourier transform) which diagonalizes the differential operators (for example, maps $\frac{d}{dx^\mu} \to k_\mu$). For generating functionals such transformation is called a path integral.

Path integrals have many virtues: they make the symmetries of the theory explicit, they help identify physically dominant configurations, and they suggest systematic ways of computing the quantum corrections to the classically dominant configurations (the saddlepoint expansion). Sometimes they can even be evaluated directly, without resorting to perturbative expansions, by Monte Carlo methods.

A. A Fourier transform

To illustrate the idea, let us get rid of $\frac{d}{dJ_i}$ derivatives by going from generating functionals to their Fourier transforms:

$$Z[J] = \int [d\phi]\tilde{Z}[\phi]e^{i\phi_i J_i} \quad , \tag{3.1}$$

$$[d\phi] = \frac{d\phi_1}{\sqrt{2\pi}} \frac{d\phi_2}{\sqrt{2\pi}} \cdots \quad , \tag{3.2}$$

$$- i\frac{d}{dJ_i} Z[J] = \int [d\phi]\phi_i \tilde{Z}[\phi]e^{i\phi_i J_i} \ . \tag{3.3}$$

Fields ϕ_i are dual to sources J_i in the same sense that momenta k^μ are dual to space coordinates x^μ. As the indices i, j, ... can take continuous values, these integrals are functional integrals. $\tilde{Z}[\phi]$ can be determined by taking a Fourier transform of the DS equation (2.15):

$$0 = \int [d\phi]\left(\frac{dS[\phi]}{d\phi_i} + J_i\right)\tilde{Z}[\phi]e^{i\phi_i J_i} \quad ,$$

$$-i\frac{d}{d\phi_i}\,\tilde{Z}[\phi] = \frac{dS[\phi]}{d\phi_i}\,\tilde{Z}[\phi] \; .$$

This is again an easy differential equation to solve. The solution is called the path integral representation of generating functionals:

$$Z[J] = \int [d\phi]e^{i(S[\phi]+\phi_i J_i)} \; . \tag{3.4}$$

In this "derivation" we were rather cavalier about factors of "i" and questions of convergence. As Jens, the serious young student of field theory, objects, we try one more time.

B. Gaussian integrals

It has probably not escaped your notice that the only integral an average physicist can do is the Gaussian integral

$$\int [d\phi]e^{-\frac{\phi^2}{2\lambda}} = \sqrt{\lambda} \; , \qquad [d\phi] = \frac{d\phi}{\sqrt{2\pi}} \; . \tag{3.5}$$

This is the Gaussian integral in one dimension. In more dimensions, Gaussian integrals make their appearance in a slightly jazzed-up form

$$\int [d\phi]e^{-\frac{1}{2}\phi_i \Delta^{-1}_{ij}\phi_j} = \sqrt{\mathrm{Det}\,\Delta} \quad . \tag{3.6}$$

Derivation: Take $\Delta_{ij} = \Delta_{ji}$. A real symmetric matrix can be diagonalized by a rotation R:

$$(R^{-1}\Delta R)_{ij} = \lambda_i \delta_{ij} \; .$$

Volume is rotationally invariant: $[d(R\phi)]=[d\phi]$. Diagonalization reduces the integral to a product of one-dimensional integrals (3.5):

$$\prod_i \int \frac{d\phi_i}{\sqrt{2\pi}} e^{-\frac{\phi_i^2}{2\lambda_i}} = \prod_i \lambda_i^{\frac{1}{2}}$$

The result can be expressed as a determinant:

$$\operatorname{Det} \Delta = \operatorname{Det}(R^{-1}\Delta R) = \left\| \begin{matrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \end{matrix} \right\| = \prod_i \lambda_i \quad .$$

Using the invariance of the measure under translation $\phi_i \to \phi_i + J_k\Delta_{ki}$, we can add sources and rederive the generating functional (2.23) for the free field theory:

$$Z_0[J] = \int [d\phi] e^{-\frac{1}{2}\phi_i\Delta^{-1}_{ij}\phi_j + \phi_i J_i} = \sqrt{\operatorname{Det}\Delta}\, e^{\frac{1}{2}J_i\Delta_{ij}J_j} \quad . \tag{3.7}$$

The square-root factor is an overall normalization (vacuum bubbles) which does not contribute to the connected diagrams and is (in this case) without physical significance. Remember that the collective index $\underline{i}$ can take both discrete and continuous values; (3.6) is the definition of the <u>functional</u> Gaussian integral.

The point of this whole exercise is that Gaussian integrals give us the desired fields-sources duality:

$$\frac{d}{dJ_i} \int [d\phi] e^{-\frac{\varepsilon\phi^2}{2} + \phi_i J_i} = \int [d\phi] \phi_i e^{-\frac{\varepsilon\phi^2}{2} + \phi_i J_i} \quad . \tag{3.8}$$

Now we can go back to our definition of the path integral, and make it slightly more respectable by introducing a Gaussian damping factor:

$$Z[J] = e^{S[\frac{d}{dJ}]} \int [d\phi] e^{-\frac{\varepsilon\phi^2}{2} + \phi_i J_i} \quad , \qquad \varepsilon \to 0_+ \quad .$$

This defines the path integral, at least as a formal power series in ϕ or d/dJ:

$$Z[J] = \int [d\phi] e^{S[\phi] + \phi_i J_i} \quad , \tag{3.9}$$

irrespective of whether the action is real or imaginary, or whether we have statistical or quantum mechanics in mind. In the above we have absorbed the damping factor into propagators:

$$S[\phi] = -\tfrac{1}{2}\, \phi_i (\Delta^{-1}_{ij} + \varepsilon) \phi_j + S_I[\phi] \quad . \tag{3.10}$$

This gives the correct imaginary parts for Feynman propagators in Minkowski space (this prescription is sometimes referred to as the Euclidicity postulate).

It will become quite apparent in the discussion of fermionic Green functions that the path integrals should not be taken too literally as "integrals". They are mostly tricks for replacing differential operators $\frac{d}{dJ}$ by number-valued fields ϕ. That should not give you sleepless nights. The history of the subject is that the problems are almost always first recognized and solved in the diagrammatic formalism, and later formulated elegantly in the language of path integrals.

In the path integral formalism, the full Green functions are field expectation values:

$$G_{ij..k} = \langle\phi_i\phi_j..\phi_k\rangle = \int[d\phi]\phi_i\phi_j..\phi_k\, e^{S[\phi]}/\int[d\phi]e^{S[\phi]} \quad . \tag{3.11}$$

In statistical mechanics, the action is a real number which assigns the probability (the Boltzmann weight) to a given field configuration. In quantum mechanics, the action is an imaginary phase which determines the amplitude of a given field configuration.

Exercise 3.B.1 Extend Gaussian integration to complex fields

$$\psi_k = \frac{1}{\sqrt{2}}\left(\phi_{2k-1} + i\phi_{2k}\right),\ \psi^k = \frac{1}{\sqrt{2}}\left(\phi_{2k-1} - i\phi_{2k}\right) \ .$$

Take the propagator Δ^i_j to be a hermitian matrix. Show that for complex fields the free field generating functional (3.7) is given by

$$\begin{aligned} Z[\eta,\bar{\eta}] &= \int[d\psi d\bar{\psi}]e^{-\bar{\psi}\Delta^{-1}\psi + \bar{\eta}\psi + \bar{\psi}\eta} \\ &= \operatorname{Det}\Delta\, e^{\bar{\eta}\Delta\eta} \ , \end{aligned} \tag{3.12}$$

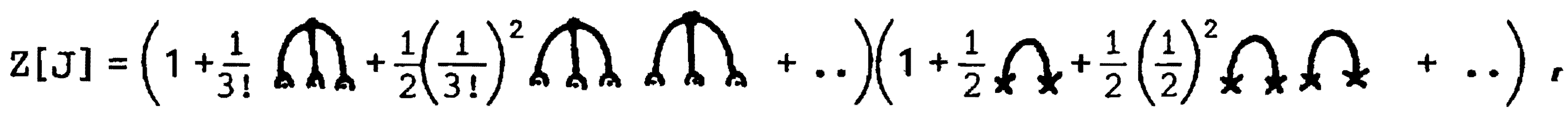
where η_k, η^k are complex sources.

C. Wick expansion

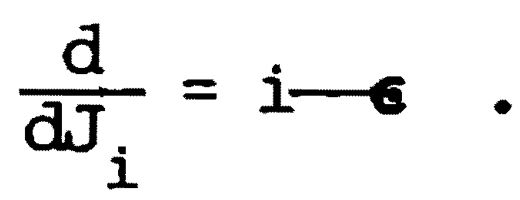
Splitting of the action into a quadratic part and an interaction part, as in (2.13) and (3.10), provides another way of generating the perturbation expansion:

$$Z[J] = \int[d\phi]e^{S_I[\phi] - \frac{1}{2}\phi\,\Delta^{-1}\phi + \phi\cdot J} = e^{S_I[\frac{d}{dJ}]}Z_0[J] \ . \tag{3.13}$$

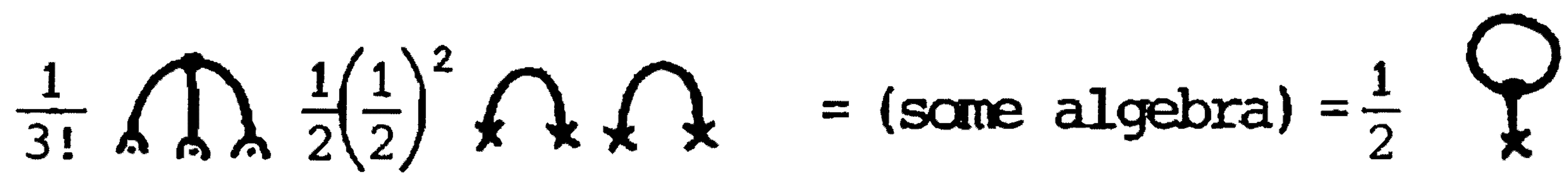
One expands both the interaction operator and the free field functional (3.7) as power series, and collects the nonvanishing terms:



where



For example,

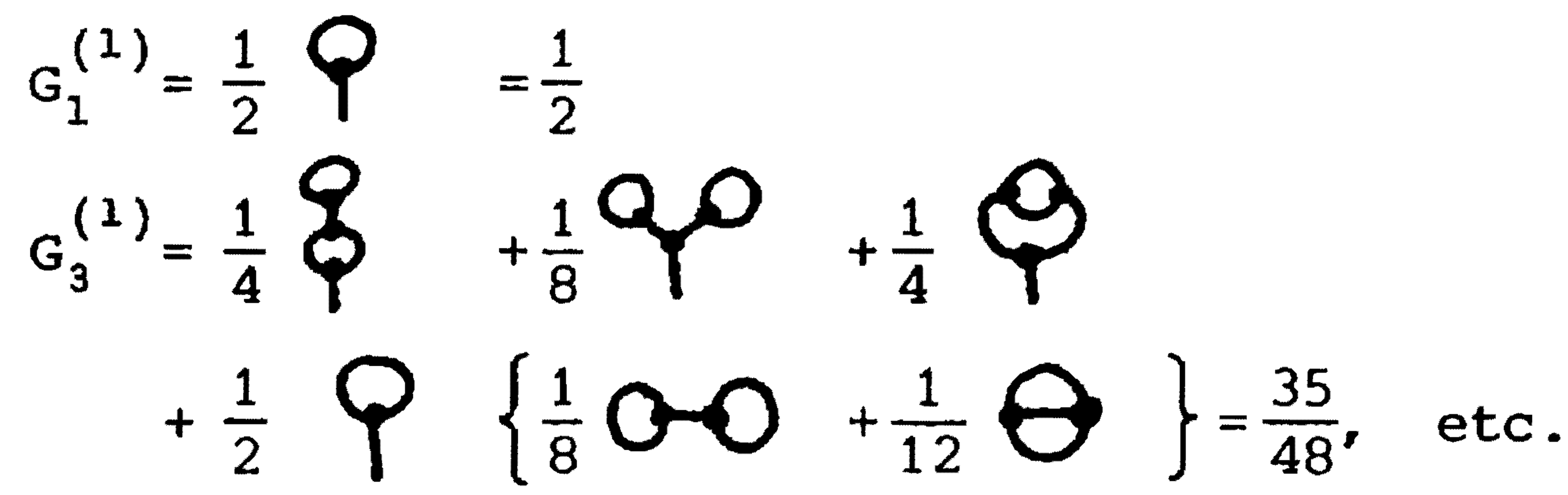


This is called the <u>Wick expansion</u>. It gives all the diagrams with the correct combinatoric factors, but is quite tedious. In practice, I prefer the DS equations.

<u>Exercise 3.C.1</u> Use the Wick expansion (3.13) to show that for zero-dimensional ϕ^3 theory (exercise 2.E.3):

$$G_k^{(m)} = \frac{(3k+m-1)!!}{k!\,(3!)^k} \qquad \text{if } 3k+m \text{ even}$$
$$= 0 \qquad \text{otherwise}$$

For example,



Hint: use the combinatorial identity

$$\left.\frac{d^k}{dJ^k} e^{J^2/2}\right|_{J=0} = (k-1)!!, \ k \text{ even} .$$

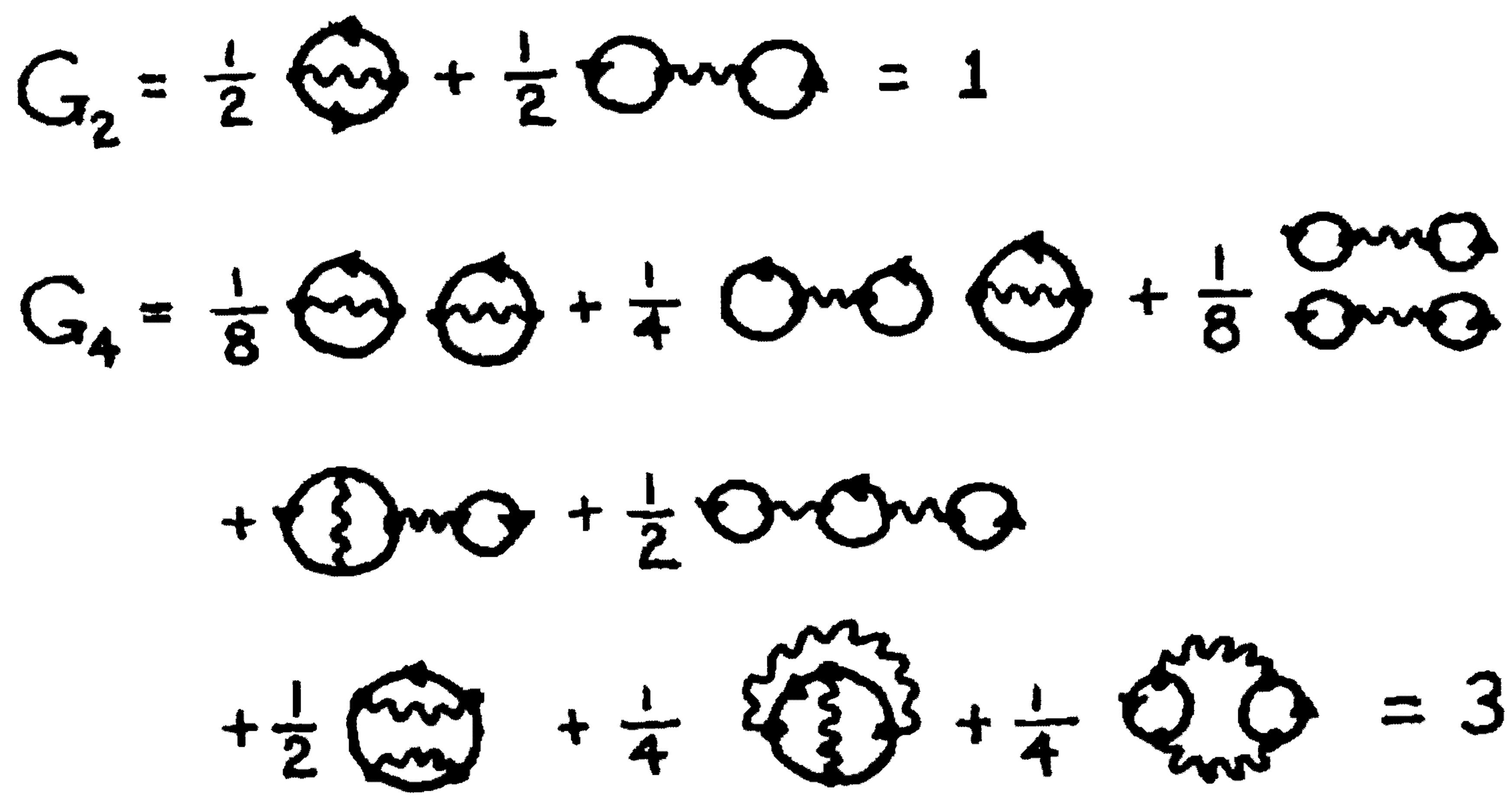
<u>Exercise 3.C.2</u> <u>Counting QED diagrams</u>. Consider a zero-dimensional QED-like action

$$S = -\ \bar\psi\psi - \tfrac{1}{2}A^2 + g\bar\psi A\psi + \bar\eta\psi + \bar\psi\eta + JA .$$

Show by Wick expansion that

$$G_k^{(e,p)} = \frac{(k+e)!\,(k+p-1)!!}{k!} \ , \ k+p \text{ even},$$

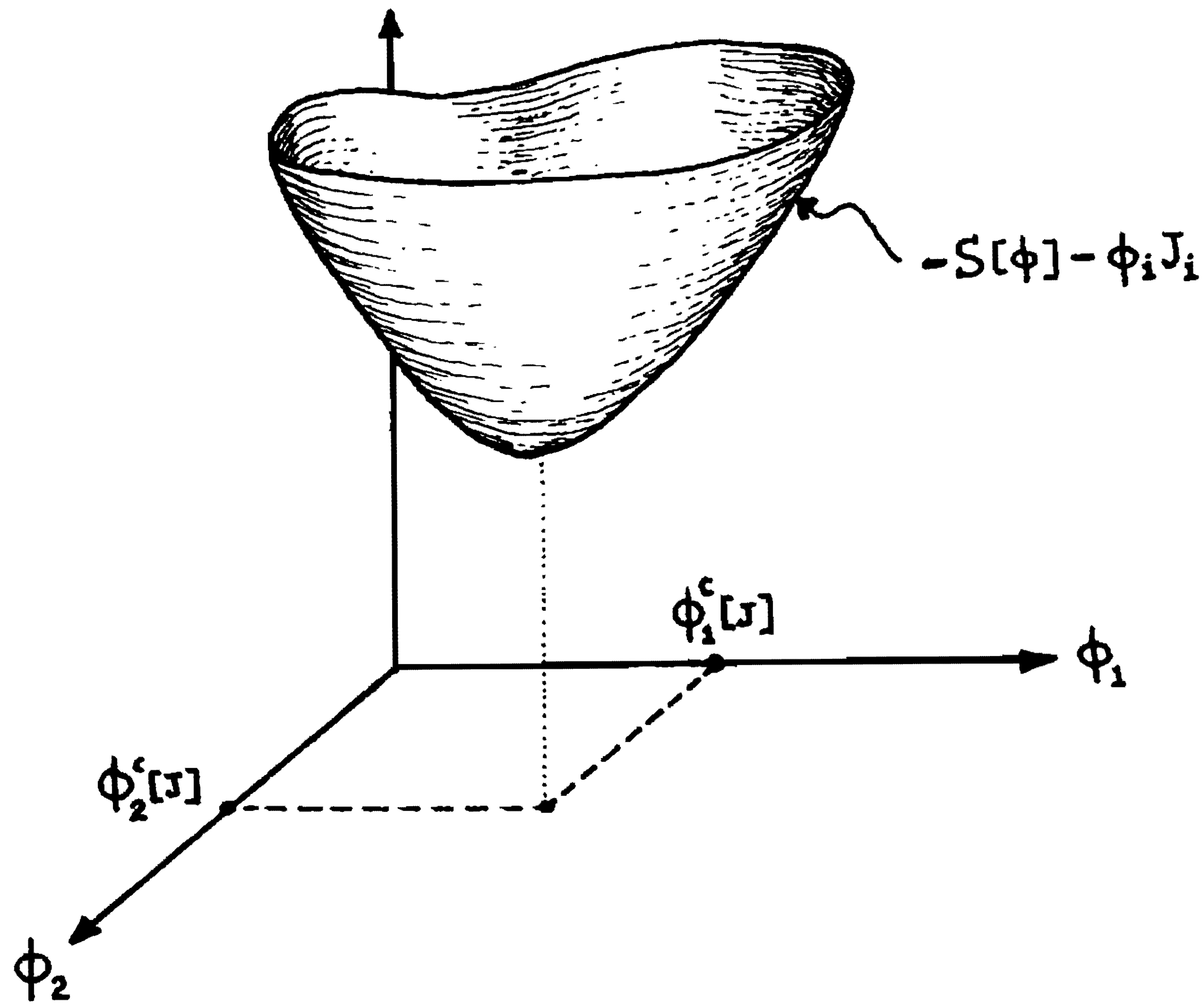
where e is the number of electron lines traversing the diagram, p is the number of photon legs, and k is the number of vertices. For example:



Hint: use $\left(\frac{d}{d\bar{\eta}}\frac{d}{d\eta}\right)^k e^{\eta\bar{\eta}} = k!$

D. Tree expansion

Let us take the path integral (3.9) very literally, and look at it as an ordinary multidimensional integral. We take ϕ_i to be real variables, and action a real function. The integral is finite only if the action is large and negative (high price of straying from the beaten path) almost everywhere, except for some localized regions of the ϕ-space. Highly idealized, the action looks something like this (we have suppressed an infinity of other coordinates):



(3.14)

The path integral will be dominated by the value of the action at the maximum (or maxima). ϕ^c, the location of the maximum, is determined by the extremum condition (cf. (3.9))

$$\frac{dS[\phi^c]}{d\phi_i} + J_i = 0 \ . \qquad (3.15)$$

Hence the path integral is dominated by the solutions of the classical equations of motion. That these are really the familiar classical equations of motion can be seen by abandoning for a moment the collective index notation and writing out the integrations in the euclidean ϕ^3 action explicitly:

$$S[\phi] = -\int dx \left[\tfrac{1}{2}\phi(x)\,(-\partial^2 + m^2)\,\phi(x) + \frac{g}{3!}\phi^3(x)\right]$$

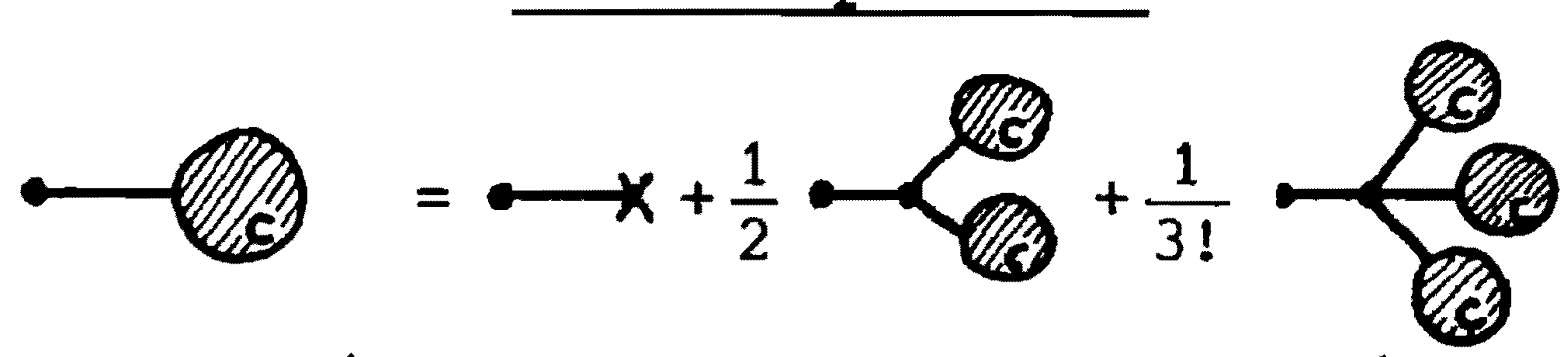
$$-\frac{\delta S[\phi]}{\delta\phi(x)} = (-\partial^2 + m^2)\,\phi(x) + \frac{g}{2}\phi^2(x) = J(x) \ . \qquad (3.16)$$

The classical equations of motion differ from the quantum equations of motion (the DS equations (2.33)) by the absence of $d/d\phi$ terms. To interpret the classical solutions diagrammatically, we split the action into a quadratic part and an interaction part, as in (2.13):

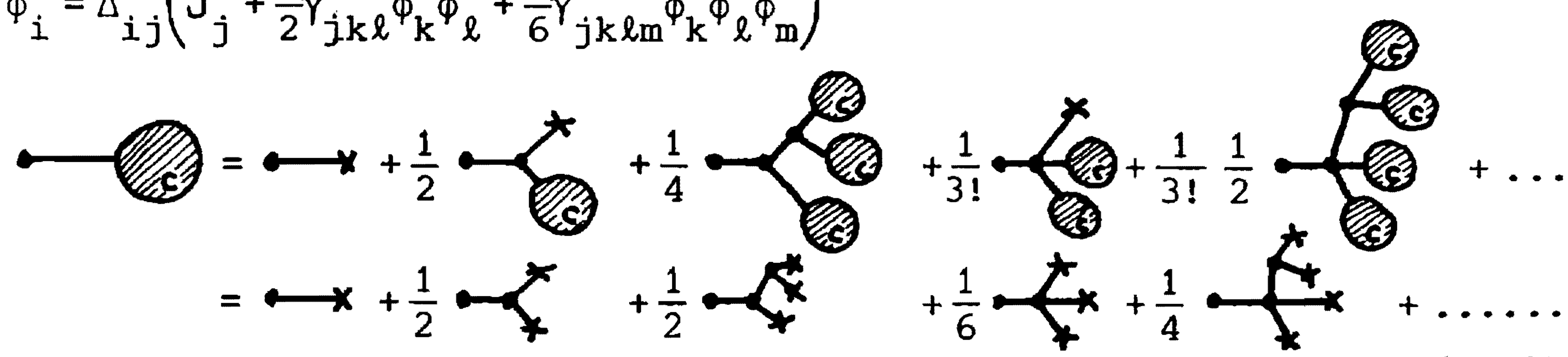
$$-\Delta^{-1}_{ij}\phi^c_j + \frac{dS_I[\phi^c]}{d\phi_i} + J_i = 0 \ ,$$

$$\phi^c_i[J] = \Delta_{ij}\left(J_j + \frac{dS_I[\phi^c]}{d\phi_j}\right) \ . \qquad (3.17)$$

Unlike the quantum DS equations (2.21), the classical equations involve no loop terms. The iteration of the classical equations results in the tree expansion:



$$\phi^c_i = \Delta_{ij}\left(J_j + \frac{1}{2}\gamma_{jk\ell}\phi^c_k\phi^c_\ell + \frac{1}{6}\gamma_{jk\ell m}\phi^c_k\phi^c_\ell\phi^c_m\right)$$



(3.18)

This expression for the expectation value of a field is classical or deterministic in the sense that it involves no summations over virtual excitations, so it does not "feel" the probabilistic (quantum) aspects of the theory. It is also a way of getting at non-perturbative effects (such as spontaneous symmetry breaking): ϕ^c represents an infinite resummation which replaces the false vacuum ($\langle\phi\rangle \neq 0$) by the true ground state ($\langle\phi-\phi^c\rangle = 0$).

E. Legendre transformations

The classical approximation to a path integral is the value of the integrand at its extremum (3.15) (up to an irrelevant overall normalization factor):

$$Z_c[J] = e^{S[\phi^c]+\phi_i^c J_i} ,$$

$$W_c[J] = S[\phi^c] + \phi_i^c J_i \ . \tag{3.19}$$

The 1PI generating functional $\Gamma[\phi]$ satisfies extremum condition (2.27), analogous to the classical equations of motion (3.15). Indeed, the diagrammatic relation (2.24) between the connected and the 1PI Green function is a tree expansion of the connected Green functions, with all quantum loops confined to 1PI Green functions. Hence the 1PI generating functional $\Gamma[\phi]$ can be interpreted as an <u>effective</u> (or quantum) <u>action</u>, which satisfies the classical equations of motion (3.15), and where all quantum (or fluctuation) effects are incorporated into effective (proper) vertices, i.e. 1PI Green functions. Equation (3.19) becomes a relation between the connected and the 1PI Green functions:

$$W[J] = \Gamma[\phi] + \phi_i J_i \ . \tag{3.20}$$

This is just the Legendre transformation (2.28).

F. Saddlepoint expansion

The classical (tree, Born) approximation to Green functions is given by (3.19). The first quantum (or statistical fluctuation) correction is obtained by approximating the bottom of the potential (3.14) by a parabola, i.e. by keeping the quadratic term in the Taylor expansion

$$S[\phi] + \phi_i J_i = S[\phi^c] + \phi_i^c J_i + \overbrace{(\phi_i - \phi_i^c)\left(\frac{dS[\phi^c]}{d\phi_i} + J_i\right)}^{=\,0}$$
$$+ \frac{1}{2}(\phi_i - \phi_i^c)\frac{d^2S[\phi^c]}{d\phi_i d\phi_j}(\phi_j - \phi_j^c) + \frac{1}{3!}\;\ldots\ldots \tag{3.21}$$

The linear term vanishes because we are expanding around an extremum, and the quadratic term can be integrated by the Gaussian integration (3.6):

$$Z[J] \simeq e^{S[\phi^c] + \phi_i^c J_i}\,\frac{1}{\sqrt{\mathrm{Det}\left(-\frac{d^2S[\phi^c]}{d\phi_i d\phi_j}\right)}} \;. \tag{3.22}$$

To interpret the determinant diagrammatically, we use

$$\mathrm{Det}\,M = e^{\mathrm{tr}\,\ell n M} \;. \tag{3.23}$$

Derivation:

$$\begin{aligned}
\delta(\ell n\,\mathrm{Det}\,M) &= \ell n\,\mathrm{Det}(M+\delta M) - \ell n\,\mathrm{Det}\,M \\
&= \ell n\,\mathrm{Det}(1+\delta M/M)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Det}(1+\Delta) &= (1+\Delta_{11})(1+\Delta_{22})\ldots\ldots \\
&\quad - \Delta_{21}\Delta_{12}(1+\Delta_{33})\ldots\ldots + \ldots \\
&= 1+\mathrm{tr}\Delta+0(\Delta^2)
\end{aligned}$$

$$\begin{aligned}
&\simeq \ell n(1+\mathrm{tr}\,\delta M/M) \\
&\simeq \mathrm{tr}\,\delta M/M \\
&= \mathrm{tr}\,\delta(\ell n M) \\
&= \delta(\mathrm{tr}\,\ell n\,M)
\end{aligned}$$

hence

$$\ell n\,\mathrm{Det}\,M = \mathrm{tr}\;\ell n\,M \;.$$

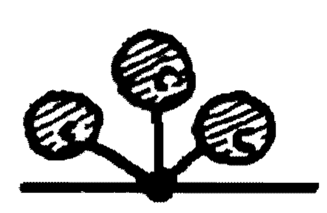
This is obvious for diagonalizable matrices:

$$\mathrm{Det}\,M = \prod_i \lambda_i = e^{\sum_i \ell n \lambda_i} = e^{\mathrm{tr}\;\ell n\,M} \;.$$

QED

Splitting S″ into the bare propagator and the interactions with the classical background field

$$\frac{d^2S[\phi^c]}{d\phi_i d\phi_j} = -\,\Delta_{ij}^{-1} + \gamma_{ij}[\phi^c] \;,$$

$\gamma_{ij}[\phi^c] =$  $+\frac{1}{2}$  $+\frac{1}{6}$  $+\;\ldots\ldots\ldots$ (3.24)

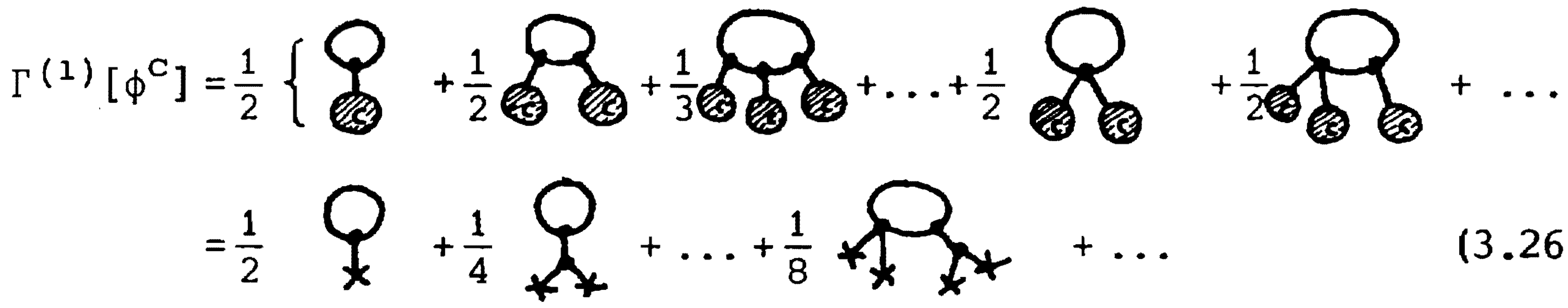
we can write the first approximation to Z[J] as

$$Z[J] \simeq e^{S[\phi^c] - \frac{1}{2}\mathrm{tr}\,\ell n(1-\Delta\gamma[\phi^c]) + \phi_i^c J_i} \cdot \sqrt{\mathrm{Det}\Delta} \quad ,$$

(the overall $\sqrt{\mathrm{Det}\Delta}$ factor can be dropped). In this approximation the effective action is given by

$$\Gamma[\phi^c] = S[\phi^c] + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\,\mathrm{tr}(\Delta\gamma[\phi^c])^k \ . \tag{3.25}$$

This is called the one-loop effective action, as its diagrammatic expansion consists of all one-loop diagrams:



(3.26)

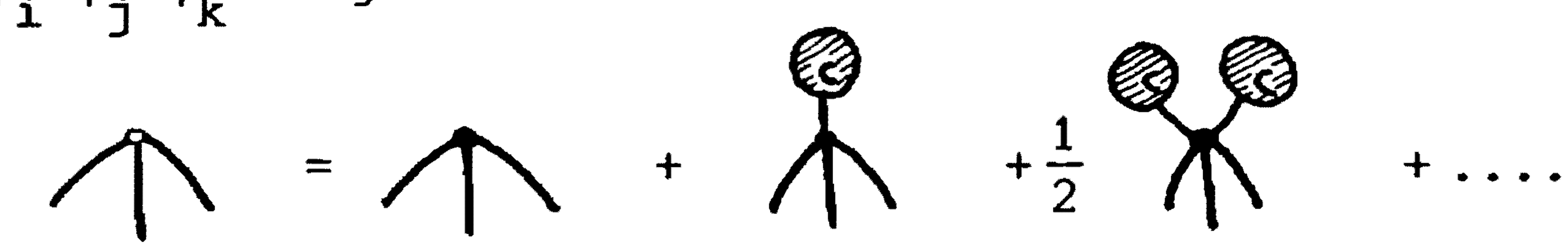
The higher loop contributions to the effective action can be computed by the ordinary perturbation expansion, with ϕ^c playing the role of a "background field", i.e. the field which describes the classical background configuration in which the propagation and the interactions take place. This expansion is carried out in the next exercise.

Exercise 3.F.1 The loop expansion for effective action. Introduce an auxiliary source K_i in the saddlepoint expansion (3.21)

$$Z[J] = e^{S[\phi^c]+\phi_i^c J_i}\int[d\phi]e^{\frac{1}{2}\phi_i\frac{d^2S[\phi^c]}{d\phi_i d\phi_j}\phi_j + S_I^c[\phi] + \phi_i K_i}\Big|_{K=0}$$

$$S_I^c[\phi] = \frac{1}{3!}\,\frac{d^3S[\phi^c]}{d\phi_i d\phi_j d\phi_k}\,\phi_k\phi_j\phi_i + \frac{1}{4!}\,\frac{d^4S[\phi^c]}{d\ldots\ldots}\ \ldots\ldots$$

$$\frac{d^3S[\phi^c]}{d\phi_i d\phi_j d\phi_k} =$$



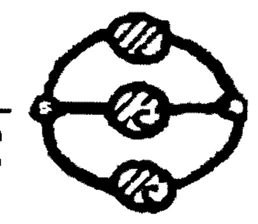
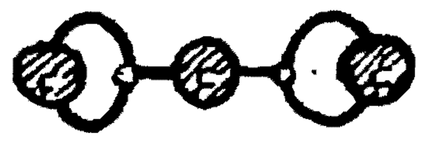
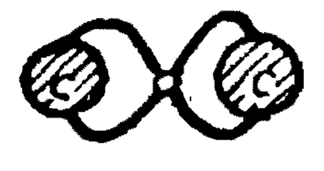
Now we can use the Wick expansion (3.13) to write the loop expansion for J:

$$Z[J] = e^{S[\phi^c]+\phi_i^c J_i + \frac{1}{2}\,\mathrm{tr}\,\ell n\left(-\frac{d^2S[\phi_c]}{d\phi_i d\phi_j}\right)}$$

$$\times\, e^{S_I^c[\frac{d}{dK}]}e^{\frac{1}{2}K_i\left(-\frac{1}{\frac{d^2S[\phi^c]}{d\phi_i d\phi_j}}\right)K_j}\Bigg|_{K=0} \ .$$

We can interpret this expansion as an ordinary perturbation expansion for vacuum bubbles, with propagators and vertices dependent on the classical background field ϕ^c. All possible insertions of sources J_i are summed up into tree insertions by $\phi^c[J]$. Compute the beginning of this expansion

$$W[J] = S[\phi^c] + \tfrac{1}{2}\,\mathrm{tr}\,\ln(\Delta\gamma[\phi^c])$$

$$+\frac{1}{12}$$  $$+\frac{1}{8}$$  $$+\frac{1}{8}$$  $$+ \ldots \qquad (3.27)$$

Compare with the results of exercise 2.I.2. Write down the beginning of the loop expansion for the effective action $\Gamma[\phi]$.

Exercise 3.F.2 Consider a QED-like theory from exercise 2.D.1. The path integral can be written as

$$Z[J,\bar{\eta},\eta] = \int [dA] e^{-\frac{1}{2}A_\mu D^{-1}_{\mu\nu}A_\nu + J_\mu A_\mu}\, Z[\bar{\eta},\eta]_A$$

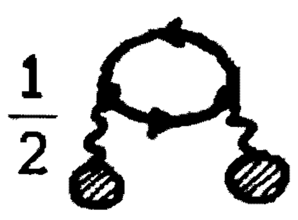
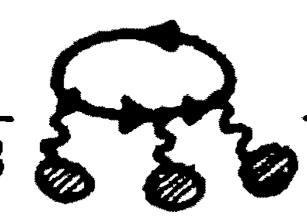
$$Z[\bar{\eta},\eta]_A = \int [d\bar{\psi}][d\psi] e^{-\bar{\psi}(\Delta^{-1} - \not{A})\psi + \bar{\eta}\psi + \bar{\psi}\eta}$$

$$\not{A}_{\alpha\beta} = A_\mu(\gamma^\mu)_{\alpha\beta}\ .$$

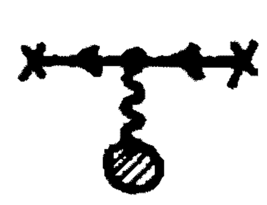
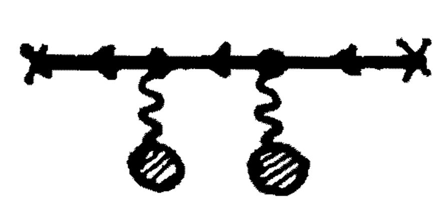
$Z[\bar{\eta},\eta]_A$ can be interpreted as the generating functional for the free electrons propagating in the background field $\not{A}_{\alpha\beta}$. Show that

$$Z[\bar{\eta},\eta]_A = e^{-\ln \mathrm{tr}(1-\Delta\not{A}) + \bar{\eta}\Delta \frac{1}{1-\not{A}\Delta}\eta}\ . \qquad (3.28)$$

The trace part accounts for all virtual electron loops:

$$-\ln \mathrm{tr}(1-\Delta\not{A}) =$$  $$+\frac{1}{2}$$  $$+\frac{1}{3}$$  $$+ \ldots \qquad (3.29)$$

while the source term describes the propagation of the electron in the background A_μ field:

$$\bar{\eta}\Delta\frac{1}{1-\not{A}\Delta}\eta =$$  $$+$$  $$+$$  $$+ \ldots . \qquad (3.30)$$

Exercise 3.F.3 Counting QED diagrams. (Continuation of exercise 3.C.2). Integrate over "photon" fields to obtain

$$Z[J,\bar{\eta},\eta] = e^{-\ln(1-g\frac{d}{dJ}) + \bar{\eta}\frac{1}{1-g\frac{d}{dJ}}\eta}\, e^{J^2/2}\ .$$

Show that the number of full electron propagator diagrams without electron loops is

$$D_k = (k-1)!!\ , \qquad k \text{ even}$$

$$D_2 =$$ 

$$D_4 =$$  $$+$$  $$+$$  , etc.

What is the number of the photon self-energy graphs with only one electron loop? Furry's theorem says that all diagrams with electron loops with odd numbers of photon legs vanish. They can be eliminated from the loop expansion by replacement

$$\ell n(1-gA) \to \tfrac{1}{2}\,\ell n(1-gA) + \tfrac{1}{2}\,\ell n(1+gA) \ .$$

Show that the number of full electron propagators (electron loops included) is

$$D_k = \frac{(k+1)!!\,(k-1)!!}{k!!}, \quad k \text{ even} \ .$$

Check that $D_4 = \frac{45}{8}$ (this is not an integer, as disconnected graphs like

$\frac{1}{4}$ 

are included).

G. Point transformations

One of the main advantages of the path integral formalism is the compactness of Ward identities. The key idea is simple. In

$$Z[J] = \int [d\phi] e^{S[\phi]+J_i\phi_i}$$

the left-hand side is independent of ϕ, hence invariant under infinitesimal point transformations

$$\phi_i \to \phi_i + \varepsilon F_i[\phi]$$

$$F_i[\phi] = f_i + f_{ij}\phi_j + f_{ijk}\phi_j\phi_k + \ldots\ldots \ . \tag{3.31}$$

The Jacobian for this change of variables is (dropping terms of order ε^2 and higher):

$$[d\phi] \to [d\phi]\det\left|\delta_{ij} - \varepsilon\frac{dF_i[\phi]}{d\phi_j}\right| = [d\phi]\left(1 - \varepsilon\frac{dF_i[\phi]}{d\phi_i}\right) .$$

Collecting all terms up to order ε we obtain

$$Z[J] = \int [d\phi]\left(1-\varepsilon\frac{dF_i[\phi]}{d\phi_i}\right)e^{S[\phi]+J_i\phi_i}\left(1+\varepsilon\left(\frac{dS[\phi]}{d\phi_i} + J_i\right)F_i[\phi]\right) ,$$

$$0 = \int [d\phi]\left\{\left(\frac{dS[\phi]}{d\phi_i} + J_i\right)F_i[\phi] - \frac{dF_i[\phi]}{d\phi_i}\right\}e^{S[\phi]+J_i\phi_i} \ .$$

Remembering the equivalence $\phi_i \leftrightarrow d/dJ_i$ we can write this as

$$\left\{\left(\frac{dS}{d\phi_i}\left[\frac{d}{dJ}\right] + J_i\right)F_i\left[\frac{d}{dJ}\right] - \frac{dF_i}{d\phi_i}\left[\frac{d}{dJ}\right]\right\}Z[J] = 0 \ . \tag{3.32}$$

We have already, unknowingly, used a special case of this identity. If $F_i[\phi] = f_i$ = constant (a translation), then

$$\left(\frac{dS}{d\phi_i}\left[\frac{d}{dJ}\right] + J_i\right)Z[J] = 0 \quad . \tag{2.15}$$

The Dyson-Schwinger equations are consequences of the translational invariance of path integrals. A more interesting situation arises if (3.31) is a symmetry of the action

$$\frac{dS[\phi]}{d\phi_i} F_i[\phi] = 0 \quad . \tag{3.33}$$

If this transformation also leaves invariant the measure $[d\phi]$, then (3.32) reduces to a <u>Ward identity</u>:

$$J_i F_i\left[\frac{d}{dJ}\right]Z[J] = 0 \ . \tag{3.34}$$

The Ward identities are immensely important. They tell us how the symmetries of the action (classical theory) relate various Green functions (quantum theory). About this - later.

<u>Exercise 3.G.1</u> <u>Derivative interactions</u>. Throughout these notes we treat the sums over discrete indices and the integrals over continuous variables as the one and the same thing. However, for derivative interactions we must be more careful. Consider a one-dimensional example with action

$$S = \int dt\ \mathcal{L}(t)$$

where the Lagrangian density includes derivatives:

$$\mathcal{L}(t) = \tfrac{1}{2}\dot{\phi}_i K_{ij}\dot{\phi}_j + L_i\dot{\phi}_i - V(\phi) .$$

Show that the correct definition of the path integral is

$$Z[J] = \int [d\phi]\,(\mathrm{Det}\,K)^{\frac{1}{2}}\, e^{S+\int dt J_i\phi_i} \ .$$

Hint: the path integral must be invariant under variable change

$$K \to \frac{\partial\phi}{\partial\bar{\phi}}\,\bar{K}\,\frac{\partial\phi}{\partial\bar{\phi}}, \quad [d\phi] \to [d\bar{\phi}]\mathrm{Det}\left(\frac{\partial\phi}{\partial\bar{\phi}}\right) .$$

H. Summary

The basic assumption of the statistical (quantum) mechanics is that the physical processes can be described additively, as sums of probabilities (amplitudes). Whether we describe these sums by diagrams (generating functional formalism) or field configurations (path integral formalism) is largely a matter of convenience. The two formalisms offer two ways of visualising the relation between the classical and the quantum physics.

In the path integral picture, the transition rates are dominated by the valleys of the potential, and the quantum effects are the heavily penalized forays up the hillsides. In the statistical mechanics they are suppressed by small Boltzmann weights; in quantum mechanics they are suppressed by destructive interference of phases.

In the Feynman diagram picture, physical processes are dominated by classical propagation (tree diagrams) and the quantum effects are represented by internal loops (virtual excitations).

The two pictures are related by

$$G_{ij..k} = \langle\phi_i\phi_j..\phi_k\rangle = \frac{1}{Z[0]}\int[d\phi]\phi_i\phi_j..\phi_k e^{S[\phi]} \ .$$

A path integral is dominated by the extremal solutions of the classical equations of motion

$$\frac{dS[\phi^c]}{d\phi_i} + J_i = 0 \ .$$

The quantum effects can be included systematically by the loop expansion of the effective (quantum) action:

$$\Gamma[\phi^c] = S[\phi^c] - \tfrac{1}{2}\mathrm{tr}\,\ln\,(1-\Delta\gamma[\phi^c]) + \ldots.$$

$$\gamma_{ij}[\phi^c] = \frac{d^2S_I[\phi^c]}{d\phi_i d\phi_j} \ .$$

The classical symmetries of the action

$$F_i[\phi]\frac{dS[\phi]}{d\phi_i} = 0$$

imply the quantum symmetries, or Ward identities

$$J_iF_i\left[\frac{d}{dJ}\right]Z[J] = 0 \ .$$

4. FERMIONS

A. Pauli principle

In chapter 2 we have assumed that the Green functions are symmetric, i.e. that the particles we are describing are bose particles. What happens if the Pauli principle is at work? The Pauli principle is the quantum mechanical version of Archimedes' law. Archimedes' law says that two bodies cannot be in the same place at the same time; the Pauli principle does not allow existence of more than one particle in a given quantum-mechanical state.

In the Green function formalism the state of a particle is specified by its collective index (particle type, spin, position ...). Take a source which produces a particle in a definite quantum-mechanical state, i.e. a source which is nonvanishing only for one value of the collective index:

$$J_i = \delta_{im} \quad .$$

If the Pauli principle is at work, the Green functions must vanish any time two or more of their indices take the same value:

$$G_{ijk..\ell} J_i J_j = 0 \quad . \tag{4.1}$$

The basic assumption of the whole scheme that we are expounding here is that the amplitudes are additive. A linear superposition of state is also a state, and it too must satisfy the Pauli principle (here $K_i = \delta_{i\ell}$ is a source for a particle in state $\underline{\ell}$):

$$G_{ijk..}(J_i + K_i)(J_j + K_j) = 0$$

$$\Rightarrow (G_{ijk..} + G_{jik..}) J_i K_j$$

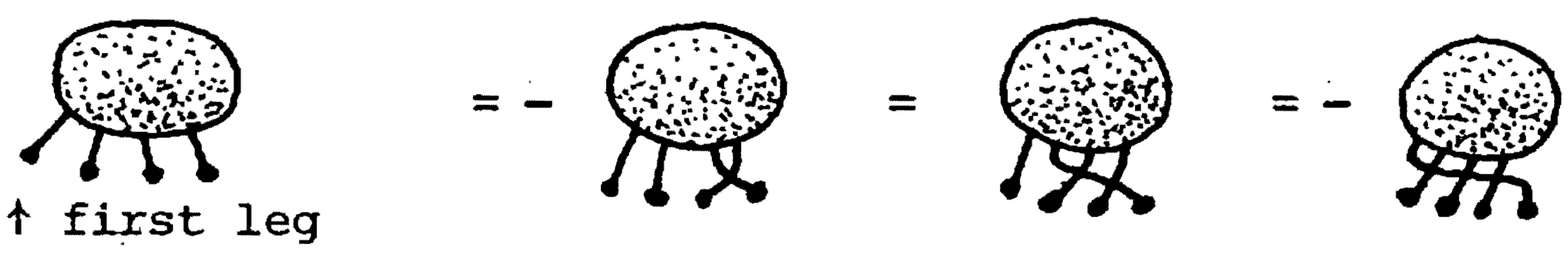
Consequently, the Green functions must be antisymmetric under interchange of fermionic indices:

$$G_{ijk...} = - G_{jik...} = G_{jki...} = \ldots \quad . \tag{4.2}$$

(In the compact index notation a multiplet can include both bosons and fermions: for example, for QED (cf. equation (3.27)) $\phi_i = (\psi, \bar{\psi}, A_\mu)$ stands for electrons, positrons and photons. In such cases we have to distinguish between the fermionic and the bosonic indices.)

From now on I will consider only the theories in which all Green functions have even numbers of fermionic legs. Another way of saying this is that we shall always assume that the action is a commuting number.

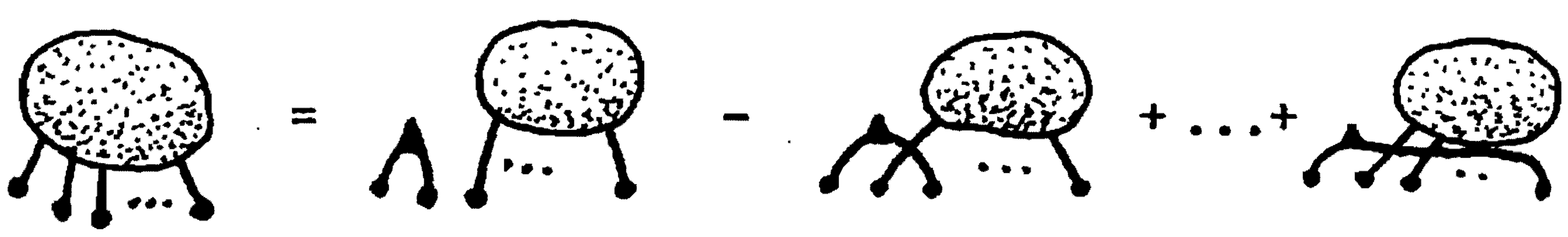
Fermionic Green functions with even numbers of legs are anti-cyclic:



$$\langle\psi_i\psi_j\psi_k\psi_\ell\rangle = -\langle\psi_i\psi_j\psi_\ell\psi_k\rangle = \langle\psi_i\psi_\ell\psi_j\psi_k\rangle = -\langle\psi_\ell\psi_i\psi_j\psi_k\rangle \quad . \tag{4.3}$$

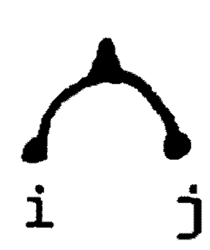
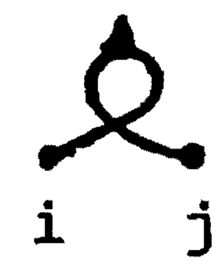
In order to keep track of signs, the diagrammatic notation must indicate which leg is the first leg. We do it by always drawing the fermionic legs below the Green function blobs, and taking the leftmost leg to be the first one. This fixes all relative signs. The overall sign is physically irrelevant.

The perturbation expansion can be generated by the Dyson-Schwinger equations, just as in the bosonic case. The diagrams and the combinatoric factors are the same; the only difference is the signs due to the antisymmetry of Green functions. For example, the free fermion field theory DS equations are



(4.4)

Fermionic propagators are antisymmetric, so the first and the second legs must be distinguished. We do this diagrammatically by drawing a little wart on the propagator:

$\Delta_{ij} =$  $= -$  $= -\Delta_{ji}$. (4.5)

Exercise 4.A.1 Can you prove that fermionic Green functions must have an even number of legs?

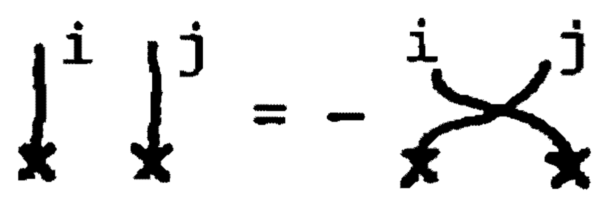
Exercise 4.A.2 Can you prove that fermionic Green functions need not have an even number of legs?

B. Anticommuting sources

In the bosonic case, the discussion of the general properties of Green functions was greatly facilitated by the introduction of generating functionals. In the fermionic case we cannot simply add scalar source functions (2.4) and form the vacuum Green function (2.10), because this would yield zero, identically:

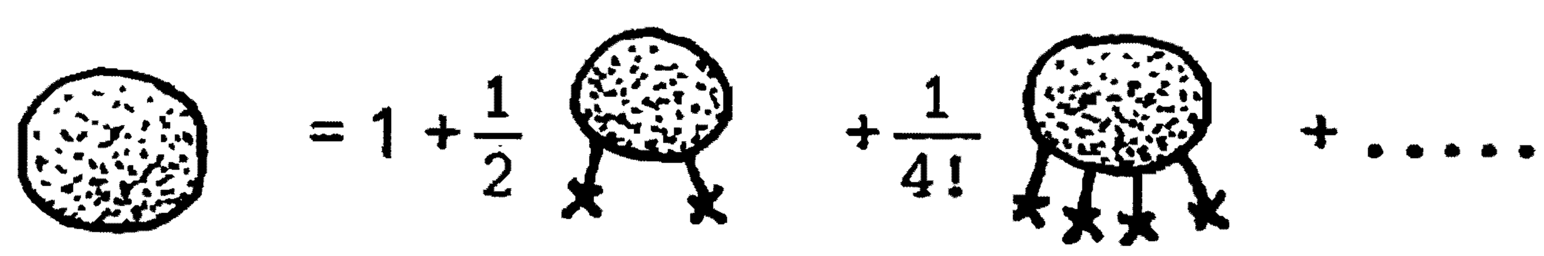
$$G_{ijk\ldots} J_i J_j J_k \ldots = G_{ijk\ldots} \frac{1}{2}(J_i J_j - J_j J_i) J_k = 0 \quad .$$

However, a simple trick provides a way out; we replace J_i by anticommuting sources:



$$\eta_i \eta_j = - \eta_j \eta_i \quad . \tag{4.6}$$

Then the fermionic generating functional can be defined as



$$Z[\eta] = \sum_{m=0} \frac{1}{(2m)!} G_{ij..k} \eta_k \ldots \eta_j \eta_i \quad . \tag{4.7}$$

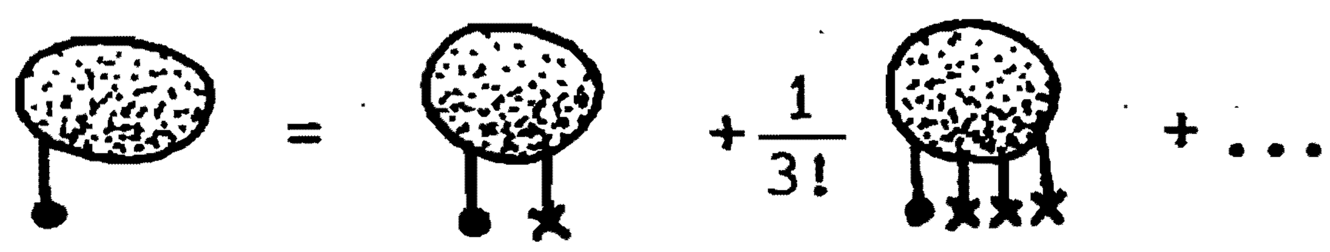
(Remember, our Green functions always have even numbers of legs.)

The signs due to sources are kept track of by drawing the sources ordered along the bottom of the diagram. Green functions can be retrieved from the generating functional by differentiation, just as in the bosonic case (2.11). However, the derivatives must also be anticommuting:

$$\frac{d}{d\eta_i} \eta_j = \delta_{ij} - \eta_j \frac{d}{d\eta_i} \quad ,$$

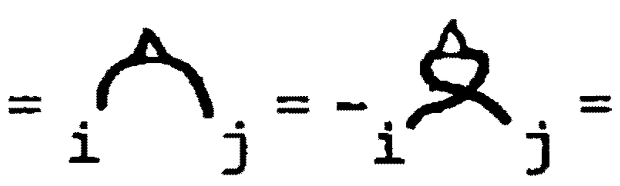
$$\frac{d}{d\eta_i} \frac{d}{d\eta_j} = - \frac{d}{d\eta_j} \frac{d}{d\eta_i} \quad . \tag{4.8}$$

All the relations between the full, connected and 1PI generating functionals that we have derived for the bosonic case take the same form for the fermionic generating functionals. There is only one sign subtlety. As all the terms in (4.7) involve even numbers of sources, all generating functionals are commuting numbers, and the sources implicit in them lead to no sign confusion. However, if a leg is pulled out by differentiation, the relative ordering of the implicit sources is important for the sign determination. Diagrammatically we fix the sign by requiring that all the implicit sources lie to the right of the pulled legs:

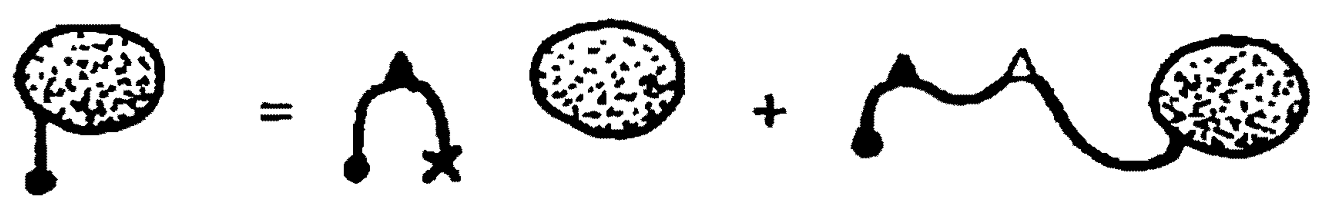
$$\frac{d}{d\eta_i} Z[\eta] =$$  $$(4.9)$$

Exercise 4.B.1 Fermionic loops. (This exercise is a convoluted attempt to prove the minus sign rule for fermions by diagrammatic means.) The simplest interacting fermionic field theory has only a bilinear interaction term:

$$S_I[\psi] = \frac{1}{2}\psi_i \not{A}_{ji}\psi_j$$

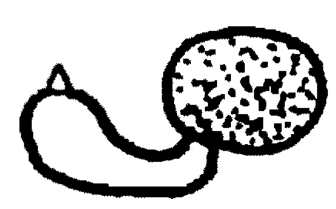
$\not{A}_{ij} =$  $= -\not{A}_{ji}$.

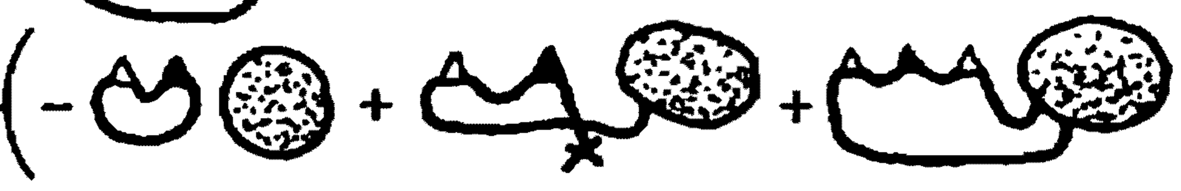
Here $\not{A}$ could be an external background photon field $\not{A}_{ij} = gA_\mu(\gamma^\mu)_{ij}$, as in (3.27). The DS equations corresponding to (4.4) are



$$\frac{d}{d\eta_i} Z[\eta] = \Delta_{ij}\left(\eta_j + \not{A}_{kj}\frac{d}{d\eta_k}\right)Z[\eta] \ . \qquad (4.10a)$$

Construct the DS equation for pulling out a "photon" $\not{A}$. This can be done by differentiating $Z[\eta]$ with respect to the coupling constant; a 2-leg vertex gets pulled out. Pull the first fermion leg. It either ends in the second leg, on a source, or on a 2-leg vertex:

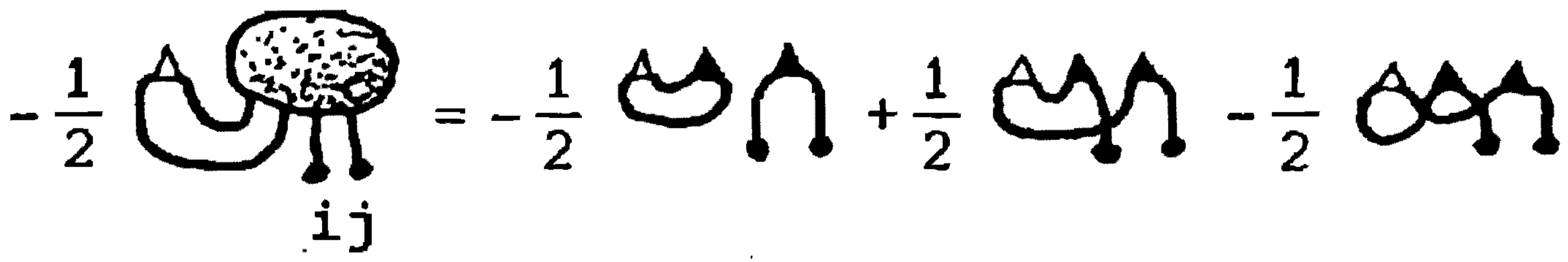
$g\frac{d}{dg}Z[\eta] = -\frac{1}{2}$ 

$= \frac{1}{2}\Big($  $\Big)$

$$= \frac{1}{2}\left(-\mathrm{tr}\not{A}\Delta + \eta_i\Delta_{ij}\not{A}_{jk}\frac{d}{d\eta_k} + (\not{A}\Delta\not{A})_{ij}\frac{d}{d\eta_j}\frac{d}{d\eta_i}\right)Z[\eta]$$

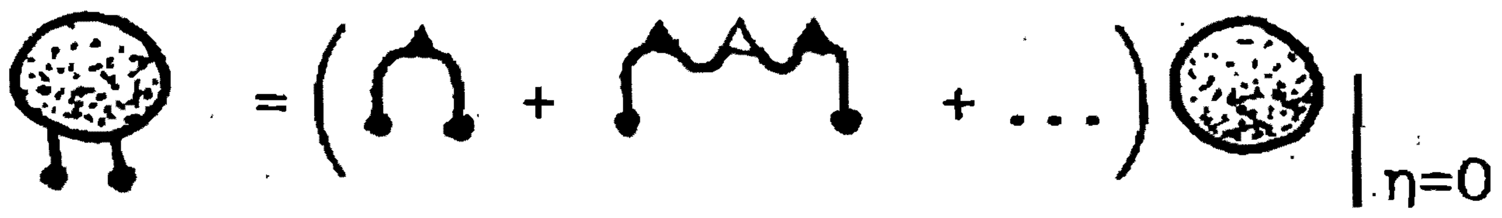
(4.10b)

According to our convention (4.9) all implicit sources lie to the right of the explicit legs. The real trick consists of getting the signs straight. The relative sign between the first and second term is due to the antisymmetry of fermionic Green functions. The overall sign is fixed by requiring consistency with the DS equations (4.10a). For example, if we substitute the 4-leg free fermionic Green function (4.4) into the above, we obtain



$$= -\frac{1}{2}(\mathrm{tr}\Delta\cancel{A})\Delta_{ij} + \Delta_{ij}\cancel{A}_{jk}\Delta_{kj} \ .$$

The sign of the connected term must be consistent with the expansion (4.11):



Show by iterating (4.10b) that

$$W[\eta] = \frac{1}{2}\mathrm{tr}\,\ln(1-\Delta\cancel{A}) + \frac{1}{2}\eta_i\left(\frac{1}{\Delta^{-1}-\cancel{A}}\right)_{ij}\eta_j \ . \tag{4.11}$$

Compare with (3.25). The difference between the bosonic and the fermionic theories is that each fermionic loop carries a factor -1.

Exercise 4.B.2 Derive the relations between the full, connected and 1PI fermionic generating functionals. Write down the Dyson-Schwinger equations such as

$$\left(\frac{dS}{d\psi_i}\left[\frac{d}{d\eta}\right] + \eta_i\right)Z[\eta] = 0 \tag{4.12}$$

without getting confused about fermionic signs.

C. Fermion arrows

In the literature, fermionic generating functionals are never defined in terms of a single source, as in (4.7). We have introduced them in this way to parallel the bosonic formalism. However, usually a pair of sources is used; one for fermions, and one for antifermions. We shall now rewrite the fermionic generating functionals in this more conventional form.

We start by considering the most trivial fermionic theory; we take the range of the collective index to be $i = 1,2$. The propagator is a (2×2) antisymmetric matrix:

$$\Delta = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$$

and the action (2.13) takes the form

$$S[\psi] = -\frac{1}{2}\psi_i\Delta^{-1}_{ij}\psi_j = -\frac{1}{\lambda}\,\psi_1\psi_2 \ .$$

The (2×2) matrix Δ has eigenvalues $\pm i\lambda$. We can eliminate this matrix by replacing ψ_1,ψ_2 by

$$\psi=\frac{1}{\sqrt{2}}\left(\psi_1+\psi_2\right),\bar{\psi}=\frac{1}{\sqrt{2}}\left(\psi_1-\psi_2\right),$$

(this is reminiscent of the introduction of charged bosonic fields, equation (3.12)). The propagator is now just a number:

$$S[\bar{\psi},\psi]=-\bar{\psi}\frac{1}{\lambda}\psi\ .$$

Matrix Δ is invertible only if $\operatorname{Det}\Delta\neq0$. For an antisymmetric matrix this is possible only in even dimensions. A real antisymmetric $(2m\times2m)$ matrix Δ_{ij} can always be brought to form

$$G^{-1}\Delta G=\begin{bmatrix} 0 & -\lambda_1 & & & & & & \\ \lambda_1 & 0 & & & & & & \\ & & 0 & -\lambda_2 & & & & \\ & & \lambda_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & -\lambda_m \\ & & & & & & \lambda_m & 0 \end{bmatrix} \tag{4.13}$$

by means of a sympletic rotation $G\varepsilon Sp(2m)$. (This is the fermionic analogue of the diagonalization which leads to (3.6).) Defining

$$\psi_i=\frac{1}{\sqrt{2}}\left(\psi_{2i-1}+\psi_{2i}\right),\ \psi^i=\frac{1}{\sqrt{2}}\left(\psi_{2i-1}-\psi_{2i}\right) \tag{4.14}$$

we can write the free action as

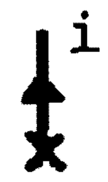
$$S[\bar{\psi},\psi]=-\bar{\psi}\Delta^{-1}\psi=-\ \psi^i\Delta_i^{-1j}\psi_j\quad, \tag{4.15}$$

where the propagator is now an $(m\times m)$ matrix which in the diagonalized form looks like

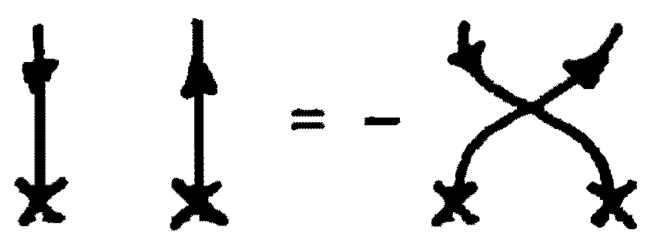
$$\Delta_i^j=\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_m \end{bmatrix}.$$

In this way a 2m-dimensional fermionic field ψ_i can always be replaced by a pair of m-dimensional fields $\bar{\psi},\psi$. Diagrammatically we distinguish the upper and the lower indices by drawing arrows flowing <u>away</u> from upper indices and <u>into</u> the lower indices:

$\Delta_i^j =$ 

$\eta_i =$  , $\eta^j =$  . (4.16)

One advantage of the fermion-antifermion formalism is that the antisymmetric propagator (4.13) is replaced by (4.16) which carries no funny signs. However, it still follows from the definition (4.14) that the fermion and the antifermion fields and sources are anticommuting:



$$\eta^i\eta_j = -\eta_j\eta^i \quad . \tag{4.17}$$

The fermionic generating functionals are now a double series in terms of the fermion, antifermion sources:

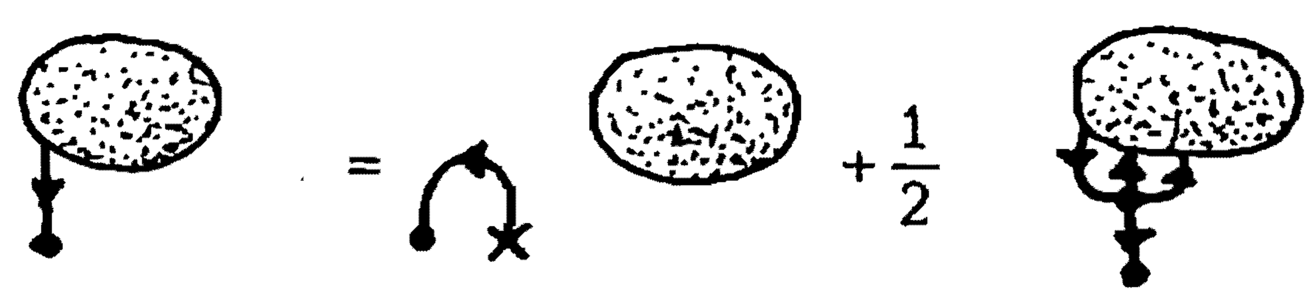
$$Z[\bar{\eta},\eta] = \sum_{m,n} Z_{\underbrace{ijk\ldots\ell}_{m\text{ out-legs}}}{}^{\overbrace{i'j'\ldots k'}^{n\text{ in-legs}}} \frac{\eta_{k'}\ldots\eta_{j'}\eta_{i'}}{n!}\frac{\eta^{\ell}\ldots\eta^j\eta^i}{m!} \quad . \tag{4.18}$$

<u>Exercise 4.C.1</u> <u>Fermionic loops</u>. Show that the connected generating functional for fermion propagation in a background "photon" field is given by:

$$W[\bar{\eta},\eta] = \ell n\ \mathrm{tr}(1-\Delta A\!\!\!/) + \bar{\eta}\frac{1}{\Delta^{-1} - A\!\!\!/}\eta \ . \tag{4.19}$$

Compare with the bosonic case (3.28). The difference between the bosonic and the fermionic theories is that each fermionic loop carries a factor -1.

<u>Exercise 4.C.2</u> <u>Dyson-Schwinger equations</u>. The fermionic $(\bar{\psi}\psi)^2$ theory DS equations for full Green functions are given diagrammatically by



Show that the DS equations for directed fermions can be written as

$$\left(\frac{dS}{d\psi_i}\left[\frac{d}{d\eta},\frac{d}{d\bar{\eta}}\right]+\eta^i\right)Z[\bar{\eta},\eta]=0 \ . \tag{4.20}$$

Exercise 4.C.3 QED DS equations. The four vertex in the preceeding exercise could be a phenomenological approximation to a boson exchange (Fermi theory of weak interactions is of this type)

(is this consistent with fermionic symmetry?). Add a boson propagator to the theory and write the boson and fermion DS equations for this theory.

D. Fermionic path integrals

We have seen in chapter 3 that a lot can be gained by defining a "Fourier" transform which diagonalizes the differential operators:

$$\frac{d}{d\eta_i}Z[\eta]\rightarrow\psi_i\tilde{Z}[\psi] \ . \tag{4.21}$$

For fermions the derivatives anticommute (4.8) so ψ_i have to be anticommuting numbers. Let us blindly imitate the bosonic case and write down

$$Z[\eta]=\int[d\psi]e^{\eta_i\psi_i}\tilde{Z}[\psi] \ .$$

What is this "integral"? Consider first the one-dimensional case. The left-hand side must be independent of ψ and, in particular, invariant under translations $\psi\rightarrow\psi+\theta$:

$$\int[d\psi]\psi=\int[d\psi](\psi+\theta) \ .$$

This works only if

$$\int[d\psi]=0$$

$$\int[d\psi]\psi\neq 0 \ .$$

We take $\int[d\psi]\psi=1$ (just a normalization convention). As $\psi^2=\psi^3=\ldots=0$, there are no other integrals to be evaluated. The integration operation must be anticommutative because $\psi\theta = -\theta\psi$ implies

$$\int [d\psi]\theta\psi = -\theta\int [d\psi]\psi = -\left[\int [d\psi]\psi\right]\theta \ .$$

The generalization to many dimensions is

$$\int [d\psi_i]\psi_j = \delta_{ij} \ . \tag{4.22}$$

Curiously, the fermionic "integration" is indistinguishable from the fermionic "differentiation" (4.8). It is really no integration at all; it is simply an operational rule which implements the desired diagonalization (4.21):

$$\begin{aligned}\frac{d}{d\eta_i} Z[\eta] &= \int [d\psi]\frac{de^{\eta_j\psi_j}}{d\eta_i}\tilde{Z}[\psi] \quad - \frac{d}{d\eta} Z[\eta] \\ &= \int [d\psi]\psi_i\tilde{Z}[\psi] \ ,\end{aligned} \tag{4.23}$$

(as usual, we assume that the number of fermionic dimensions is even). Now, just as in the bosonic case (3.4), we can compute $\tilde{Z}[\psi]$ from (4.20) by solving the fermionic Dyson-Schwinger equation:

$$Z[\eta] = \int [d\psi]e^{S[\psi] + \eta_i\psi_i} \ . \tag{4.24}$$

This is the path integral representation for the fermionic Green functions.

<u>Exercise 4.D.1</u> Can you think of a simple argument which will give the correct $i\varepsilon$ prescription for fermionic propagators, analogous to (3.10) for the bosonic theory?

<u>Exercise 4.D.2</u> Check (4.23).

E. Fermionic determinants

The simplest fermionic analogue to the bosonic gaussian integral (3.5) is the 2-dimensional integral

$$\int[d\psi_1 d\psi_2]e^{-\frac{1}{2}\psi_i\Delta^{-1}_{ij}\psi_j} = \int[d\psi_1 d\psi_2](1-\frac{1}{\lambda}\psi_1\psi_2)$$

$$= \frac{1}{\lambda} = \mathrm{Det}\,\Delta^{-\frac{1}{2}} \quad , \tag{4.25}$$

where

$$\Delta_{ij} = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix} \quad .$$

In odd dimensions such integrals always vanish, as at least one $\int[d\psi_i]$ is unmatched. In even dimensions

$$\int[d\psi]e^{-\frac{1}{2}\psi_i\Delta^{-1}_{ij}\psi_j} = \mathrm{Det}\,\Delta^{-\frac{1}{2}} \quad . \tag{4.26}$$

<u>Derivation</u> (analogous to (3.6)): $\Delta_{ij} = -\Delta_{ji}$, hence there exists a sympletic rotation G such that Δ_{ij} can be brought to form (4.13). Sympletic rotations are volume preserving, so $d(G\psi) = d\psi$. This rotation reduces the 2m dimensional integral to a product of m two-dimensional integrals (4.25): the result is

$$\prod_{i=1}^{m} \frac{1}{\lambda_i} = \mathrm{Det}\,\Delta^{-\frac{1}{2}} \quad . \qquad \text{QED.}$$

The important thing to note is that the fermionic "gaussian" integral yields inverse determinant, in contrast to the bosonic integral (3.6). If you repeat the saddlepoint analysis of sect. 3.F and use $\ln(\det M) = \mathrm{tr}(\ln M)$ rule (3.23), you will find that in the fermionic case the effective action (3.25) is given by

$$\Gamma[\psi^c] = S[\psi^c] - \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\mathrm{tr}(\Delta\gamma[\psi^c])^K \quad . \tag{4.27}$$

As we have already shown diagrammatically in exercises 4.B.1 and 4.C.1, each fermion loop carries a factor -1.

<u>Exercise 4.E.1</u> Introduce a source term $\eta_i\psi_i$ in (4.26) and compute the generating functional (cf. (3.7)) for the free fermionic field theory.

<u>Exercise 4.E.2</u> Show that for directed fermions, sect.4.C, the fermionic gaussian integral is given by

$$\int[d\psi d\bar{\psi}]e^{-\bar{\psi}\Delta^{-1}\psi} = \frac{1}{\mathrm{Det}\,\Delta} \quad . \tag{4.28}$$

F. Fermionic jacobians

The only possible redefinition of a one-dimensional fermionic integration variable is

$$\psi \to \psi' = a\psi + \theta \ .$$

The jacobian $d\psi = Jd\psi'$ must be such that the integration rule (4.22) is preserved

$$1 = \int [d\psi]\psi = \int [d\psi'] J \frac{\psi' - \theta}{a} = \frac{J}{a} \ .$$

Hence the jacobian is $J = d\psi'/d\psi$, the <u>inverse</u> of the bosonic jacobian. That is easy to understand if one remembers that the fermionic "integration" is the same thing as the fermionic differentiation:

$$\int [d\psi] = \frac{d}{d\psi_1}\frac{d}{d\psi_2} \cdots \frac{d}{d\psi_{2m}}$$
$$= \left(\frac{d\psi_i'}{d\psi_1}\frac{d\psi_j'}{d\psi_2}\cdots\frac{d\psi_k'}{d\psi_{2m}}\right)\frac{d}{d\psi_i'}\frac{d}{d\psi_j'}\cdots\frac{d}{d\psi_k'} \ .$$

As the fermionic differentiations anticommute, the term in the brackets is fully antisymmetric; the determinant. The jacobian in 2m dimensions is therefore

$$\int [d\psi] = \int [d\psi'] \det\left(\frac{d\psi_i'}{d\psi_j}\right) \ , \qquad (4.29)$$

the inverse of a bosonic jacobian.

<u>Exercise 4.F.1</u> <u>A trivial supersymmetry</u>. Take one bose and two Fermi dimensions. Using $\det\Delta/\det\Delta = 1$, we can write

$$1 = \int [dA d\bar{\omega} d\omega] e^{-\frac{A^2}{2\lambda} - \bar{\omega}\frac{1}{\sqrt{\lambda}}\omega} \ .$$

It is very easy to find a supersymmetry of this action. A shift

$$A \to A + \varepsilon\sqrt{\lambda}\omega \ , \quad \varepsilon \text{ fermionic,}$$

produces an extra term in the action: $-A\varepsilon\omega/\sqrt{\lambda}$. This can be compensated by a shift of the antifermionic field

$$\bar{\omega} \to \bar{\omega} - \varepsilon A \ .$$

The action $S[A,\bar{\omega},\omega]$ of this free field theory is therefore invariant under supersymmetric (Fermi-bose mixing) transformations

$$A \to A + \varepsilon\sqrt{\lambda}\omega$$
$$\bar{\omega} \to \bar{\omega} - \varepsilon A \quad . \tag{4.30}$$

Add sources

$$Z_0[J,\bar{\eta}_1\eta] = \int [dAd\bar{\omega}d\omega] e^{S[A,\bar{\omega},\omega]+JA+\bar{\eta}\omega+\bar{\omega}\eta}$$

and show that the supersymmetry induces a Ward identity of type (3.34). Verify diagrammatically that the identity is satisfied. This is quite trivial, and still, the QED Ward identities amount to no more than this. In that case A is the photon field, $\sqrt{\lambda}$ longitudinal insertion k^μ, and ω the QED ghost which nobody cares about because it always decouples.
The main lesson of this exercise is this: if we (1) create fake boson degrees of freedom and (2) remove them by ghosts, the theory might have a hidden supersymmetry.

G. Summary

Fermions (or Grassmann numbers) are tricks for manipulating antisymmetric Green functions. Green functions are still ordinary numbers (real for statistical mechanics, complex for quantum mechanics), and there is no mystique in computing them (only tedium). The physical content of fermions is that they offer a way of imposing constraints. One such constraint is Pauli principle - electrons are fermions. The QCD ghosts which we will construct in chapter 6 are another example: they eat up the unphysical longitudinal gluon degrees of freedom. Physically, fermions are to be counted as negative degrees of freedom (fermion loops carry minus signs) which cancel the unphysical bose degrees of freedom.

Fermionic Green functions are antisymmetric under interchange of indices. The fermionic sources and fields anticommute;

$$\eta_i\eta_j = -\eta_j\eta_i \quad , \qquad \psi_i\psi_j = -\psi_j\psi_i \quad ,$$

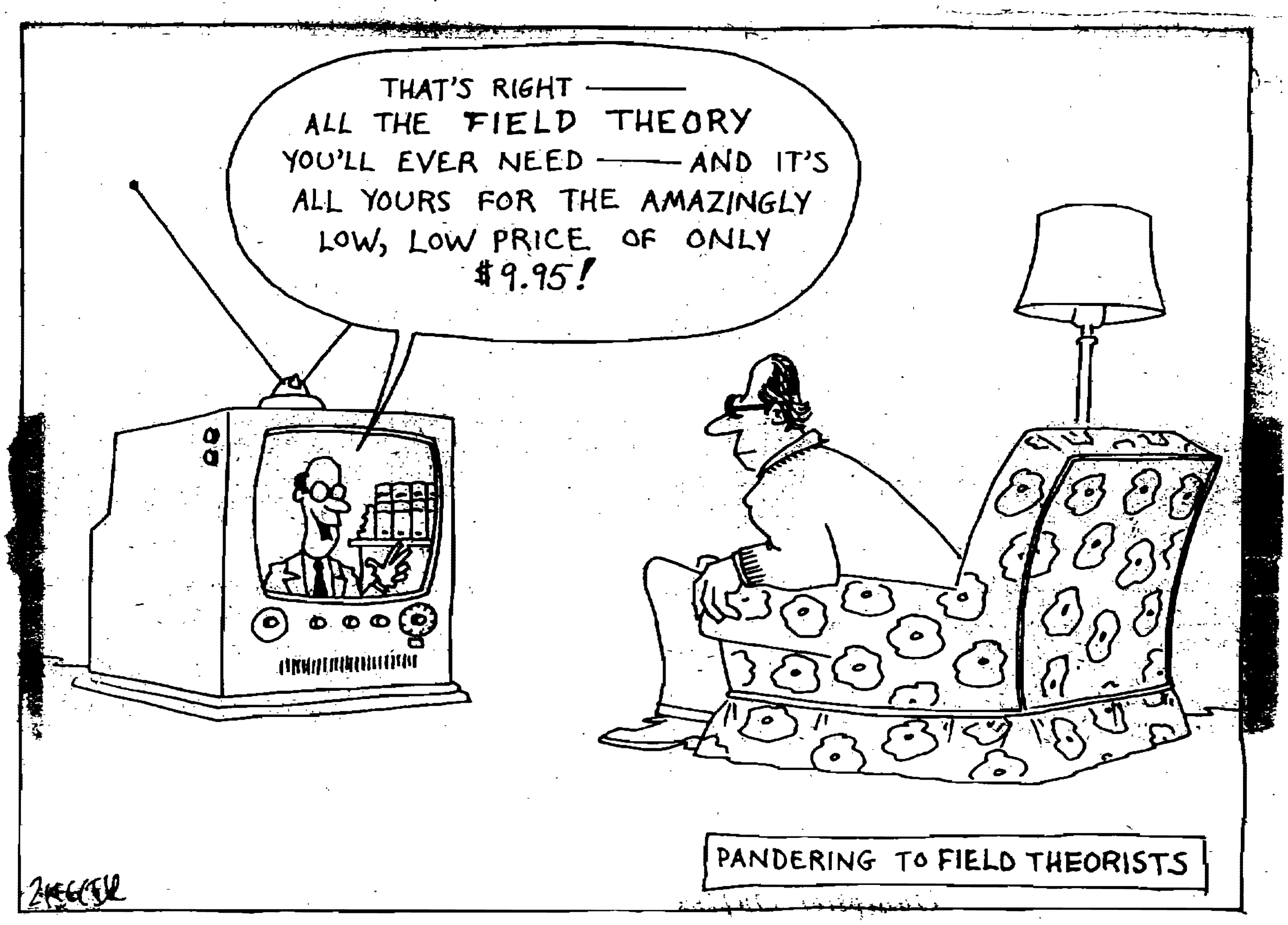
$$\frac{d}{d\eta_i}\eta_j = \delta_{ij} - \eta_j\frac{d}{d\eta_i} \quad ,$$

$$\frac{d}{d\eta_i}\frac{d}{d\eta_j} = -\frac{d}{d\eta_j}\frac{d}{d\eta_i} \quad .$$

The fermionic integrals are defined by

$$\int [d\psi_i]d\psi_j = \delta_{ij} \quad .$$

The entire machinery developed for bose fields applies to Fermi fields, modulo few irrelevant sign confusions and one relevant sign; factor -1 for each fermionic loop.



5. SPACETIME PROPAGATION

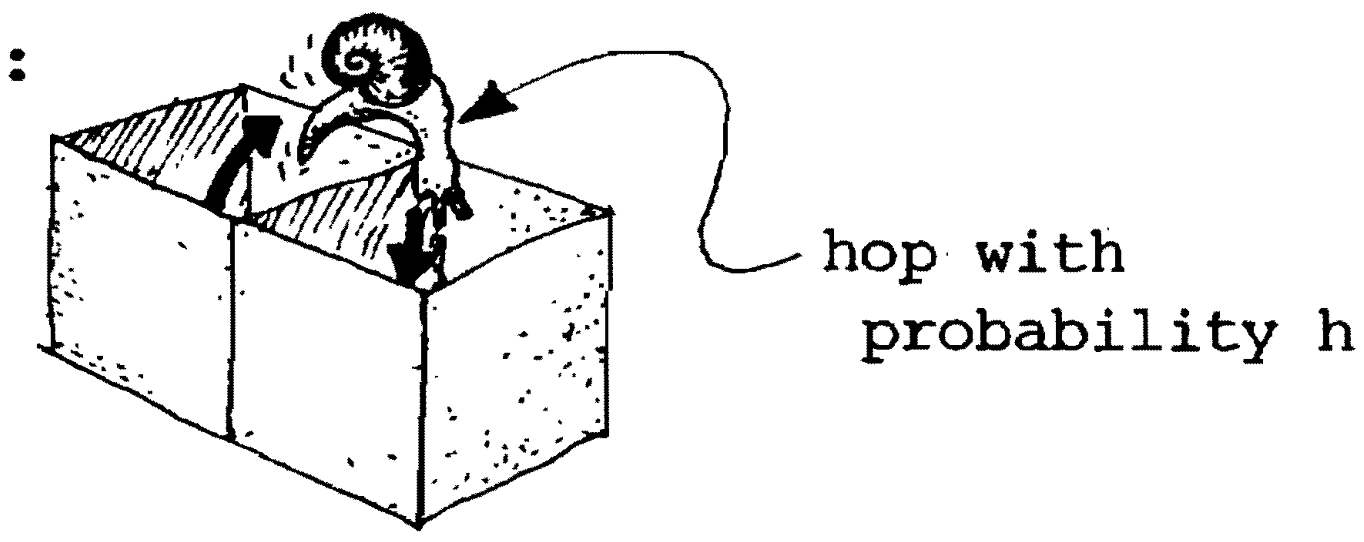
Until now the collective indices have stood for all particle labels; spacetime location, spin, particle type and so on. To apply field theory to particle physics we have to describe propagation of particles through the spacetime. I find it most convenient to formulate the field theory in our spacetime as an analytic continuation from a Euclidean world in which there is no distinction between time and space. What do we mean by propagation in such a space?

Our formulation is inevitably phenomenological: we have no idea what the structure of our spacetime on distances much shorter than nuclear sizes might be. The spacetime might be discrete rather than continuous, or it might have geometry different from the one we observe at the accessible distance scales. The formalism we use should reflect this ignorance. We will deal with this problem by subdividing the space into small cells and requiring that our theory be insensitive to distances comparable to or smaller than the cell sizes.

Our next problem is that we have no idea why there are particles, and why or how they propagate. The most we can say is that there is some probability that a particle hops from one spacetime cell to another spacetime cell. At the beginning of the century, the discovery of Brownian motion showed that matter was not continuous but was made up of atoms. In particle physics we have no indication of having reached the distance scales in which any new spacetime structure is being sensed: hence for us this hopping probability has no direct physical significance. It is simply a phenomenological parameter: in the continuum limit it will be replaced by the mass of the particle.

A. Free propagation

We assume for the time being that the state of a particle is specified by its spacetime position, and that it has no further labels (such as spin or color): $i = (x_1, x_2, \ldots, x_d)$. What is it like to be free? A free particle exists only in itself and for itself; it neither sees nor feels the others; it is, in this chilly sense, free. But if it is not at once paralyzed by the vast possibilities opened to it, it soon becomes perplexed by the problems of realizing any of them alone. Born free, it is constrained by the very lack of constraint. Sitting in its cell, it is faced by a choice of doing nothing (s = stopping probability) or hopping into any of the 2d neighboring cells (h = hopping probability):

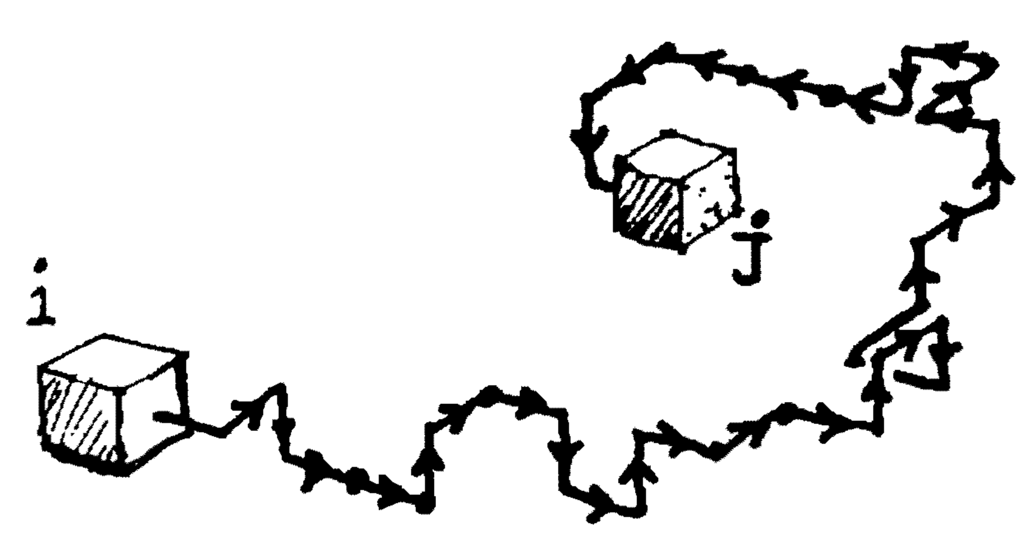


The number of neighboring cells defines, if you wish, the dimension of the spacetime. The hopping and stopping probabilities are related by the probability conservation: $1 = s + 2dh$. Taking the hopping probability to be the same in all directions means that we have assumed that the space is isotropic.

Our next assumption is that the spacetime is homogeneous, i.e. that the hopping probability does not depend on the location of the cell. (Otherwise the propagation is not free, but is constrained by some external geometry.) This can either mean that the spacetime is infinite, or that it is compact and periodic (a torus). That is again something beyond our ken - we proceed in the hope that the predictions of our theory will be insensitive to very large distances.

The isotropy and homogeneity assumptions imply that our theory should be invariant under rotations and translations. The requirement of insensitivity to the very short and very long distances means that the theory must have nice ultraviolet and infrared properties.

A particle can start in a spacetime cell i and hop along until it stops in the cell j. The probability of this process is $h^L s$, where L is the number of steps in the corresponding path:



The total probability that a particle wanders from the i-th cell and stops in the j-th cell is the sum of probabilities associated with all possible paths connecting the two cells:

$$\Delta_{ij} = s \sum_L h^L N_{ij}(L) \; . \tag{5.1}$$

$N_{ij}(L)$ is the number of all paths of length L connecting i and j. Define a stepping matrix

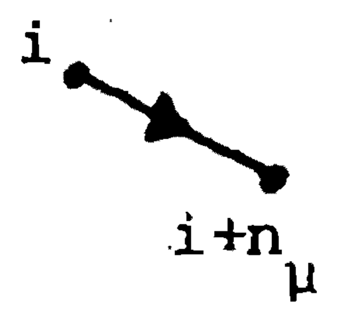
$$(S^\mu)_{ij} = \delta_{i+n_\mu , j} \; . \tag{5.2}$$

If a particle is introduced into the i-th cell by a source

$$J_k = \delta_{ik} \; ,$$

the stepping matrix moves it into a neighboring cell:

$$(S^\mu J)_k = \delta_{i+n_\mu , k} \rightarrow$$



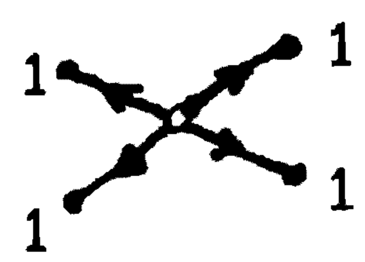
The operator

$$(h \cdot S)_{ij} = \sum_{\mu=1}^{d} h_\mu [(S^\mu)_{ij} + (S^\mu)_{ji}] \; ,$$

$$h_\mu = (h,h,\ldots,h) \tag{5.3}$$

generates all paths of length 1 with probability h:

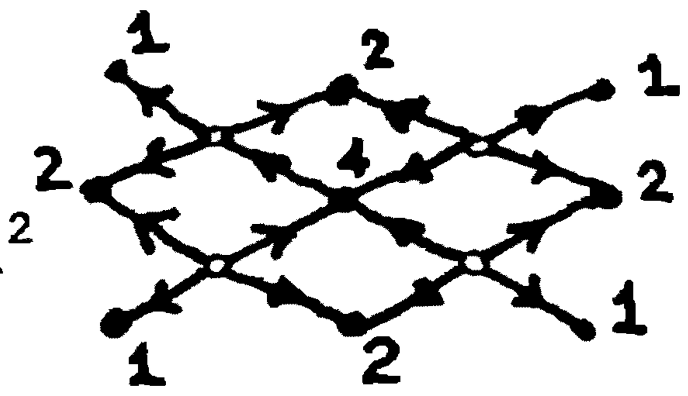
$$(h \cdot S) J = h$$



i-th cell

(The examples are drawn in two dimensions). The paths of length 2 are generated by

$$(h \cdot S)^2 J = h^2$$



and so on. Note that the k-th component of the vector $(h \cdot S)^L J$ counts the number of paths of length L connecting the i-th and the k-th spacetime cells. The total probability that the particle stops in the k-th cell is given by

$$\phi_k = s \sum_L (h \cdot S)^L_{ki} J_i \quad ,$$

$$\phi = \frac{s}{1-h\cdot S} J \ . \tag{5.4}$$

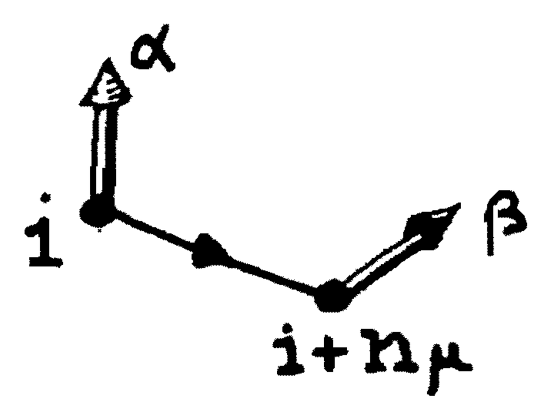
The value of the field[†] ϕ_k at a spacetime point k measures the probability of observing the particle introduced into the system by the source J. The Euclidean free scalar particle propagator (5.1) is given by

$$\Delta_{ij} = \left(\frac{s}{1-h\cdot S}\right)_{ij} \quad , \tag{5.5}$$

or, in the continuum limit (do exercise 5.A.1) by

$$\Delta(x,y) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik\cdot(x-y)}}{k^2+m^2} \quad . \tag{5.6}$$

So far we have assumed that the particle hops to any neighboring cell with the same probability. What happens if the particle hiding in the spacetime cell is not a small spherical object, but something long and shapely? In that case, we have to introduce spin labels to define the particle orientation: $i = (x_\mu, \alpha)$. Such a particle will hop and retain its orientation with some probability, and hop and change its orientation with a different probability. The hopping probability h is now replaced by a hopping matrix

$(h^\mu)_{\alpha\beta} =$  (5.7)

which describes the probability that a particle with the spin label α hops one step in the direction μ and flips its spin to β. We do not want to give up the isotropy and homogeneity of spacetime, so the hopping matrix can depend only on the relative orientations of the two spins. In other words, the hopping matrix must be an invariant tensor under spacetime translations and rotations.

[†] Interpreting ϕ as a field is consistent with the previous definition of a free field, equations (2.22) and (2.25).

How does one describe orientation of a particle? That depends on the particle type. For example, if the particle orientation can be specified by a d-dimensional vector, we need d spin labels. We shall always assume that the range of the spin index is finite. In the language of group theory this means that we shall consider only the finite dimensional representations of the rotation group. Furthermore, we shall be interested only in irreducible representations. The physical reason is that reducible representations are resolved into irreducible components by quantum corrections. For example, if a free propagator contains both an isotropic part which propagates as a scalar (5.5) and a non-isotropic remainder, one-loop corrections will be in general different for the two parts.

If a particle of spin α is introduced into i-th cell by means of a source

$$J_{k\beta} = \delta_{\alpha\beta}\delta_{ik} \quad ,$$

the stepping matrix (5.2) generates the probabilities associated with all paths of length one:

$$(h\cdot S)J_{k\beta} = h^{\mu}_{\beta\gamma}\left(S^{\mu}_{k\ell} + S^{\mu}_{\ell k}\right)J_{\ell\gamma} \quad .$$

The probabilities associated with all paths of length two are given by $(h\cdot S)^2 J$, and so on. Hence the propagator for a free spinning particle is given by

$$\Delta_{i\alpha,j\beta} = s\delta_{ij}\delta_{\alpha\beta} + s\sum_{L>0}(h\cdot S)^{L}_{i\alpha,j\beta}$$

$$= \left(\frac{s}{1-(h\cdot S)}\right)_{i\alpha,j\beta} \quad . \tag{5.8}$$

To make further headway, one has to be more specific about the hopping probability h^{μ}. This would get us too deep into group theory, and (if we started thinking about fermions), lead to ulcers. We stop now.

Exercise 5.A.1 Continuum propagator. Define the finite difference operator by

$$\partial f(x) = \frac{f(x+\frac{a}{2}) - f(x-\frac{a}{2})}{a}$$

where a is the lattice spacing. Show that

$$\frac{1}{h}\sum^{d} h_\mu (S^\mu_{ij} + S^\mu_{ji}) = 2d + a^2\partial^2 \quad ,$$

where $\partial^2 = \partial_\mu \partial_\mu$ is the finite difference Laplacian. Show that the Euclidean scalar lattice propagator (5.5) is given by

$$\Delta^{-1}_{ij} = 1 - \frac{ha^2}{s}\partial^2 .$$

The mass in the continuum propagator (5.6) is related to the hopping parameter by

$$m^2 = \frac{s}{ha^2} \quad . \tag{5.9}$$

If the particle does not like hopping ($h \to 0$), the mass is infinite and there is no propagation. If the particle does not like stopping ($s \to 0$), the mass is zero and the particle zips all over the space.
Diagonalize ∂^2 by Fourier transforming and derive (5.6).

B. A leap of faith

We have constructed the Euclidean free-particle propagator from a few basic notions such as addition of probabilities and spacetime homogeneity and isotropy. At some point we have to face two non-intuitive facts: our world is Minkowskian, not Euclidean, and the theory of elementary particles is quantum mechanics, not statistical mechanics. Usually somebody tells you that the quantum mechanics is obtained from the classical mechanics by replacing Poisson brackets by commutators (canonical quantization). This gives me no intuition about quantum mechanics. With my present (lack of) understanding, I find it easier to think of field theory in terms of probabilities, as we have done up to now, and then make a leap of faith by saying: our world is a Wick rotation of the Euclidean world,

$$x_4 = ix_0 \quad . \tag{5.10}$$

This gives us

1) special relativity $\qquad g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ (5.11)

2) quantum mechanics; Boltzmann weight e^S is replaced by a phase factor $e^{iS/\hbar}$.

For example, Euclidean action

$$S[\phi] = \int d^d x \tfrac{1}{2}\phi(x)\,(\partial^2 + m^2)\,\phi(x) \quad , \tag{5.12}$$

is replaced by the Minkowski action

$$\frac{i}{\hbar}\, S[\phi] = \frac{i}{\hbar}\int d^d x \tfrac{1}{2}\phi(x)\,(g_{\mu\nu}\partial^\mu\partial^\nu + m^2)\,\phi(x) \quad , \tag{5.13}$$

where the imaginary factor i is the jacobian from the change of variables (5.10).

3) correspondence principle; Planck constant $\hbar$ is the scale of quantum fluctuations, and the classical mechanics is the large action limit of the quantum theory.

It is not good enough†, but it will get us through the night.

C. Scattering matrix

A run-of-the-mill particle scattering experiment looks something like this

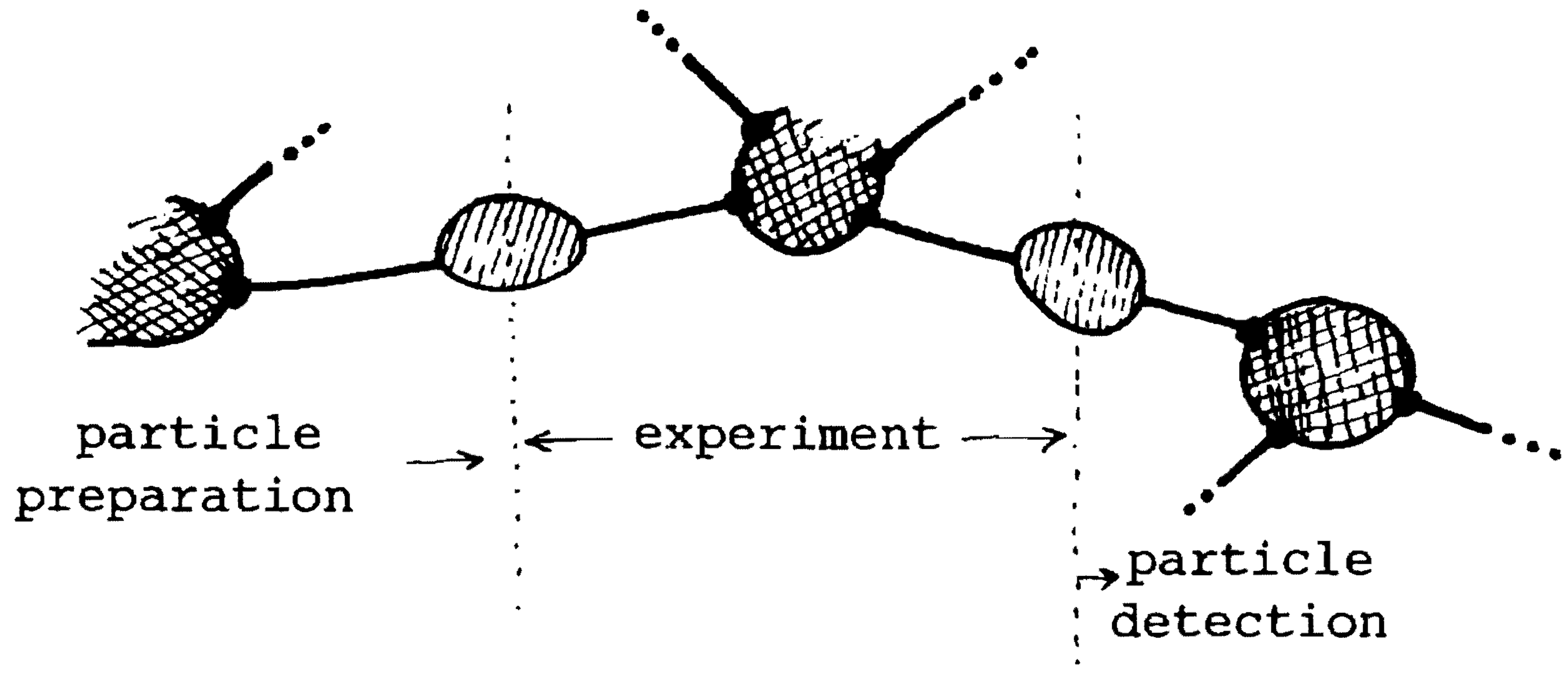
(5.14)

Particles with sharply defined 4-momentum are accelerated over kilometer distances, collide in regions of nuclear size and the

† There is a little problem with interpreting measurements.

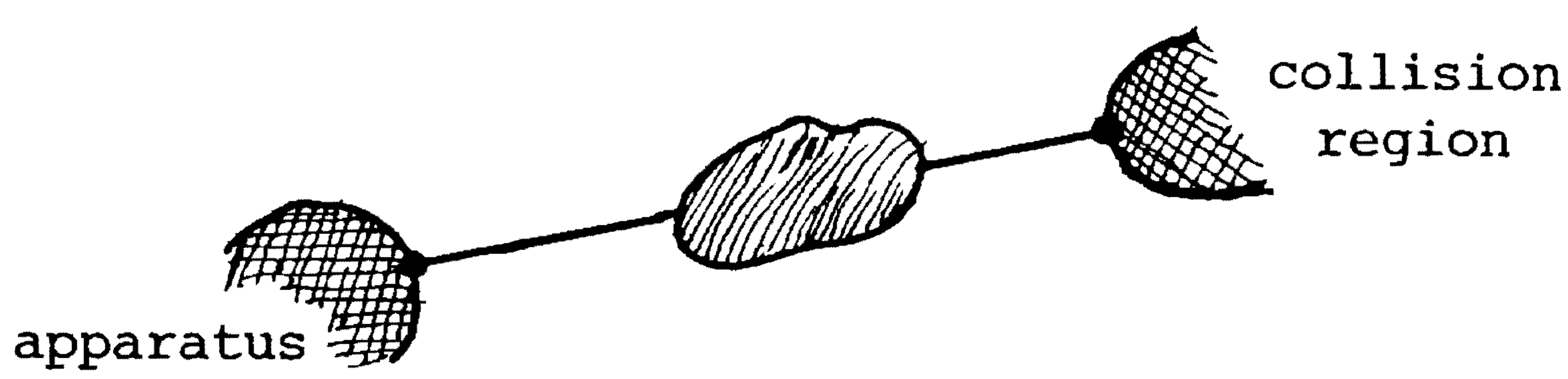
resulting particles fly tens of meters to detectors. The theoretical predictions for such experiments are expressed in terms of connected Green functions. If you think about it, you will realize that the experiments measure the effective vertices, or the 1PI Green functions.

If you <u>really</u> think about it, our formulation in terms of sources is a brave idealization. In reality the entire experiment is one large system



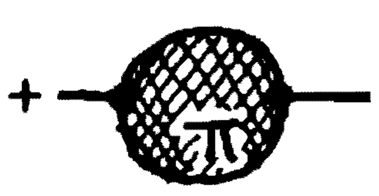
(5.15)

and approximating the experimental apparatus by sources makes sense only when the interaction region can be well separated. The particles which traverse the macroscopic distances between the interaction region and the experimental apparatus are classical, mass-shell particles with $k^2 = m^2$:

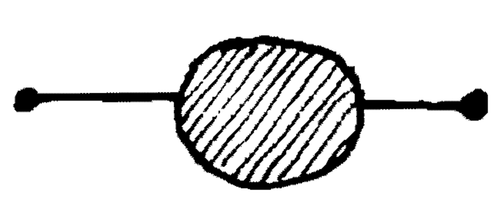


(5.16)

We can measure the mass of these particles by measuring their four-momenta. The theory predicts a mass-shift

$m^2 = m_0^2 +$  $\Big|_{k^2 = m^2}$ (5.17)

This relates the bare mass (mass with all interactions turned off) to the physical mass. The theory also predicts a wave-function renormalization

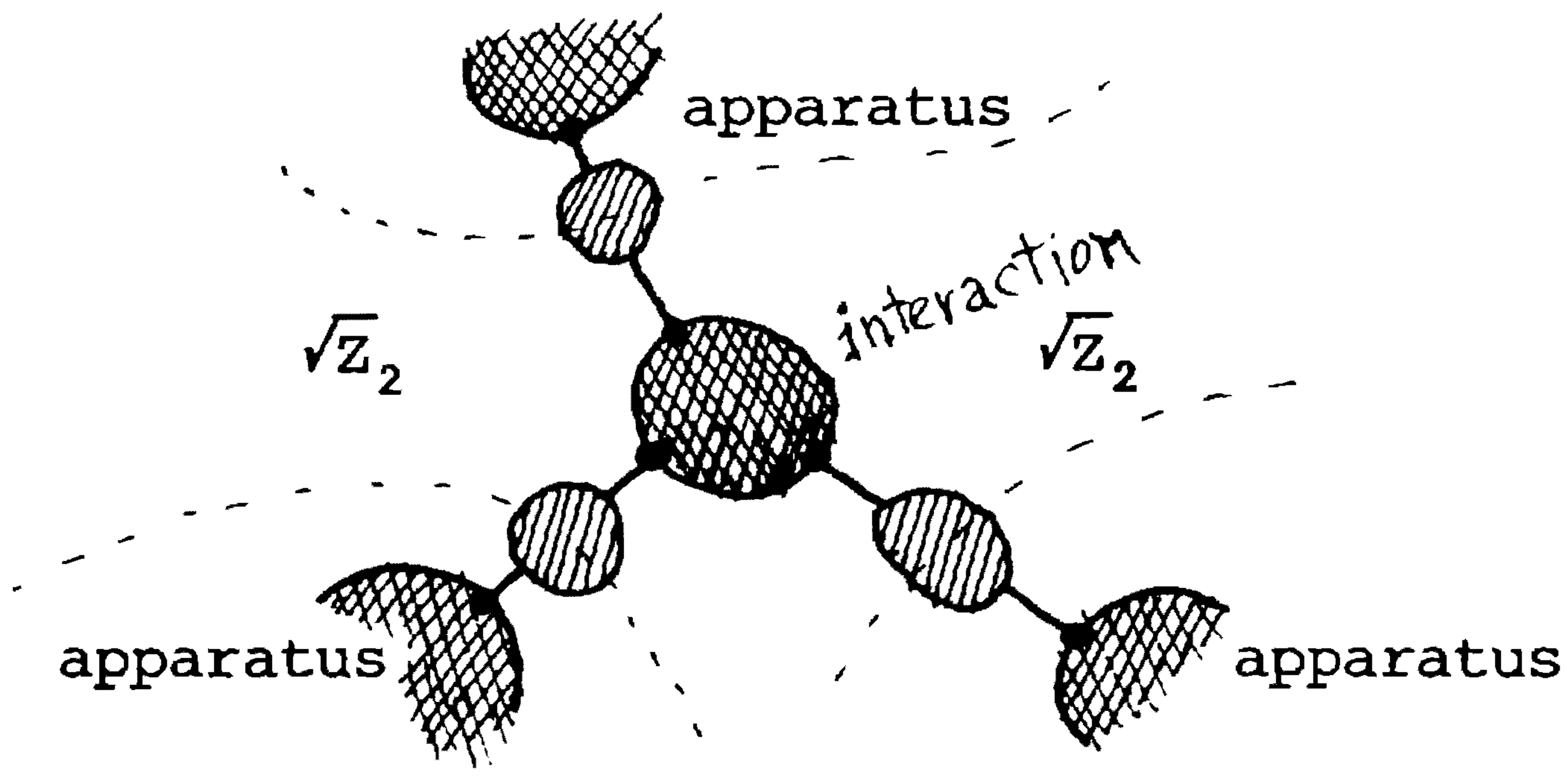
 $\Big|_{k^2 = m^2} = \dfrac{Z_2}{k^2 - m^2}$. (5.18)

If the particles also carry spin, there will be further mass-shell constraints. They are expressed in terms of polarizations $\varepsilon^{\mu}(k)$, spinor wave functions $u_{\alpha}(k)$, etc.; we shall soon see such objects. They are the reason why Z_2 is called the "wave function renormalization constant".

A connected Green function (2.17) has a propagator on each external leg. These propagators develop poles if the corresponding particles traverse macroscopic distances, and what is probed in an experiment is not the entire Green function, but only its mass-shell amputation

$$\prod_i (k_i^2 - m_i^2) G^{(c)}(k_1,k_2,\ldots)\big|_{k^2=m^2} \quad .$$

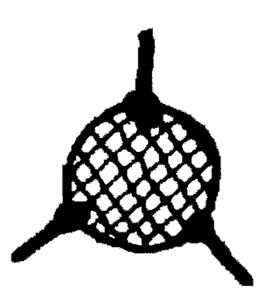
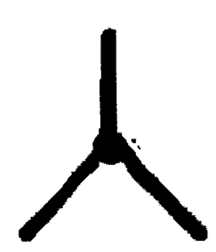
The renormalization constants Z_2 survive all such amputations, and cannot be disentangled from the measurements of the physical coupling constants:



The resolution of this problem is to absorb Z_2 into the definitions of the physical coupling constants by

$$g = Z_2^{k/2} Z_1^{-1} g_0 \ , \tag{5.19}$$

where g_0 is the bare coupling constant (for a vertex with k legs), and the vertex renormalizations Z_1 are computed from

 $\Big|_{k^2=m^2} \equiv \frac{1}{Z_1}$  (5.20)

(and so on for higher vertices). The wave function renormalizations contribute factors of $\sqrt{Z_2}$ because they must be shared in a sisterly fashion between the two ends of each propagator. So, the quantities that are really measured in experiments, and therefore called the S-matrix (scattering matrix) elements, are

$$S(k_1,k_2,\ldots) = \prod_i \frac{k_i^2 - m_i^2}{\sqrt{Z_{2,i}}} G^{(c)}(k_1,k_2,\ldots)\Big|_{\text{mass-shell}} , \tag{5.21}$$

(for particles with spin we should also add polarization wave functions on the external legs). Here the $Z_2^{-\frac{1}{2}}$ factors account for the bits of renormalization constants absorbed by the experimental apparatus, and the bare masses and couplings are to be re-expressed in terms of the physical ones by (5.17) and (5.19).

This is called renormalization. It is not here because of (possible) ultraviolet divergences, but because it is inevitable. The only way to compare our theory with nature is to relate our Green functions to physically measurable parameters, and then re-express all predictions of the theory in terms of those parameters.

Renormalization should not be confused with regularization. Regularization is a mathematical problem of defining infinite sums in the intermediate steps of field theory calculations; renormalization is a unique, physically determined procedure of expressing the physical predictions of a theory in terms of physically measurable parameters.

6. FROM GHOULIES TO GHOSTIES

A physical photon is massless and has only transverse degrees of freedom; still, in relativistic calculations it is convenient to pretend that the photon is a vector particle. Decoupling of the extra degree of freedom is guaranteed by Ward identities. We shall use the requirement of the decoupling of the extra degrees of freedom as the guiding principle for constructing the QCD action. In retrospect it will be clear that this diagrammatic derivation corresponds step by step to the textbook local gauge invariance arguments. Still, this kind of derivation has its charms - it shows rather explicitly how the ghosts eat up the unphysical gluon degrees of freedom, and how the Ward identities guarantee their decoupling.

A. Massless vector particles

A massive vector particle is characterized by its mass M and its polarization $\varepsilon_\mu^\lambda(k)$. There are $\lambda = 1,2,\ldots,d-1$ independent polarizations; in the rest frame $k^\mu = (M,\vec{0})$, so a vector particle can point in d-1 directions. Another way to see this is to observe that k^μ, the direction of propagation of a free spinning particle, reduces the symmetry from SO(1,d-1) to SO(d-1), the rotations in the transverse spacetime directions.

In the rest frame a vector particle points in a direction $\vec{\varepsilon}$. The choice of the coordinates is quite arbitrary; one can choose any d-1 independent basis vectors $\hat{e}_\lambda$ (circular polarizations, for example) and express the polarization in this basis

$$\varepsilon_i = \sum_\lambda \varepsilon_i^\lambda \hat{e}_\lambda \qquad \lambda, i = 1,2,\ldots,d-1 \quad .$$

To describe the polarizations covariantly, we add a fake d-th polarization ε_o^λ and set it equal to zero by the <u>transversality condition</u>

$$k^\mu \varepsilon_\mu^\lambda(k) = 0 \ , \quad \begin{array}{ll} \lambda = 1,2\ldots,d-1; & \text{polarization} \\ \mu = 1,2\ldots,d; & \text{Minkowski} \end{array} \tag{6.1}$$

This reduces to $\varepsilon_o^\lambda = 0$ in the rest frame. Being explicitly covariant, the transversality condition also describes the d-1 vector polarizations in any frame.

The momentum of a physical massive particle satisfies the mass-shell condition:

$$k^2 - M^2 = 0 \quad . \tag{6.2}$$

If the particle is massless

$$k^2 = 0 \tag{6.3}$$

it is not possible to bring it to a rest frame. The best we can do is to align it along the lightcone: $k^{\mu} = (E,0,0,\ldots,E)$. A physical massless spinning particle is always whizzing along a spatial direction $\vec{k} = (0,0,\ldots,E)$, and the symmetry is reduced from $SO(1,d-1)$ to $SO(d-2)$, the rotations in the transverse space directions. Hence a massless vector particle has d-2 polarizations. The trouble is that there is no nice way of imposing the masslessness condition on the polarizations. We can, however, see that there is one degree of freedom less than in the massive case, because we can freely vary the polarizations along the longitudinal direction

$$\varepsilon_{\mu}(k) \to \varepsilon_{\mu}(k) + k_{\mu}\omega(k) \quad , \tag{6.4}$$

($\omega(k)$ arbitrary function) without violating the transversality condition (6.1). (Remember that $k^2 = 0$). For somewhat obscure historical reasons, this kind of transformation is called a gauge transformation†.

Under the gauge transformation (6.4) the transition amplitudes pick up extra contributions from the longitudinal bits, or "gaugeons". We denote gaugeons diagrammatically by

$$^{\mu} = \frac{-i}{k^2} k^{\mu} \quad . \tag{6.5}$$

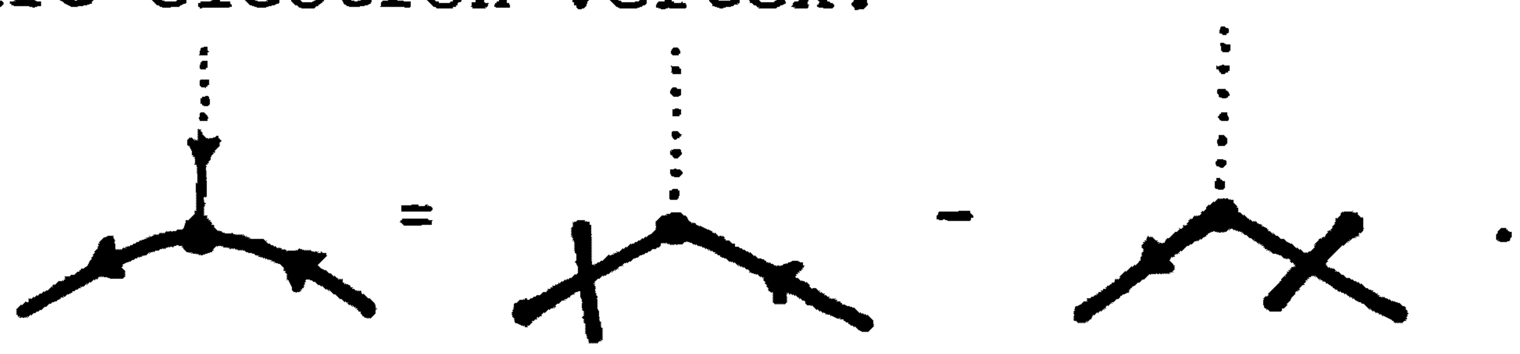
---

†The term "gauge symmetry" was introduced by James Joyce in *Ulysses* (p.490 of the Modern Library 1934 edition). Bloom is standing at the entrance of a whorehouse "feeling his occiput dubiously with the unparalleled embarassment of a harassed pedlar gauging the symmetry of her peeled pears".

(The diagrammatic rules are summarized in appendix D.) At first glance, gaugeons seem like bad news because they change the transition amplitudes. However, the only thing that matters are the physical S-matrix elements (5.23), and they are unaffected by the gaugeons. In QED this follows from the trivial momentum-conservation identity

$$\not{k} = (\not{p} + \not{k} - m) - (\not{p} - m) \quad . \tag{6.6}$$

Diagrammatically (cf. appendix D) this is the Ward identity for the bare electron vertex:



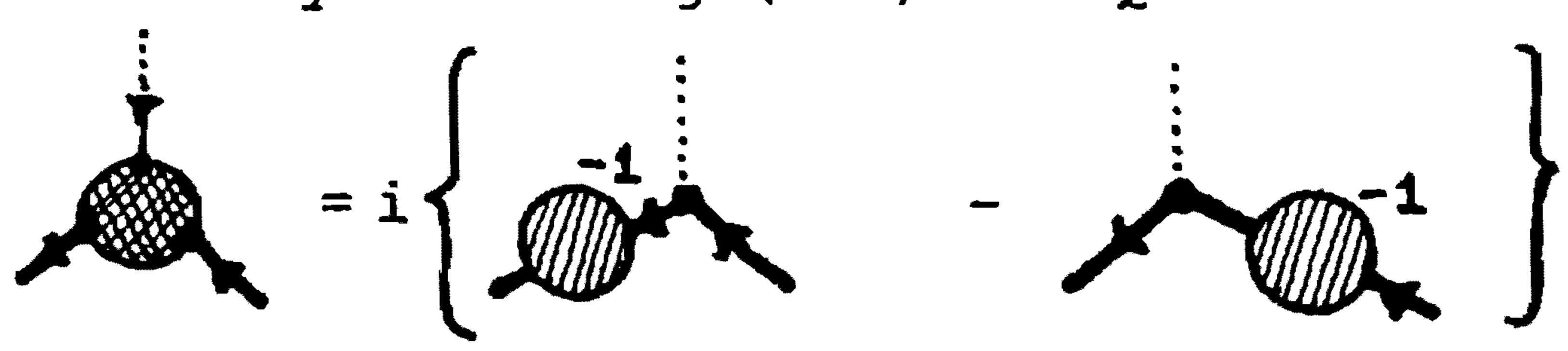
(6.7)

The slashed lines indicate factors of $(\not{p} - m)$. They vanish on the mass-shell by the Dirac equation

$$(\not{p} - m)u(p) = 0 \quad . \tag{6.8}$$

It is easy to show (next exercise) that all QED diagrams with gaugeons lead to mass-shell vanishing contributions. The QCD Ward indenties are not so trivial - their derivation will be the main subject of this and the next chapter.

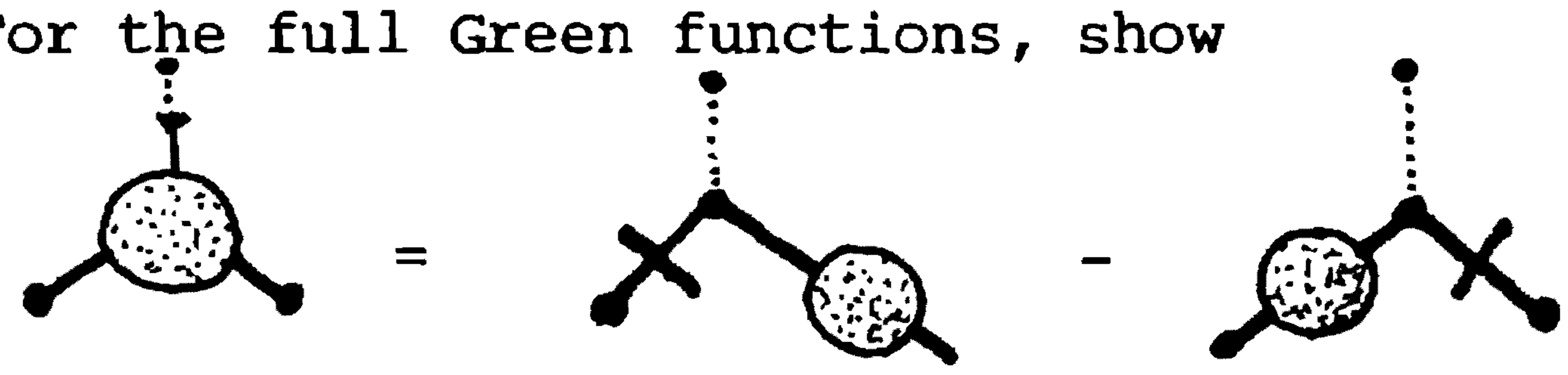
Exercise 6.A.1 Derive by iterating (6.6) the QED Ward identity



$$(p' - p)_{\mu}\Gamma^{\mu}(p,p') = e[S^{-1}(p') - S^{-1}(p)] \quad . \tag{6.9}$$

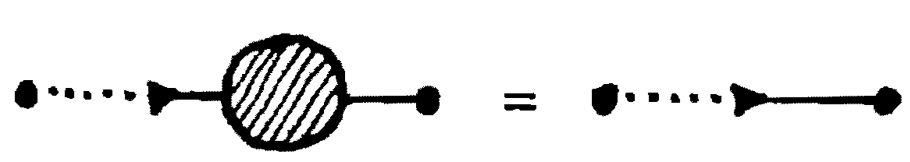
Hints:

1. For the full Green functions, show



Rewrite this for connected Green functions.

2. Show that



3. Finally, use the result of exercise 2.H.1 for the 1PI Green function.

B. Photon propagator

We have shown that QED gaugeons are innocuous; they do not affect the physical predictions of QED. One could even claim that the gaugeons are actually good news, as the gauge invariance (6.3) gives us great flexibility in defining the bare photon propagator $\langle A_\mu(x) A_\nu(y) \rangle$. Whatever your favorite way of deriving propagators may be (I like random walks of the preceeding chapter), the end result for the vector particles must be

$$D_{\mu\nu}(k) = \frac{-i}{k^2} \sum_{\lambda}^{\text{polar.}} \varepsilon_\mu^\lambda(k) \varepsilon_{\lambda\nu}(k) \quad . \tag{6.10}$$

The polarization tensors ε_μ^λ are Clebsch-Gordan coefficients which project the physical d-1 (or d-2) transverse polarizations out of the space of d-dimensional vectors. Explicit construction of Clebsch-Gordan coefficients is a tedious and unrewarding business. Fortunately we do not need them: we need only their sum in (6.10).

For massive vector particles this is easy to evaluate. We write all rank-two tensors available and fix the constants by the mass-shell conditions (6.1) and (6.2):

$$\begin{aligned} \sum_\lambda \varepsilon_\mu^\lambda(k) \varepsilon_{\lambda\nu}(k) &= A g_{\mu\nu} + B k_\mu k_\nu \\ &= g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2} \quad . \end{aligned} \tag{6.11}$$

For massless vector particles there is no such unique choice. One's first impulse is to replace (6.11) by

$$g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \quad .$$

However, any gauge-transformed polarization (6.4) should lead to an equally good propagator, so we are lead to propagators of general form

$$\sim\!\sim\!\sim = D_{\mu\nu}(k) = -\frac{i}{k^2}(g_{\mu\nu} + k_\mu f_\nu + f_\mu k_\nu) \quad , \tag{6.12}$$

where $f_\nu(k)$ is an arbitrary function. The most popular gauge choices of this type are listed in appendix C; which one is the most convenient depends on the application. More perverse gauges can be thought up, and are[†]. Each gauge choice generates its gaugeons - and if the theory is to make any sense, we must insist on their decoupling from physical processes. This is the principle from which we shall presently construct the QCD action.

More precisely, the sacred principle is the gauge invariance, which in the language of Feynman diagrams comes in two guises:

(a) external gauge invariance, or invariance under transformation (6.4):

$$\varepsilon_\mu \to \varepsilon_\mu + \delta\omega k_\mu \quad . \tag{6.13}$$

(b) internal gauge invariance, or invariance under variation of gauge-fixing parameters:

$$D_{\mu\nu} \to D_{\mu\nu} + k_\mu \delta f_\nu + \delta f_\mu k_\nu \quad . \tag{6.14}$$

Exercise 6.B.1 Gauge fixing. Any not too pathological function f in the propagator (6.12) will do, as it must decouple anyway. One usually fixes $f_\nu(k)$ by some physically motivated condition. For interactions of nearly static particles, Coulomb gauge is the natural choice. For highly relativistic situations the covariant, planar or lightcone gauges might be convenient, and so on. The gain is of purely computational nature - the physical results must be the same in all gauges. The Coulomb gauge condition

$$\sum_{i=1}^{3} \partial_i A^i(x) = 0 \tag{6.15}$$

is a typical example. This condition introduces a spacetime direction $n^\mu = (1,0,0,0)$, so the most general form of f is

$$f^\mu = Bk^\mu + Cn^\mu \quad .$$

The coefficients B and C are fixed by substituting f into the gauge condition on the propagator:

$$\begin{aligned} 0 = \langle \vec{k}\cdot\vec{A}\ A^\mu\rangle &= (k^\nu - (n\cdot k)n^\nu)\langle A^\nu A^\mu\rangle \\ &= (k^\nu - (n\cdot k)n^\nu)D^{\nu\mu} \quad . \end{aligned}$$

Here the three-vector $\vec{k}$ is expressed covariantly by $(0,\vec{k}) = k^\mu - (n\cdot k)n^\mu$. Compute the propagators listed in appendix C by this method. Observe that it is sufficient to do one calculation; once the axial gauge propagator is known, the others are obtained by special choices of the vector n^μ.

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† useful in some contexts.

Exercise 6.B.2 Physical polarizations. In (6.1) we have insisted on the transversality of the physical polarizations. This seems to be in conflict with imposing a noncovariant gauge condition such as (6.15). (a) Straighten out this confusion. (b) Communicate the resolution to the author.

C. Colored quarks

We start the construction of Quantum Chromodynamics by attempting a simple generalization of QED: we replace the electron by a set of quarks† of n different "colors", and the photon by N gluons. A free quark or gluon propagates without changing color, so the spacetime propagators are the same as in QED, while the color factors are simply Kronecker deltas. However, a quark can change color by emitting a gluon, and the QED coupling constant e generalizes to quark-antiquark-gluon ($q\bar{q}G$) coupling matrices T_i

$$= ig(T_i)_a^{\;b}(\gamma^\mu)_{\alpha\beta}$$

$a,b = 1,2,\ldots,n$ quark colors
$i,j = 1,2,\ldots,N$ gluon colors .

In QED the strength of radiative corrections is measured by the fine structure constant $\alpha = e^2/(4\pi)$. In QCD the corresponding quantity (color weight for 1-quark loop correction to the gluon propagator) is $\mathrm{Tr}(T_iT_j)$. If T_i is a hermitian matrix, this can be diagonalized

$$\mathrm{tr}(T_iT_j) = a_i\delta_{ij} \;, \qquad a_i \geq 0 \;; \quad (\text{no sum on } i) \;. \tag{6.16}$$

The a_i is the "fine structure constant" with which the i-th color gluon couples. If T_i are not hermitian, we might be in trouble, because some a_i could be negative (that is like taking imaginary e in QED). Henceforth we shall always take coupling matrices T_i to be hermitian.

Thinking exercise 6.C.1: What could go wrong if $q\bar{q}G$ couplings were not hermitian?

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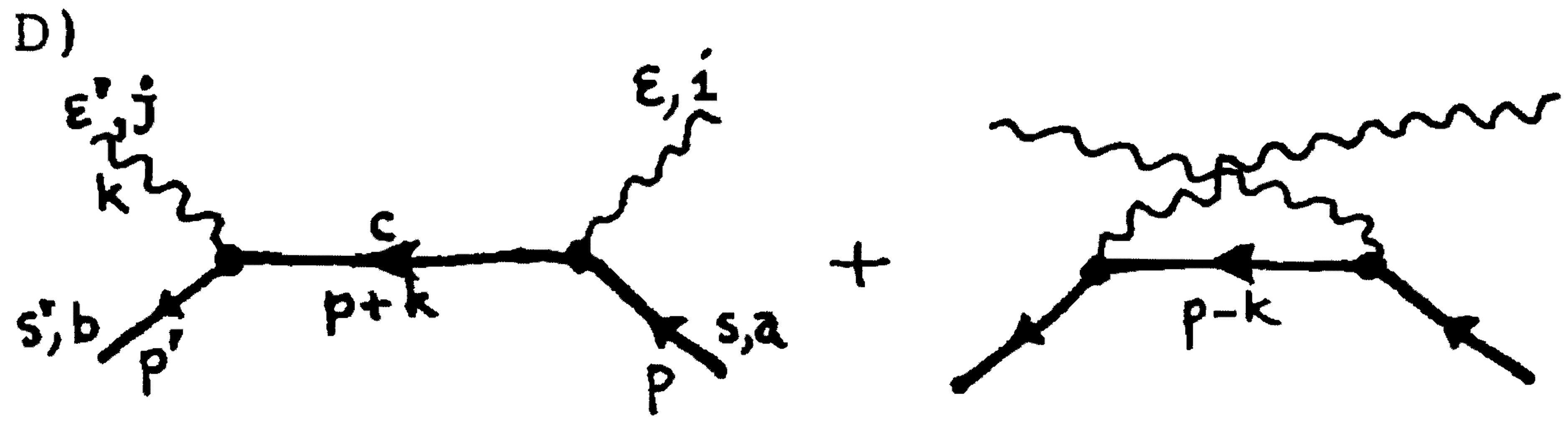
†Quarks have also been introduced by James Joyce: "tree quarks for Muster Mark", *Finnegans Wake*, II.iv.

D. Compton scattering

We assume that the gluons are massless vector particles, just like photons. They should be transverse, and the gaugeons introduced by the longitudinal polarizations (6.4) must not contribute to the S-matrix.

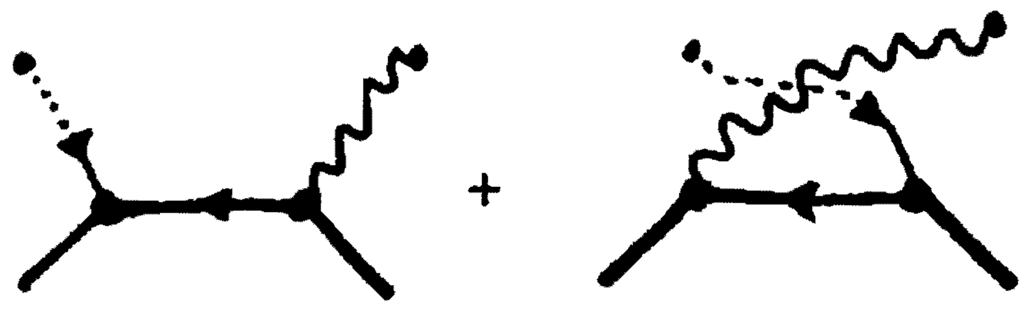
Let us check this by considering the simplest conceivable process: the Compton scattering in the lowest order. The contributing (QED-like) Feynman diagrams are (the rules are summarized in appendix D)

$\mathcal{M} =$ 

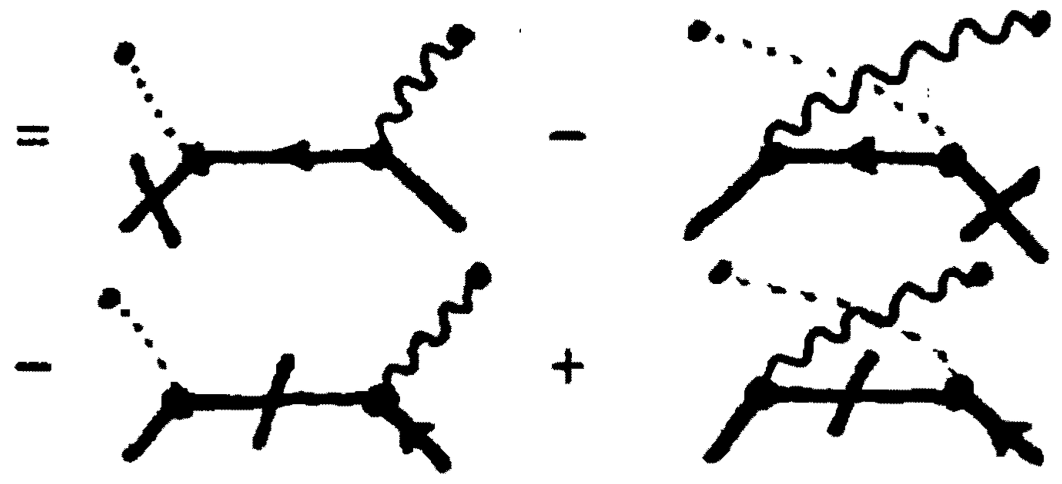
$$= (ig)^2\,\bar{u}(\bar{p}',s')\left[\not\epsilon'\left(T_j\right)^c_b \frac{i}{\not p+\not k-m}\left(T_i\right)^b_a \not\epsilon + \not\epsilon\left(T_i\right)^c_b \frac{i}{\not p-\not k-m}\left(T_j\right)^b_c \not\epsilon'\right]u(p,s) \quad , \tag{6.17}$$

(from now on we shall suppress the polarization and spinor wave functions $\varepsilon_\mu, u, \bar{u}$).

The gaugeon insertions from (6.4) lead to extra contributions to the S-matrix:

$\delta\mathcal{M} =$ 

The bare Ward identity (6.7) yields

$\delta\mathcal{M} =$  (6.18)

The first two terms vanish on the mass-shell. The last two terms differ only in the color factors and yield

$$-\left(T_iT_j - T_jT_i\right)^b_a\left(-i\gamma^\mu\right) \quad . \tag{6.19}$$

In QED $T_i \to e$, and this vanishes, ensuring the gauge invariance of the Compton scattering. What happens in QCD?

E. Color algebra

So far we have put no restrictions on the color couplings other than that T_i be hermitian. A gluon can change an initial quark of any color into a final quark of any color, so there are $i = 1,2,\ldots,n^2$ gluon colors, and there should be n^2 linearly independent coupling matrices T_i. In other words, T_i form a complete basis for expanding hermitian matrices†

$$M_a^b = \sum_{i=1}^{n^2} m_i (T_i)_a^b , \quad \text{real } m_i \quad . \tag{6.20}$$

The color factor $i(T_iT_j - T_jT_i)$ in (6.19) is also a hermitian matrix, so it can be expanded in the T_i basis (repeated indices summed over)

$$T_iT_j - T_jT_i = i\, C_{ij}{}^k\, T_k \tag{6.21}$$

with real constants $C_{ij}{}^k$. This is a Lie algebra, and $C_{ij}{}^k$ are called structure constants. It is convenient to choose the generators T_i in such a way that the Killing-Cartan metric (6.16) is diagonal. We take all $a_i > 0$ (if any a_i were vanishing, the corresponding gluons would not couple at all). If $a_i \neq a_j$, the corresponding gluons couple with different strengths, and the generators T_i can be divided into mutually commuting subsectors (the Lie algebra is semi-simple). The interesting case is the simple Lie algebras, for which all gluons couple with the same strength. (6.16) becomes a normalization convention for Lie algebra generators

$$\mathrm{tr}(T_iT_j) = a\delta_{ij} \qquad i = 1,2\ldots N \leq n^2 \ . \tag{6.22}$$

Physically a is the (unrenormalized) "fine structure constant". With this normalization convention, the structure constants $C_{ij}{}^k = C_{ijk}$ are fully antisymmetric

---

† This is the completeness relation for U(n) generators. In general the color group can be any subgroup of U(n), in which case (6.20) should be replaced by the appropriate completeness relation.

$$C_{ijk} = - C_{jik} = - C_{ikj} \quad . \tag{6.23}$$

So much for the color algebra. The important result is that (6.19) has no reason to vanish, so the QCD gaugeons do not (yet) decouple.

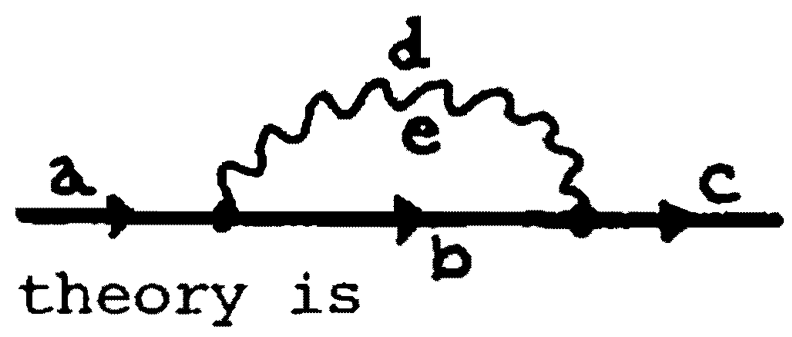
Exercise 6.E.1 Evaluation of color weights. Instead of labeling the gluon colors by $i = 1,2,\dots,N$, it is often more convenient to label them by the colors (a,b), $a,b = 1,2,\dots n$ of the corresponding quark-antiquark pairs. It is very easy to construct generators T^a_b explicitly; for example, $U(n)$ is generated by

$$\left(T^b_a\right)^d_c = \delta^b_c \delta^d_a \quad , \tag{6.24}$$

and $SU(n)$ (the Lie algebra of all traceless hermitian matrices) by

$$\left(T^b_a\right)^d_c = \delta^b_c \delta^d_a - \frac{1}{n} \delta^b_a \delta^d_c \quad . \tag{6.25}$$

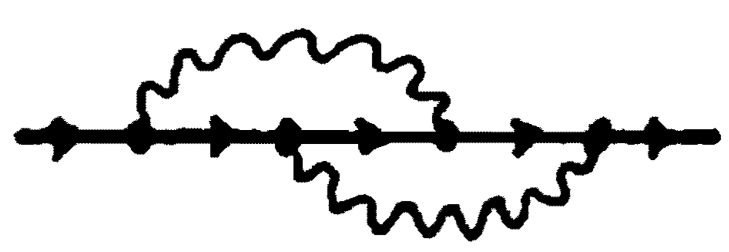
These explicit expressions for the generators enable us to compute the color weights associated with various QCD-graphs. For example, the color weight for the graph



in $U(n)$ gauge theory is

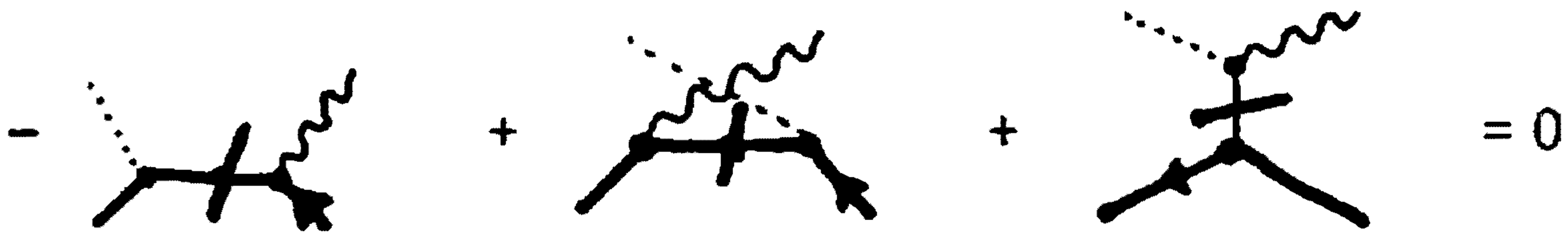
$$\left(T^e_d\right)^a_b \left(T^d_e\right)^b_c = \delta^e_b \delta^a_d \delta^d_c \delta^b_e = \delta^a_c \delta^b_b = 1 \times \mathrm{Tr}\ \underline{\underline{1}} = n \quad .$$

Color weights have a very simple physical interpretation. The momentum space integral is the same for any choice of the external and internal quark and gluon colorings, and each coloring contributes the same amount. The color weight is the number of distinct colorings. In the example above the color weight is n, because the internal quark line can be colored in n ways. What is the $SU(n)$ color weight for the above diagram? Compute the $U(n)$ and $SU(n)$ color weights for



F. Three-gluon vertex

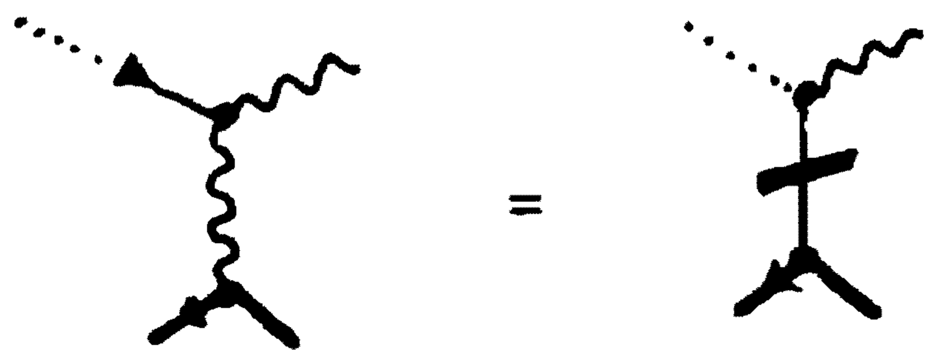
We are in trouble; gaugeons do contribute to the Compton scattering. That is not acceptable, as they are unphysical. We shall now show that the theory can be repaired by introducing a 3-gluon vertex. The physical reason why 3-gluon couplings are needed is that gluons are charged (they carry quark-antiquark colors). A 3-gluon coupling is also suggested by the form of the uncancelled term in (6.19). The Lie algebra (6.21) relates this to emission of a single gluon with coupling T_i, followed by splitting into two gluons with coupling strength iC_{ijk}. We can cancel the extra terms in (6.18) by adding such diagram:



$$= 0 \tag{6.26}$$

The three terms have the same momentum space structure (diagrammatics is explained in appendix D), so this is simply a diagrammatic statement of the Lie algebra.

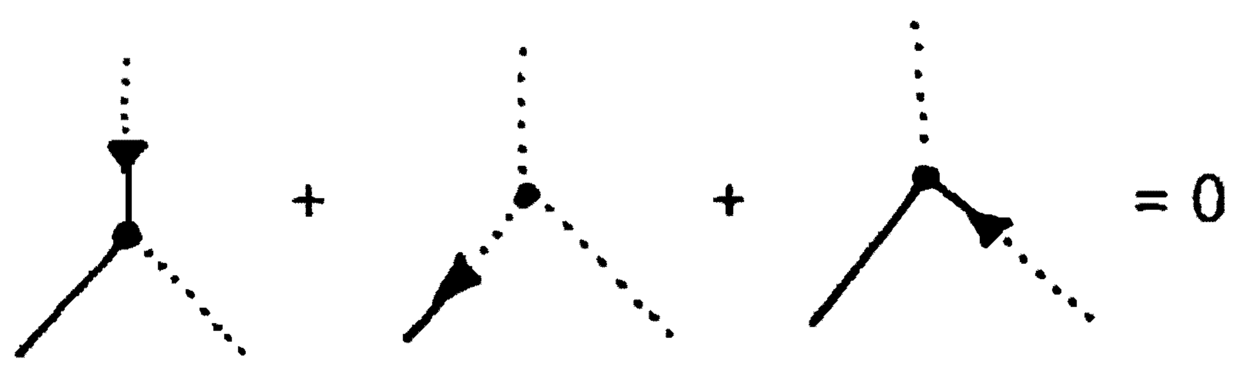
Now we have to invent a 3-gluon vertex which will, upon a gaugeon insertion, yield the desired term



+ (terms vanishing on the mass-shell) .

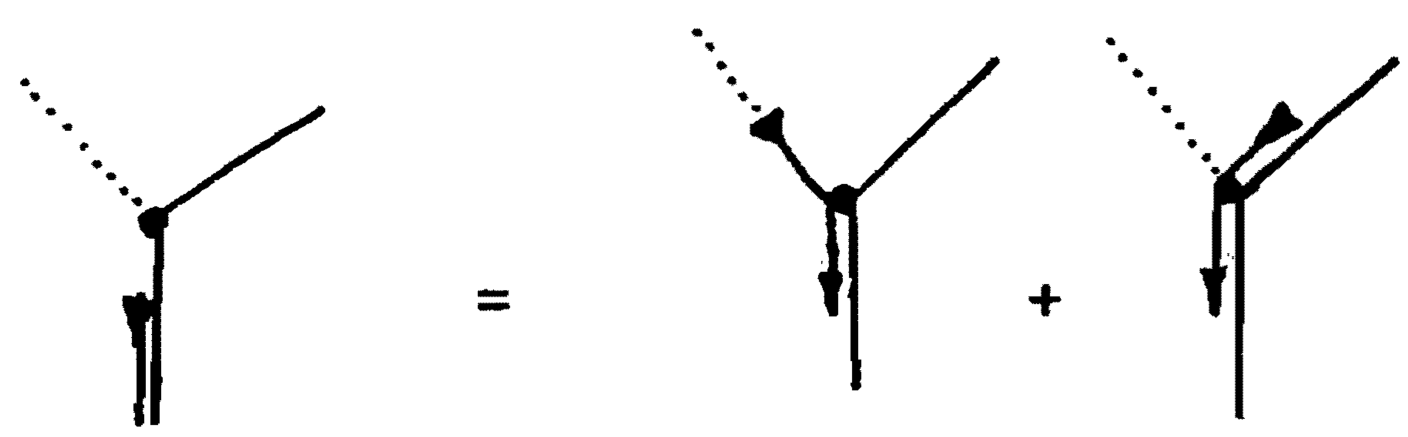
(6.27)

This is reminiscent of the bare quark vertex Ward identity (6.7). That identity is simply a statement of momentum conservation. For vectors, the momentum conservation can be diagrammatically stated as



$$i(-iC_{ijk})(k_1^\rho + k_2^\rho + k_3^\rho) = 0 \quad . \tag{6.28}$$

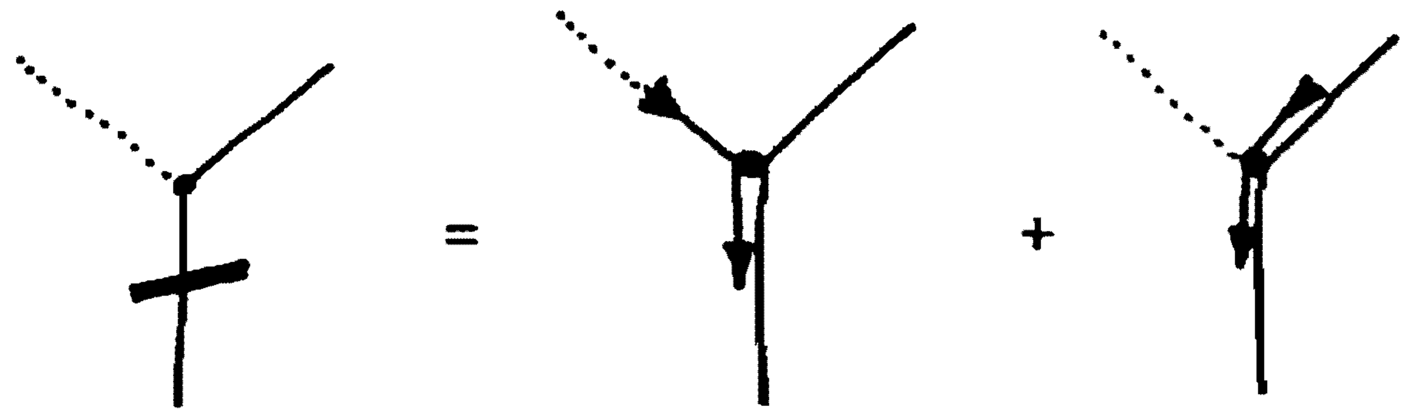
To get something that has a hope of becoming a 3-gluon vertex, we need two more Minkowski indices: the only candidates are $g^{\mu\nu}$ and $k^\mu k^\nu$. $k^\mu k^\nu$ is no good (see exercise 6.G.1), so we try multiplying by $g^{\mu\nu}$:



(6.29)

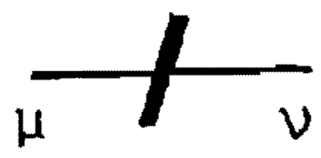
$$(-iC_{ijk})ig^{\mu\nu}k_2^\rho = (-iC_{ijk})ig^{\mu\nu}(k_1^\rho + k_3^\rho) \quad .$$

Contracting with k_2^ρ gives

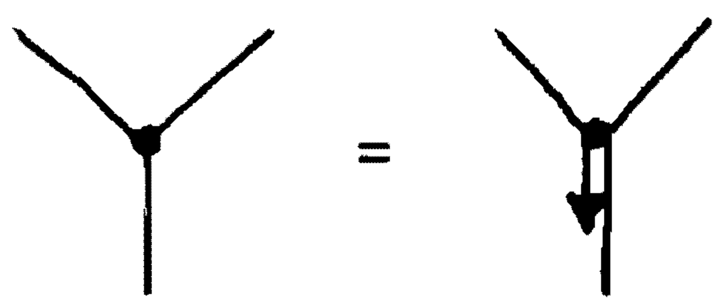


(6.30)

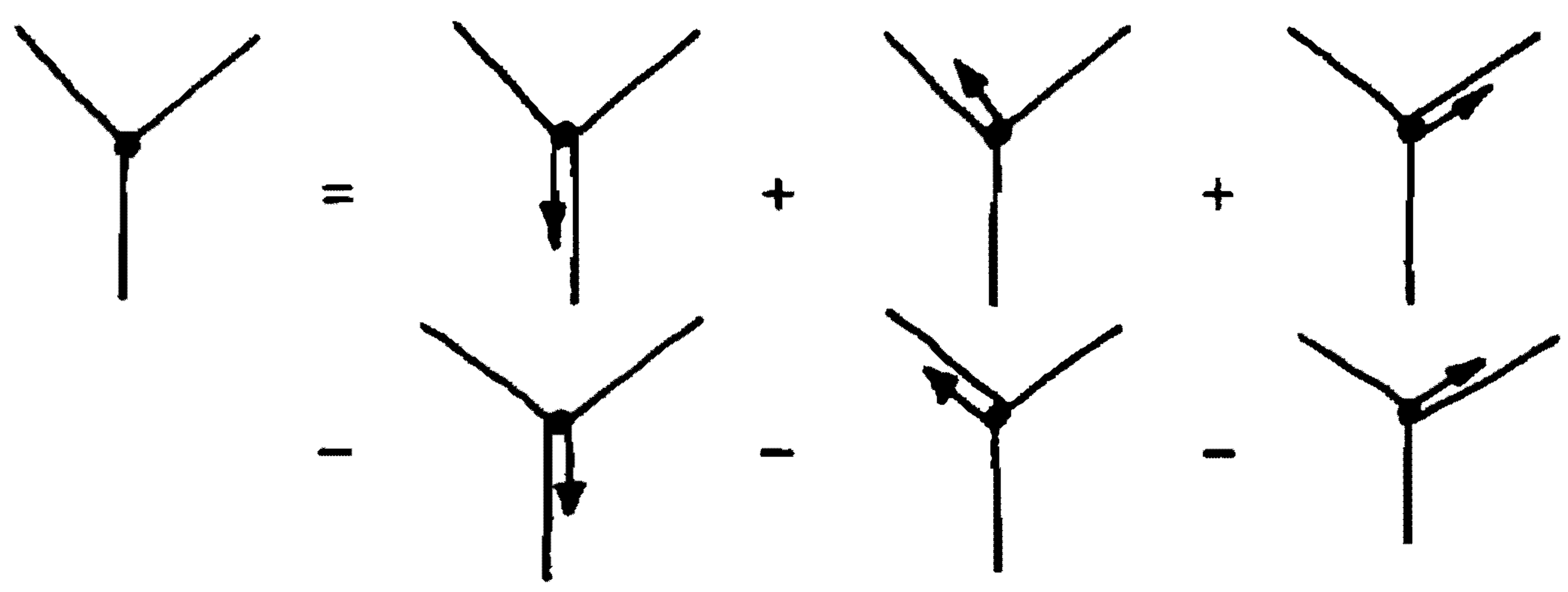
where

 $= k^2 g^{\mu\nu}$ (6.31)

As this is very reminiscent of (6.27), we are tempted to define a three-gluon vertex by



This cannot be right. Gluons are bosons, and the vertex must be symmetric. So we symmetrize our guess:

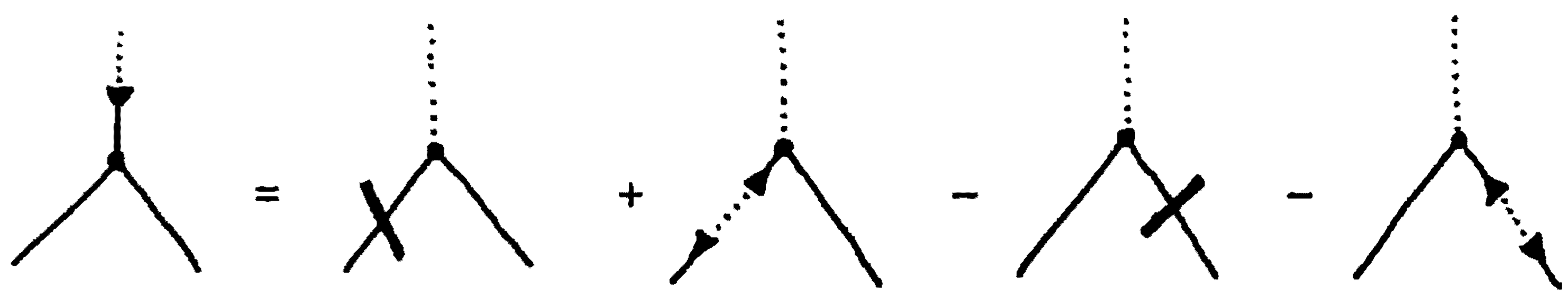


(6.32)

$$\gamma^{ijk}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) = (-iC_{ijk})\,(ig^{\mu_2\mu_3}(k_3-k_2)^{\mu_1} + ig^{\mu_1\mu_2}(k_2-k_1)^{\mu_3}$$

$$+ ig^{\mu_1\mu_3}(k_1-k_3)^{\mu_2}) \quad .$$

Does this satisfy the condition (6.27)? A simple computation yields



(6.33)

$$k_\mu \gamma^{\mu\nu\sigma}(k,k_2,k_3) = \left(-iC_{ijk}\right) i\left[g^{\mu_2\mu_3}\left(k_2^2-k_3^2\right) - \left(k_2^{\mu_2}k_2^{\mu_3} - k_3^{\mu_2}k_3^{\mu_3}\right)\right] \; .$$

This looks right, at least in the Feynman gauge. For gluons in the arbitrary gauge (6.12) we use the identity

$$-\frac{i}{k^2}(g^{\mu\sigma} + f^\mu k^\sigma + k^\mu f^\sigma)\,(g^\nu_\sigma k^2 - k_\sigma k^\nu)$$

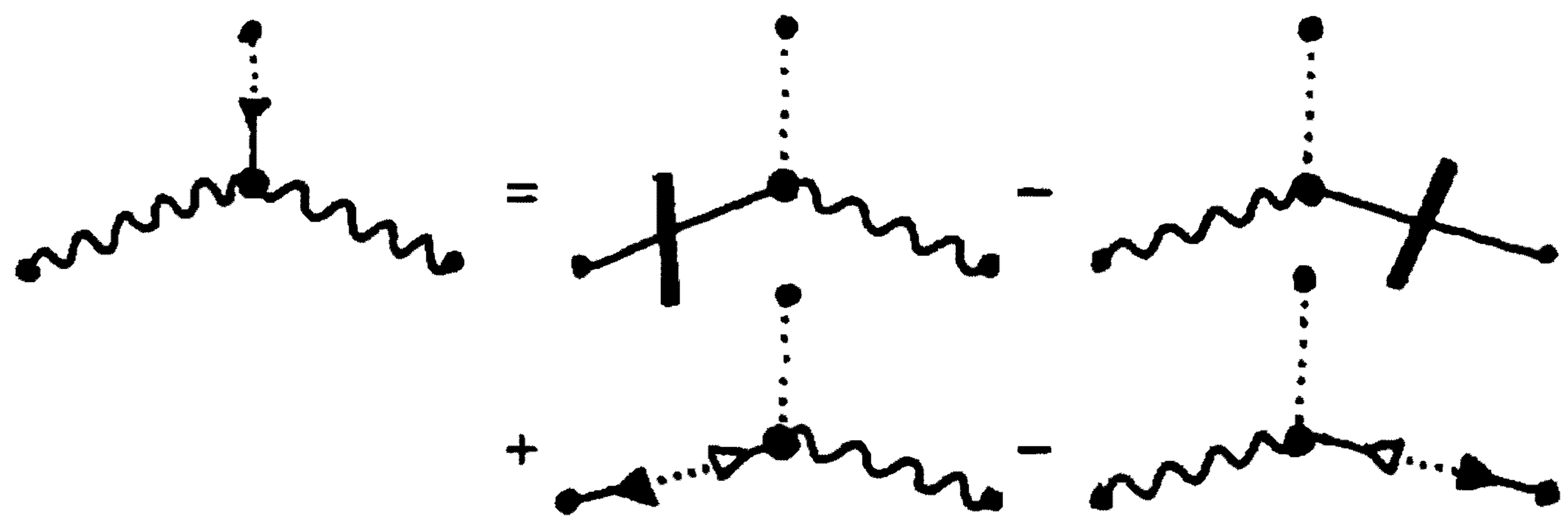
$$= \frac{i}{k^2}(g^{\mu\nu}k^2 - k^\mu h^\nu) \qquad (6.34)$$

where

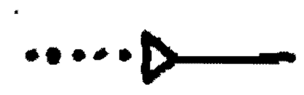
$$h^{\nu} = k^{\nu} - k_{\mu}(k^{\mu}f^{\nu} - f^{\mu}k^{\nu}) \tag{6.35}$$

$$k_{\mu}h^{\mu} = k^{2} \ ; \tag{6.36}$$

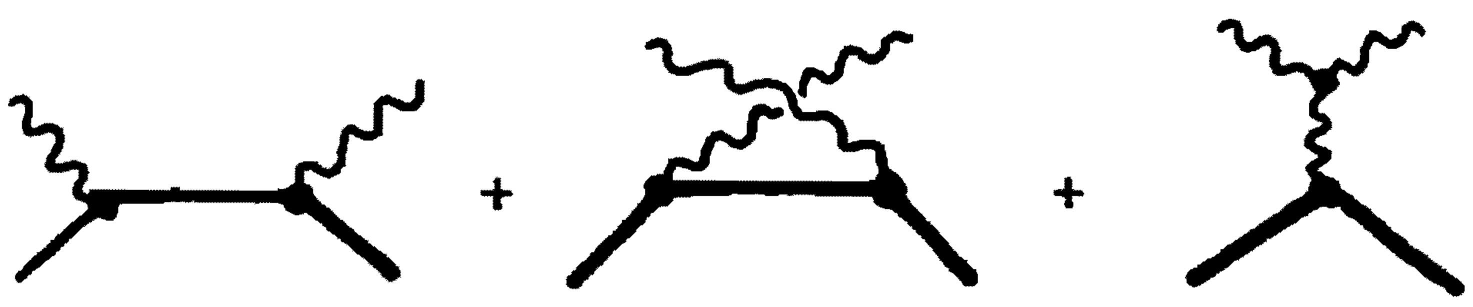
to rewrite (6.33) as the <u>bare 3-gluon vertex Ward identity:</u>



(6.37)

Here  stands for h^{ν}, and the wiggly lines are gluon propagators (see appendix D for diagrammatics). This identity and the three-gluon vertex (6.32) are the main results of this section.

With the three-gluon vertex, the Compton scattering is given by

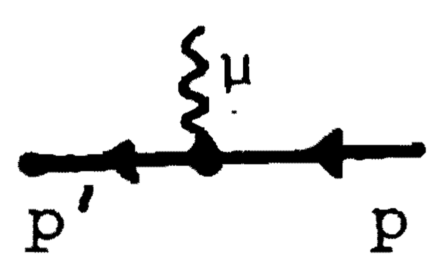
$\mathcal{M} =$ 

(6.38)

rather than by (6.17). One can easily check that the gaugeons now decouple. The Ward identity (6.37) generates 3 extra terms beyond the desired (6.27), but they all vanish on the mass-shell by the transversality condition (6.1) and the mass-shell condition (6.3).

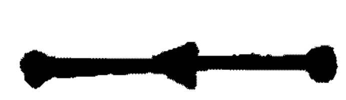
<u>Exercise 6.F.1</u> A three-gluon vertex has three Minkowski indices. Show that they cannot all be carried by the momentum vectors: $\gamma^{\mu\nu\sigma} \neq c k_1^{\mu} k_2^{\nu} k_3^{\sigma}$. Hint: the color factor is antisymmetric.

<u>Exercise 6.F.2</u> <u>Scalar QED vertices</u>. For scalar charged particles the only available vectors are p_{μ} and p'_{μ} and the photon vertex is given by



$$= -ie(p+p')^{\mu} \tag{6.39}$$

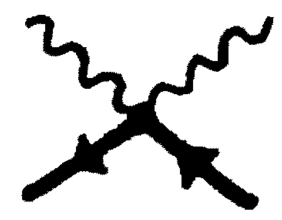
The propagator is the usual scalar propagator



$$= \frac{i}{p^{2} - m^{2}} \quad . \tag{6.40}$$

Show that gaugeons do not decouple if (a) we add a $(p-p')^{\mu}$ part to the vertex (6.39), or

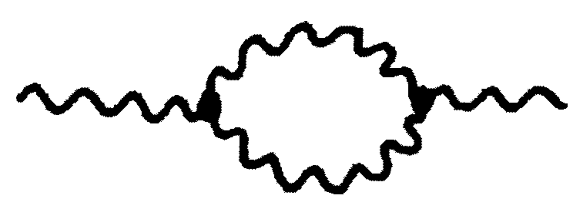
(b) the Compton amplitude is given by diagrams in (6.17). Save the day by devising a 2-photon vertex

 = ?

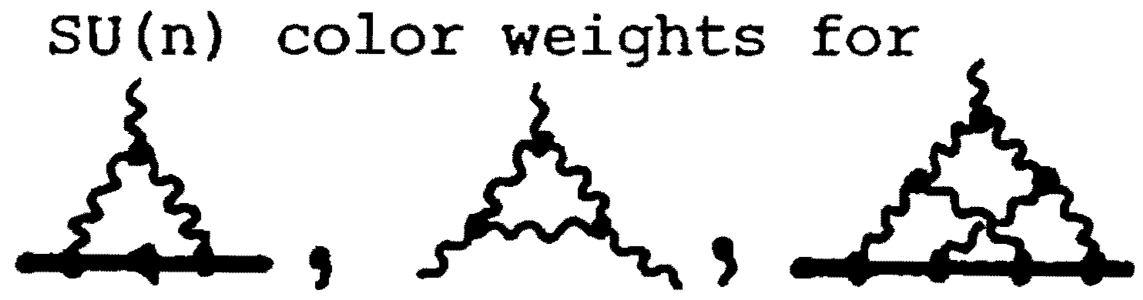
which ensures the decoupling of gaugeons.

<u>Exercise 6.F.3</u> Derive the bare three-gluon vertex Ward identity (6.34). Check the gaugeon decoupling in (6.35).

<u>Exercise 6.F.4</u> (Continuation of exercise 6.D.1). Show that for SU(n) the color weight for the gluon self-energy diagram is

 = 2n .

What is it for U(n)? U(n) is non-semisimple - how does that manifest itself? Is color weight reducible to (6.22)? Compute also U(n), SU(n) color weights for

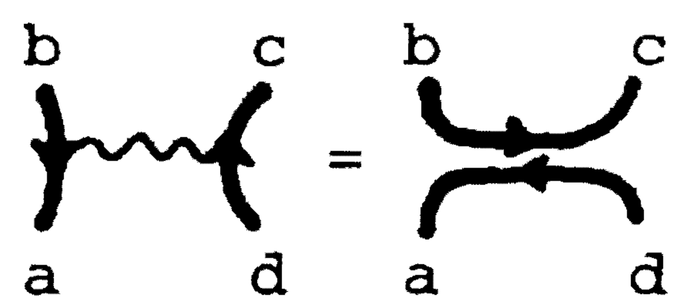


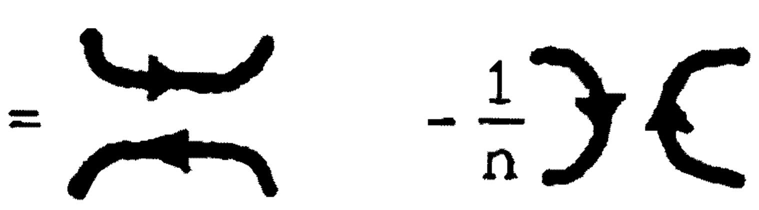
Hints: Lie algebra (6.21) together with normalization (6.22) implies that

$$(-iC_{ijk}) = \frac{1}{a}\,\mathrm{tr}(T_iT_jT_k - T_kT_jT_i) \quad .$$

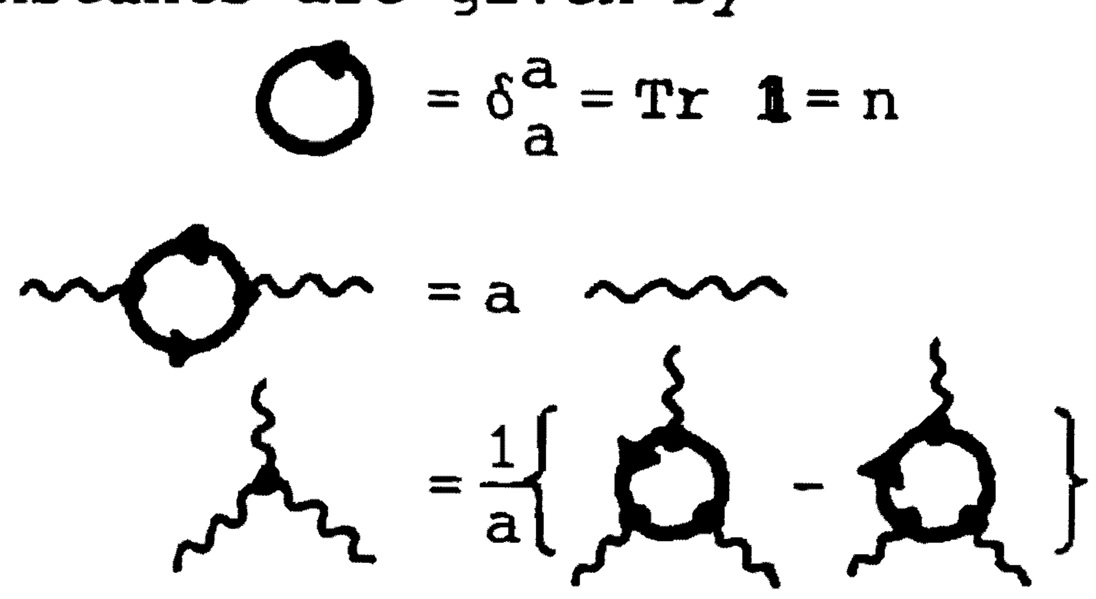
Use this to eliminate the 3-gluon color factors. Resulting color weights can be evaluated by (6.24) and (6.25).

"Birdtracks" are a convenient method for evaluating color weights. In this formalism the gluon projection operators (6.24) and (6.25) are replaced by diagrams:

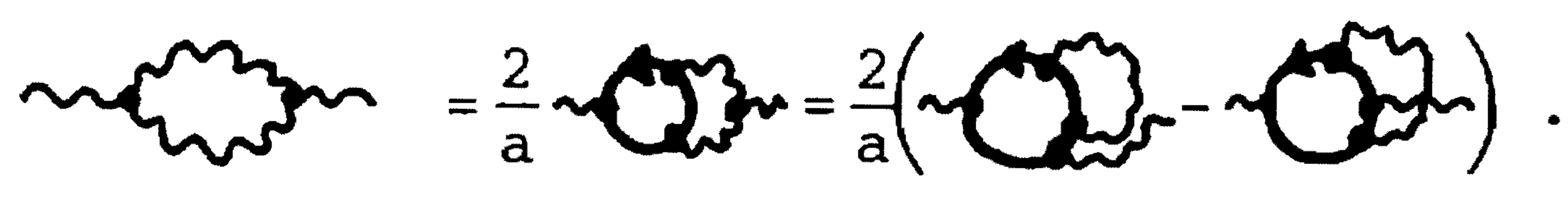
U(n): $(T_a^b)_c^{\ d} = \frac{1}{a}$ 

SU(n): $(T_a^b)_c^{\ d} =$ 

The number of quark colors, the normalization (6.16), and the structure constants are given by

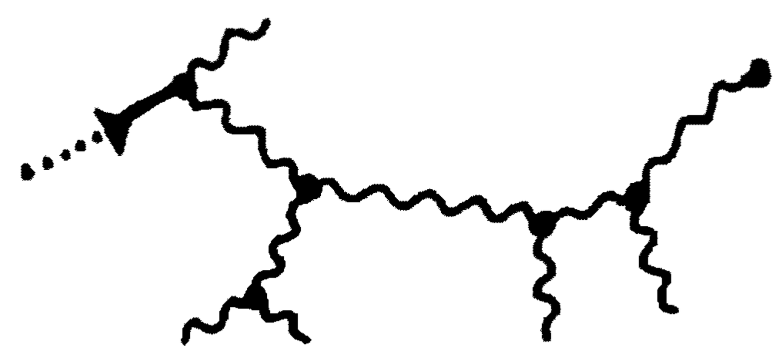


For example, the above gluon self-energy is evaluated by substituting the diagrammatic gluon projection operators in

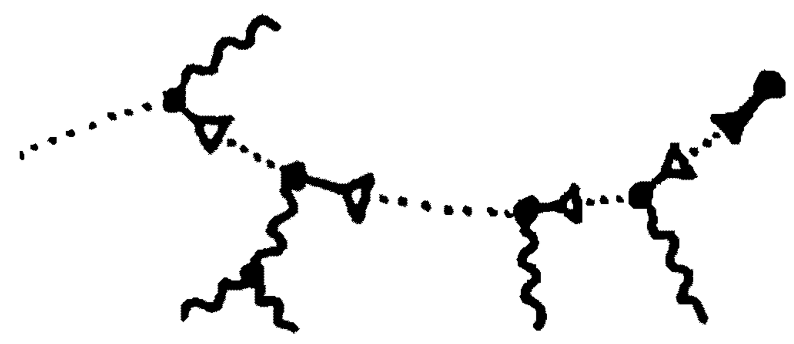


G. Ghosts

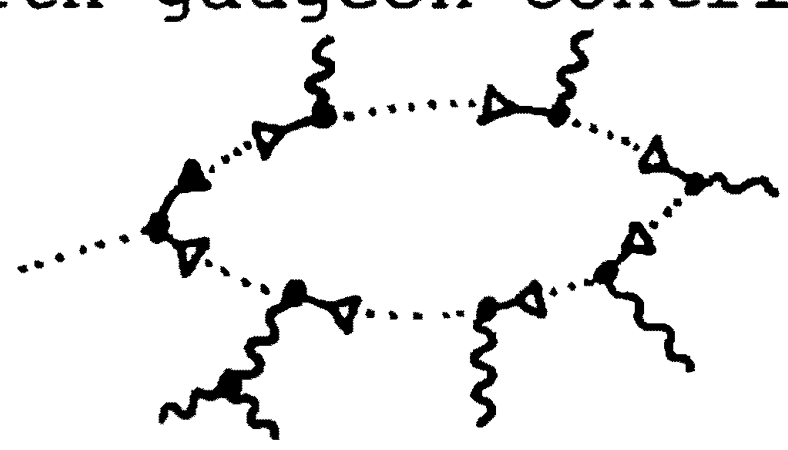
So far so good - we have repaired the Compton scattering by introducing a 3-gluon vertex. However, the 3-gluon bare Ward identity is rather complicated; beyond the terms analogous to the spinor Ward identity (6.7) there are two extra terms with $k^\mu h^\nu$ numerators. If a diagram has a number of 3-gluon vertices



a k^μ insertion will (after repeated applications of the Ward identity (6.37)) yield contributions like

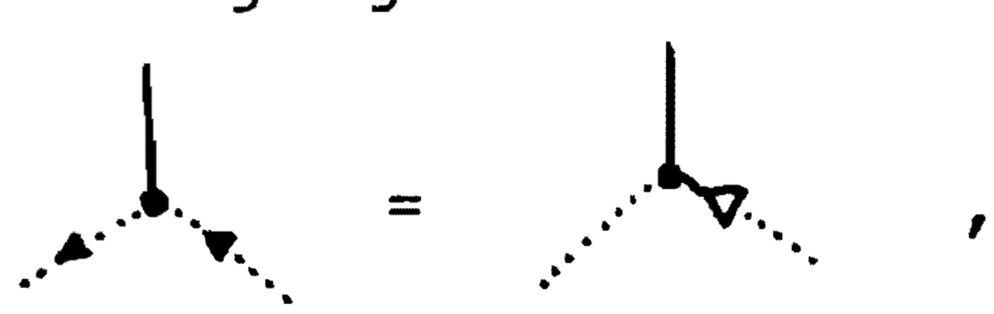


If such a gaugeon line ends up on a quark line, it will (by applications of the fermion Ward identity (6.7)) eventually yield mass-shell vanishing contributions. But if it loops onto itself, we are stuck with gaugeon contributions of the type

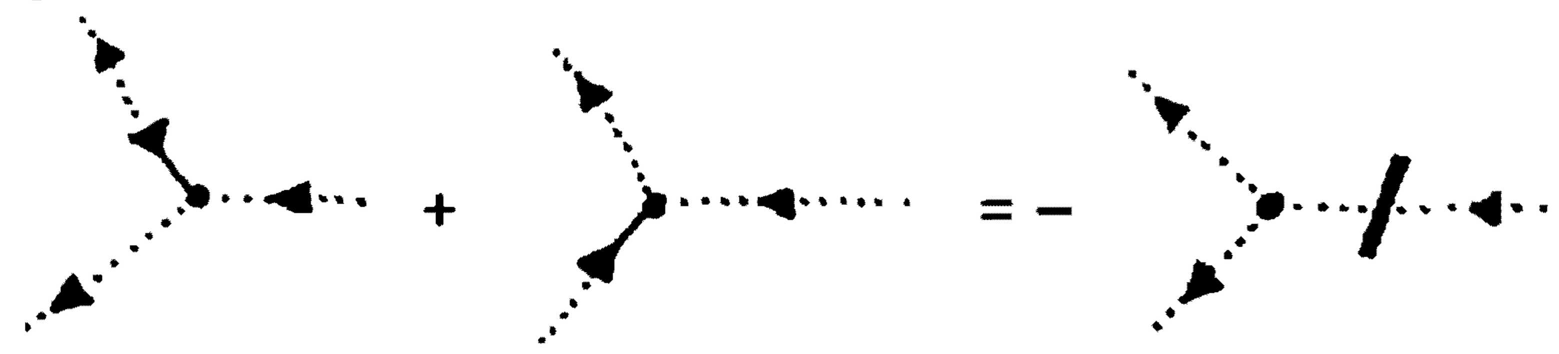


which have no reason to vanish. The problem is that the physical gluon has only d-2 degrees of freedom, but with our Feynman rules, all d components contribute; there are too many degrees of freedom circulating along the loops.

This disease has a drastic cure. We introduce a new particle, called a ghost, whose sole purpose is to (in the manner of ghoulies) eat up the longitudinal degrees of freedom. It couples to gluons just like the gaugeon

 (6.41)

(cf. appendix D), but each ghost loop carries a minus sign and thus cancels the corresponding gaugeon loop. As we have seen in chapter 4, such particles must obey Fermi statistics. The arrow on the ghost line keeps track of the $h^\mu(k)$ factors in (6.41). We shall prove in the next chapter that ghosts indeed cure the gaugeon loop problem. For that we shall also need the bare ghost vertex Ward identity, which, as always, is simply a statement of the momentum conservation, this time combined with the identity (6.36):

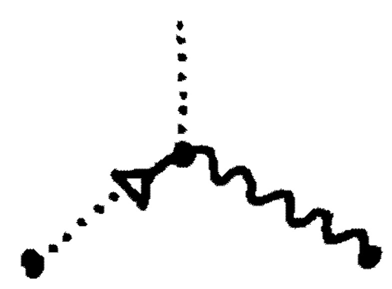


$$C_{ijk}\left[-k_\mu h^\mu(k+k') - k'_\mu h^\mu(k+k')\right] = -\,C_{ijk}(k+k')^2 \ . \tag{6.42}$$

(Note that because of the color factor the "vertex" on the right-hand side is antisymmetric - this is another indication that the ghosts must be treated as fermions.)

In order to verify the correctness of the ghost prescription we shall have to go through some algebra. However, the physics of ghosts should already be clear; gaugeons are unphysical degrees of freedom, and the ghosts are here to cancel them. Neither "particle" has any physical meaning by itself.

Exercise 6.G.1 Show that for axial gauges $n^\mu D_{\mu\nu} = 0$, so that the gaugeons decouple

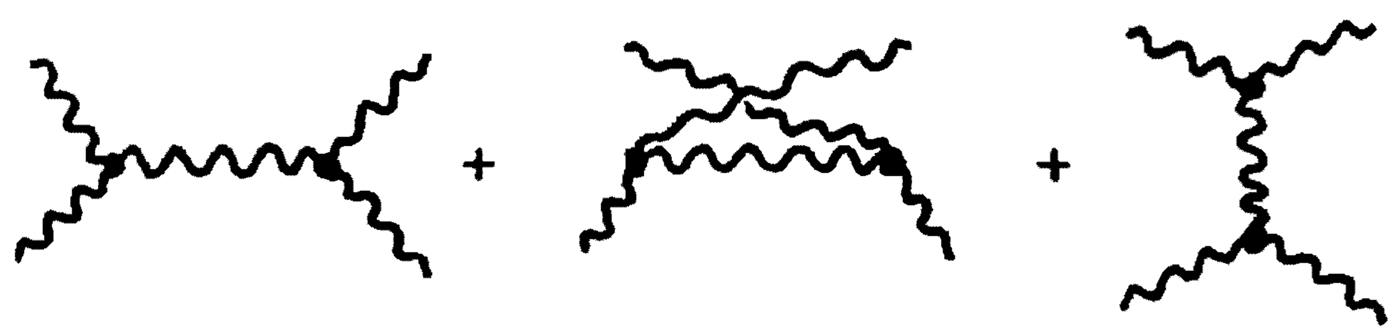


$$= 0 \ . \tag{6.43}$$

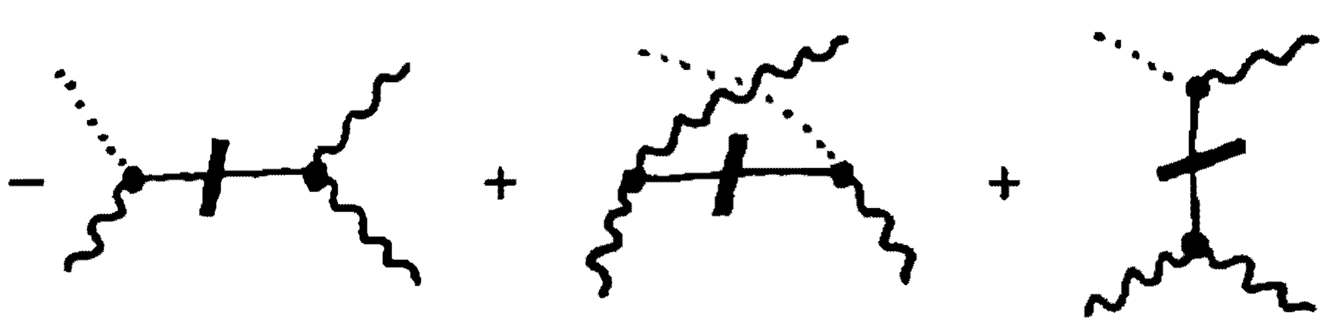
This means that the axial gauges are "ghost-free"; the Ward identities will turn out to be no more complicated than the QED ones. This is the reason that the axial gauges are often used in general diagrammatic gauge invariance arguments. Computationally they are horrid.

H. Four-gluon vertex

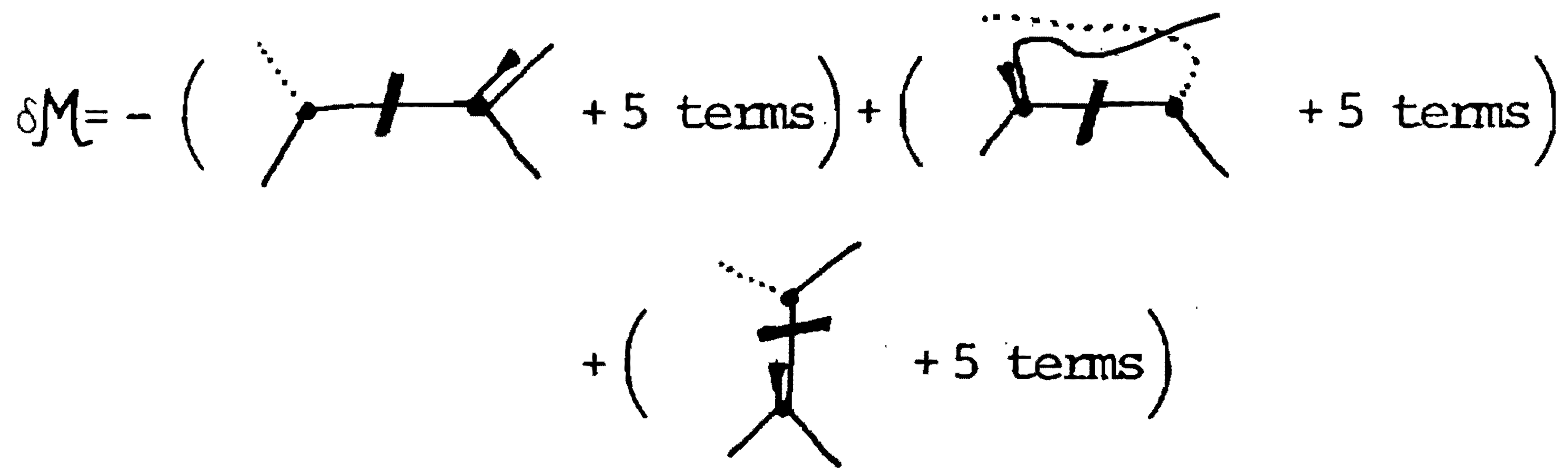
The next thing we have to check is the gauge invariance of the gluon-gluon Compton scattering:

$\mathcal{M} =$  (6.44)

Inserting a gaugeon, using the gluon Ward identity (6.37), and discarding the mass-shell vanishing contributions, we end up with

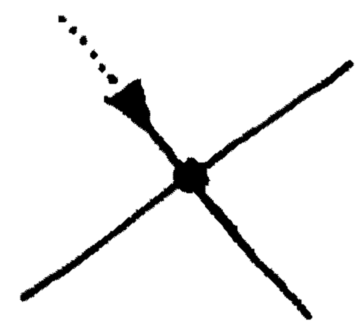
$\delta\mathcal{M} = -$  (6.45)

Replacing each 3-gluon vertex by (6.32) yields lots of terms



(6.46)

There is no reason for this to vanish. To rescue the theory we have to devise a 4-gluon vertex for which a gaugeon insertion

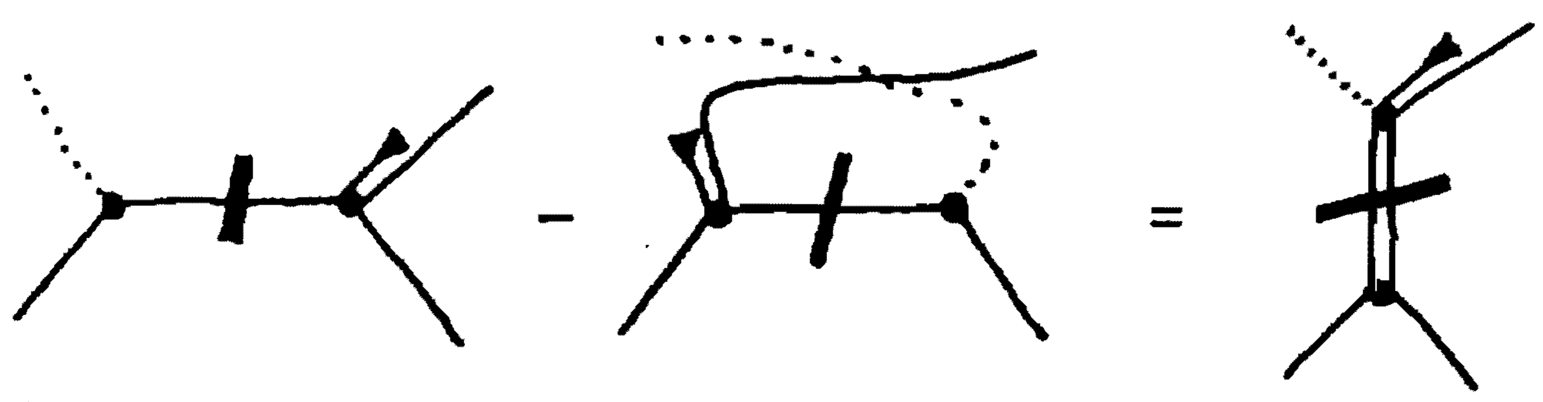


(6.47)

precisely cancels (6.46). We do this by rewriting (6.46) in a form that resembles a gaugeon insertion into a 4-vertex. The tools that we have at our disposal are the momentum conservation (6.28) and the Lie algebra commutator (6.21), which, for 3-gluon couplings, is the Jacobi relation

$$C_{ijm}C_{mk\ell} - C_{jm\ell}C_{kim} = C_{jkm}C_{m\ell i} \quad . \tag{6.48}$$

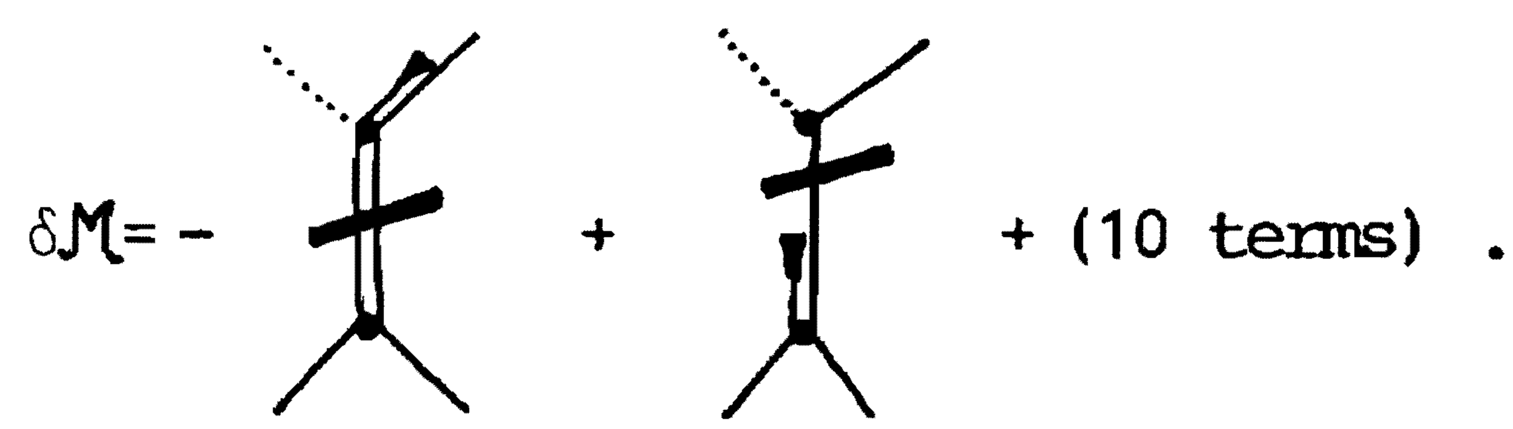
We can use the Jacobi identity to combine the (6.46) terms with the same Minkowski structure. For example,



$$ig^{\nu\rho}ik_4^{\mu}((-iC_{ijm})(-iC_{mk\ell})-(-iC_{imk})(-iC_{jm\ell}))(-i)$$

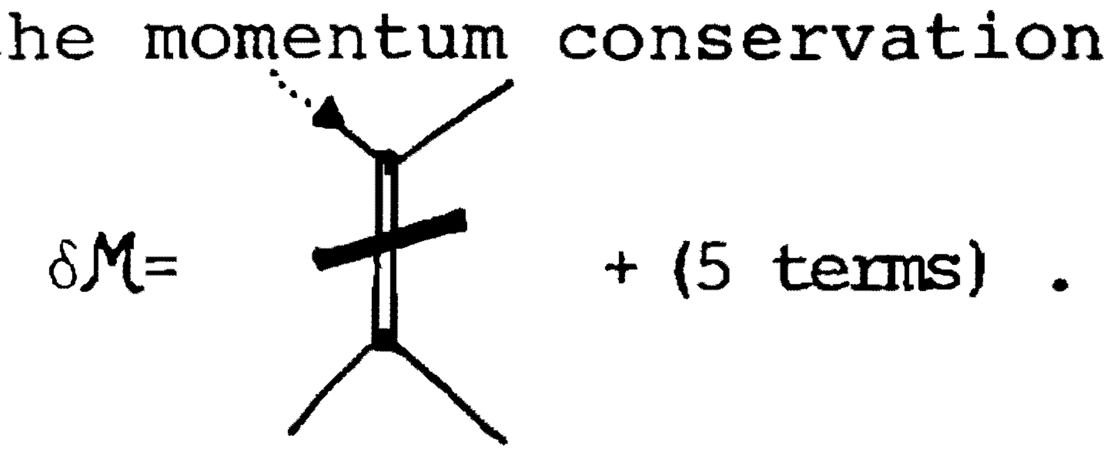
$$= ig^{\nu\rho}ik_4^{\mu}(-C_{im\ell})(-iC_{mjk})(-i) \quad . \tag{6.49}$$

This reduces the number of terms in (6.46) to twelve:



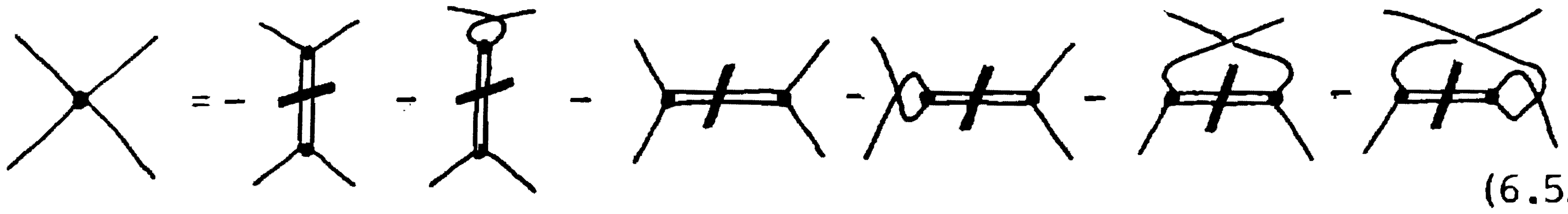
(6.50)

By the momentum conservation (6.28) these add up to six terms



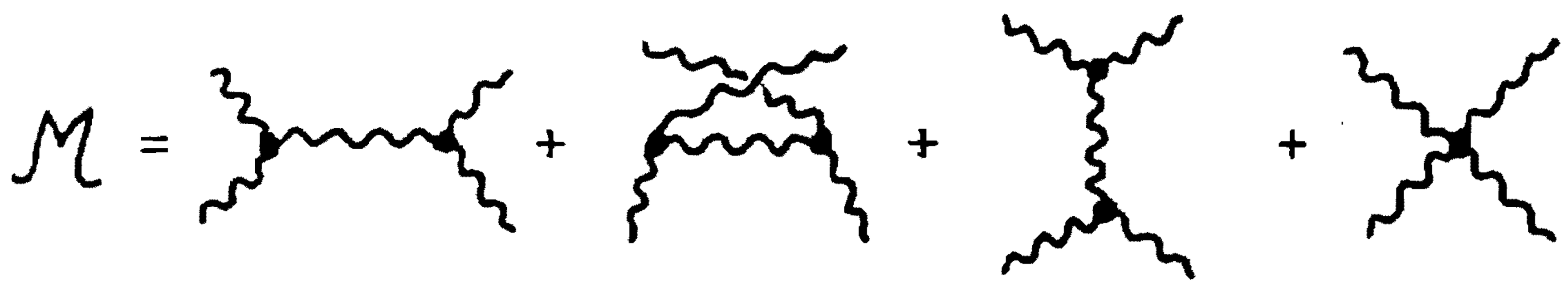
(6.51)

Now the gaugeon contribution is of the desired form (6.47). If we define the four-gluon vertex by



(6.52)

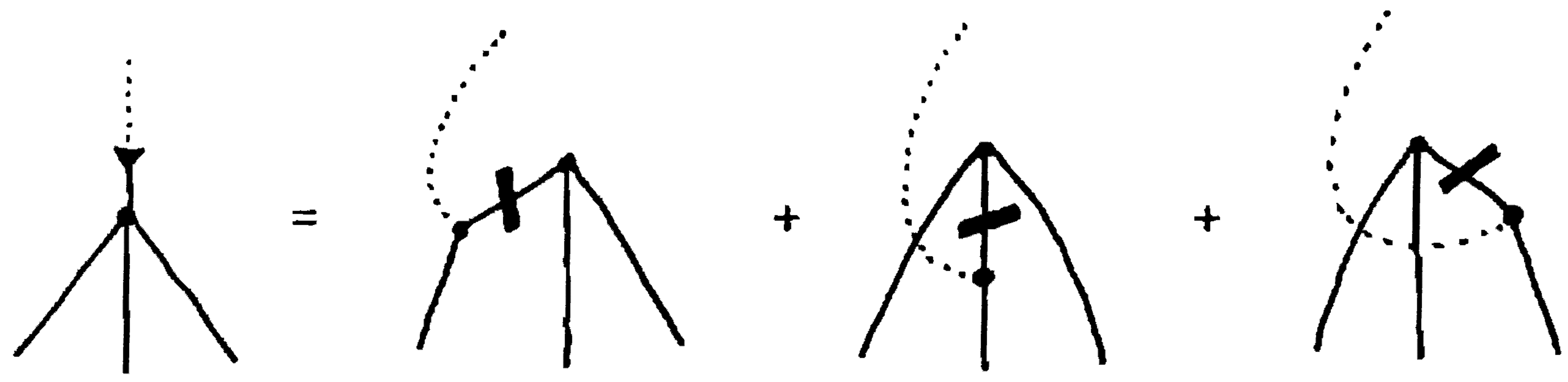
the gluon-gluon scattering amplitude



(6.53)

is gauge invariant.

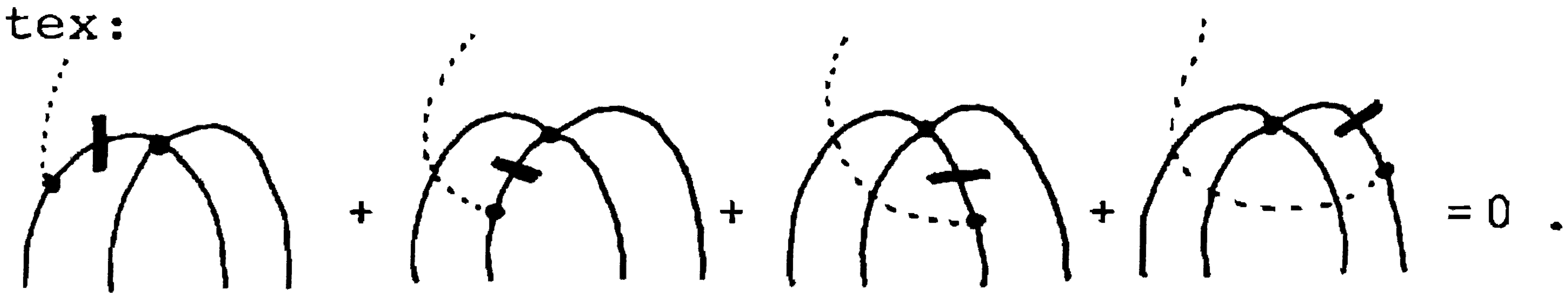
Definition of the four-gluon vertex (6.52), together with the <u>bare four-gluon vertex Ward identity</u>



(6.54)

are the main results of this section. (The Ward identity follows from (6.45)).

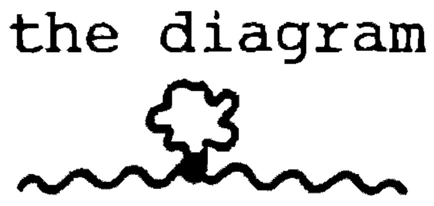
So far we have succeeded in making the quark-gluon and the gluon-gluon tree level scattering amplitudes gauge invariant, but at what a price: three new kinds of vertices and even ghosts. This looks like a story without end; next one might need a 5-gluon vertex to fix up the five-leg Green functions, etc. Indeed, in theories like gravity, one would find 5-graviton vertex, 6-graviton vertex,..... . For QCD the buck stops here - we shall prove that in the next chapter. To carry out the proof, we shall also need the following <u>invariance condition</u> for the four-gluon vertex:



(6.55)

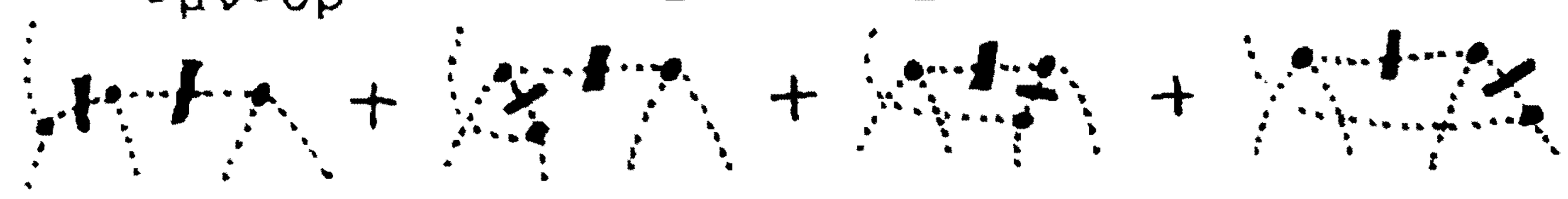
This is simply a statement that $C_{ijk}C_{k\ell m}$ is an invariant tensor (exercise 6.H.2).

<u>Exercise 6.H.1</u> (Continuation of exercise 6.F.4). Compute the SU(n) color weight for the diagram



Hint: the 4-gluon vertex (6.52) is really composed of pairs of 3-vertices, so group-theoretically there are no 4-vertices.

Exercise 6.H.2 Prove the invariance of the 4-gluon vertex, (6.55). Hint: note that nothing in (6.55) depends on the momentum. Substituting (6.52) you will observe that each Minkowski factor $g_{\mu\nu}g_{\sigma\rho}$ is multiplied by color factor



Prove (by using the Jacobi identity (6.48)) that this vanishes.

I. QCD action

As explained in chapter 2, Feynman rules can be compactly summarized by the action functional (2.13). Carrying this out for the QCD Feynman rules is a straightforward but somewhat tedious continuation of exercises 2.E.2 and 2.D.1. The compact indices are replaced by the full set of explicit indices:

$$\phi_i \to \left(A^i_\mu(k), \bar{\omega}^i(k), \omega^i(k), \bar{q}^a_\alpha(k), q_{a\alpha}(k)\right) \quad , \tag{6.56}$$

where A is the gluon field, ω and $\bar{\omega}$ is the ghost and antighost fields, and q and $\bar{q}$ the quark and antiquark fields.

The result is known to everybody:

$$\begin{aligned}
S[\phi] &= i\int dx\mathcal{L} \\
\mathcal{L} &= \mathcal{L}_{YM} + \mathcal{L}_{fix} + \mathcal{L}_{ghost} + \mathcal{L}_{quark} \\
\mathcal{L}_{YM} &= -\tfrac{1}{4}\left(F^i_{\mu\nu}\right)^2 \\
&\qquad F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + gC_{ijk}A^i_\mu A^k_\nu \\
\mathcal{L}_{fix} &= -\frac{1}{2a}\left(\partial^\mu A^i_\mu\right)^2 && \text{(covariant gauges)} \\
\mathcal{L}_{ghost} &= \bar{\omega}^i \partial^\mu D^{ij}_\mu \omega^j && \text{(covariant gauges)} \\
&\qquad D^{ij}_\mu = \delta^{ij}\partial_\mu + gC_{ikj}A^k_\mu \\
\mathcal{L}_{quark} &= i\bar{q}_a \not{D}^a_b q^b - m\bar{q}_a q^a \\
&\qquad D^a_{b\mu} = \delta^a_b \partial_\mu - ig(T_i)^a_b A^i_\mu \quad .
\end{aligned} \tag{6.57}$$

Checking the equivalence between the above action and our Feynman rules is dullness embodied (though nothing compared to doing the same for the supergravity actions). The only nontrivial step is the inversion of the gluon propagator (this is needed for the quadratic part of the action (2.13)). The general case is unilluminating and we relegate it to the exercise 7.H.1; the problem can be understood by looking just at the covariant gauges. The covariant propagator (appendix C) can be decomposed into the transverse and longitudinal parts

$$ik^2 D^{\mu\nu} = (g^{\mu\nu} - k^\mu k^\nu/k^2) + a k^\mu k^\nu/k^2 \quad . \tag{6.58}$$

The inverse is simply

$$-ik^{-2}(D^{-1})^{\mu\nu} = (g^{\mu\nu} - k^\mu k^\nu/k^2) + \frac{1}{a} k^\mu k^\nu/k^2 \quad . \tag{6.59}$$

However, if $a = 0$ the propagator is purely transverse, and it cannot be inverted. There is nothing wrong in using $a = 0$ (Landau) gauge in evaluating Feynman diagrams, but non-invertibility is a problem for the path integral formulation: a zero eigenvalue for the propagator (6.58) means that the path integral (3.7) has no gaussian damping factor for integrations over longitudinal fields $A^i_\mu \propto \omega^i(k)k_\mu$. These troublesome directions are just our old gaugeons in a new guise, and the cure is gauge fixing.

Multiplying (6.59) by the momentum conservation delta function and $A^i_\mu(k)A^i_\nu(k')$, summing over color and Minkowski indices, integrating over momenta and Fourier transforming, we obtain the quadratic part of $\mathcal{L}_{YM}$

$$S_{\text{transverse}} = -\frac{i}{4}\int dx \left(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu\right)^2 \tag{6.60}$$

and the gauge fixing term $\mathcal{L}_{fix}$ in (6.57). The remainder of (6.57) is obtained in the same way.

Exercise 6.I.1 Inverting gluon propagators. Under the gauge transformation (6.4) the polarization sum (6.10) transforms into

$$\varepsilon^\mu \cdot \varepsilon^\nu \to \varepsilon^\mu \cdot \varepsilon^\nu + k^\mu(\omega\cdot\varepsilon^\nu) + (\varepsilon^\mu\cdot\omega)k^\nu + (\omega\cdot\omega)k^\mu k^\nu .$$

Define functions $h^\mu_T(k)$, $B(k)$ by

$$\varepsilon^\mu \cdot \omega = -h^\mu_T/k^2 \quad , \qquad \omega\cdot\omega = B/k^2 \quad .$$

With transverse ε_μ, equation (6.1), the gluon propagator (6.10) can be written as

$$ik^2 D^{\mu\nu}(k) = (g^{\mu\nu} - k^\mu k^\nu/k^2) - (h^\mu_T k^\nu + k^\mu h^\nu_T)/k^2 + Bk^\mu k^\nu/k^2$$

$$k_\mu h^\mu_T = 0 \quad . \tag{6.61}$$

This is nothing but a rewrite of (6.12) in terms of the transverse, mixed and longitudinal parts, convenient for inversion. Check that h_T is the transverse part of the ghost vertex (6.35), $h = k + h_T$. Show that the inverse propagator is given by

$$(ik^2D^{-1})_{\mu\nu} = (g_{\mu\nu} - k_\mu k_\nu/k^2) + h_\mu h_\nu/(B - h_T^2/k^2) \quad . \tag{6.62}$$

Show that the gauge fixing terms in the action are given by

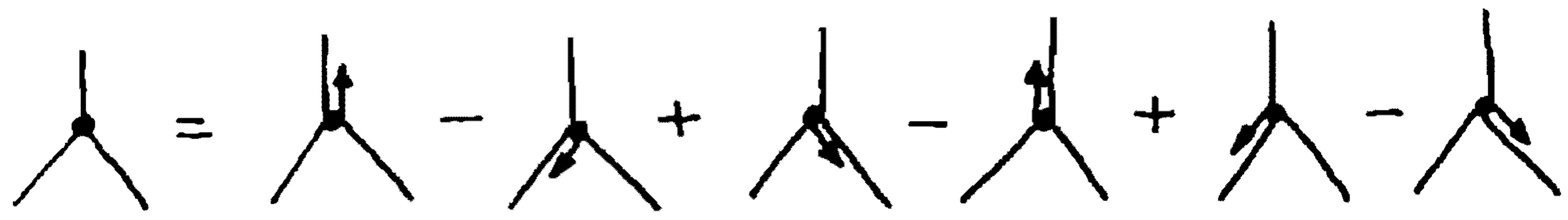
$$\begin{aligned}
&\text{Covariant:} && \mathcal{L}_{fix} = -\frac{1}{2a}(\partial_\mu A^\mu)^2 \\
&\text{Axial} \quad : && \mathcal{L}_{fix} = -\frac{1}{2a}(n_\mu A_a^\mu)^2 \\
&\text{Planar} \quad : && \mathcal{L}_{fix} = -\frac{1}{2an^2}(n_\mu A_a^\mu)\,\partial^2\,(n_\nu A_a^\nu) \\
&\text{Coulomb} \quad : && \mathcal{L}_{fix} = -\frac{1}{2a}[(n^2\partial_\mu - n\cdot\partial n_\mu)A_a^\mu]^2 \quad .
\end{aligned} \tag{6.63}$$

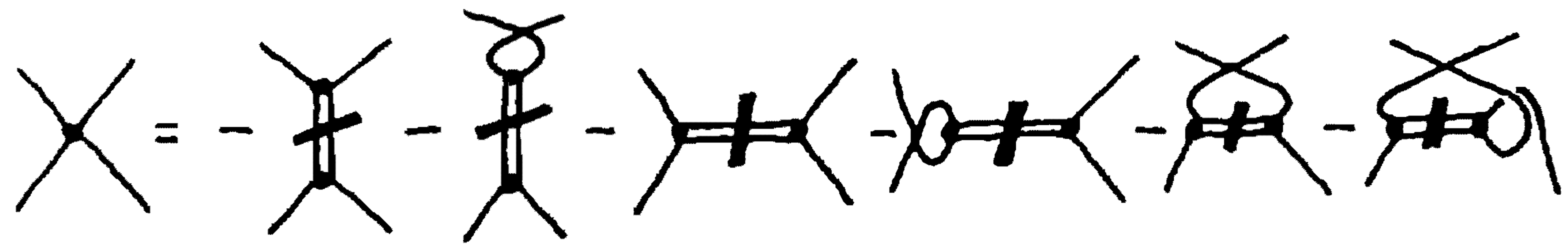
Corresponding propagators are given in appendix C. Note that most of the popular gauges (Landau, axial, etc.) correspond to the singular $a \to 0$ and/or $n^2 \to 0$ limits. Do you feel uncomfortable? Reflect briefly upon whether you are really enjoying this.

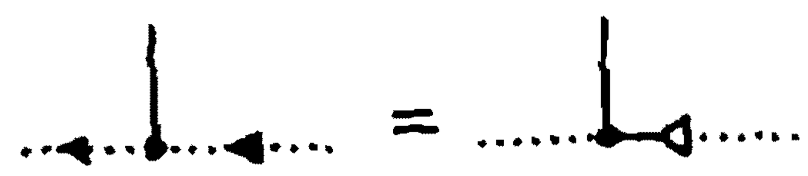
Exercise 6.I.2 Construct (6.57) from our Feynman rules, or verify the Feynman rules from (6.57), whichever is more to your taste. Note that the ghost propagators and vertices differ in the two formulations. Is that a problem?

J. Summary:

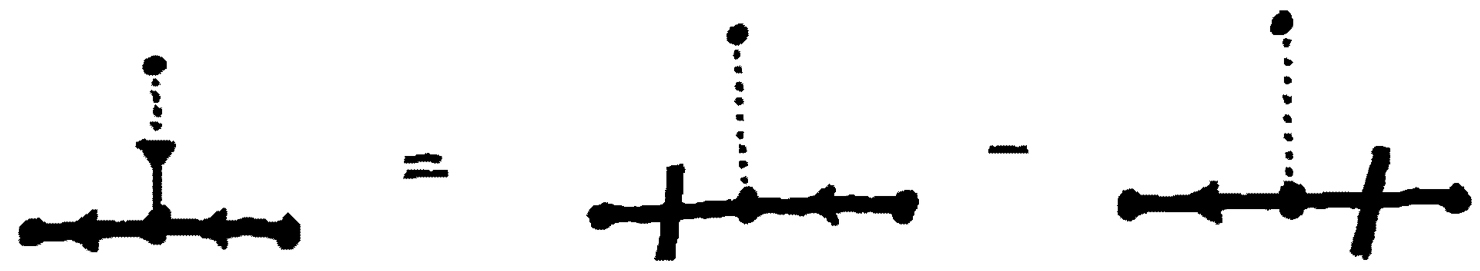
bare vertices:

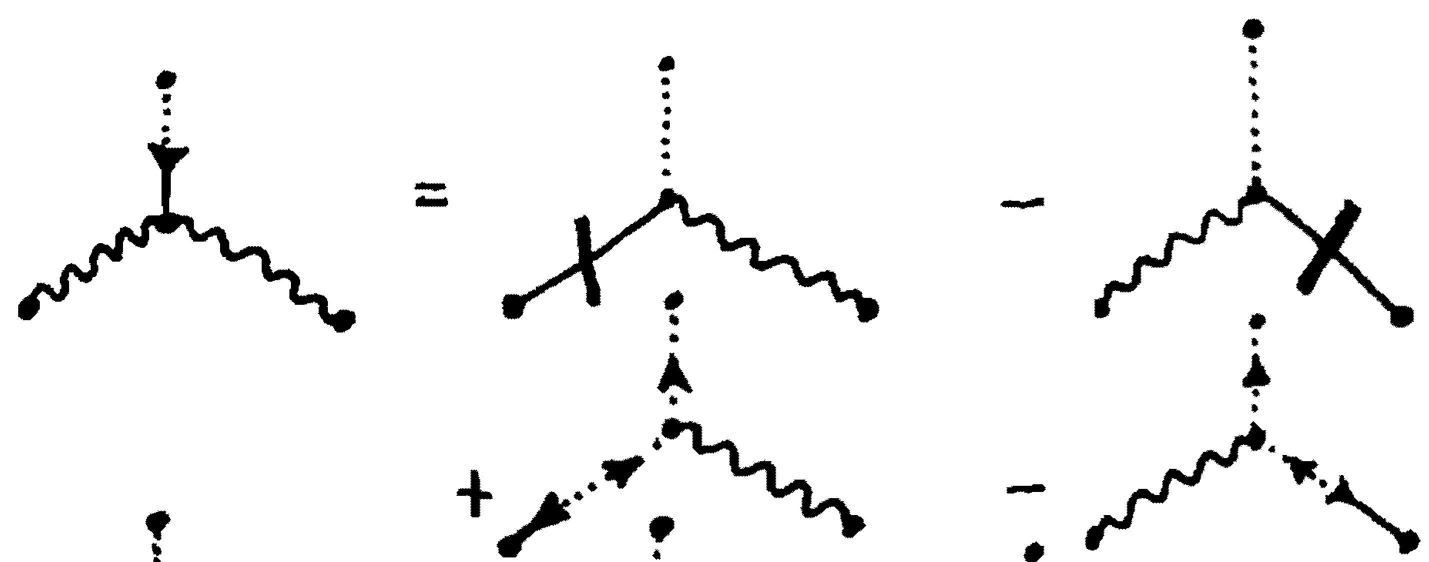
3-gluon  (6.64a)

4-gluon  (6.64b)

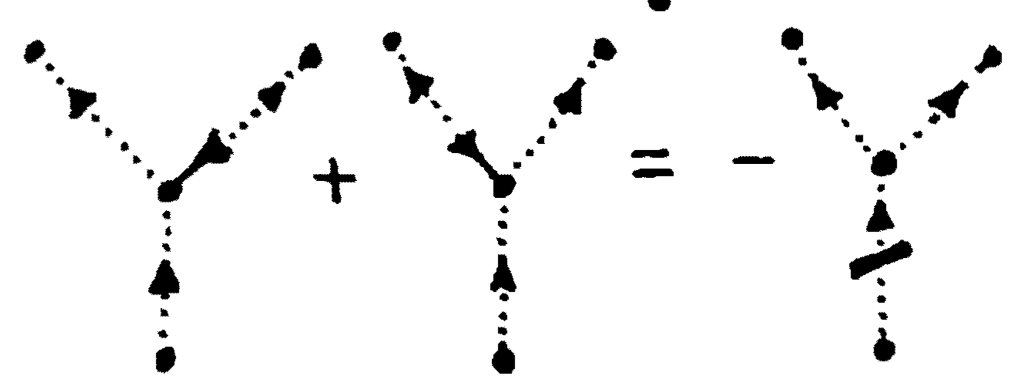
ghost-gluon  (6.64c)

bare vertex Ward identities:

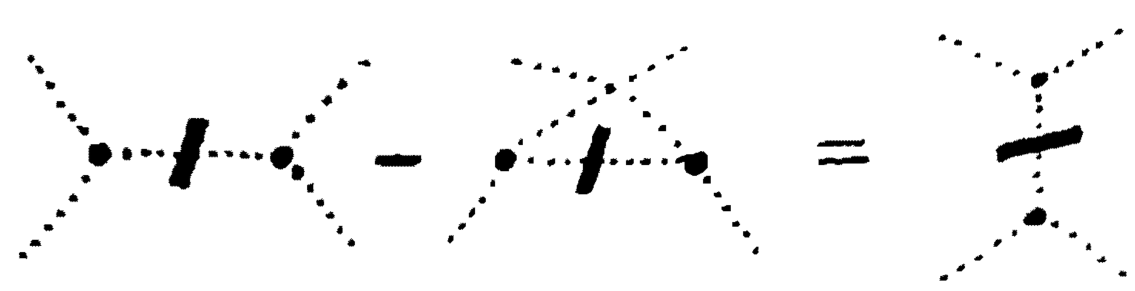
quark-gluon  (6.65a)

3-gluon  (6.65b)

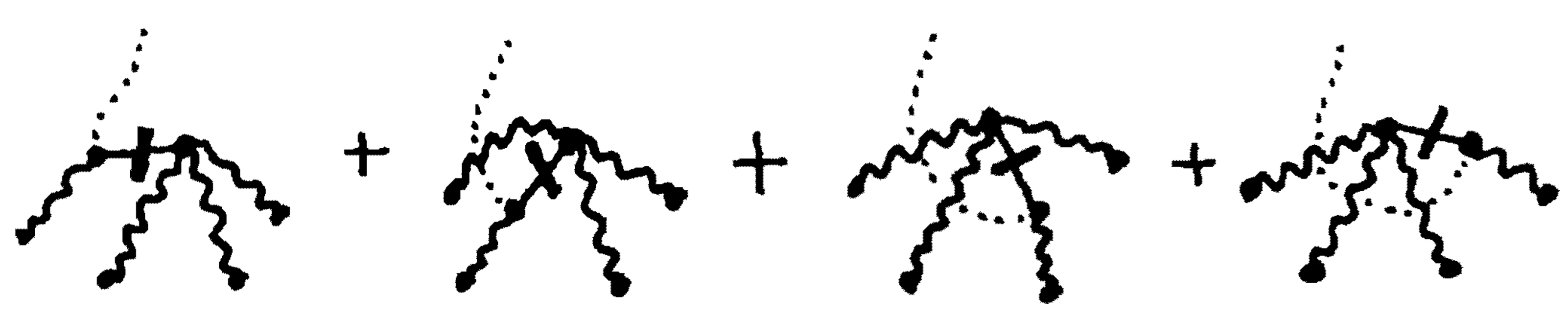
4-gluon  (6.65c)

ghost-gluon  (6.65d)

invariance conditions:

Jacobi identities  (6.66a)

(and similarly with Minkowski factors)

4-gluon  (6.66b)

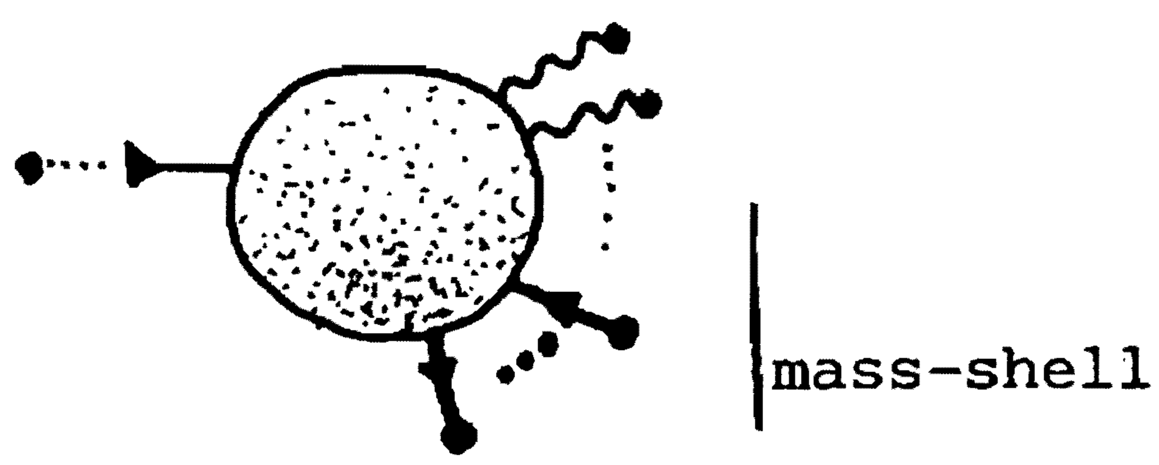
The diagrammatic rules are explained in appendix D.

7. QCD WARD IDENTITIES

We tried to put a little color into QED and we got into a considerable mess. It seems as though one has to introduce a new vertex or particle for each process one looks at - a dismal prospect. Fortunately things are not that bad - we shall now prove that with the QCD vertices constructed in the last chapter the gaugeons decouple from all S-matrix elements. Regardless of their later guises, the requisite identities are contained in the original Gerard 't Hooft's paper[†], so we shall call them Ward identities.

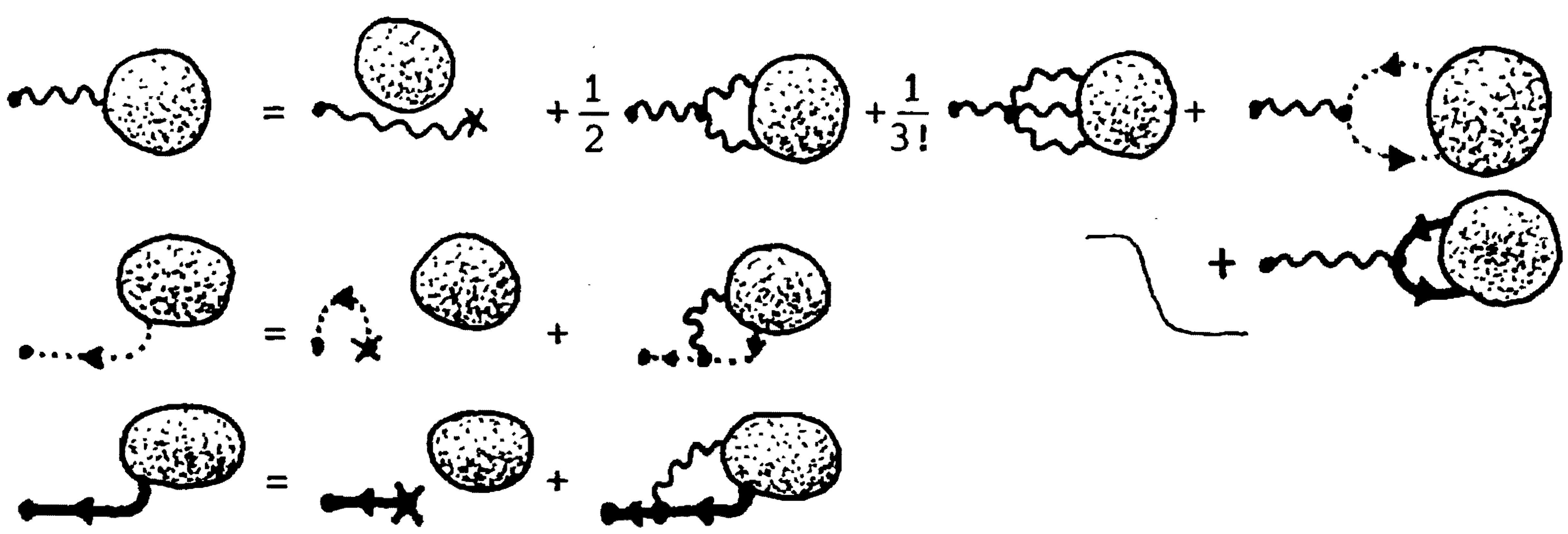
A. Ward identities for full Green functions

In this section we shall prove that the gaugeons (6.5) decouple from any QCD mass-shell process;



= 0 (7.1)

The QCD Dyson-Schwinger equations (2.12)



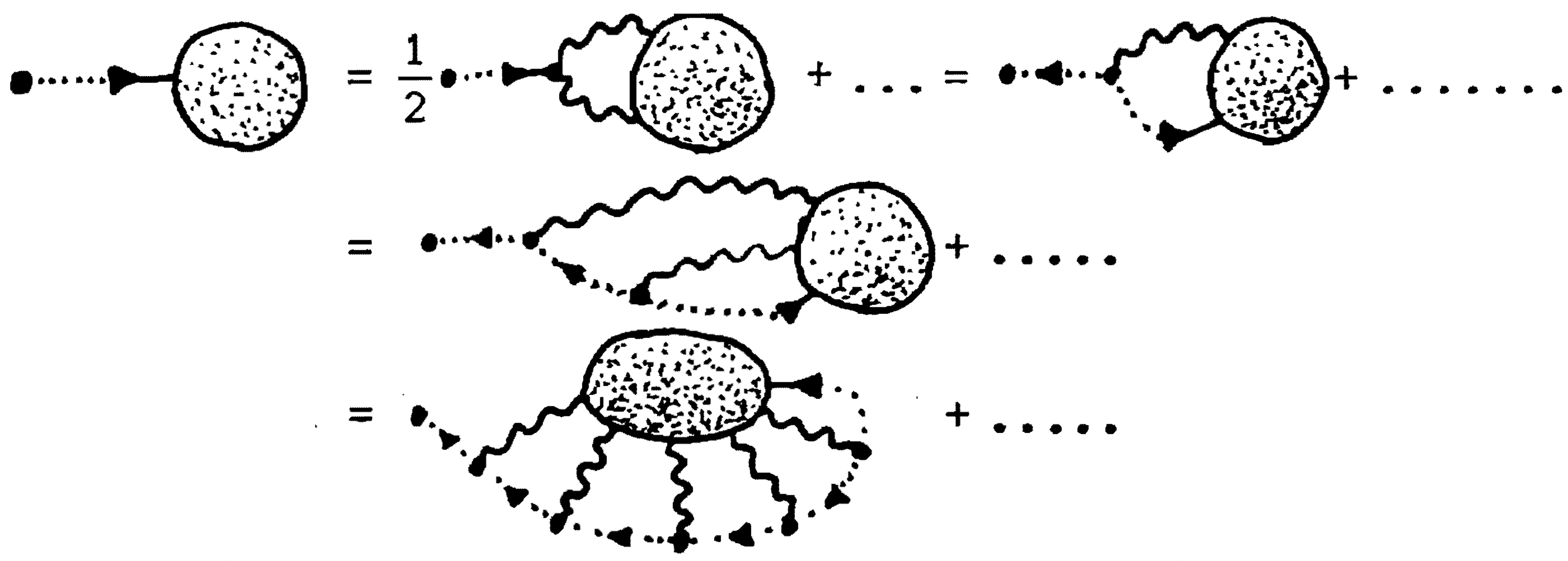
(7.2a)

(7.2b)

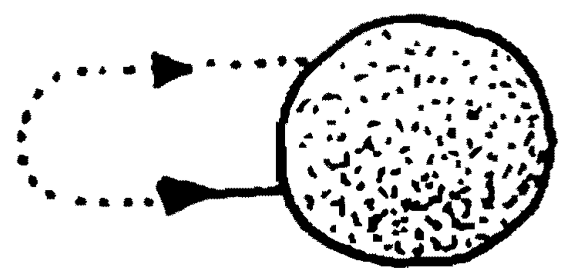
(7.2c)

enable us to follow the gaugeon into the Green functions. Because of the bare 3-gluon Ward identity (6.37), the gaugeons "propagate" into the diagrams:

[†] G. 't Hooft, "Renormalization of massless Yang-Mills fields", Nucl. Phys. B33(1971)173. These identities are also known as Lie - Engel - Schur - Wigner - Eckhart - Schwinger - Stückelberg - Feynman - Ward - Takahashi - Green - 't Hooft - Veltman - Taylor - Slavnov - Lee - Zinn-Justin - Nielsen - Kluberg - Stern - Zuber - Becchi - Rouet - Stora - Kugo-Ojima - Feigenbaum - Witten - Polyakov - Parisi - Wilson - Moffat identities.

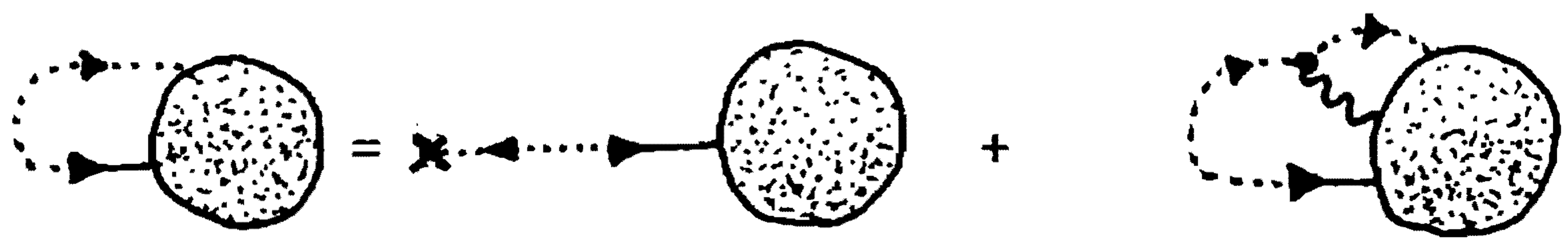


This fact (together with much hindsight) suggests that the convenient starting point for the proof is not the external leg gaugeon (7.1), but gaugeon insertion anywhere inside a Green function:



(7.3)

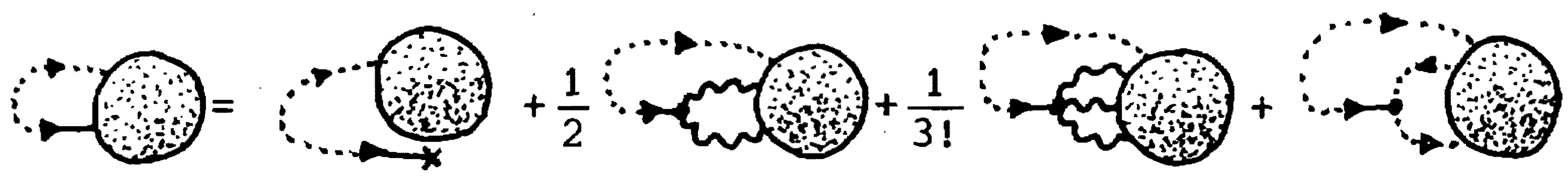
The ghost DS equation (7.2b) yields the desired external gaugeon insertion (7.1), together with an extra term



(7.4)

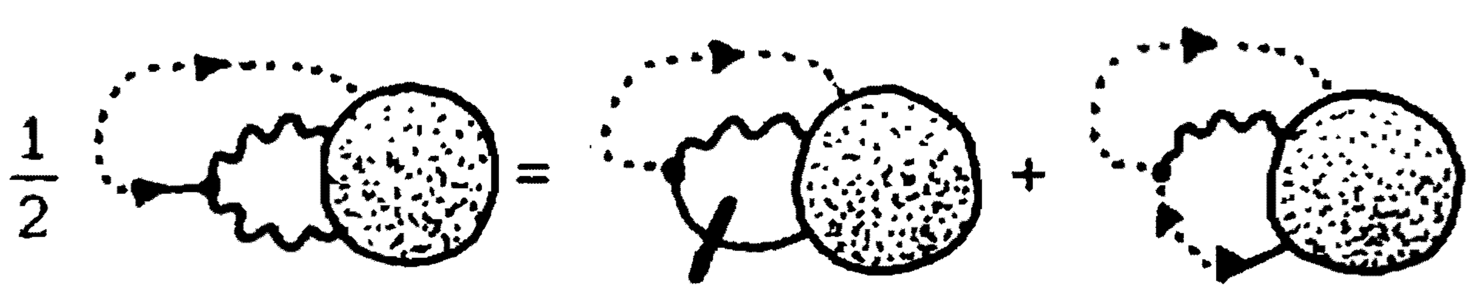
> As ghosts are fermions, the ghost equations are bound to cause sign anxieties. The best thing to do is to relax and remember that the only thing that matters is that each ghost loop carries a minus sign.

The gluon DS equation (7.2a) yields



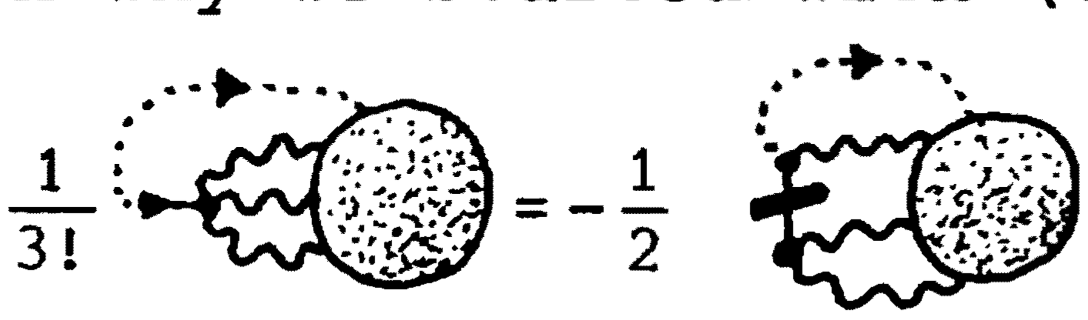
(7.5)

(We omit quarks for the time being - their inclusion is straightforward, cf. exercise 7.A.1). The last three terms are clearly there to be hit by the bare Ward identities (6.37), (6.54), and (6.42):

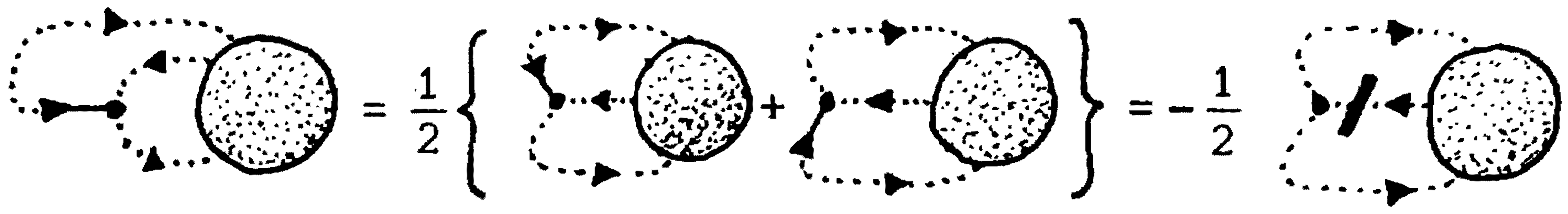


(7.6)

(The second term cancels the extra bit in (7.4); this is the reason why we started with (7.3) rather than (7.1).)

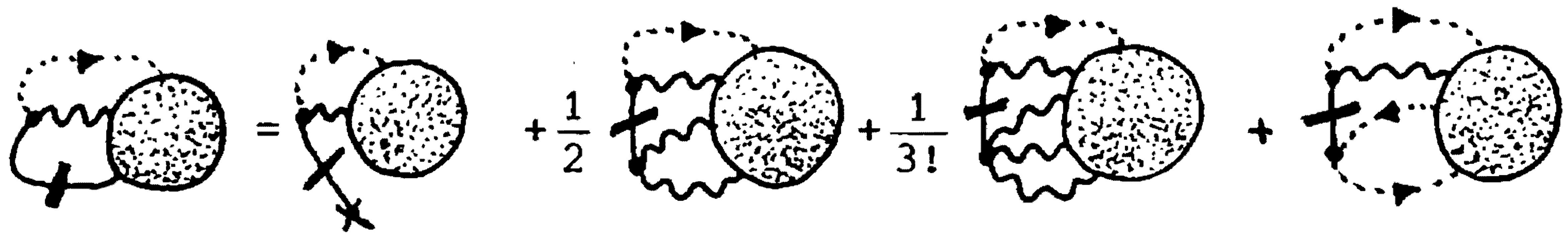


(7.7)



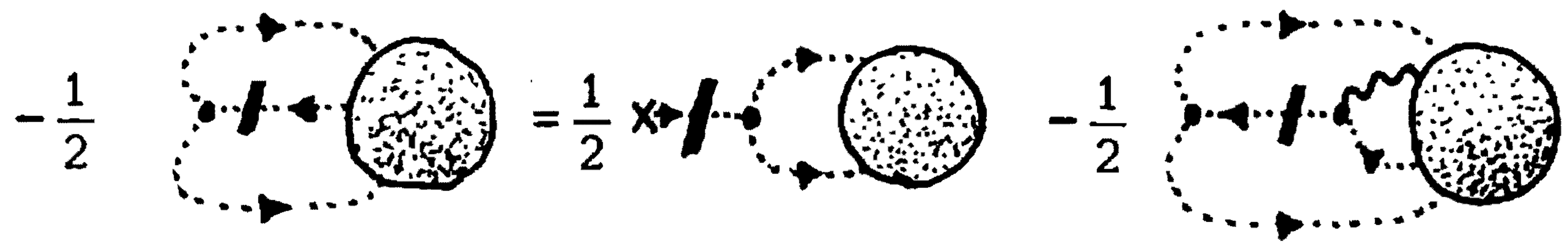
(7.8)

We turn back to DS equations to expand the surviving term in (7.6):



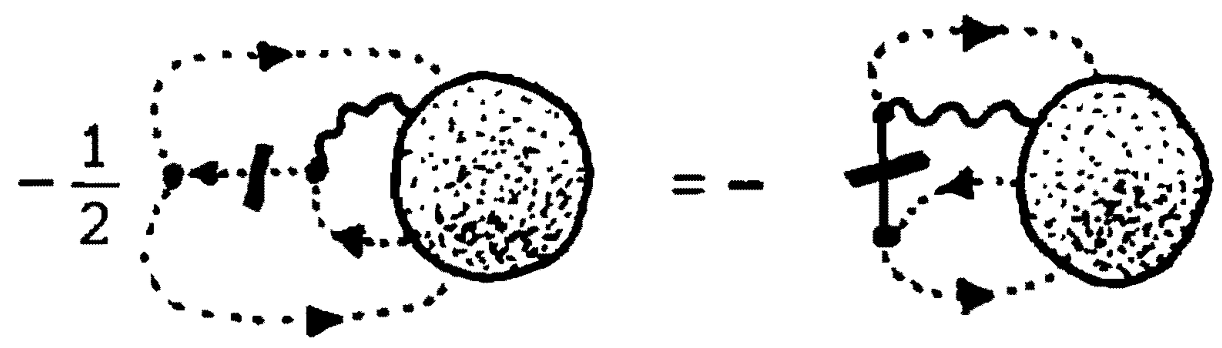
(7.9)

The second term cancels against (7.7), the third term vanishes by (6.55), and to kill the last term we expand (7.8)



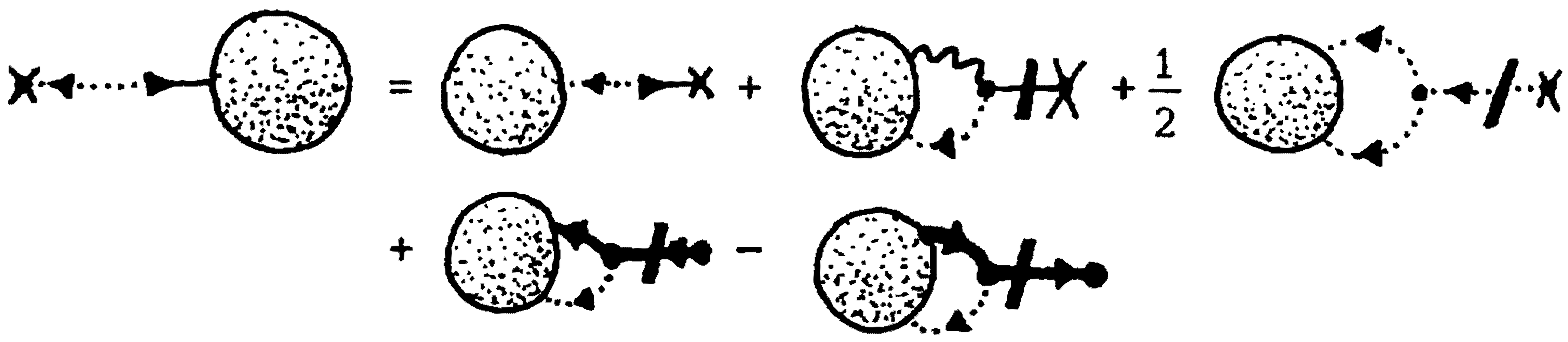
(7.10)

By the Jacobi identity (6.66) the second term cancels the last term in (7.9)



(7.11)

All the messy terms have cancelled. We collect the survivors, putting (7.4) on the left-hand side and (7.5) on the right-hand side:



(7.12)

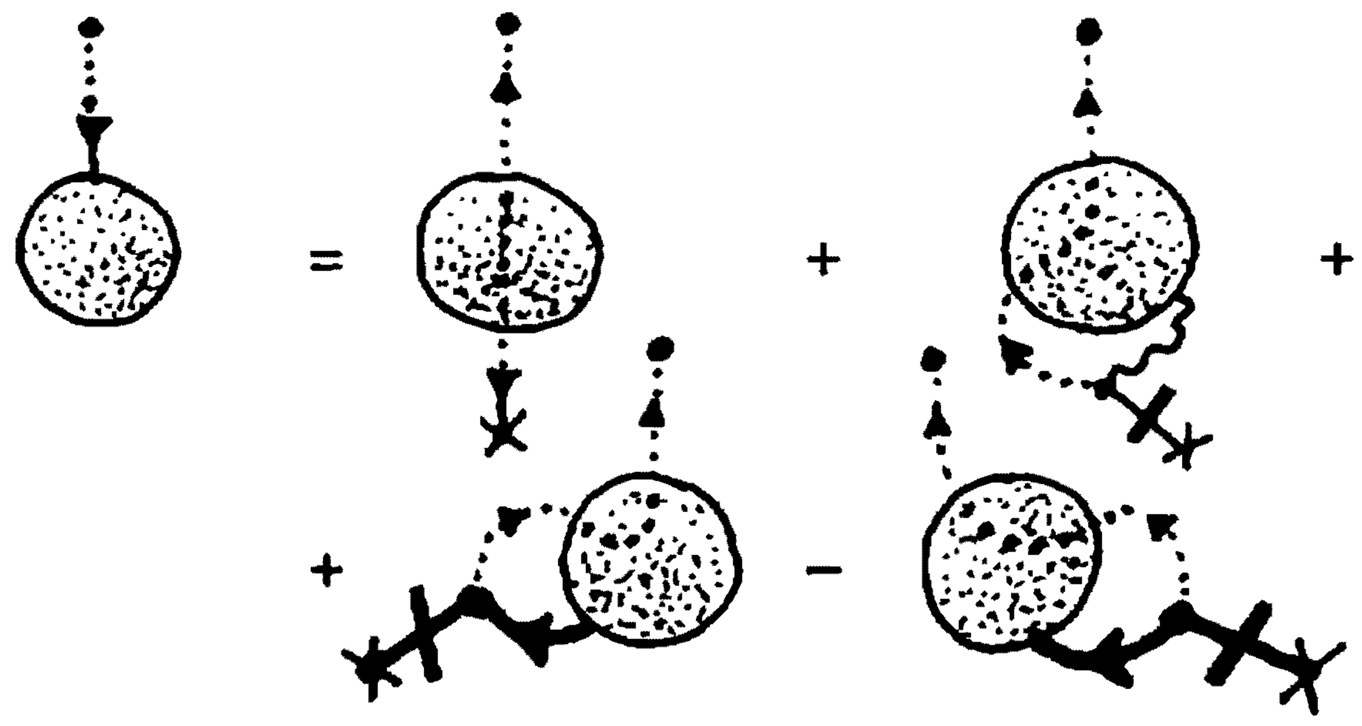
(we have included the quarks - cf. exercise 7.A.1). This is our main result; the Ward identities for the full Green functions. In (7.1) we set out to prove that the left-hand side (a gaugeon insertion) vanishes for any mass-shell process. All the terms on the right-hand side vanish on the mass-shell; the first by the polarization condition (6.1) and the remainder by the equations of motion (6.2), so the gaugeons indeed decouple.

Exercise 7.A.1 Quark Ward identities. Derive (7.12) by keeping the quark terms in DS equations and using the bare quark Ward identity (6.7).

Exercise 7.A.2 Inevitability of ghosts. Try to check the gaugeon decoupling in the theory without ghosts (drop (7.2b) and the ghost term in (7.2a)). Do the non-vanishing terms suggest introduction of ghosts?

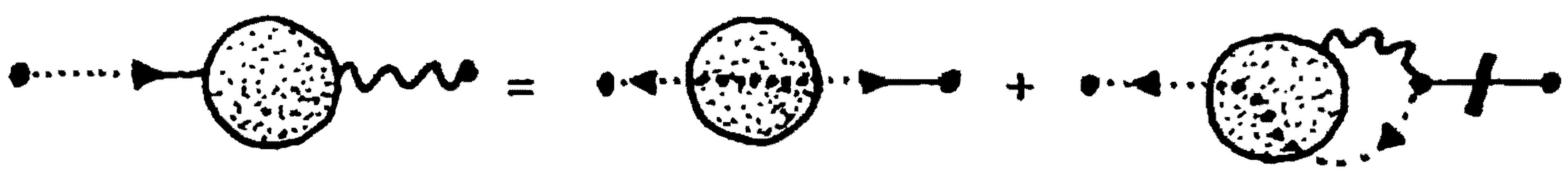
B. Examples of Ward identities

The Ward identities (7.12) can be rewritten in a more transparent form by pulling out an anti-ghost leg and setting the remaining anti-ghost sources equal to zero:



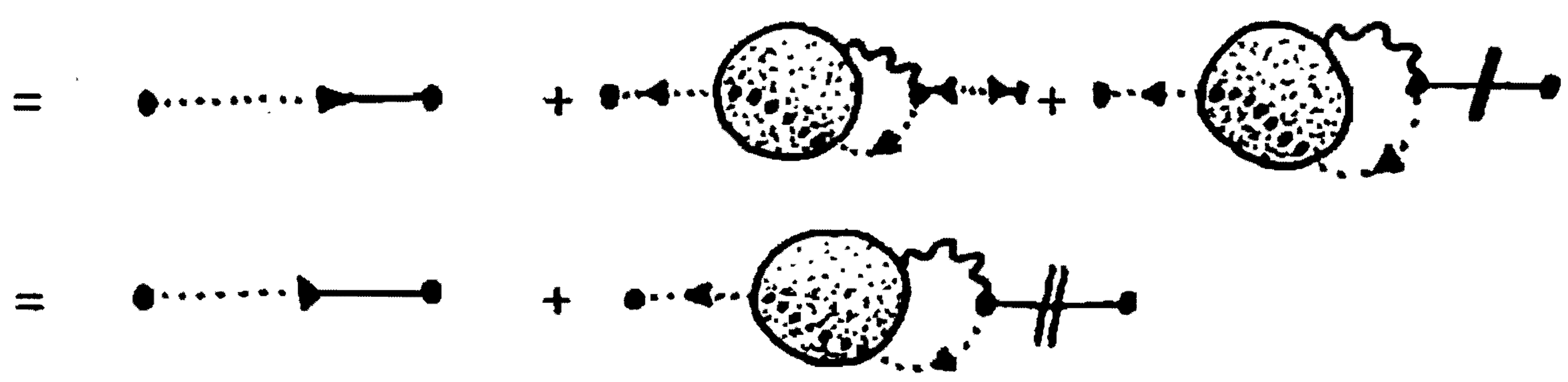
(7.13)

What happens is that as the gaugeon eats its way into a Feynman diagram, it leaves a ghost in its wake: we have indicated this by a dotted line. In QED the ghost is not coupled and Ward identities are rather simple, as in (6.9). In QCD the ghost is coupled, and the Ward identities are a more complicated affair. The simplest example is the Ward identity for the gluon self-energy:



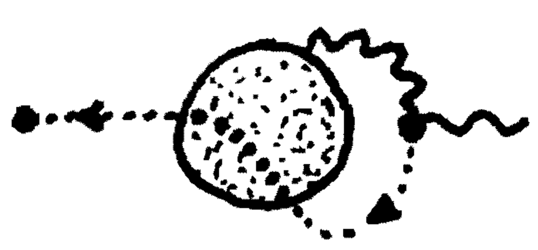
(7.14)

This takes a particularly simple form in covariant gauges, where the ghost vertex (6.41) is $h^\mu = k^\mu$. Using the ghost DS equation (7.2b) we can rewrite the above as

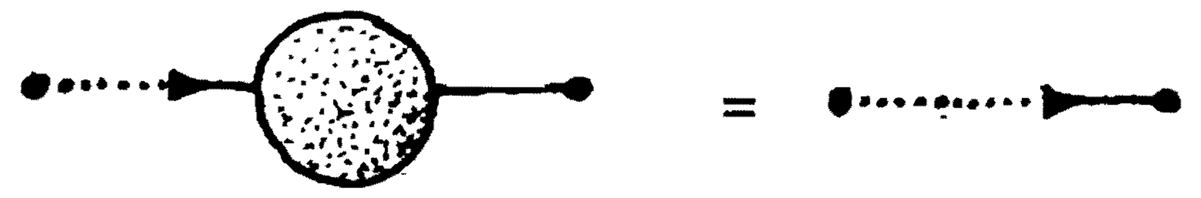


(7.15)

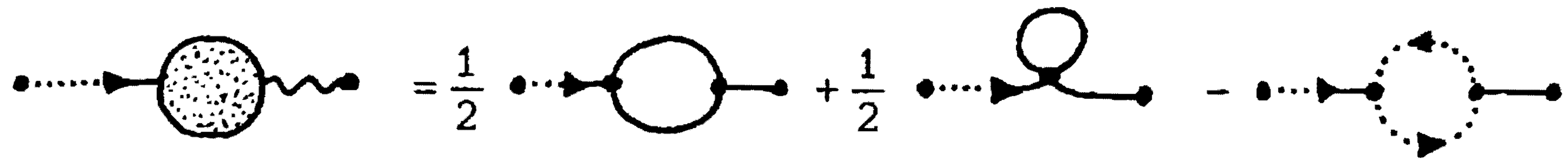
(we drop the vacuum bubbles). The double slash indicates the transverse projection factor $k^2 g^{\mu\nu} - k^\mu k^\nu$. As we are in the covariant gauges, the only invariant tensor with one index is k^μ, so

 $\mu = f(k^2)k^\mu$.

Because of the transverse projector in (7.15) such term does not contribute, and we find that the longitudinal part of the gluon propagator has no radiative corrections:

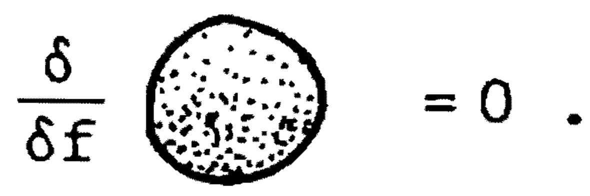
 , covariant gauges (7.16)

<u>Exercise 7.B.1</u> <u>1-loop Ward identities</u>. Check the gluon propagator at one-loop level is explicitly transverse in the Feynman gauge. Hints: Substitute diagrammatic vertices, bare Ward identities and Jacobi identities into

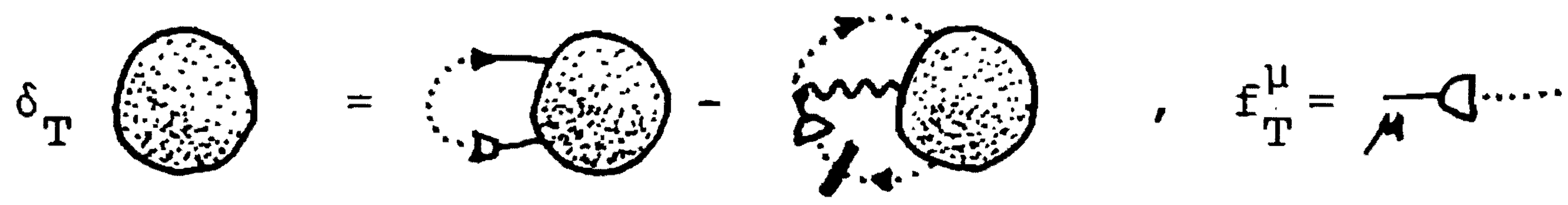


Do not drop anything because it vanishes by dimensional regularization (you are not supposed to know that yet; besides, it just messes up the proof).

<u>Exercise 7.B.2</u> Prove that the vacuum bubbles are gauge invariant:



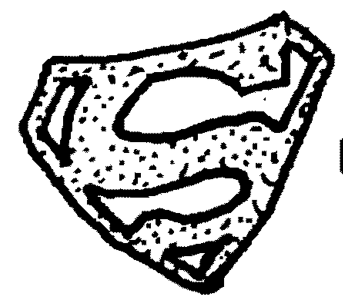
Hint: decompose f into transverse and longitudinal parts: $\delta f^{\mu} = \frac{(\delta f \cdot k)}{k^2} k^{\mu} + \delta f_T^{\mu}$. The ghost vertex variations are $\delta h_L = 0$, $\delta h_T = -k^2 \delta f_T$. A gauge variation of Z consists of two parts; variation of the gluon propagators and variation of the ghost vertices:



propagator variation   ghost vertex variation

<u>Exercise 7.B.3</u> <u>Sign anxieties</u>. It is pretty hard to keep track of signs in QCD; there are signs due to the antisymmetry of C_{ijk}'s, to the fermionic nature of ghosts, to momentum arrows in gluon vertices, to $-i$'s in propagators. One useful sign check is obtained by replacing full Green functions by their lowest order (tree) contributions. Check (7.13) by comparing its tree approximations to the bare vertex Ward identities of chapter 6.

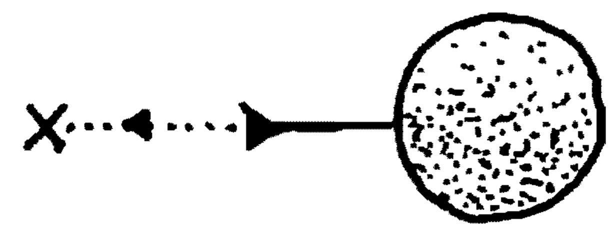
<u>Exercise 7.B.4</u> <u>Ward identities for the connected, 1PI Green functions</u> (continuation of exercise 6.A.1). Use the relations between the full, connected and 1PI Green functions developed in chapter 2 to rewrite the Ward identities (7.12) and (7.13). Work this out for the 1PI quark vertex, gluon self-energy, etc.

C. It is  upersymmetry!

The classics illustrated Ward identities (7.12) do everything we promised they would do, but Jens J. Jensen† is still

†The inventor of 3-j coefficients.

unhappy: they look different from the Ward identities in Jens' favourite textbook. What irritates Jens is the gaugeon insertion on the left-hand side of (7.12):



(7.17)

The propagator going into the blob ..... $= -ik^{\mu}/k^2$ is neither a ghost not a gluon. Well, that is no sweat. After a brief two weeks' reflection one observes that (6.61) implies

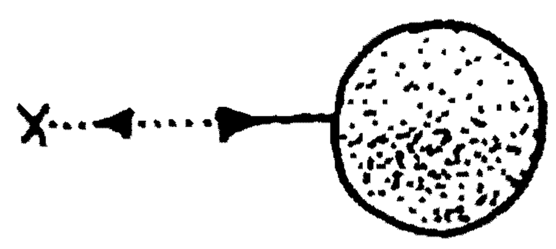
$$-\frac{ik^{\mu}}{k^2} = \frac{h_{\nu}}{B - h_T^2/k^2} D^{\nu\mu} \; . \tag{7.18}$$

We can use this identity to replace $-k^{\mu}/k^2$ by the gluon propagator. If we introduce diagrammatic notation for the "gauge fixing functional"

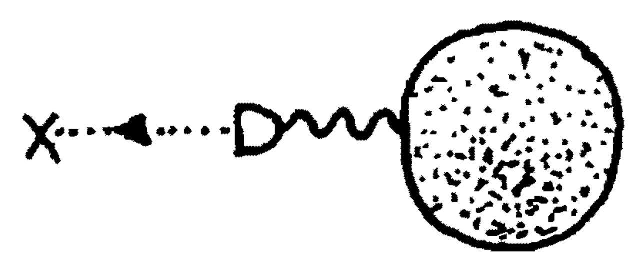
$$\frac{1}{a}\mathfrak{F}_i [A] \equiv \frac{h^{\mu}}{B - h_T^2/k^2} A^i_{\mu} =$$

(7.19)

(7.17) can be redrawn as



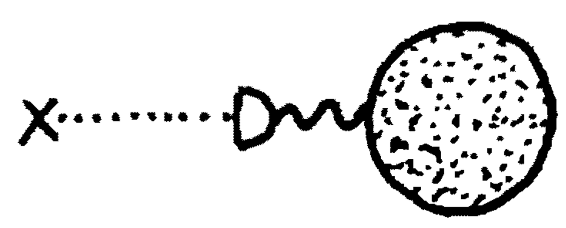
$=$



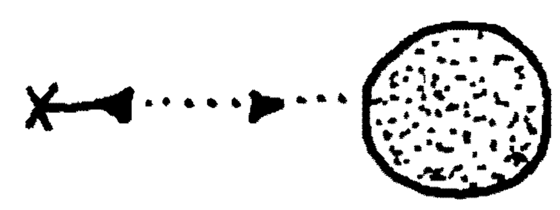
(7.20)

Written in the generating functional notation, the terms contributing to (7.12) are[†]

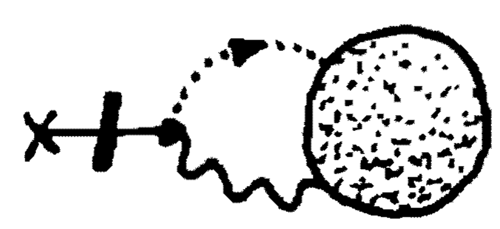
$$\int dx \xi_j(x) \frac{1}{a} F_j\left[\frac{d}{dJ(x)}\right] Z[J] =$$



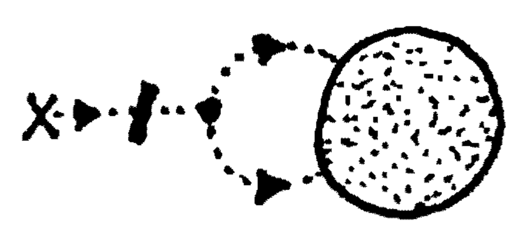
$$\int dx J^{\mu}_j(x) \partial_{\mu} \frac{d}{d\bar{\xi}_j(x)} Z[J] =$$



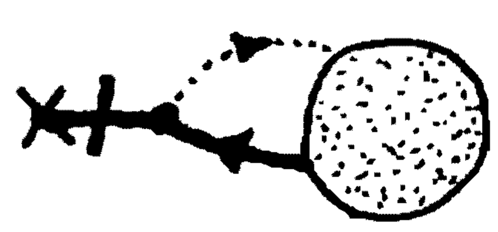
$$\int dx J^{\mu}_i(x) \left(igC_{ijk}\right) \frac{d}{dJ^{\mu}_j(x)} \frac{d}{d\bar{\xi}_k(x)} Z[J] =$$



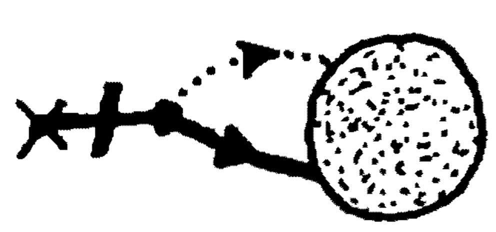
$$\int dx \bar{\xi}_j(x) \left(\frac{i}{2} gC_{ijk}\right) \frac{d}{d\bar{\xi}_j(x)} \frac{d}{d\bar{\xi}_k(x)} Z[J] = \frac{1}{2}$$



$$\int dx \eta^a(x) g\left(T_j\right)^b_a \frac{d}{d\eta^b(x)} \frac{d}{d\bar{\xi}_j(x)} Z[J] =$$



$$-\int dx \bar{\eta}_a(x) g\left(T_j\right)^a_b \frac{d}{d\bar{\eta}_b(x)} \frac{d}{d\bar{\xi}_j(x)} Z[J] = -$$



(7.21)

† No contractual obligation by Nordita regarding correctness of signs or factors of i is either expressed or implied in this or any other equation in this document.

This is as good a demonstration as any that one diagram is better than 50 symbols. In a slightly more compact notation, the Ward identities (7.12) are given functionally by

$$\left(J\cdot D\frac{d}{d\bar{\xi}} + \frac{1}{2}\bar{\xi}\cdot\left[\frac{d}{d\bar{\xi}},\frac{d}{d\bar{\xi}}\right] + \frac{1}{a}\xi\cdot F\left[\frac{d}{dJ}\right]\right)Z[J] = 0 \ . \tag{7.22}$$

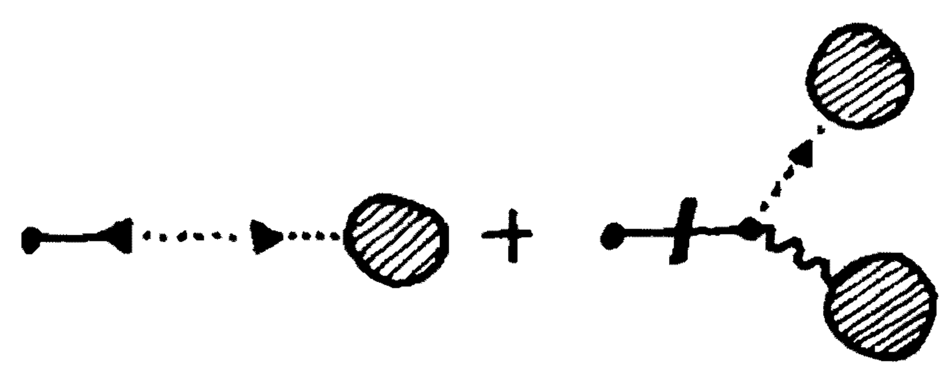
Here $D = D^{ij}_{\mu}$ is the covariant derivative from (6.57), $J, \bar{\xi}, \xi, \bar{\eta}, \eta$ are respectively the gluon, ghost, antighost, quark, antiquark sources, and we have dropped quarks - their inclusion is straighforward.

As promised in chapter 3, the Ward identities are indeed of the form

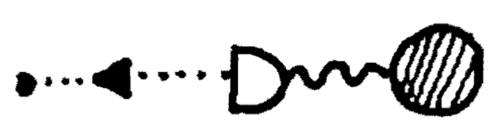
$$J_i F_i\left[\frac{d}{dJ}\right]Z[J] = 0 \quad . \tag{7.23}$$

The generators of the transformation $\delta\phi_i = \varepsilon F_i[\phi]$, equation (3.31), can be read off (7.21)

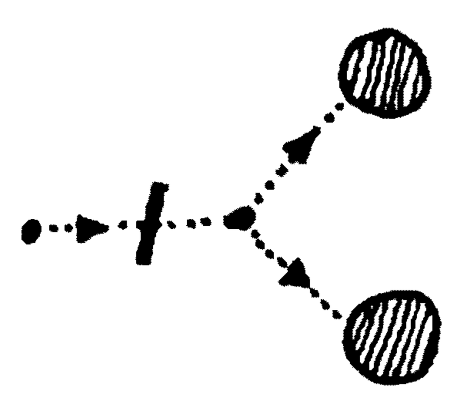
$\delta A^i_{\mu} = \varepsilon D^{ij}_{\mu}\omega^j$ =



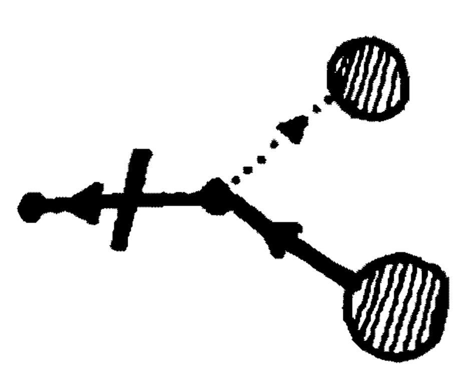
$\delta\bar{\omega}^i = -\ \varepsilon\frac{1}{a}F_i[A]$ =



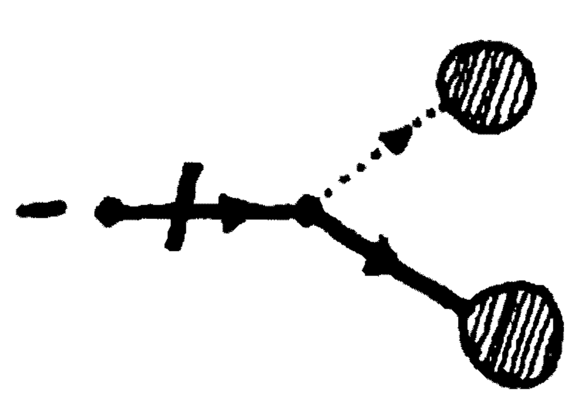
$\delta\omega^i = -\ \varepsilon\frac{g}{2}C_{ijk}\omega^j\omega^k$ $= -\frac{1}{2}$



$\delta\bar{q}_a = \varepsilon ig\left(T_j\right)^b_a\omega^j\bar{q}_b$ =



$\delta q^a = -\ \varepsilon ig\left(T_j\right)^a_b\omega^j q^b$ = −



(7.24)

According to (3.33), the action is invariant under transformations generated by $F_i[\phi]$:

$$\frac{dS[\phi]}{d\phi_i} F_i[\phi] = 0 \ .$$

This is a supersymmetry, because it mixes bosonic gluons A and fermionic ghosts $\omega, \bar{\omega}$. It is far from obvious (it was introduced by Becchi, Rouet and Stora in 1975) and it is very deep, or trivial, depending on the time of the day. In either case, the BRS symmetry is an elegant tool for proving the renormalizability of QCD, a topic that belongs to the next tome of the ultimate QCD review†. We stop here, deserting the long-leggety beasties for chaos, which, after all, is the source of all creation.

Exercise 7.C.1 BRS invariance. A discouraging aspect of hidden supersymmetries like the BRS symmetry is that they are so hard to discover. QCD suggests a systematic way to construct the generators, which goes something like this:

1. Start with $\mathcal{L}_{YM}$, which is invariant under $\delta A = \varepsilon D\omega$.
2. Problem; the gluon propagator is not invertible. Break the invariance by adding $\mathcal{L}_{fix} = -(\partial\cdot A)^2/(2a)$. This generates
$$\delta\mathcal{L}_{fix} = -\frac{\varepsilon}{a}(\partial\cdot A)(\partial\cdot D)\omega \ .$$
3. Attempt to restore the symmetry by adding a new field with variation
$$\delta\bar{\omega} = \varepsilon(\partial\cdot A)/a$$
and action term
$$\mathcal{L}_{ghost} = \bar{\omega}(\partial\cdot D)\omega \ .$$
4. This does not quite work because D is field-dependent, and $\delta\mathcal{L}_{ghost}$ generates an extra term
$$\bar{\omega}_i C_{ijk}\partial\cdot D_{k\ell}\omega_\ell\omega_j \ .$$
5. Save the day by varying ω as well
$$\delta\omega_i = -\frac{\varepsilon}{2}C_{ijk}\omega_j\omega_k .$$

Antisymmetry of C_{ijk} forces you to take ω fermionic.
Check all steps in the above argument.

Exercise 7.C.2 Ward identities for the effective action. Use the methods of chapter 2 to rewrite (7.22) in terms of 1PI functionals. Hint: introduce extra sources for the non-linear terms in (7.24).

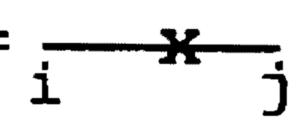
† A.D. Kennedy, in preparation.

APPENDIX A: 2-PARTICLE IRREDUCIBILITY

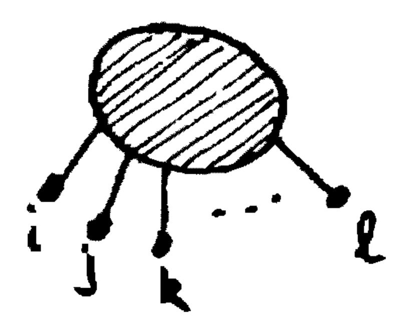
The virtue of the diagrammatic derivation of the 1PI Green functions, section 2.G, is that one does not need to prove 1P-irreducibility; it is built-in, by construction. To test the power of the method, I do it here for 2-particle irreducible Green functions, and am (almost) successful. This is a warming-up exercise for computing QCD bound states. Besides, it is crowding my notebooks.

Introduce <u>2 kinds of sources</u>: $J = (J_i, J_{ij})$

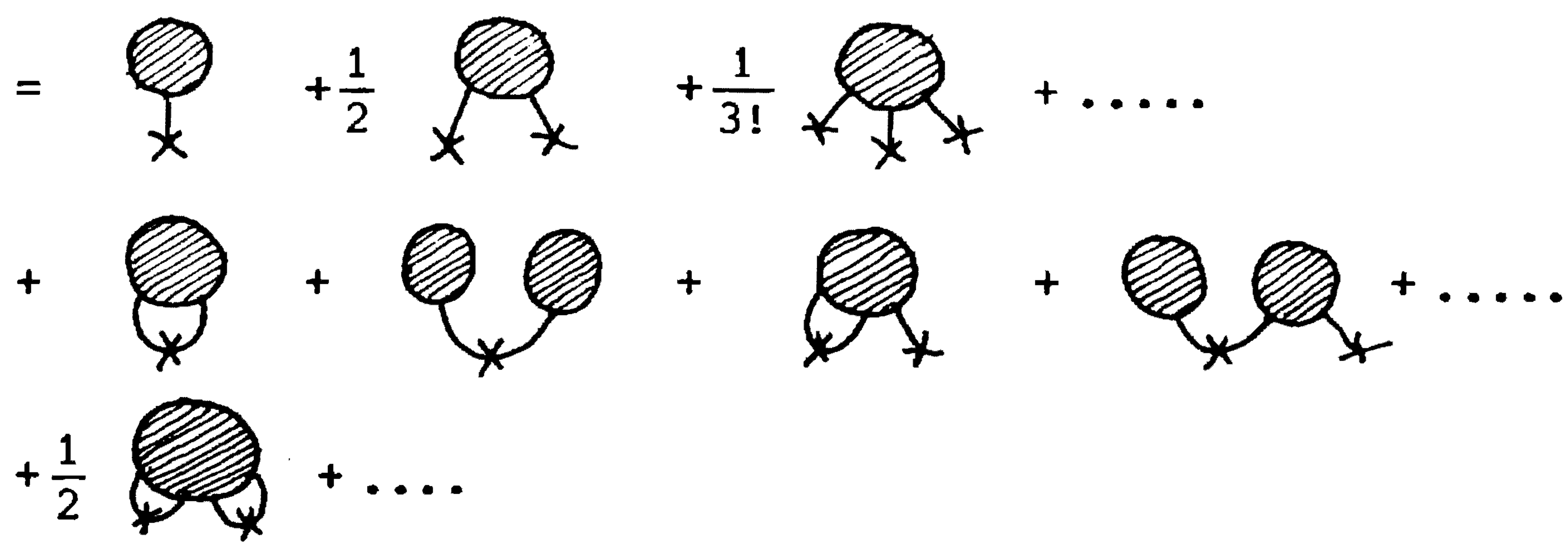
1-particle sources $J_i =$ 

2-particle sources $J_{ij} =$  $= J_{ji}$ (A.1)

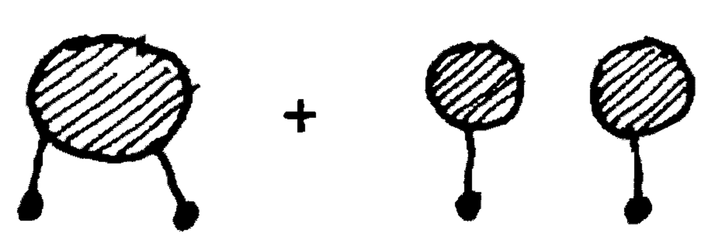
The <u>connected</u> Green functions are the same as usual

$G^{(c)}_{ijk\ldots\ell} =$  $= \left.\frac{d}{dJ_i}\frac{d}{dJ_j}\cdots\frac{d}{dJ_\ell} W[J]\right|_{J=0}$

as they are evaluated at $J_i = J_{ij} = 0$. The generating functional is a double expansion in J_i and J_{ij};

$W[J] =$  (A.2)

Removing a two-particle source can disconnect a connected diagram:

$\frac{dW[J]}{dJ_{ij}} =$  $= \frac{d^2W[J]}{dJ_i dJ_j} + \frac{dW[J]}{dJ_i}\frac{dW[J]}{dJ_j}$ (A.3)

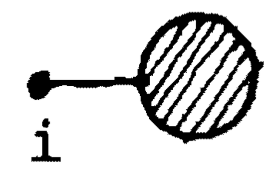
Nota bene:

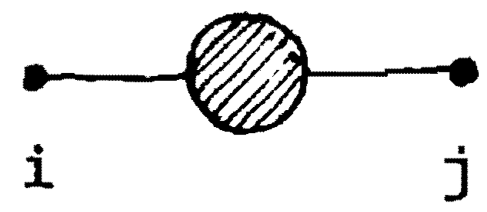
$$\frac{d}{dJ_{ij}} J_{mn} = \frac{1}{2}\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) , \qquad (A.4)$$

do not forget symmetrizations!

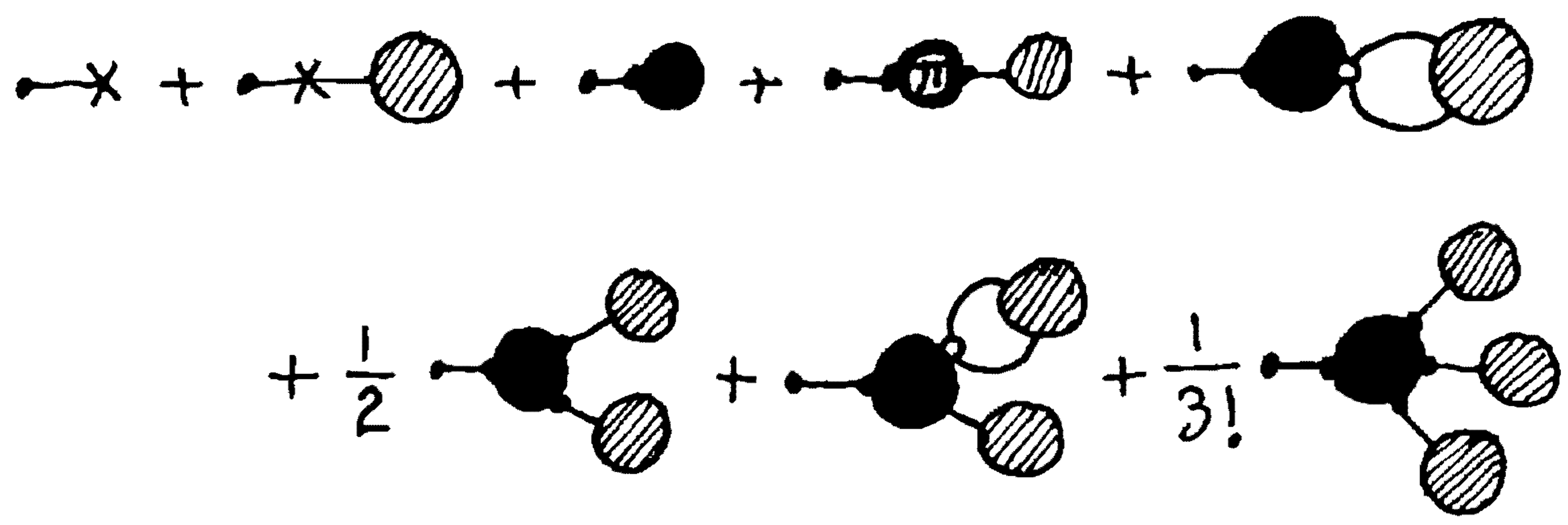
To define 2-particle irreducible graphs, we have to remove tadpoles (connected to the rest of the diagram by 1 line) and self-energy insertions (connected to the rest of the diagram by 2 lines), Hence introduce

$$\phi = \left(\phi_i, D_{ij}\right)$$

fields: $\phi_i = \dfrac{dW[J]}{dJ_i} =$ 

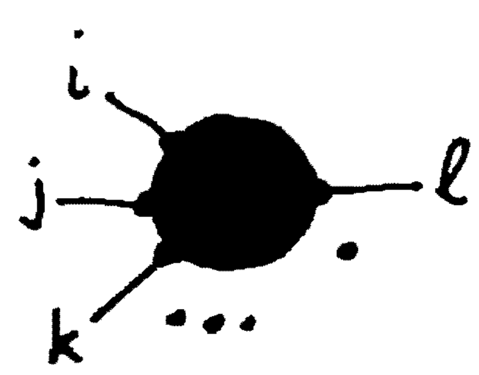
propagators: $D_{ij} = \dfrac{d^2W[J]}{dJ_i dJ_j} =$  (A.5)

If we pull out a leg, it either ends on a source, or 2PI diagram, or 2P-reducible diagram:

$\phi_i = \dfrac{dW[\vec{J}]}{dJ_i} =$  (A.6)

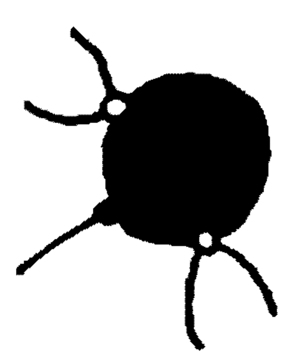
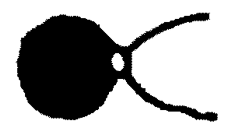
$$\phi_i = \Delta_{ij}\Big(J_j + J_{jk}\phi_k + \Gamma_j + \pi_{jk}\phi_k + \Gamma_{j\underline{ki}}D_{k\ell}$$
$$+ \tfrac{1}{2}\Gamma_{jk\ell}\phi_k\phi_\ell + \Gamma_{jk\underline{\ell m}}\phi_k D_{\ell m} + \tfrac{1}{3!}\Gamma_{jk\ell m}\phi_k\phi_\ell\phi_m + \ldots\Big) \,.$$

The <u>2-particle irreducible</u> (2PI) Green functions are drawn as black blobs, with each external line coming into a separate vertex:

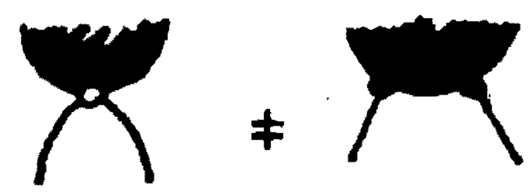
$\Gamma_{ijk..\ell} =$ 

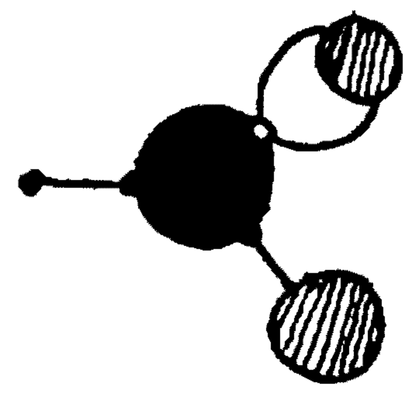
$= \dfrac{d}{d\phi_i}\dfrac{d}{d\phi_j}\dfrac{d}{d\phi_k}\cdot\cdot\dfrac{d}{d\phi_\ell}\Gamma[\phi]\Big|_{\phi_i = D_{ij} = 0}$. (A.7)

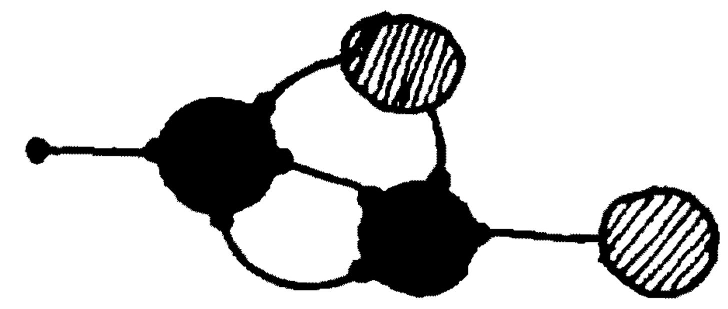
Derivatives with respect to self-energies are denoted by the corresponding pairs of lines coming into a white vertex:

$\Gamma_{\underline{ij}k\underline{\ell m}..} =$  $= \dfrac{d}{dD_{ij}}\dfrac{d}{d\phi_k}\dfrac{d}{dD_{\ell m}}\Gamma[\phi]\Big|_{\phi=0}$ (A.8) caution: 1)  can be 2-particle reducible

2) $\frac{d}{dD_{ij}} \neq \frac{d}{d\phi_i} \frac{d}{d\phi_j}$

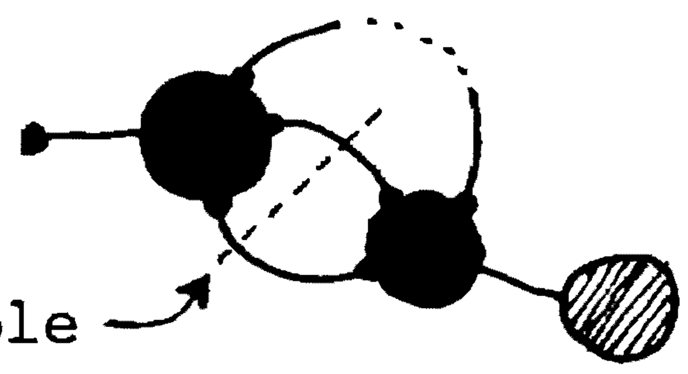


example: a term like  contains diagrams such as

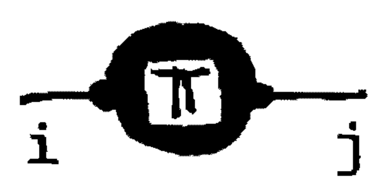


When

when we remove the propagator, the remainder is 2-particle reducible



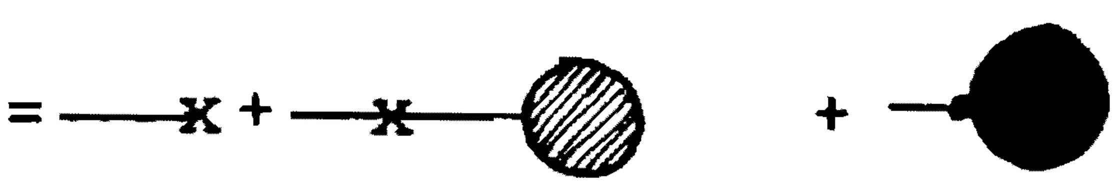
2-particle reducible

In the above expansion of dW/dJ_i, the π_{ij} =  term is 2PI. We sum up its iteration by defining

$$\Gamma_{ij} = - \Delta^{-1}_{ij} + \pi_{ij} \quad , \tag{A.9}$$

and the expansion can be rewritten as the first duality relation:

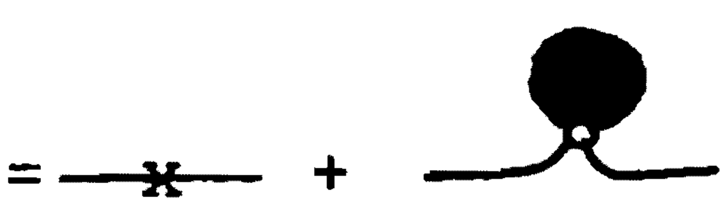
$$0 = J_j + J_{jk}\phi_k + \frac{d\Gamma[\phi]}{d\phi_j}$$

0 =  (A.10)

extra term due to 2-particle sources

The second duality relation is

$$0 = J_{ij} + \frac{d\Gamma[\phi]}{dD_{ij}}$$

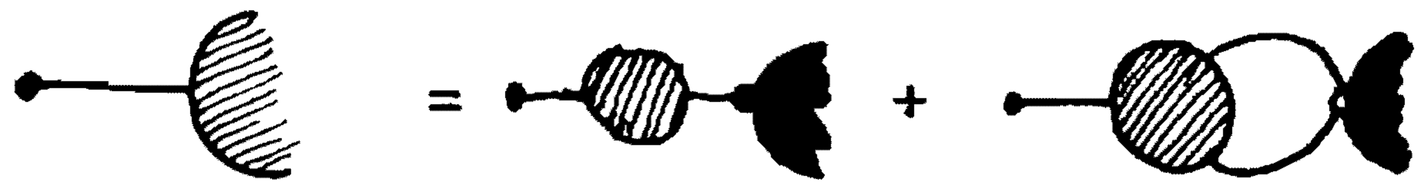
0 =  (A.11)

I do not know how to derive this diagrammatically†, but algebraically it comes from the <u>second Legendre transform</u>:

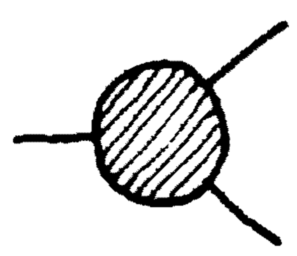
$$\Gamma[\phi] = W[J] - \frac{dW[J]}{dJ_i} J_i - \frac{dW[J]}{dJ_{ij}} J_{ij} , \tag{A.12}$$

by differentiating with respect to D_{ij}. To go from connected to 2PI Green functions, use the chain rule:

$$\frac{d}{dJ_i} = \frac{d\phi_j}{dJ_i} \frac{d}{d\phi_j} + \frac{dD_{jk}}{dJ_i} \frac{d}{dD_{jk}} = D_{ij} \frac{d}{d\phi_j} + \frac{d^3W[J]}{dJ_i dJ_j dJ_k} \frac{d}{dD_{jk}} , \tag{A.13}$$



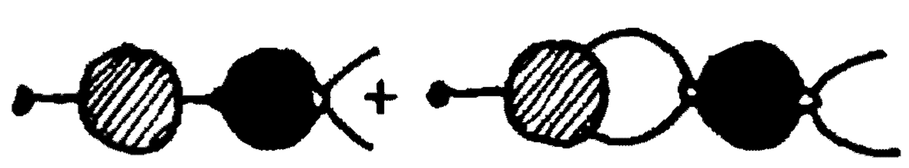
↑this has to be re-expressed in terms of ϕ_i, D_{ij}

To eliminate , use the identity

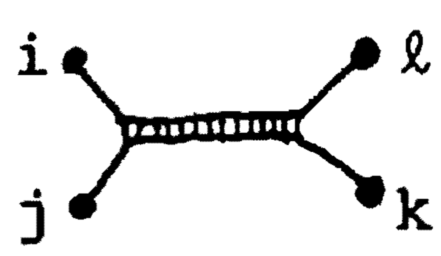
$$\frac{dJ_{mn}}{dJ_i} = 0 . \tag{A.14}$$

Substituting $d\Gamma/dD_{mn}$ for J_{mn} and using the chain rule, we obtain

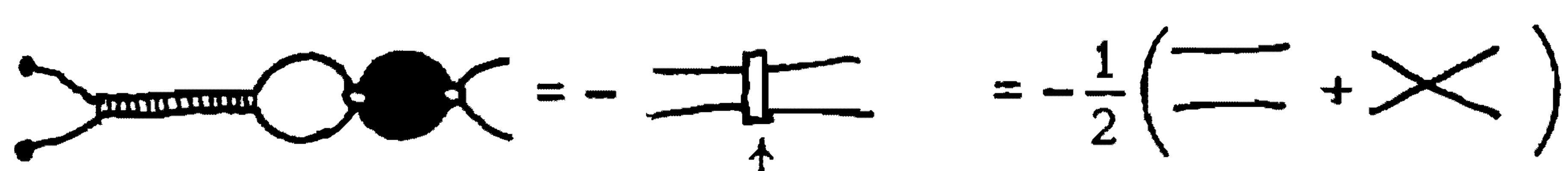
$$0 = \left(D_{ij} \frac{d}{d\phi_j} + \frac{d^3W[J]}{dJ_i dJ_j dJ_k} \frac{d}{dD_{jk}} \right) \frac{d\Gamma[\phi]}{dD_{mn}}$$

0 =  (A.15)

Define 2-particle propagator as the inverse of $\Gamma_{\underline{k\ell}\ \underline{mn}}$:

$D_{\underline{ij}\ \underline{k\ell}}$ =  (A.16)

$$D_{\underline{ij}\ \underline{k\ell}} \Gamma_{\underline{k\ell}\ \underline{mn}} = -\frac{1}{2} \left(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk} \right)$$

 (A.17)

↑symmetrized, 2-particle subspace

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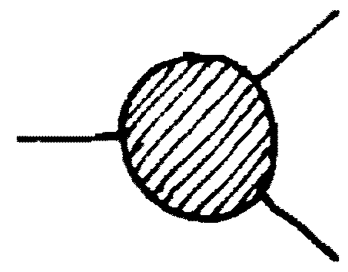
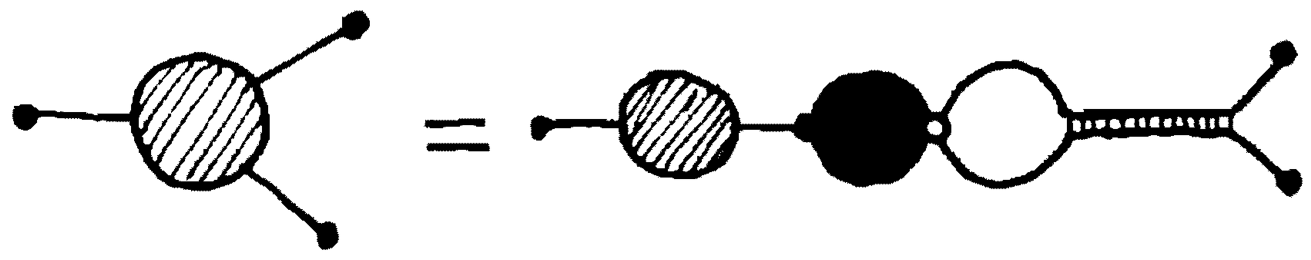
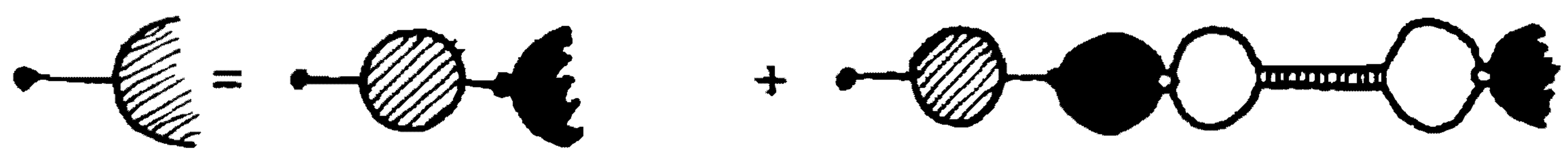
†Here is where my derivation falls flat on its face.

Now we can eliminate  in terms of ϕ_i, D_{ij} functions:

 (A.18)

and the chain rule takes a sensible form

$$\frac{d}{dJ_i} = D_{ij}\left(\frac{d}{d\phi_j} + \frac{d^2\Gamma[\phi]}{d\phi_j dD_{k\ell}} D[\phi]_{\underline{k\ell}\ \underline{mn}} \frac{d}{dD_{mn}}\right)$$

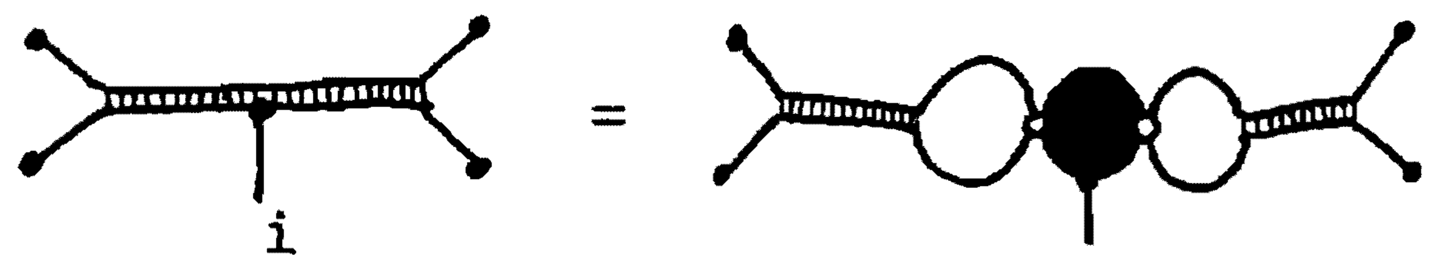
 (A.19)

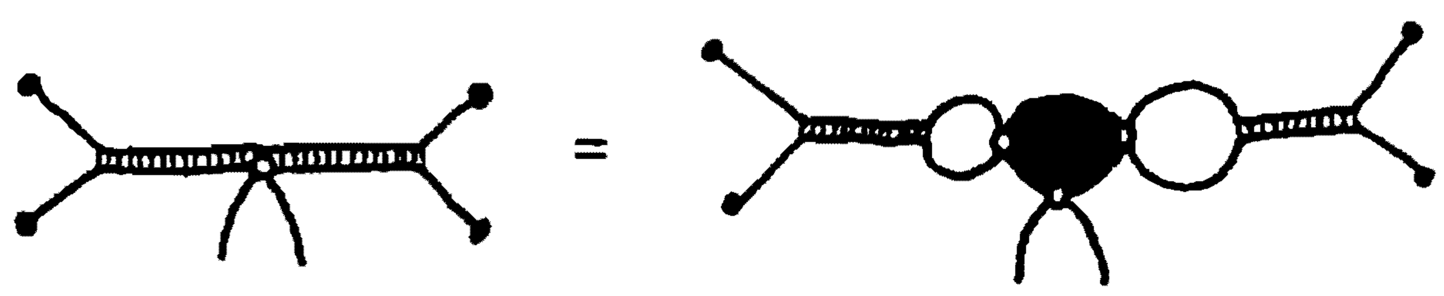
This says that if we follow a line into a connected diagram, we either encounter a 2PI piece, or a 2P-reducible piece.

To be able to evaluate

$$\frac{d}{dJ_i}\frac{d}{dJ_j}\cdot\cdot\frac{d}{dJ_k}W[J]$$

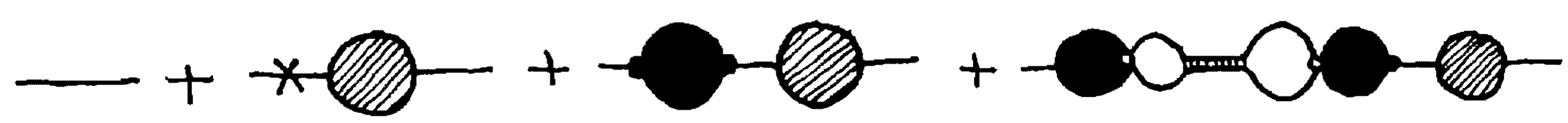
in terms of 2PI bits, we also need to compute $d/d\phi_i\ D_{\underline{k\ell}\ \underline{mn}}$ and $d/dD_{ij}\ D_{\underline{k\ell}\ \underline{mn}}$. They follow from the definition of $D_{\underline{k\ell}\ \underline{mn}}$ as the inverse of $\Gamma_{\underline{k\ell}\ \underline{mn}}$:

$\frac{d}{d\phi_i} D_{\underline{k\ell}\ \underline{mn}} =$ 

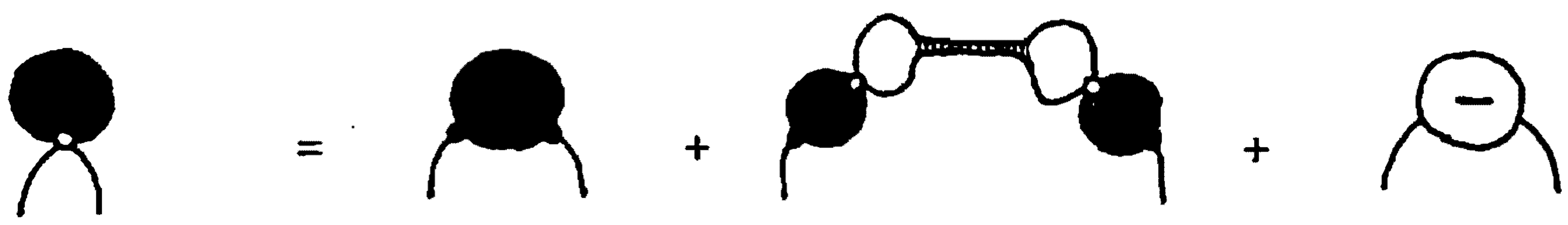
$\frac{d}{dD_{ij}} D_{\underline{k\ell}\ \underline{mn}} =$  (A.20)

This is also sensible, as will be clear from the perturbative expansion of the 2-particle propagator.

Finally, we need to relate d/dD_{ij} (a diagrammatically obscure thing) to $d/d\phi_i\ d/d\phi_j$ (an operation which yields 2PI Green functions). This we obtain by differentiating the first duality relation with respect to d/dJ and using the chain rule

$0 =$  (A.21)

Replacing J_{ij} by the second duality rule and multiplying by inverse propagator, we obtain



$$\frac{d\Gamma[\phi]}{dD_{ij}} = \frac{d^2\Gamma[\phi]}{d\phi_i d\phi_j} + \frac{d^2\Gamma[\phi]}{d\phi_i dD_{k\ell}} D[\phi]_{\underline{k\ell}\,\underline{mn}} \frac{d^2\Gamma[\phi]}{dD_{mn} d\phi_j} + D^{-1}_{ij}[\phi] \ . \tag{A.22}$$

This enables us to systematically get rid of d/dD_{ij} derivatives.

Now we can rewrite any relation between connected Green functions in terms of 1PI functions by going from

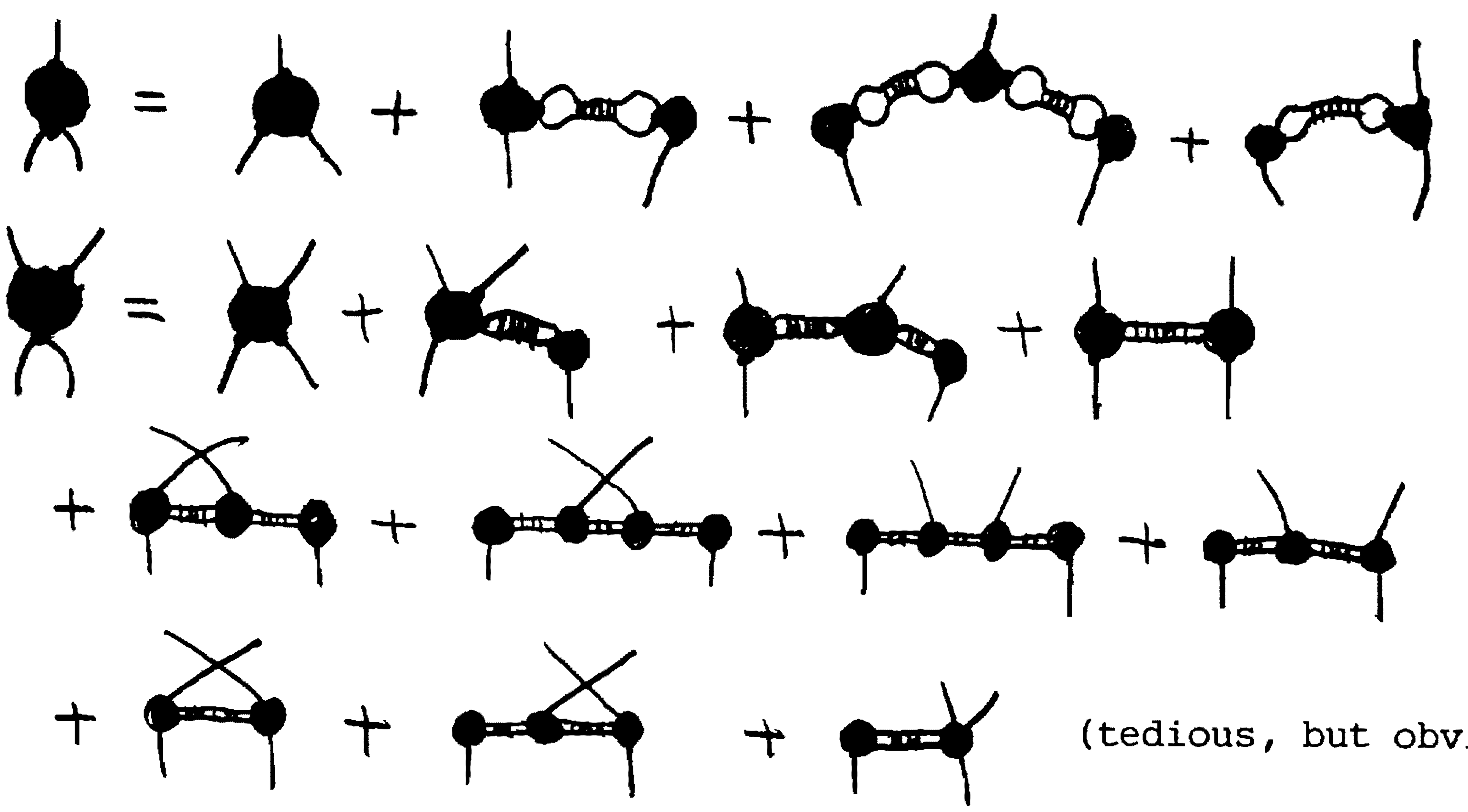
$$J_i, J_{ij}, \frac{d}{dJ_i}, W[J]$$

to dual variables and functions

$$\phi_i, D_{ij}, \frac{d}{d\phi_i}, \Gamma[\phi]$$

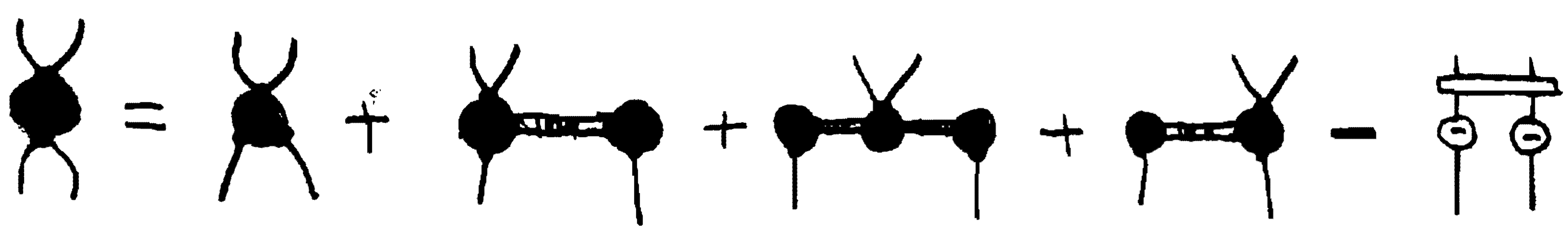
using $D_{\underline{ij}\,\underline{k\ell}}$ and d/dD_{ij} in intermediate steps.

Sundry expansions:



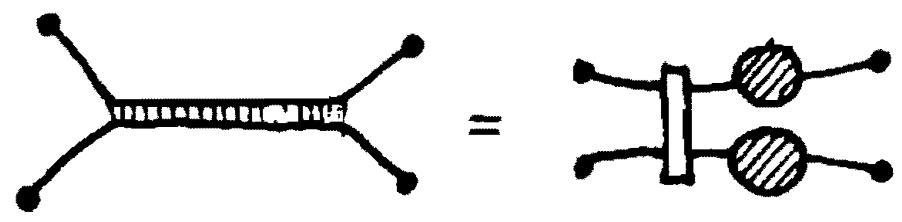
(A.23)

(tedious, but obvious)



(A.24)

The last term shows that not only is $\Gamma_{\underline{ij}\,\underline{k\ell}}$ not 2P-irreducible, it is not even connected. That is a good thing; it is necessary so that $D_{\underline{ij}\,\underline{k\ell}}$ can be the inverse of $\Gamma_{\underline{ij}\,\underline{k\ell}}$: it has to start as

 + (connected pieces) (A.25)

Perturbative expansions for 2PI graphs

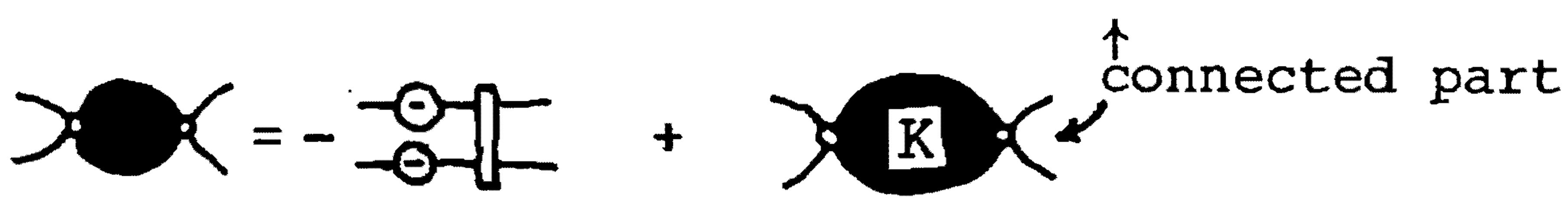
Perturbative expansions isolate the quadratic part of the action (bare propagator) and treat the rest as "interaction" parts. In the general formalism, the bare propagator is hidden in

$$\Gamma_{ij} = -\Delta^{-1}_{ij} + \pi_{ij}$$

(A.26)

It is convenient to also isolate the non-interacting part of the two-particle propagator:

$$\Gamma_{\underline{ij}\ \underline{k\ell}} = -\frac{1}{2}\left(D^{-1}_{ik}D^{-1}_{j\ell} + D^{-1}_{i\ell}D^{-1}_{jk}\right) + K_{\underline{ij}\ \underline{k\ell}}$$

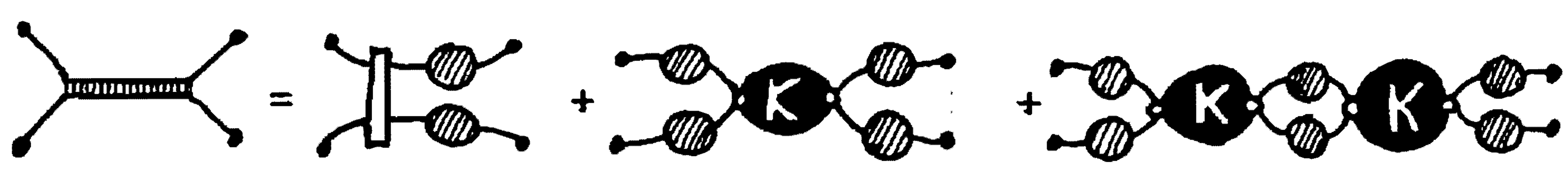


(A.27)

We implement these reshufflings by defining an "<u>interaction</u>" <u>general functional</u>

$$\Gamma[\phi] = \Gamma^{(I)}[\phi] - \frac{1}{2}\,\phi_i \Delta^{-1}_{ij}\phi_j + \frac{1}{2}\,\mathrm{tr}\ \ell n D \quad . \qquad \text{(A.28)}$$

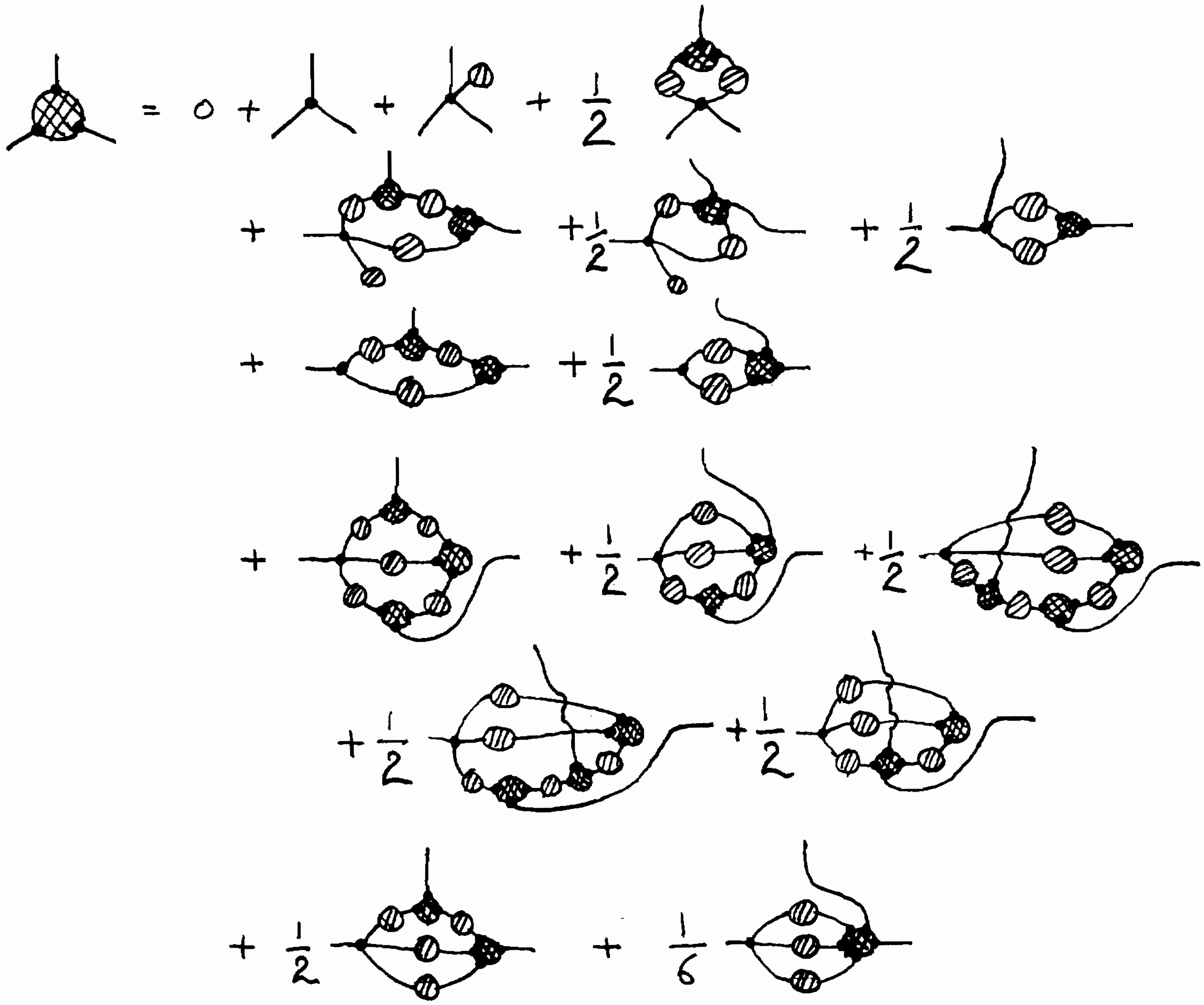
Now we can expand the two-particle propagator in terms of $K_{\underline{ij}\ \underline{k\ell}}$:



(A.29)

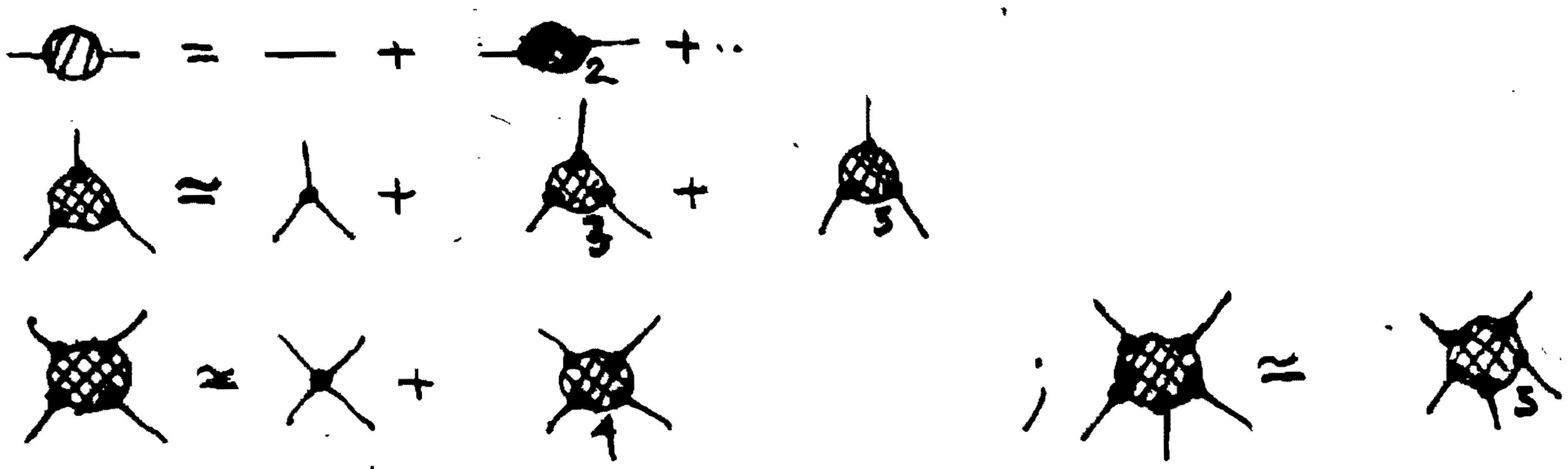
APPENDIX B: Solution of "find 7 errors" (Exercise 2.H.3)

Pull a third leg out of the equation (2.35):



As we need Γ_{ijk} only to g^5 order, truncate all subdiagram expansions

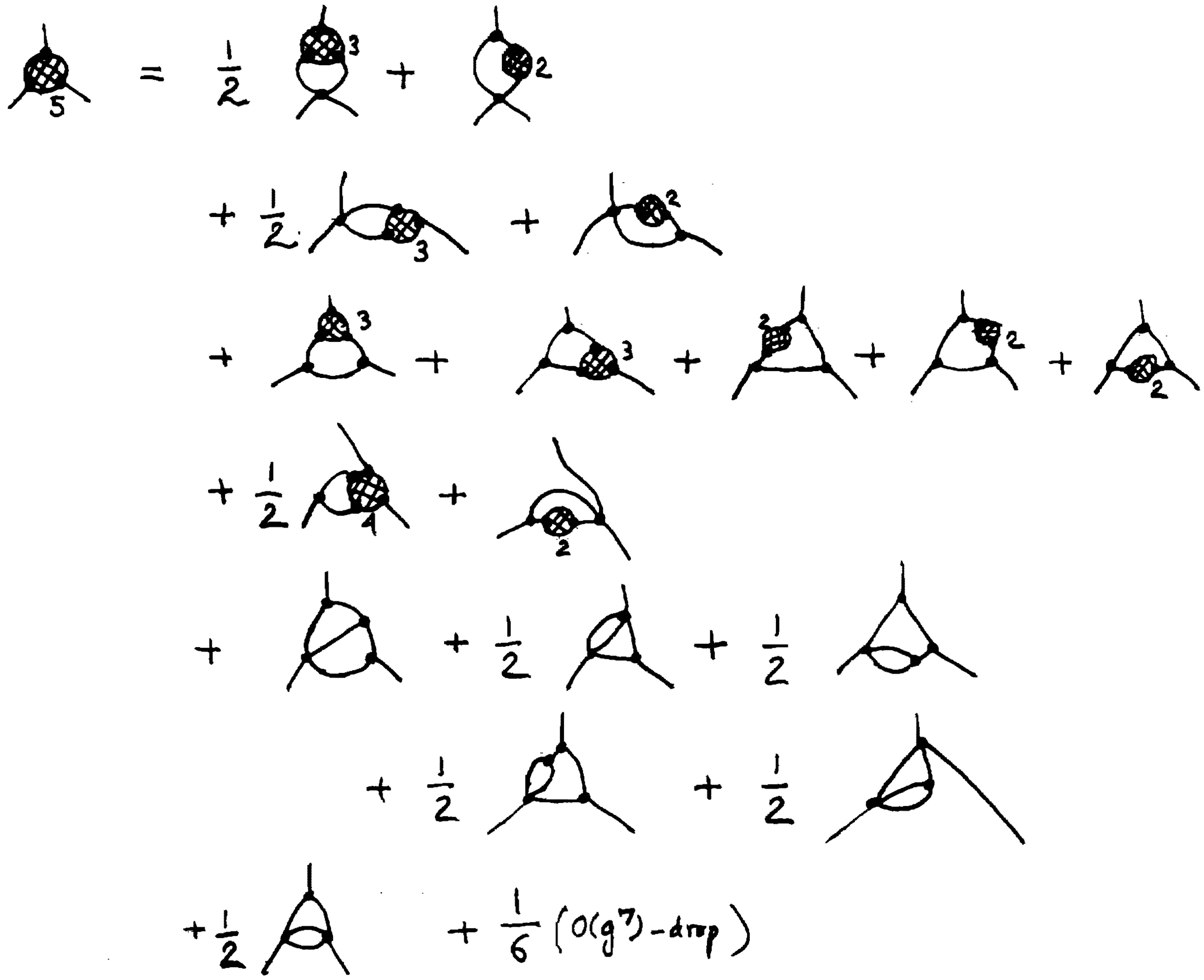
(where subscript k means all terms of order g^k)



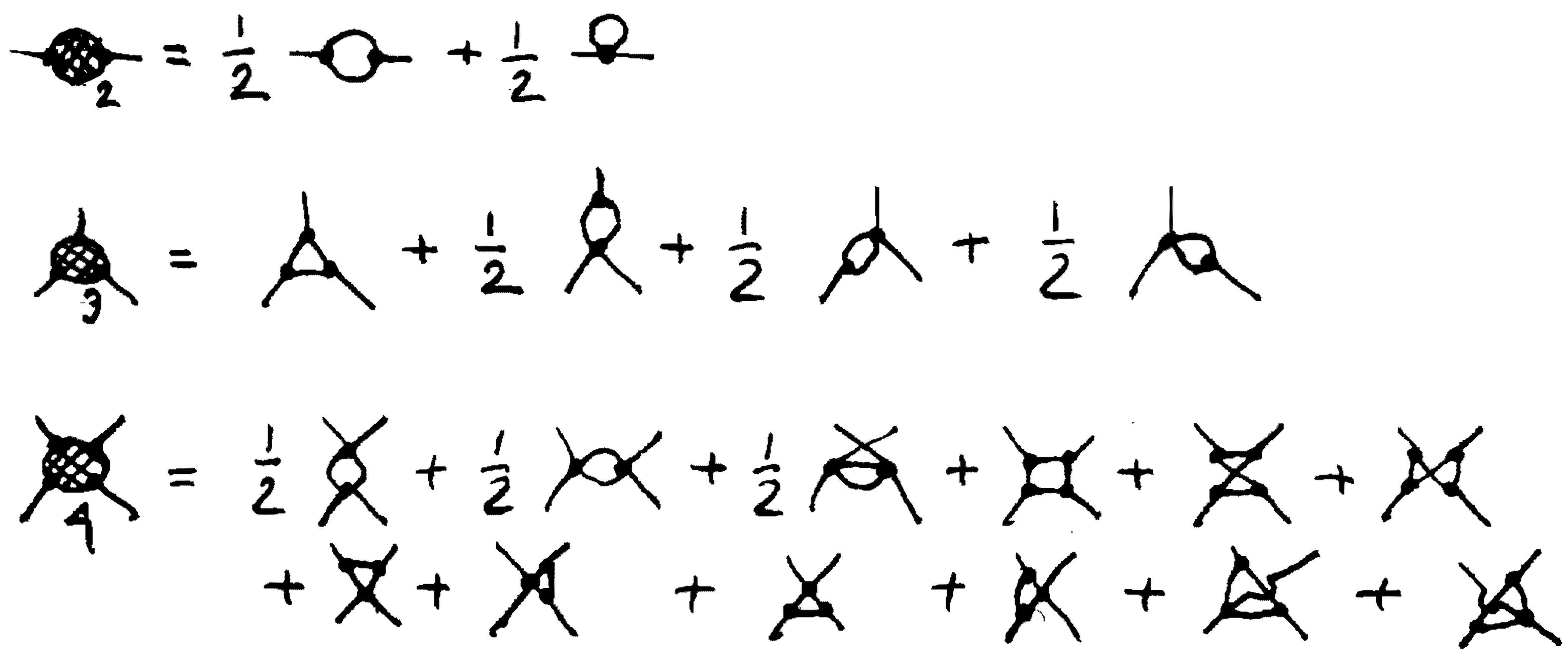
+ higher, drop all such

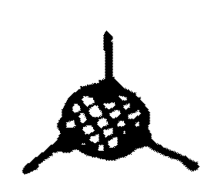
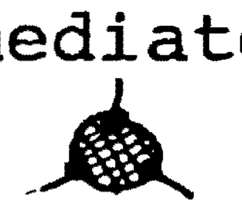
order g^2

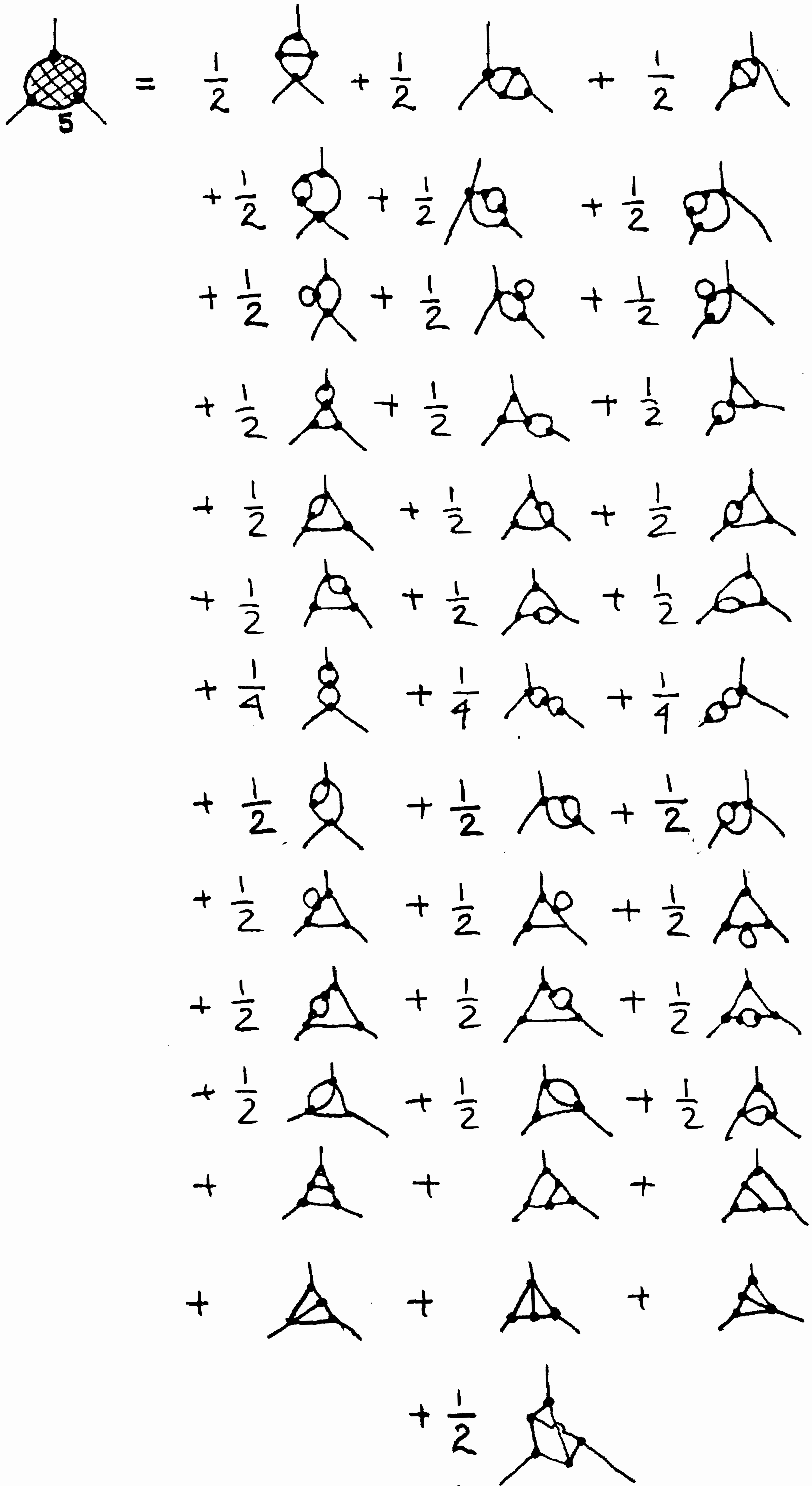
Substituting such expansions, and keeping only g^5 terms: (remember,  = 0)



Need subdiagram expansions



The last expansion comes from pulling out a leg from  and immediately dropping all terms higher than g^1 (the last 7 terms in  Dyson-Schwinger equation). Substituting, one obtains the correct expansion:



← originally the wrong factor

} originally missing

↑ originally missing

APPENDIX C: SOME POPULAR GAUGES

Covariant gauges: (Feynman $a = 1$; Landau $a = 0$):

$$D^{\mu\nu} = \frac{-i}{k^2+i\varepsilon}\left[g^{\mu\nu} - (1-a)\frac{k^\mu k^\nu}{k^2}\right] , \quad h^\mu = k^\mu \quad . \tag{C.1}$$

General axial gauges:

$$D^{\mu\nu} = -\frac{i}{k^2+i\varepsilon}\left[g^{\mu\nu} - \frac{n^\mu k^\nu + k^\mu n^\nu}{n\cdot k} + \frac{ak^2 + n^2}{(n\cdot k)^2} k^\mu k^\nu\right] , \quad h^\mu = \frac{k^2 n^\mu}{(k\cdot n)} . \tag{C.2}$$

Usually $n_\mu = (0,0,0,1)$ picks out a spatial axis.

Axial or temporal gauges ($a=0$):

$$D_{\mu\nu} = \frac{-i}{k^2+i\varepsilon}\left[g_{\mu\nu} - \frac{n_\mu k_\nu + k_\mu n_\nu}{(n\cdot k)} + \frac{n^2 k_\mu k_\nu}{(n\cdot k)^2}\right] , \quad n_\mu D^{\mu\nu} = 0 . \tag{C.3}$$

General planar gauges:

$$D_{\mu\nu} = \frac{-i}{k^2+i\varepsilon}\left[g_{\mu\nu} - \frac{n_\mu k_\nu + k_\mu n_\nu}{(n\cdot k)} + (1-a)\frac{n^2 k_\mu k_\nu}{(n\cdot k)^2}\right] , \quad h^\mu = \frac{k^2 n^\mu}{(k\cdot n)} \quad . \tag{C.4}$$

Lightcone ($a=0$, $n^2=0$) and planar ($n^2 = -ak^2 \neq 0$) gauges:

$$D_{\mu\nu} = \frac{-i}{k^2+i\varepsilon}\left[g_{\mu\nu} - \frac{n_\mu k_\nu + k_\mu n_\nu}{(n\cdot k)}\right] \quad . \tag{C.5}$$

General Coulomb gauges:

$$D^{\mu\nu} = \frac{-i}{k^2+i\varepsilon}\left[g^{\mu\nu} - \frac{n\cdot k(n^\mu k^\nu + k^\mu n^\nu)}{(n\cdot k)^2 - n^2k^2} - \frac{ak^2 - n^2((n\cdot k)^2 - n^2k^2)}{((n\cdot k)^2 - n^2k^2)^2} k^\mu k^\nu\right] ,$$

$$h^\mu = k^2\frac{(n\cdot k)n^\mu - k^\mu}{(n\cdot k)^2 - k^2} , \quad h^\mu n_\mu = 0 . \tag{C.6}$$

Usually $n^\mu = (1,0,0,0)$ picks out the time direction so that $k^\mu - (n\cdot k)n^\mu = (0,\vec{k})$.

Coulomb gauge $(a=0\ ,\ n^2=1)$:

$$D_{\mu\nu} = -\frac{i}{k^2+i\varepsilon}\left[g_{\mu\nu} - \frac{(n\cdot k)\,(k_\mu n_\nu + n_\mu k_\nu) - k_\mu k_\nu}{(n\cdot k)^2 - k^2}\right] \quad . \tag{C.7}$$

See (6.63) for the gauge-fixing terms $\mathcal{L}_{fix}$.

APPENDIX D: FEYNMAN RULES FOR QCD

- Propagators

gluons  $= \delta_{ij} \frac{-i}{k^2} \left(g^{\mu\nu} + f^\mu f^\nu + k^\mu f^\nu \right)$, (D.1a)

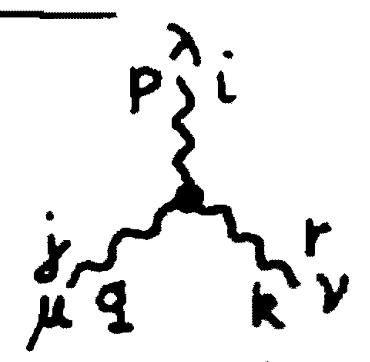
 $= \delta_{ij} \frac{-i}{k^2} g^{\mu\nu}$ Feynman gauge . (D.1b)

ghost  $= \delta_{ij} \frac{-i}{k^2}$ (D.2)

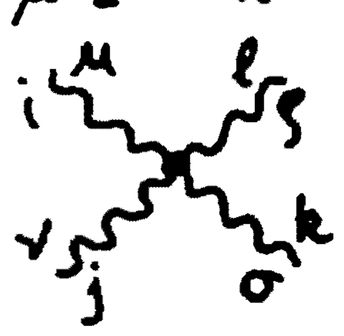
quark  $= \delta^a_b \frac{i}{\not{p} - m}$ (D.3)

(cf. appendix C for other gauges).

Vertices



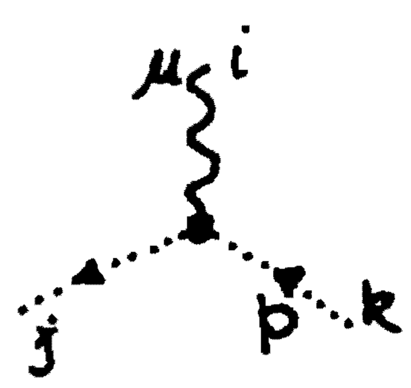
$$= (-igC_{ijk})\, i[g_{\lambda\mu}(p-q)_\nu g_{\mu\nu}(q-r)_\lambda + g_{\nu\lambda}(r-p)_\mu] \ , \qquad \text{(D.4)}$$



$$= (-g^2 C_{ijm} C_{mk\ell})\, i(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma})$$

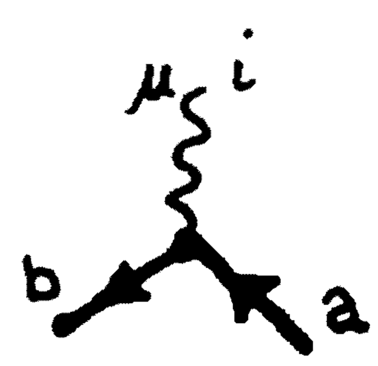
$$+ (-g^2 C_{\ell im} C_{mjk})\, i(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma})$$

$$+ (-g^2 C_{ikm} C_{mj\ell})\, i(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma}) \ . \qquad \text{(D.5)}$$



$$= (-igC_{ijk})(-ih^\mu) \qquad \text{(D.6a)}$$

$$= (-igC_{ijk})(-ip^\mu) \quad \text{(covariant gauges)} \qquad \text{(D.6b)}$$



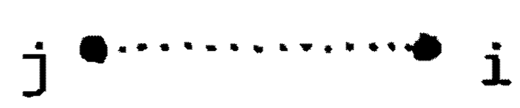
$$= g\left(T_i\right)^a_b (-i\gamma^\mu) \ . \qquad \text{(D.7)}$$

All momenta flow outward. Ghost vertices for other gauges are given in appendix C. The first factor is the color weight (cf. exercises 6.E.1 and 6.F.4). A factor $\int \frac{d^d k}{(2\pi)^d}$ for each loop.

Diagrammatic notation

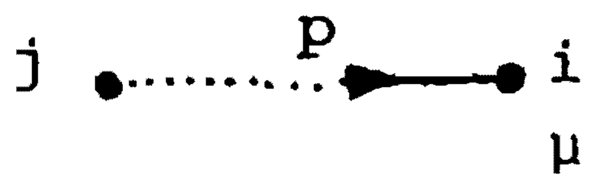
One way to avoid the proliferation of color indices, Minkowski indices, and other QCD factors is to introduce auxiliary Feynman rules:

Auxiliary propagators:

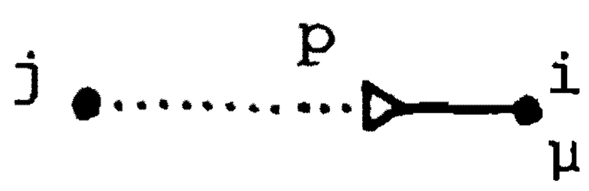


$\delta_{ij} \dfrac{-i}{p^2}$ (D.8a)

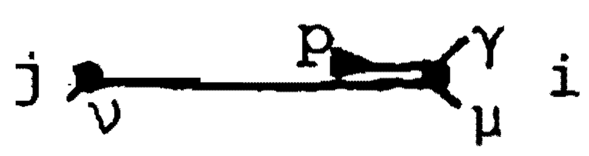
$\delta_{ij} \dfrac{-i}{p^2} g_{\alpha\mu} g_{\beta\nu}$ (D.8b)



$\delta_{ij} \dfrac{-i}{p^2} (\pm p_\mu)$ + if arrow along p, − if arrow against p (D.9a)

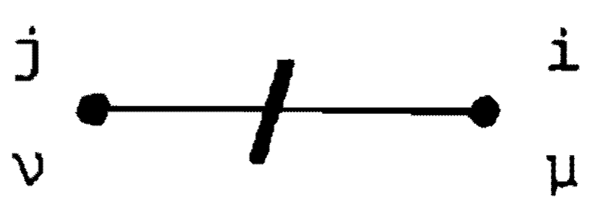


$\delta_{ij} \dfrac{-i}{p^2} (\pm h_\mu(p))$, + if arrow along p, − if arrow against p (D.9b)

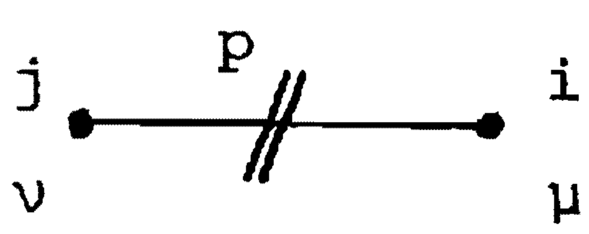


$\delta_{ij} \dfrac{-i}{p^2} g_{\nu\mu} (\pm p_\mu)$ (D.9c)

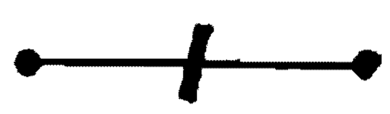
$\delta^b_a \, i$ (D.10a)



$\delta_{ij} (-i) g_{\mu\nu}$ (D.10b)



$\delta_{ij} (-i) (g_{\mu\nu} - p_\mu p_\nu / p^2)$

=  +  (D.10c)

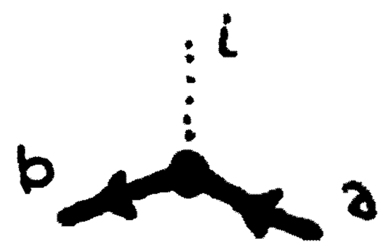
$\delta_{ij} (-i)$ (D.10d)

$\delta_{ij} (-i) g_{\alpha\mu} g_{\beta\nu}$ (D.10e)

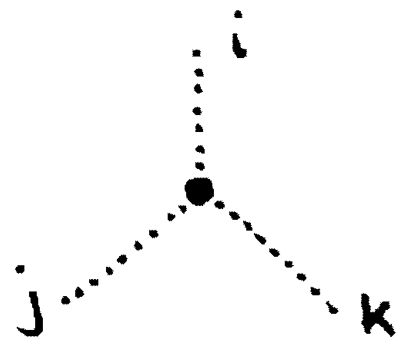
 = −  (D.10f)

Each line connecting two vertices (or an external source and a vertex) carries factor $-i/p^2$ for gluons and ghosts, and $i/(\not{p}-m)$ for quarks. Dotted lines keep track of color indices; thin lines keep track of Minkowski indices.

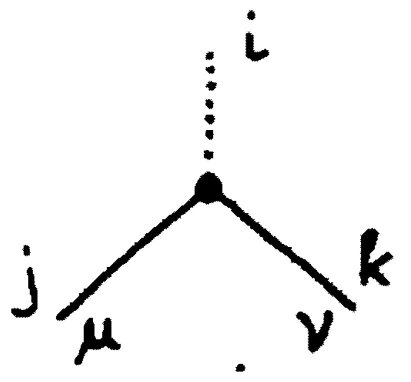
Auxiliary vertices:



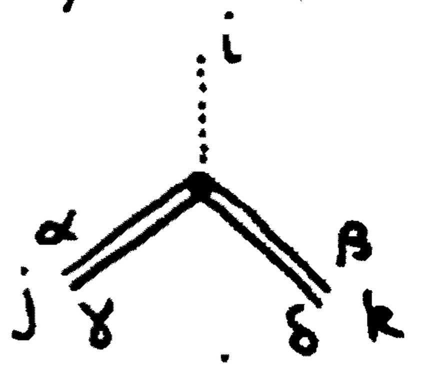
$$\left(T_i\right)^a_b\,(-i) \qquad \text{(D.11a)}$$



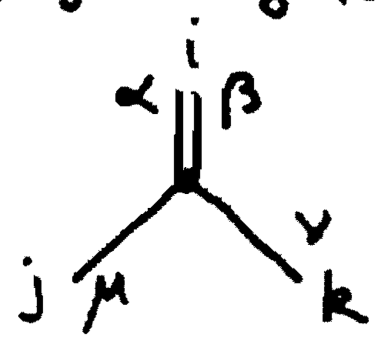
$$(-iC_{ijk})i \qquad \text{(D.11b)}$$



$$(-iC_{ijk})ig_{\mu\nu} \qquad \text{(D.11c)}$$



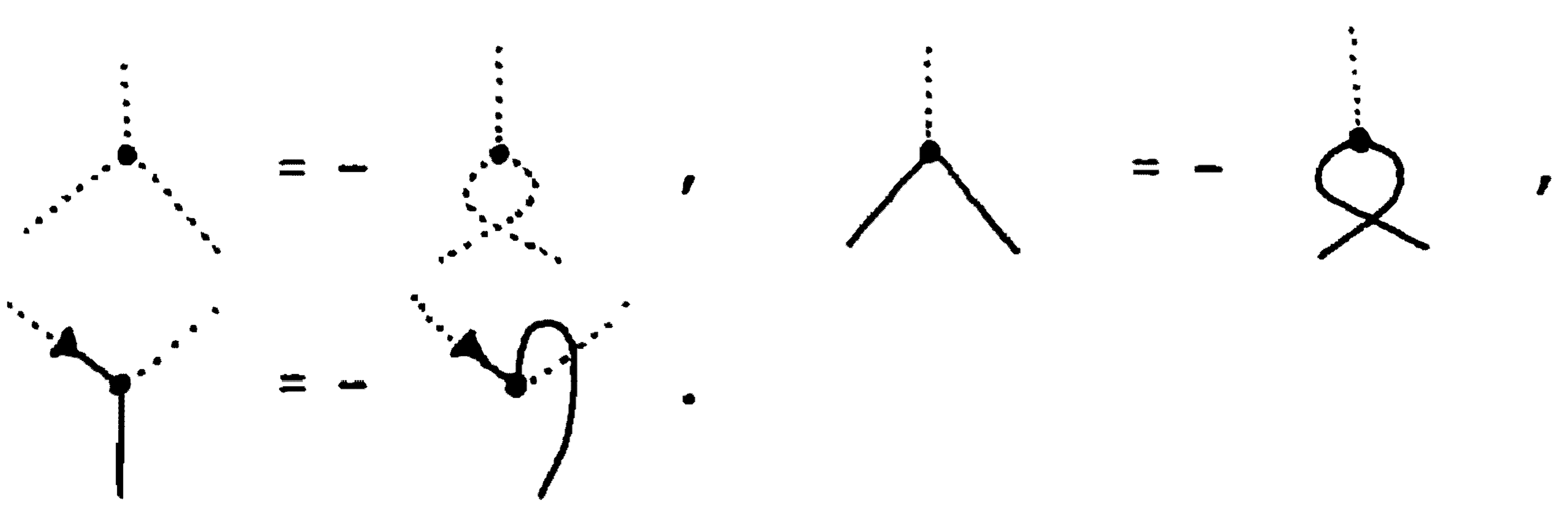
$$(-iC_{ijk})ig_{\alpha\beta}g_{\gamma\delta} \qquad \text{(D.11d)}$$



$$(-iC_{ijk})ig_{\alpha\mu}g_{\beta\nu} \qquad \text{(D.11e)}$$

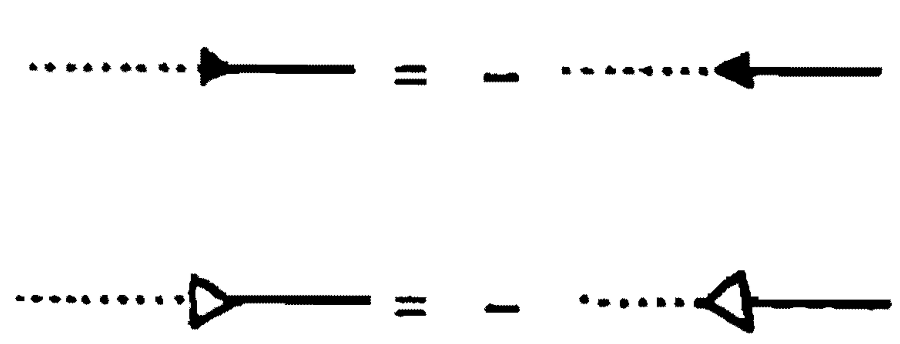
Signs

C_{ijk} indices are read anticlockwise around the vertex. Due to the antisymmetry of C_{ijk}, the corresponding vertices change sign under interchange of any two legs:



(D.12)

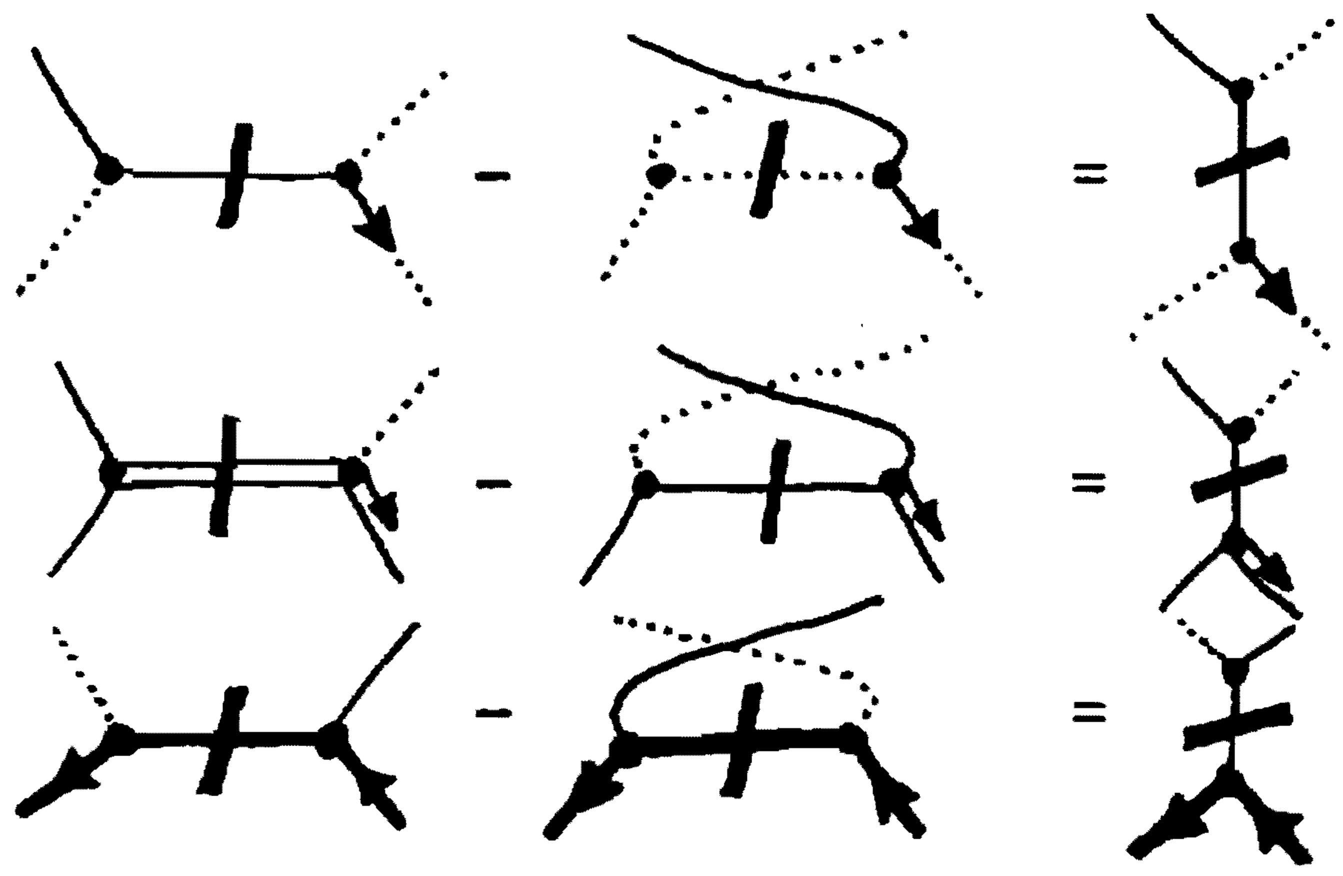
Arrows for p^μ and h^μ factors indicate the momentum flow and change sign under arrow reversal:



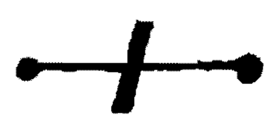
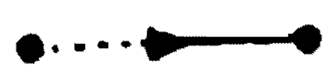
(D.13)

Jacobi identities, Lie algebra

They are all statements of (6.48) and (6.21), but decorated with different Minkowski factors:



(D.14)

Comments: It would be more consistent to treat propagators as two-leg vertices, but it is traditional to denote them by lines. This causes some unnecessary ugliness, such as slash notation  for lines without propagators, and confusion between  and  which we tried to clarify in equation (7.20).

Critics say:

... Seen in [Cvitanović's] framework, field theory books are like every other form in the universe: they are generated by changing intervals of tension between a dominant system and a competing system in a space-time continuum that is dependent on the process of competition between these two stabilities and not on any General Concept of Space and Time ... [Cvitanović's] method thus valorizes the microcosm which illuminates macrocosmic form by the high tendency of microcosmic patterns to repeat themselves and so greatly limit structural variation in the macrocosm ... But on another level, as in the sagas, the *Song of Rolland*, the *Illiad*, the *Odyssey*, the *Nibelungenlied*, the *Aeneid* and *Beowulf*, the real dynamic focus of the book is the power of anger.

Patricia Harris Stablein

A distinguished reviewer says: