chaotic field theory time reversal

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ChaosBook.org/overheads/spatiotemporal
→ Chaotic field theory slides

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chaotic / turbulent field theory?

herding cats audience reviews:[1]

Arnd Bäcker:

"even faster than Tomaž Prosen"

Martina Hentschel:

"Amazing :)"

Arnd Bäcker:

"... maybe I even understood
a single word..."

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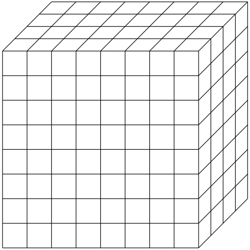
[1] the limiting speed for transfer of quantum-chaotic information is 1 Prozen.

Euclidean lattice field theory

scalar *field* $\phi(x)$

evaluated on lattice points



$$\phi_z = \phi(x)$$
$$x = a\,z = \text{lattice point}$$
$$z \in \mathbb{Z}^d$$

a periodic point per each unit cell

[2]

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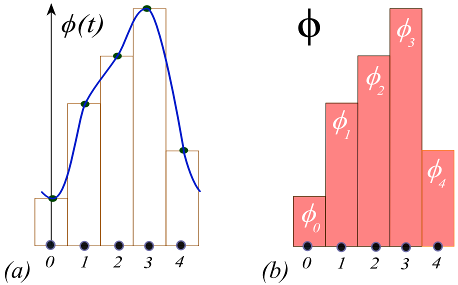
[2] G. Münster and M. Walzl, *Lattice gauge theory - A short primer*, 2000.

Discretization of a $1d$ field theory

scalar *field* $\phi(x)$ evaluated on lattice points



periodic field $\phi(t)$ is a function of continuous coordinate t

corresponding discretized period-5 lattice state $X = \overline{\phi_0\phi_1\phi_2\phi_3\phi_4}$,

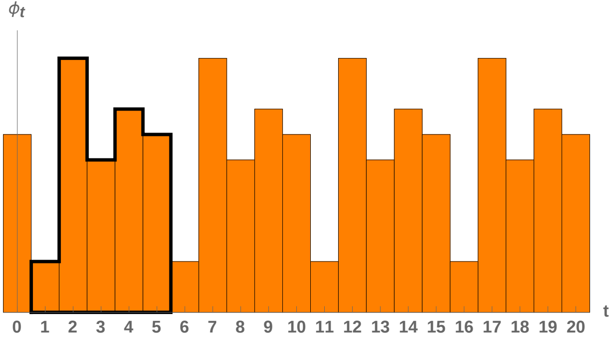
Horizontal: t coordinate, lattice sites marked by dots, labelled by $t \in \mathbb{Z}$

the value of the discretized field $\phi_t \in \mathbb{R}$ is plotted as a bar centred at lattice site t

1-dimensional lattice field theory

write a periodic field over n-sites Bravais cell as
the lattice state and the symbol block (sources)

$$X = (\phi_{t+1}, \cdots, \phi_{t+n}), \quad M = (m_{t+1}, \cdots, m_{t+n})$$



'M' for 'marching orders' : come here, then go there, $\cdots$

field theory is defined by its action

field theory

field configuration X occurs with probability

$$p(\mathrm{X}) = \frac{1}{Z}\, e^{-S[\mathrm{X}]}\,, \qquad Z = Z[0]$$

partition function = sum over all configurations

$$Z[\mathrm{M}] = \int [d\phi]\, e^{-S[\mathrm{X}]+\mathrm{X}\cdot\mathrm{M}}\,, \qquad [d\phi] = \prod_{z}^{\mathcal{L}} \frac{d\phi_z}{\sqrt{2\pi}}$$

'source' M

example : Euclidean ϕ^4 theory

continuum action

$$S = \int dx^d \left\{ \frac{1}{2} \sum_{i=1}^{d} (\partial_\mu \phi(x))^2 + \frac{\mu^2}{2} \phi(x)^2 + \frac{g}{4!} \phi(x)^4 \right\}$$

lattice action

$$S = \sum_{z,z'} \frac{1}{2} \left\{ \phi_z \left(-\Box + \mu^2 \right)_{zz'} \phi_{z'} \right\} + \sum_z \frac{g}{4!} \phi_z^4 .$$

in ‘lattice units’ : $a = 1$

QFT path integrals : semi-classical WKB quantization

WKB backbone

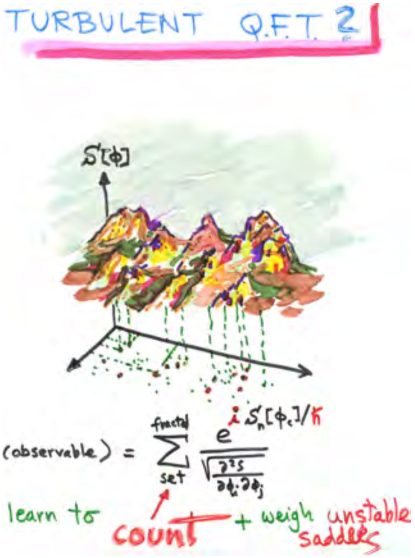
classical field theory

extremal condition → eqs

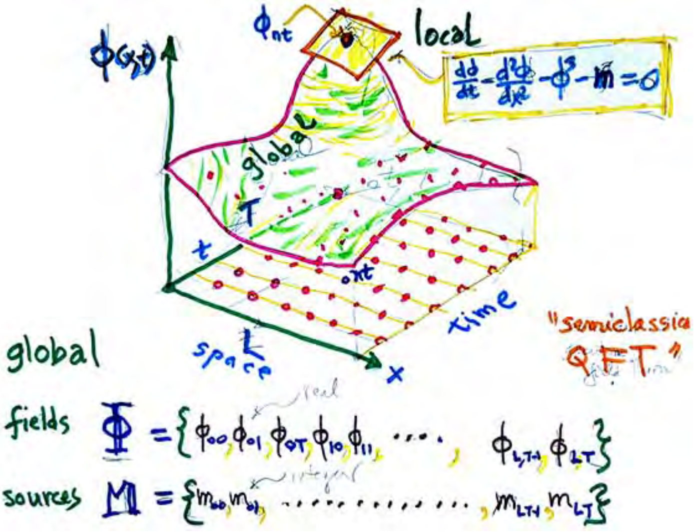
$$\frac{\delta S[X]}{\delta \phi_z} = m_z$$

classical solution X satisfies the local extremal condition on every lattice site

a fractal set of saddles



think globally, act locally



for each symbol array M, a periodic lattice state X_M

Laplacian discretization

discrete lattice

Laplacian in 1 dimension

$$\phi_{t+1} - 2\phi_t + \phi_{t-1} = \Box\,\phi_t$$

so have an (anti)oscillator chain, known as

$d = 1$ Klein-Gordon (or damped Poisson) equation

$$(-\Box + \mu^2)\,\phi_t = m_t\,, \qquad \mu^2 = s - 2$$

examples : 1d lattice field theories

spatiotemporal lattice field theory

$$-\phi_{t+1} + V'[\phi_t] - \phi_{t-1} \quad = \quad m_t$$

spatiotemporal Bernoulli

$$-\phi_{t+1} + s\,\phi_t \qquad\qquad = \quad m_t$$

spatiotemporal cat

$$-\phi_{t+1} + s\,\phi_t - \phi_{t-1} \quad = \quad m_t$$

spatiotemporal Hénon

$$-\phi_{t+1} + a\,\phi_t^2 - \phi_{t-1} \quad = \quad m_t$$

spatiotemporal ϕ^4 theory

$$-\phi_{t+1} + \frac{g}{3!}\phi_t^3 - \phi_{t-1} \quad = \quad m_t$$

herding cats in *d* spacetime dimensions : ‘spatiotemporal cat’

spatiotemporal cat : a strong coupling field theory

spatiotemporal cat symmetries :

translations $\circ$ time-reversal $\circ$ spatial reflections

point-group of the square lattice:
rotations by $\pi/2$
reflections across axes and diagonals,

$$\mathrm{D}_4 = \{1, r, r^2, r^3, \sigma, \sigma_1, \sigma_2, \sigma_3\} .$$

international crystallographic notation[3],
the square lattice space group *p*4*mm*.

not a traditional
spatially weakly coupled lattice model[4]

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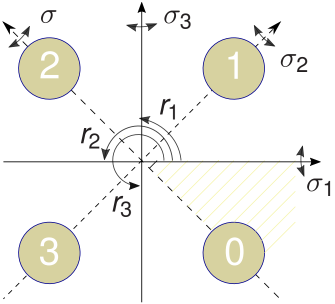
[3] M. S. Dresselhaus et al., *Group Theory: Application to the Physics of Condensed Matter*, (Springer, New York, 2007).

[4] L. A. Bunimovich and Y. G. Sinai, Nonlinearity **1**, 491 (1988).

symmetries of a square lattice unit cell



4 rotations r_j, 4 shift-reflections σ_k of dihedral group

$$\mathrm{D}_4 = \{1, r, r^2, r^3, \sigma, \sigma_1, \sigma_2, \sigma_3\}$$

overly the square onto itself.
They also tile it with the 8 copies $\hat{\mathcal{M}}_\ell$ of the fundamental domain (the shaded wedge)

retreat to : 1d lattice field theories

spatiotemporal cat

$$-\phi_{t+1} + s\,\phi_t - \phi_{t-1} \quad = \quad m_t$$

spatiotemporal Hénon

$$-\phi_{t+1} + a\,\phi_t^2 - \phi_{t-1} \quad = \quad m_t$$

spatiotemporal ϕ^4 theory

$$-\phi_{t+1} + \frac{g}{3!}\phi_t^3 - \phi_{t-1} \quad = \quad m_t$$

orbit Jacobian (Hill, Hessian, ...) matrix

each lattice state has its own

$$\mathcal{J}[\mathsf{X}] = \begin{pmatrix} s_0 & -1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ -1 & s_1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & s_2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & s_{n-2} & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & -1 & s_{n-1} \end{pmatrix},$$

stretching factor $s_t = V''[\phi_t]$ is
function of the site field ϕ_t for the given lattice state X

1. can compute Hill determinant $\mathrm{Det}\,\mathcal{J}$
2. Hill-Lindstedt-Poincaré :
   all calculations should be done on reciprocal lattice
3. toolbox : discrete Fourier transforms, irreps of D_n

Symmetries of 1-dimensional lattices

There are only two 1-dimensional space groups G:
$p1$ *infinite cyclic group* C_∞ of all lattice translations,

$$\mathrm{C}_\infty = \{\cdots, r_{-2}, r_{-1}, 1, r_1, r_2, r_3, \cdots\}$$

$p1m$, the *infinite dihedral group* D_∞ of all translations and reflections[5],

$$\mathrm{D}_\infty = \{\cdots, r_{-2}, \sigma_{-2}, r_{-1}, \sigma_{-1}, 1, \sigma, r_1, \sigma_1, r_2, \sigma_2, \cdots\}$$

group multiplication $g_i g_j$

	r_j	σ_j
r_i	r_{i+j}	σ_{j-i}
σ_i	σ_{i+j}	r_{j-i}

either adds up translations,
or shifts and then reverses their direction

[5] Y.-O. Kim et al., Pacific J. Math. **209**, 289–301 (2003).

Symmetries of 1-dimensional Bravais sublattices

Bravais cell of *period* n : given by vector **a** of length n
Bravais sublattice generated by translations

$$r_j \to r_{j\mathbf{a}}$$

symmetry : translation subgroup of C_∞

$$H_{\mathbf{a}} = \{\cdots, r_{-2\mathbf{a}}, r_{-\mathbf{a}}, 1, r_{\mathbf{a}}, r_{2\mathbf{a}}, \cdots\},$$

and

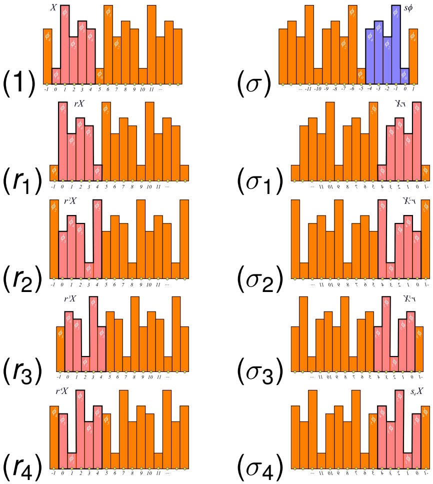
$$r_j \to r_{j\mathbf{a}}, \qquad \sigma \to \sigma_k \quad 0 \leq k < n,$$

symmetry : n infinite dihedral subgroups of D_∞

$$H_{\mathbf{a},k} = \{\cdots, r_{-2\mathbf{a}}, \sigma_k r_{-2\mathbf{a}}, r_{-\mathbf{a}}, \sigma_k r_{-\mathbf{a}}, 1, \sigma_k, r_{\mathbf{a}}, \sigma_k r_{\mathbf{a}}, r_{2\mathbf{a}}, \sigma_k r_{2\mathbf{a}}, \cdots\},$$

Bravais cell of period n,
with reflection across a symmetry point shifted k 1/2 steps

D_∞ orbit of a generic lattice state

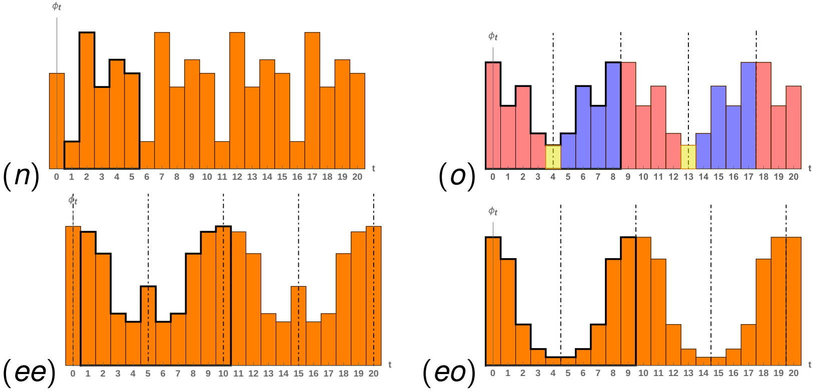


lattice state $X = \overline{\phi_0\phi_1\phi_2\phi_3\phi_4}$, no reflection symmetry

translation group H_5 invariant

D_∞-orbit is isomorphic to D_5 : 10 distinct lattice states

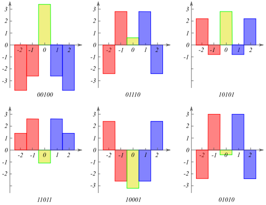
4 kinds of Bravais lattice states



(n) *no reflection symmetry:* H_5 invariant period-5 lattice state
(o) *odd period, symmetric:* an $H_{9,8}$ invariant period-9
(ee) *even period, even symmetric:* $H_{10,0}$ invariant period-10
(eo) *even period, odd symmetric:* $H_{10,9}$ invariant period-10

example : 5-period Bravais lattice site, with reflection

scalar *field* $\phi(x)$



temporal Hénon period-5 lattice state $\phi_{-2}\phi_{-1}\phi_0\,\phi_1\phi_2$

$$\phi_i = \phi_{i+5}\,, \quad \phi_{-i} = \phi_i\,.$$

reflection symmetric; fixed lattice field ϕ_0 colored gold

$$\begin{aligned} -S'[\phi_0] + 2\phi_1 &= -m_0 \\ \phi_0 - S'[\phi_1] + \phi_2 &= -m_1 \\ \phi_1 - S'[\phi_2] + \phi_2 &= -m_2 \end{aligned}$$

with a 3-dimensional orbit Jacobian matrix

$$\mathcal{J} \;=\; \begin{pmatrix} s_0 & -2 & 0 \\ -1 & s_1 & -1 \\ 0 & 1 & s_2 - 1 \end{pmatrix}$$

zeta functions unlike 1980's

periodic orbit theory, version (1) : counting lattice states[6]

Lind zeta function

$$\zeta_{Lind}(t) = \exp\left(\sum_H \frac{N_H}{|G/H|} t^{|G/H|}\right)$$

sum is over all subgroups H of space group G
N_H is the number of fixed points of H
$|G/H|$ is the number of states in H orbit

1. Lind zeta function counts group orbits, one per each set of equivalent lattice states

[6] D. A. Lind, "A zeta function for Z^d-actions", in *Ergodic Theory of Z^d Actions*, edited by M. Pollicott and K. Schmidt (Cambridge Univ. Press, 1996), pp. 433–450.

zeta functions unlike 1980's

periodic orbit theory, version (1) :
counting lattice states for reflection-symmetric systems[7,8]

Kim-Lee-Park zeta function

$$\zeta_\sigma(t) = \sqrt{\zeta_{top}(t^2)}\; e^{h(t)},$$

where ζ_{top} is the Artin-Mazur zeta function, and the counts of the 3 kinds of symmetric orbits are

$$h(t) = \sum_{m=1}^{\infty} \left\{ N_{2m-1,0}\, t^{2m-1} + \left(N_{2m,0} + N_{2m,1}\right)\, \frac{t^{2m}}{2} \right\}$$

[7] M. Artin and B. Mazur, Ann. Math. **81**, 82–99 (1965).
[8] Y.-O. Kim et al., Pacific J. Math. **209**, 289–301 (2003).

what we still do not understand today

1. solved so far only 1-dimensional spatiotemporal lattice, point group D_1
2. should all time-reversal symmetric systems be analyzed this way ?
3. should all dynamical systems should be solved on reciprocal lattice ?
4. for 2-dimensional spatiotemporal chaotic field theory, still have to do this for square lattice point group D_4
5. then, solve the problem of turbulence (Navier-Stokes, Yang-Mills, general relativity)

time reversal

Fejér [4, 8] (1916) Fejér and Riesz lemma:
every positive trigonometric polynomial can be represented by the square of the absolute value of another trigonometric polynomial whose coefficients are, in general, complex

factoring the orbit Jacobian matrix

$$\mathcal{J} \;=\; \Box - \mu^2 1 = \left(r^{-1} - 1\right)(r - 1) - \mu^2 1\,,$$

where

$$\mu = \sqrt{s-2}\,.$$

is the Yukawa mass parameter

XXX

centered, reflection (anti)symmetric difference operators

$$\begin{aligned} \tilde{\partial} &= \tilde{r} - \tilde{r}^{-1}, \qquad \tilde{r} = r^{1/2} \\ &= -\tilde{\partial}^{\top}, \end{aligned}$$

interpolate 1/2-unit spacing lattice $\tilde{\mathcal{L}}$ points between the integer lattice $\mathcal{L}$ points
$\tilde{r} = r^{1/2}$ is the shift operator on the 1/2 lattice spacing. So two applications of 1/2 lattice shift operator give you one full lattice spacing.

1/2-unit spacing lattice $\tilde{\mathcal{L}}$ points between the integer lattice $\mathcal{L}$ points, with the derivatives written as

$$\begin{aligned} (r-1) &= \tilde{r}\tilde{\partial} \\ \left(r^{-1}-1\right)(r-1) &= -\tilde{\partial}^2 = \Box \\ \mathcal{J} &= \Box - \mu^2 1 = \tilde{\mathcal{J}}^\top \tilde{\mathcal{J}} \\ \tilde{\mathcal{J}} &= \tilde{\partial} - \mu 1 = \tilde{r} - \mu 1 - \tilde{r}^{-1} \\ \tilde{\mathcal{J}}^\top &= \tilde{\partial} + \mu 1 = \tilde{r} + \mu 1 - \tilde{r}^{-1} \end{aligned} \tag{1}$$

XXX

in the matrix form, the $\mathcal{J} = \tilde{\mathcal{J}}^\top \tilde{\mathcal{J}}$ factorization can be checked by matrix multiplication

$$\mathcal{J} = \begin{pmatrix} -s & 0 & 1 & 0 & \dots & 1 & 0 \\ 0 & -s & 0 & 1 & \dots & 0 & 1 \\ 1 & 0 & -s & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -s & 0 & 1 \\ 1 & 0 & \dots & 1 & 0 & -s & 0 \\ 0 & 1 & \dots & 0 & 1 & 0 & -s \end{pmatrix}$$

$$\tilde{\mathcal{J}} = \begin{pmatrix} -\mu & -1 & 0 & 0 & \dots & 0 & 1 \\ 1 & -\mu & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -\mu & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & -\mu & -1 \\ -1 & 0 & \dots & \dots & \dots & 1 & -\mu \end{pmatrix},$$

where $\mathcal{J}, \tilde{\mathcal{J}}^\top, \tilde{\mathcal{J}}$ act on the 1/2-unit spacing lattice $\tilde{\mathcal{L}}$

"Dirac" "square root"

the metal map takes a temporal lattice form

$$\tilde{\phi}_{t+1} - \mu\,\tilde{\phi}_t - \tilde{\phi}_{t-1} = -\tilde{m}_t\,,$$

or, in terms of a lattice state X, the corresponding symbol block' M, and the $[n \times n]$ shift operator r,

$$(\tilde{r} - \mu\,\mathbb{1} - \tilde{r}^{-1})\,\tilde{\mathsf{X}} = -\tilde{\mathsf{M}}\,,$$