Finite Groups

Nonlinear Dynamics 2: Week 13

some very very basic stuff about group theory……

Melody Lu

What we have learned so far

- Symmetric system

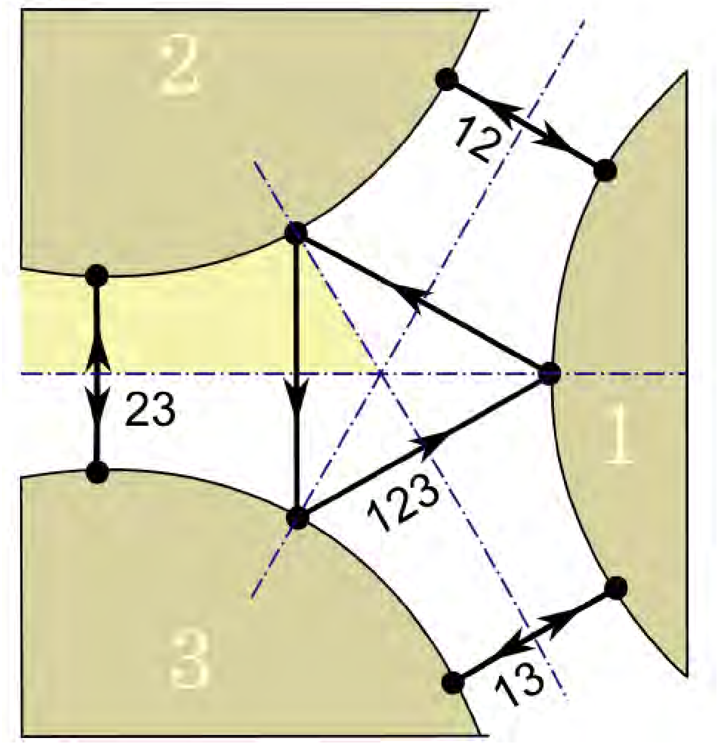
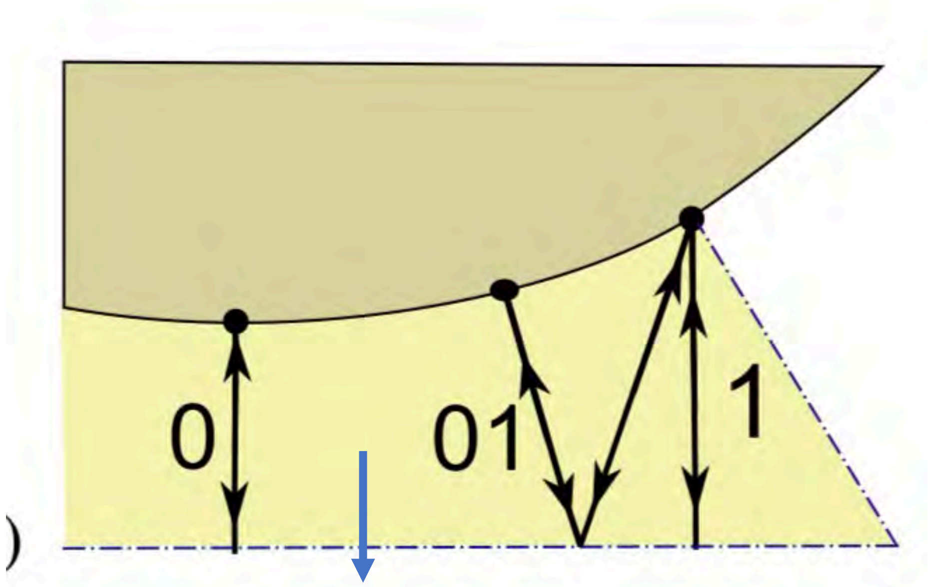


Figure 11.1 from The Chaos Book: The 3-disk pinball cycles



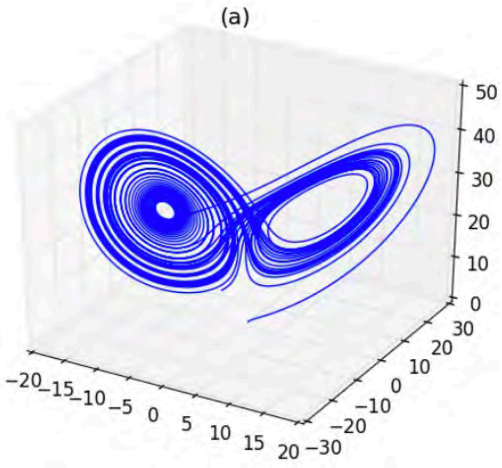
fundamental domain

The 14 full state space cycles can be restricted to the **fundamental domain** and recorded in binary sequences

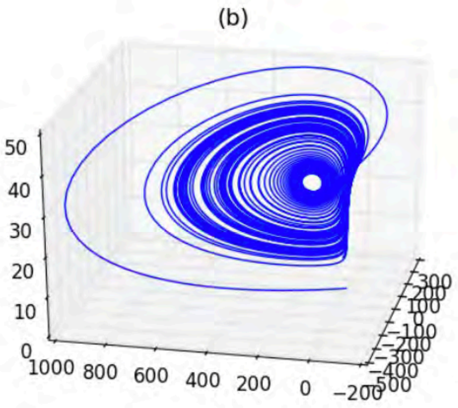
What we have learned so far

- Symmetry of Lorenz flow (from Homework 4)

$$
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= \rho x - y - xz \\
\dot{z} &= xy - bz
\end{aligned}
$$



Reduce to



Lorenz system is equivariant under the action of rotation by π about the z-axis

The trajectory in the full state space

The trajectory in the symmetry reduced space

What we have learned so far

- Symmetric reduction by the method of slices

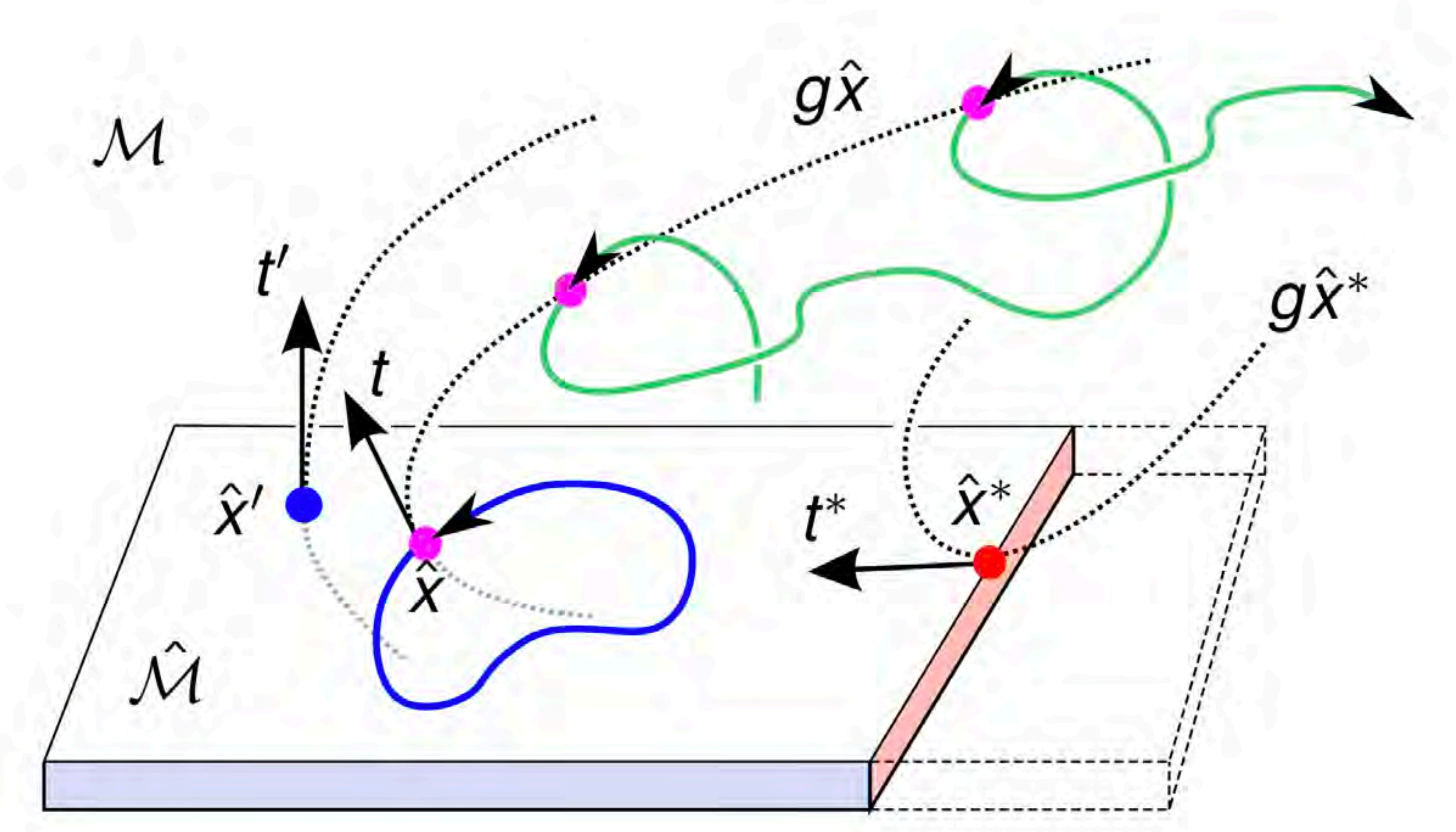


Figure 13.4 from the Chaos book: slice $\widehat{M}$ = M/G : a (d −1)-dimensional slab transverse to the template group tangent t'

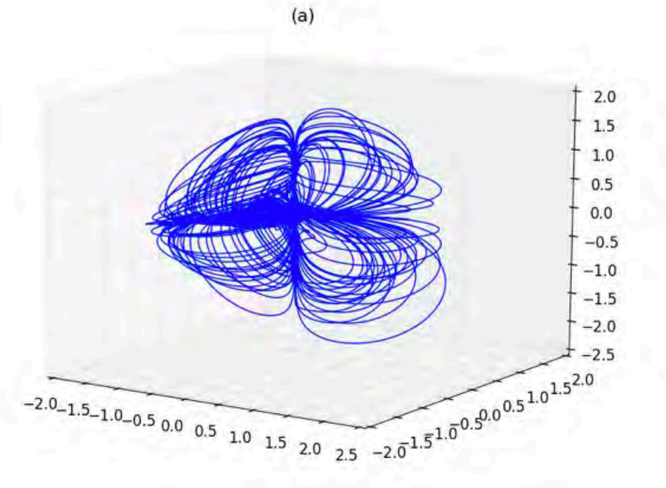
What we have learned so far

- Symmetric reduction by the method of slicing

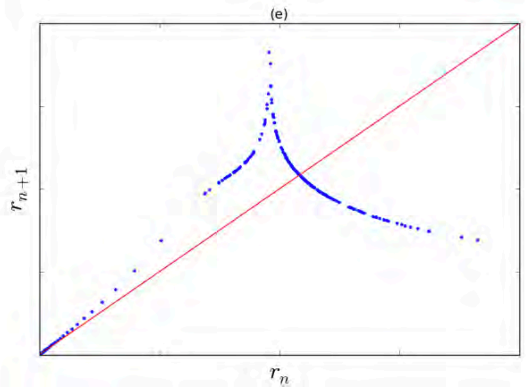
Two-modes system (From Homework 5)

$$
\begin{aligned}
\dot{x}_1 &= (\mu_1 - r^2)\,x_1 + c_1\,(x_1 x_2 + y_1 y_2)\,, \qquad r^2 = x_1^2 + y_1^2 \\
\dot{y}_1 &= (\mu_1 - r^2)\,y_1 + c_1\,(x_1 y_2 - x_2 y_1) \\
\dot{x}_2 &= x_2 + y_2 + x_1^2 - y_1^2 + a_2 x_2 r^2 \\
\dot{y}_2 &= -x_2 + y_2 + 2\,x_1 y_1 + a_2 y_2 r^2
\end{aligned}
$$

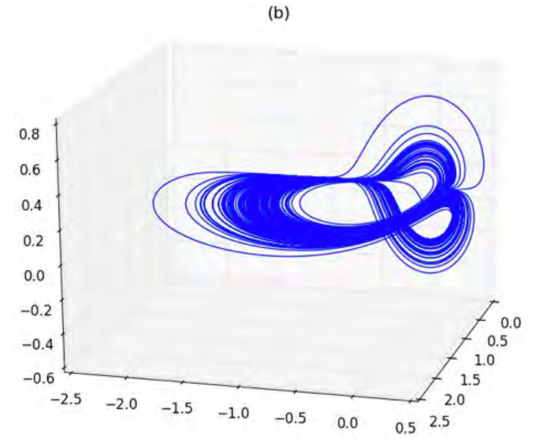
we try to reduce its continuous symmetry through method of slicing



Full state space



1-d return map



Reduced state space

Definition of a Group

A collection of elements form a group when satisfying the following conditions:

1. The product of any two elements of the group is itself an element of the group. For example, relations of the type AB = C are valid for all members of the group.

2. The associative law is valid – i.e., (AB)C = A(BC).

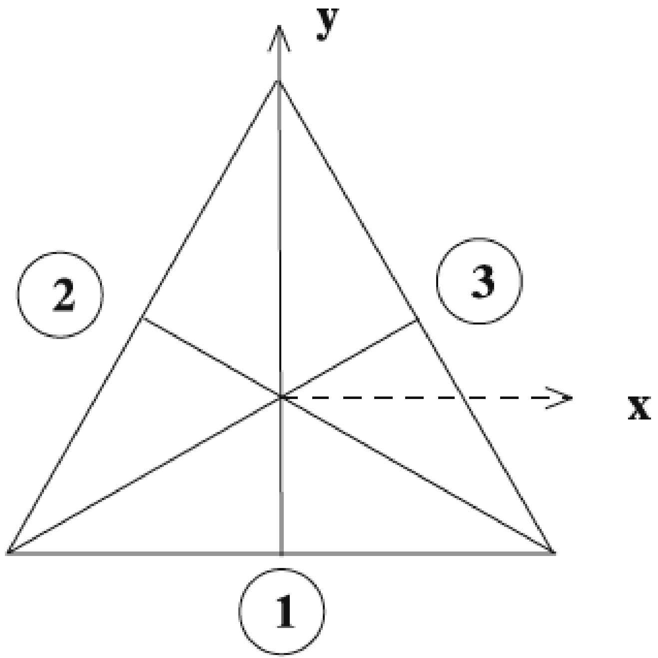
3. There exists a unit element E (also called the identity element) such that the product of E with any group element leaves that element unchanged AE = EA = A.

4. For every element A there exists an inverse element A^{-1} such that $A^{-1}A = AA^{-1} = E$.

5. operation doesn't have to be **commutative**, so it is possible AB≠BA, but if it does for every pair A,B $\in G$, we say this group is an **Abelian group**.

A Simple Example of a Group

Consider a permutation group for three numbers, P(3). We can have 3!=6 possible permutations



top row: the initial state
bottom row: the final position of each number

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad F = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

The symmetry operations on an equilateral triangle are the rotations by $\pm\, {}^{2\pi}/_{3}$ about the origin and the rotations by π about the three twofold axes.

We can call each symmetry operation an element E, A, B,,, of the group as an element of the group P(3)

A Simple Example of a Group

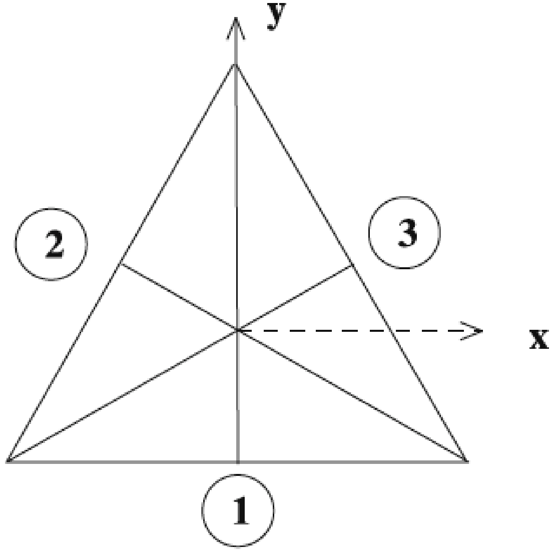
- Multiplication table

AD=B

First operation A: flip 2 and 3 along y

Second operation D: clockwise rotate by $2\pi/3$

Operation B: flip 1 and 3



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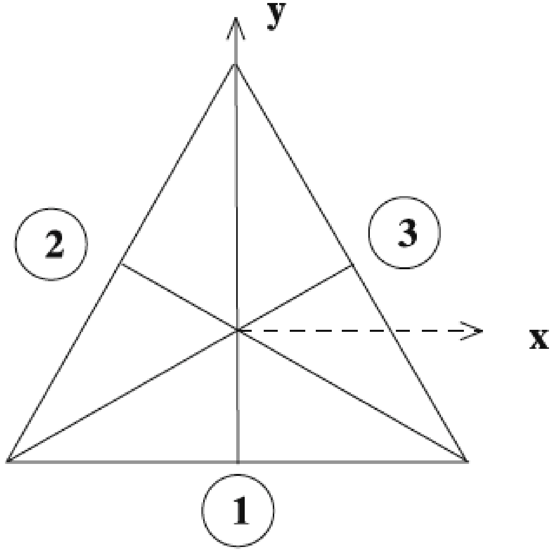
	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

Relations of the type AD = B defines use of multiplication table

A Simple Example of a Group

- Multiplication table



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$$C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad F = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Verify the associative law:
(AB)C = A(BC)

(AB)C = DC = B

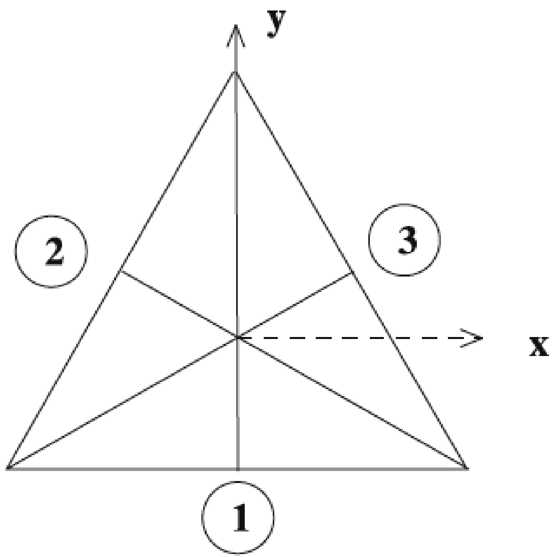
A(BC) = AD = B

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	$\boxed{B}$	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

Relations of the type AD = B defines use of multiplication table

A Simple Example of a Group

- Multiplication table



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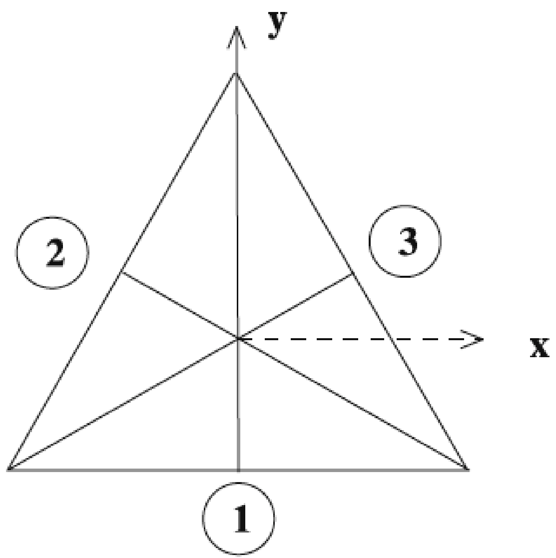
Each element of the permutation group P(3) has a one-to-one correspondence to the symmetry operations of an equilateral triangle, and we therefore say that these two groups are **isomorphic** to each other

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

Relations of the type AD = B defines use of multiplication table

Matrix Representations



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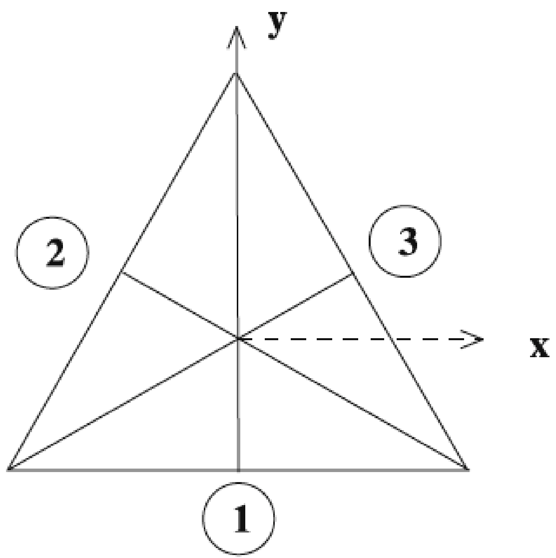
If we can associate each element with a matrix that obeys the same multiplication table as the elements, we can carry out all geometrical operations analytically in terms of arithmetic operations on matrices

i.e. If the elements obey AD=B, then the matrices representing the elements must obey

$$M(A)M(D) = M(B)$$

Matrix Representations



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For example, a set of matrices that satisfy the multiplication table for group P(3) are:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad D = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad F = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

How to understand it?

In x-y plane, we have a vector $x = \binom{0}{1}$, if we want to rotate it clockwise by 120^o

$$x' = Dx = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}$$

Basic definitions

- ***Definition 1.*** The order of a group ≡ the number of elements in the group (for finite groups)

  Ex. P(3) is of order 6

- ***Definition 2.*** A subgroup ≡ a collection of elements within a group that by themselves form a group.

  Examples of subgroups in $P(3)$:

$$\begin{array}{lll} E & (E,A) & (E,D,F) \\ & (E,B) & \\ & (E,C) & \end{array}$$

Basic definitions

- ***Theorem.*** If in a finite group, an element X is multiplied by itself enough times (n), the identity $X^n = E$ is eventually recovered.

Proof: If the group is finite, and any arbitrary element is multiplied by itself repeatedly, the product will eventually give rise to a repetition.

Let Y be a repetition

$$Y = X^p = X^q\,, \quad \text{where} \quad p > q\,.$$

and $p = q + n$:

$$X^p = X^{q+n} = X^q X^n = X^q = X^q E$$

$$X^n = E$$

Basic definitions

- ***Definition 3.*** The order of an element ≡ the smallest value of n in the relation of: $X^n = E$

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

For example

- E is of order 1,
- A, B, C are of order 2,
- D, F are of order 3.

Rearrangement Theorem

- ***Rearrangement Theorem.*** If E, $A_1, A_2, \dots, A_h$ are the elements of a group, and if A_k is an arbitrary group element, then the assembly of elements

$$A_kE, A_kA_1, \dots, A_kA_h$$

contains each element of the group once and only once.

Proof (part 1): We show first that every element is contained

Let X be an arbitrary element, If the elements form a group there will be an element $A_r = A_k^{-1}X$. Then $A_kA_r = A_kA_k^{-1}X = X$. Therefore, we can always find X in the group after multiplication of appropriate group elements.

Rearrangement Theorem

- ***Rearrangement Theorem.*** If $E, A_1, A_2, \dots, A_h$ are the elements of a group, and if A_k is an arbitrary group element, then the assembly of elements

$$A_k E, A_k A_1, \dots, A_k A_h$$

contains each element of the group once and only once.

Proof (part 2): We show that X occurs only once

Suppose X appears twice, so we have $\mathrm{X} = A_k A_r = A_k A_s$. Multiply both sides by A_k^{-1}, Therefore, we have $A_r = A_s$, which implies that two elements in the original group are identical

Cosets

- ***Definition 5.*** If $\mathcal{B}$ is a subgroup of the group G, and X is an element of G, then the assembly EX, $\mathcal{B}_1$X, $\mathcal{B}_2$X, … , $\mathcal{B}_g X$ is the right coset of $\mathcal{B}$, where $\mathcal{B}$ consists of E, $\mathcal{B}_1, \mathcal{B}_2, \ldots\ \mathcal{B}_g$.

Note: a coset does not to be a subgroup

Example: cosets of P(3)

Let $\mathcal{B} = (\mathrm{E}, \mathrm{A})$ be a subgroup

subgroup

Not subgroups

$$
\begin{array}{ll}
(E, A)E \rightarrow \boxed{E, A} & (E, A)C \rightarrow C, F \\
(E, A)A \rightarrow A, E & (E, A)D \rightarrow D, B \\
(E, A)B \rightarrow B, D & (E, A)F \rightarrow F, C
\end{array}
$$

Cosets

- ***Theorem.*** Two right cosets of a given subgroup either contain the same elements, or else have no elements in common.

  Proof: Clearly two right cosets either contain no elements in common or at least one element in common. We show that if there is one element in common, all elements are in common.

  Let $\mathcal{B}X$ and $\mathcal{B}Y$ be two right cosets. If $\mathcal{B}_k X$ = $\mathcal{B}_l Y$ = one element that the two cosets have in common, then

$$B_\ell^{-1} B_k = Y X^{-1}$$

  and YX^{-1} is in $\mathcal{B}$, therefore, EYX^{-1}, $B_1 YX^{-1}$,…, $B_g YX^{-1}$ is in $\mathcal{B}$. By the rearrangement theorem, these elements are identical to subgroup $\mathcal{B}$. Therefore, $\mathcal{B}X$ is identical to $\mathcal{B}Y$ and identical to $\mathcal{B}YX^{-1}$ so that all elements are in common.

Cosets

- ***Theorem.*** The order of a subgroup is a divisor of the order of the group

  Proof: If an assembly of all the distinct cosets of a subgroup is formed (n of them), then n multiplied by the number of elements in a coset, C, is exactly the number of elements in the group. Each element must be included since cosets have no elements in common.

  .

  Example: for the group P(3), the subgroup (E,A) is of order 2, the subgroup (E,D,F) is of order 3 and both 2 and 3 are divisors of 6, which is the order of P(3).

Conjugation and Class

Definition 6. An element B conjugate to A is by definition $B \equiv XAX^{-1}$, where X is an arbitrary element of the group.

Theorem. If B is conjugate to A and C is conjugate to B, then C is conjugate to A.

Proof:

$$B = XAX^{-1}$$
$$C = YBY^{-1}$$

$$C = YXAX^{-1}Y^{-1} = YXA(YX)^{-1}$$

Conjugation and Class

Definition 7. A class is the totality of elements which can be obtained from a given group element by conjugation.

Example: In P(3), there are three classes:

1. E;
2. A, B, C;
3. D, F.

Note that each class corresponds to a physically distinct kind of symmetry operation

Conjugation and Class

Theorem. All elements of the same class have the same order.

Proof: The order of an element n is defined by $A^n = E$. An arbitrary conjugate of A is $B = XAX^{-1}$. Then $B^n = (XAX^{-1})(XAX-1) \dots$ n times gives $XA^nX^{-1} = XEX^{-1} = E$.

Conjugation and Class

Definition 8. A subgroup B is self-conjugate (or invariant, or normal) if XBX^{-1} is identical with B for **all possible** choices of X in the group.

Definition 9. A simple group ≡ a group with no self-conjugate subgroups.

Theorem. The multiplication of the elements of two right cosets of a self-conjugate subgroup gives another right coset.

Proof: If N_i is an arbitrary element of the subgroup N, then the left coset is found by elements $XN_i = \mathrm{X}N_iX^{-1}X = N_jX$ where the right coset is formed by the elements N_jX , where $N_jX = XN_kX^{-1}$.

Factor Groups (Quotient Groups)

Definition 10. The factor group is constructed with respect to a self-conjugate subgroup as the collection of cosets of the self-conjugate subgroup, each coset being considered an element of the factor group.

Definition 11. The index of a subgroup ≡ total number of cosets = (order of group)/ (order of subgroup).

Questions?