PHYSICAL REVIEW D VOLUME 18, NUMBER 6 15 SEPTEMBER 1978

Number and weights of Feynman diagrams

Predrag Cvitanović*
The Institute for Advanced Study, Princeton, New Jersey 08540

B. Lautrup
Niels Bohr Institute, Copenhagen, Denmark

Robert B. Pearson
The Institute for Advanced Study, Princeton, New Jersey 08540
(Received 19 June 1978)

The functional techniques of field theory are adapted to the problem of evaluating sums of combinatoric and group-theoretic weights of Feynman diagrams in ϕ^N, quantum electrodynamics and non-Abelian theories. Considered are various classes of diagrams such as connected, one-particle-irreducible, and skeleton diagrams. For finite orders exact sums are given by compact recursion formulas. For higher orders estimates are obtained from the exact results or by steepest-descent methods.

I. INTRODUCTION

In trying to understand the behavior of field theory at large orders in perturbation theory, one finds that the number of diagrams contributing is an important effect. It is the cause of the combinatorial growth of amplitudes for superrenormalizable theories.[1] However, for renormalizable theories there are single diagrams which can cause combinatorial growth[2,3] and for gauge theories there could exist strong cancellations between diagrams.[4] In this paper we determine the number of diagrams contributing to various amplitudes. We find that it is both computationally efficient and instructive to treat the whole problem as a *zero-dimensional field theory*. This approach enables us not only to count diagrams, but also to treat a more general class of problems where the zero-dimensional fields transform under some local symmetry group.

In a perturbative expansion in any field theory the kth-order term in some amplitude has the form

$$A_k=\sum_G C_G W_G F_G, \tag{1.1}$$

where the sum extends over all topologically distinct kth-order Feynman diagrams. C_G is the symmetry factor of the diagram, W_G is a symmetry group weight associated with the diagram, and F_G is the integral over loop momenta in the diagram. For a zero-dimensional field theory $F_G=1$, and if there is no symmetry group, $W_G=1$ as well, so we have

$$A_K=\sum_G C_G \tag{1.2}$$

the weighted sum of diagrams. We loosely refer to this as the "number of diagrams" or to evaluating this sum as "diagram counting." In the interesting cases for QED all of the C's are 1 so this is just the number of diagrams. If there is a symmetry group we obtain the sum of symmetry factors times group-theoretic weights.

In Sec. II we introduce the notation that we will use in the rest of the paper and state the standard definitions[5] and identities of the generating functions for various classes of Green's functions. In Sec. III we apply this to ϕ^N and QED to obtain recursive formulas for the number of diagrams. In Sec. IV these results are generalized to theories with global symmetries. In Sec. V we determine the asymptotic behavior of the number of diagrams in large orders.

II. NOTATION AND DEFINITIONS

A field theory is defined by the *vacuum generating function*

$$\begin{aligned} \mathfrak{z} &= \int [d\phi]\, e^S, \\ S &= S_I+S_0, \\ S_0 &= -\tfrac{1}{2}\phi_i \Delta^{-1}_{ij}\phi_j + J_i\phi_i, \end{aligned} \tag{2.1}$$

which generates *full Green's functions*

$$(Z^m)_{ij\cdots}=\left.\frac{\partial^m \mathfrak{z}}{\partial J_i \partial J_j \cdots}\right|_{J=0}. \tag{2.2}$$

The *vacuum bubbles* are given by

$$Z=\mathfrak{z}|_{J=0}. \tag{2.3}$$

The *expectation value* of a product of m fields is

$$\langle\phi_i\phi_j\cdots\phi_k\rangle=(Z^m)_{ij\ldots k}/Z. \quad (2.4)$$

We often single out the expectation value of a single field which is called a *tadpole*

$$\Phi_i=\langle\phi_i\rangle. \quad (2.5)$$

The logarithm of the vacuum generating function

$$\mathcal{W}=\ln \mathfrak{Z} \quad (2.6)$$

is the *generator of connected Green's functions*

$$(W^m)_{ij\ldots}=\frac{\partial^m \mathcal{W}}{\partial J_i \partial J_j\cdots}. \quad (2.7)$$

In particular the *connected vacuum bubbles* are given by

$$W=\mathcal{W}|_{J=0}, \quad (2.8)$$

while the *exact propagator* is given as

$$D_{ij}=(W^2)_{ij}. \quad (2.9)$$

The Legendre transform of the generator of connected Green's functions

$$\tilde{\Gamma}=\mathcal{W}-\mathcal{J}_i\phi_i \quad (2.10)$$

is the *generator of one-particle-irreducible* (1*PI*) *Green's functions*

$$(\Gamma^m)_{ij\ldots}=\left.\frac{\partial^m\tilde{\Gamma}}{\partial\phi_i\partial\phi_j\cdots}\right|_{\phi=0}. \quad (2.11)$$

We distinguish the tadpole of (2.5), which has $J=0$, from the variable ϕ in (2.9) which is related to $\tilde{\Gamma}$ through

$$\phi_i=\frac{\partial\mathcal{W}}{\partial J_i},\quad \mathcal{J}_i=-\frac{\partial\tilde{\Gamma}}{\partial\phi_i}, \quad (2.12)$$

Particular 1PI or proper Green's functions are the *proper tadpole*

$$J_i=-(\Gamma^1)_i, \quad (2.13)$$

the *proper self-energy*

$$\pi_{ij}=(\Gamma^2)_{ij}+\delta_{ij}, \quad (2.14)$$

and the *proper vertex* (for theories with a trilinear coupling)

$$\Gamma_{ijk}=(\Gamma^3)_{ijk}. \quad (2.15)$$

The integral in (2.1) gives formally

$$\mathfrak{Z}=\text{const}\times\exp(S_I)\exp(\tfrac{1}{2}J_i\,\Delta_{ij}J_j), \quad (2.16)$$

where now in S_I the fields ϕ_i stand for the derivatives $\partial/\partial J_i$. Differentiating this equation with respect to J_i leads to the Dyson-Schwinger equation

$$\phi_i\mathfrak{Z}=\Delta_{ij}\left(J_j+\frac{\partial S_I}{\partial\phi_j}\right)\mathfrak{Z}, \quad (2.17)$$

While differentiating it with respect to the coupling constant g gives the simple identity

$$\frac{d}{dg}\mathfrak{Z}=\frac{dS_I}{dg}\mathfrak{Z}, \quad (2.18)$$

which will allow the Dyson-Schwinger equations to be converted into recursion relations for the number of diagrams. The form of the Dyson-Schwinger equation for $\mathcal{W}$ and $\tilde{\Gamma}$ depends on the model, but will have a form similar to those for ϕ^3 which are

$$\phi\mathcal{W}=J+\frac{g}{2}\phi^2\mathcal{W}+\frac{g}{2}(\phi\mathcal{W})^2 \quad (2.19)$$

and, with $J=-\partial/\partial\phi$

$$-J\tilde{\Gamma}=-\phi+\frac{g}{2}\phi^2-\frac{g}{2}\frac{1}{J^2\tilde{\Gamma}}. \quad (2.20)$$

One further identity which will be useful later is

$$\left.\frac{\partial\mathcal{W}}{\partial g}\right|_J=\left.\frac{\partial\tilde{\Gamma}}{\partial g}\right|_\phi. \quad (2.21)$$

The expansion of the exponential in powers of g in (2.16) gives the usual perturbation series. In our applications the expansion can be carried out explicitly for $\mathfrak{Z}$ by means of the combinatorial identities

$$\left.\frac{d^k}{dJ^k}e^{J^2/2}\right|_{J=0}=\begin{cases}(k-1)!!, & k \text{ even}\\ 0, & k \text{ odd},\end{cases} \quad (2.22)$$

$$\left.\left(\frac{d}{d\eta}\frac{d}{d\bar{\eta}}\right)^k e^{\bar{\eta}\eta}\right|_{\bar{\eta}=\eta=0}=k!, \quad (2.23)$$

$$\left.\left(\frac{d}{dJ_i}\frac{d}{dJ_i}\right)^k e^{(J_lJ_l)/2}\right|_{J=0}=\frac{(n+2k-2)!!}{(n-2)!!},$$

$$i,l=1,2,\ldots,n, \quad (2.24)$$

where $m!!=m(m-2)(m-4)\cdots$, and repeated indices are summed.

Even in zero dimensions we may consider renormalization effects. Here they take on a graph theoretic significance. If we renormalize the fields so that the exact propagators are normalized to 1, then the above formalism generates Green's functions without self-energy insertions. If we also renormalize the coupling constant so that the proper vertex $\Gamma = g$, then we count diagrams without vertex corrections as well. In other words we are counting *skeleton Green's functions* which are related to ordinary Green's functions through

$$\sum_{k=0}^{\infty} W_k^m g^k = z_2{}^{m/2} \sum_{l=0}^{\infty} S_l^m g_R^l , \tag{2.25}$$

where

$$g_R = z_2{}^{3/2} g/z_1, \quad \frac{g}{z_1} = \Gamma, \quad z_2 = D \tag{2.26}$$

for the case of ϕ^3 as an example. In (2.25) a superscript refers to the number of external legs and a subscript to the order in perturbation theory.

III. EXACT COUNTING

In this section we apply the formalism of the previous section to count the number of diagrams in theories with no internal symmetries. To illustrate we will consider the simplest possible case of a pure scalar interaction, and then consider the more interesting case of QED.

A. ϕ^N theories

Consider the action

$$S[\phi] = -\tfrac{1}{2}\phi^2 + \frac{g}{N!}\phi^N + J\phi. \tag{3.1}$$

By (2.2) and (2.16), the sum of the combinatoric weights of diagrams contributing to a full m-particle Green's function is

$$Z^m = \phi^m e^{(g/N!)\phi^N} e^{J^2/2}\big|_{J=0} , \tag{3.2}$$

which by (2.22) gives in kth order

$$Z_k^m = \begin{cases} \dfrac{(Nk+m-1)!!}{k!(N!)^k} & \text{if } Nk+m \text{ is even}, \\ 0 \quad \text{if } Nk+m \text{ is odd.} \end{cases} \tag{3.3}$$

Constants have been adjusted so that $Z_0{}^2 = 1$ and $W_1{}^N = 1$. In principle everything is now known or can be computed from the defintions of Sec. II.

In practice this would be a very painful process. The application of the Dyson-Schwinger equation leads to considerable simplification. (2.17) and (2.18) may be combined to give

$$\phi^2 \mathfrak{z} = \left(1 + J\phi + Ng\frac{d}{dg}\right)\mathfrak{z} \tag{3.4}$$

(N.B., $\phi = \partial/\partial J$ here) or in terms of m-point functions

$$Z^m = \left(m - 1 + Ng\frac{d}{dg}\right)Z^{m-2}. \tag{3.5}$$

Repeated application gives every Green's function in terms of either Z^0 or Z^1. If N is even, all odd Z^m vanish. If N is odd, direct application of (2.17) gives

$$Z^1 = \frac{g}{(N-1)!}Z^{N-1} , \tag{3.6}$$

which with (3.5) eventually gives all Green's functions in terms of $Z \equiv Z^0$. Now using (2.18) we obtain a differential equation which is satisfied by Z. If N is even we have

$$\frac{dZ}{dg} = \frac{1}{N!}\left(N - 1 + Ng\frac{d}{dg}\right)\cdots\left(1 + Ng\frac{d}{dg}\right)Z , \tag{3.7}$$

while if N is odd

$$\frac{dZ}{dg} = \frac{1}{N!(N-1)!}\left(N - 1 + Ng\frac{d}{dg}\right)\cdots\left(2 + Ng\frac{d}{dg}\right)$$

$$\times g\left(N - 2 + Ng\frac{d}{dg}\right)\cdots\left(1 + Ng\frac{d}{dg}\right)Z . \tag{3.8}$$

If we substitute $Z = e^W$ it is clear that we obtain a nonlinear differential equation for W' of degree $[\frac{1}{2}(N+1)]$. From here on the details depend on N so we will study ϕ^3 for the rest of this section for clarity. In this case the differential equation which results is

$$gW' = g^2[\tfrac{5}{12} + \tfrac{9}{4}gW' + \tfrac{3}{4}g^2(W'' + W'^2)] . \tag{3.9}$$

To obtain the connected Green's functions we must relate them to $W = W^0$. Making the substitution $\mathfrak{z} = e^{\mathfrak{W}}$ in (2.17) and (2.18) gives the *linear* partial differential equation

$$\phi\mathfrak{W} = \frac{g}{2} + J + \frac{g}{2}\left(J\phi + 3g\frac{d}{dg}\right)\mathfrak{W} , \tag{3.10}$$

which gives the analog of (3.5) for connected Green's functions

$$W^{m+1}=\frac{g}{2}\delta_{0,m}+\delta_{1,m}+\frac{g}{2}\left(m+3g\frac{d}{dg}\right)W^{m}. \qquad (3.11)$$

In particular for tadpoles and the exact propagator

$$\Phi=\frac{g}{2}+\tfrac{3}{2}g^{2}\frac{dW}{dg},$$
$$D=1+\frac{g}{2}\left(1+3g\frac{d}{dg}\right)\Phi. \qquad (3.12)$$

Combining this with (2.19) we obtain a differential equation for Φ

$$\Phi=\frac{g}{2}\left[1+\frac{g}{2}\Phi+6\left(\frac{g}{2}\right)^{2}\Phi'+\Phi^{2}\right],$$
$$\Phi_{k+2}=\frac{3k+1}{4}\Phi_{k}+\frac{1}{2}\sum_{i=1}^{k}\Phi_{i}\Phi_{k+1-i},\quad k \text{ odd} \qquad (3.13)$$

which gives for Φ and D the expansions

$$\Phi=\frac{g}{2}+5\left(\frac{g}{2}\right)^{3}+60\left(\frac{g}{2}\right)^{5}+1105\left(\frac{g}{2}\right)^{7}+27\,120\left(\frac{g}{2}\right)^{9}+\cdots, \qquad (3.14)$$
$$D=1+g^{2}+\tfrac{25}{8}g^{4}+\tfrac{390}{32}g^{6}+\tfrac{8840}{128}g^{8}+\cdots.$$

This can be compared to Fig. 1 where we show the low-order diagrams contributing to Φ and D with their associated symmetry factors. From (3.10) and (2.10) and (2.21) we obtain the analog of (3.10) for 1PI Green's functions

$$-J\tilde{\Gamma}=\frac{g}{2}-\phi+\frac{g}{2}\left(\phi J+3g\frac{\partial}{\partial g}\right)\tilde{\Gamma},$$
$$\Gamma^{k+1}=\frac{g}{2}\delta_{0,k}-\delta_{1,k}-\frac{g}{2}\left(k-3g\frac{d}{dg}\right)\Gamma^{k} \qquad (3.15)$$

(N.B., here $J=-\partial/\partial\phi$). Combining this with (2.20) we obtain the proper tadpoles, self-energies, and vertices

$$J=-\frac{g}{2}+\frac{g}{2}\left(1-\frac{3}{2}g\frac{d}{dg}\right)J^{2}$$
$$=-\tfrac{1}{2}g-\tfrac{1}{4}g^{3}-\tfrac{5}{8}g^{5}-\cdots, \qquad (3.16)$$

$$\pi=\frac{g}{2}\left(1-3g\frac{d}{dg}\right)J$$
$$=\tfrac{1}{2}g^{2}+g^{4}+\tfrac{35}{8}g^{6}+\cdots, \qquad (3.17)$$

$$\Gamma=g+g\left(-1+\frac{3}{2}g\frac{d}{dg}\right)\pi$$
$$=g+g^{3}+5g^{5}+35g^{7}+\cdots, \qquad (3.18)$$

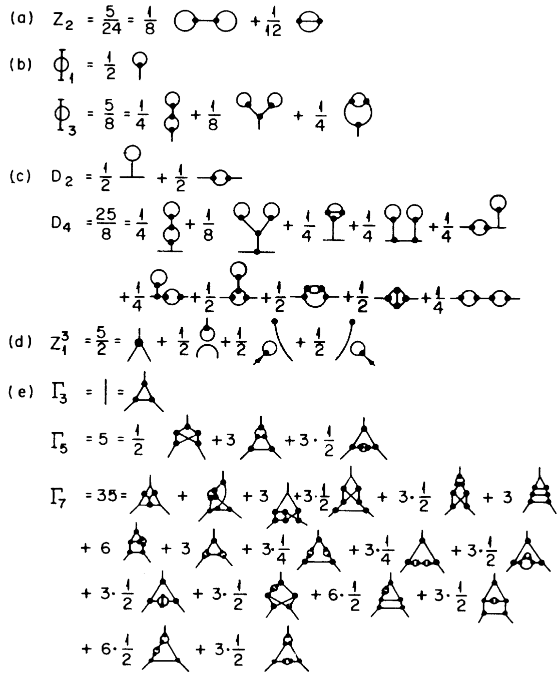

FIG. 1. Combinatoric weights for ϕ^3 diagrams (a) vacuum bubbles, (b) tadpoles, (c) exact propagator, (d) full vertex, and (e) proper vertex (the first factor is the number of diagrams with the same structure).

which may again be compared with Fig. 1. In 1PI amplitudes the expansion around $\phi=0$ eliminates all tadpole subdiagrams.

To count diagrams with no self-energy insertions consider

$$S=-\tfrac{1}{2}z_{2}\phi^{2}+\frac{g\phi^{3}}{3!}+J\phi, \qquad (3.19)$$

where the renormalization constant z_2 is determined by the requirement that

$$D=1. \qquad (3.20)$$

The same considerations as before now give

$$\left(z_{2}-\frac{3}{2}g\frac{dz_{2}}{dg}\right)\phi^{2}\mathfrak{z}=\left(1+J\phi+3g\frac{d}{dg}\right)\mathfrak{z}, \qquad (3.21)$$

$$z_{2}=1+\frac{g}{2}\Gamma, \qquad (3.22)$$

$$-J\tilde{\Gamma}=-\left(1+\frac{g}{2}\Gamma\right)\phi+\frac{\Gamma}{2}\left[1+\left(\phi J+3g\frac{d}{dg}\right)\tilde{\Gamma}\right],$$
$$\Gamma^{m+1}=-\left(1+\frac{g}{2}\Gamma\right)\delta_{m,1}+\frac{\Gamma}{2}\left[\delta_{m,0}+\left(-m+3g\frac{d}{dg}\right)\Gamma^{m}\right], \qquad (3.23)$$

$$\Gamma=g+\frac{g}{4}\left(1+\frac{3}{2}g\frac{d}{dg}\right)(\Gamma)^{2}$$
$$=g+g^{3}+\tfrac{7}{2}g^{5}+20g^{7}+\tfrac{611}{4}g^{9}+\cdots, \qquad (3.24)$$

which is checked for low orders in Fig. 1(e).

To count skeleton diagrams consider

$$S=-\tfrac{1}{2}z_2\phi^2+\frac{gz_1\phi^3}{3!}+J\phi\,, \tag{3.25}$$

where the renormalization constants are fixed by requiring (3.20) and

$$\Gamma=g. \tag{3.26}$$

This leads to

$$z_2=1+\frac{g^2}{2}z_1 \tag{3.27}$$

and a differential equation for z_1

$$1-z_1=g^2-g^2\left(2+\frac{g}{4}\frac{d}{dg}\right)(1-z_1)+\left[1+\frac{g}{2}\frac{d}{dg}+g^2\left(1+\frac{g}{8}\frac{d}{dg}\right)\right](1-z_1)^2, \tag{3.28}$$

which yields

$$1-z_1=g^2+\tfrac{1}{2}g^4+4g^6+19g^8+\cdots\,. \tag{3.29}$$

The coefficients in this series count proper vertices with no self-energy or vertex insertions. This is checked in Fig. 1(e).

In the remainder of this and in the next section we will give results similar to the above for a variety of more complex models. Before continuing let us summarize the nature of the results given in this section. For each class of Green's functions the key formula is a nonlinear differential equation for some low order Green's function which gives a recursion relation for its expansion as a power series. Higher m-point functions are obtained from this one by formulas which relate $(m+1)$-point functions to m-point functions and their derivatives. The efficiency of the formulas for computing the number of diagrams is such that one may easily compute by hand the first 10 or 20 orders while expanding the logarithms in the definitions would be prohibitive. In Sec. V we will see another use for them as they easily give the asymptotic expansion for large orders.

B. $\phi^*A\phi$ theory

As a crude model of QED consider a theory of a complex scalar coupled to an ordinary scalar with an action[6]

$$S=-\phi^*\phi-\tfrac{1}{2}A^2+g\phi^*A\phi+\eta^*\phi+\phi^*\eta+AJ. \tag{3.30}$$

This gives the generating function

$$\mathfrak{z}=\exp\left(g\frac{d}{d\eta^*}\frac{d}{d\eta}\frac{d}{dJ}\right)\exp(\eta^*\eta+\tfrac{1}{2}J^2) \tag{3.31}$$

with series coefficients

$$Z_k^{ep}=\frac{(k+e)!\,(k+p-1)!!}{k!}\,,\quad k+p\text{ even.} \tag{3.32}$$

Here e is the number of "electron" lines traversing the diagram, and p is the number of "photons" entering the diagram. Repeating the analysis for ϕ^3 we obtain

$$gW'=g^2[2+4gW'+g^2(W''+W'^2)]\,,$$
$$W=g^2+\tfrac{5}{2}g^4+\tfrac{37}{3}g^6+\tfrac{353}{4}g^8+\cdots\,, \tag{3.33}$$

$$D_e=1+gW'=1+2g^2+10g^4+74g^6+706g^8+\cdots\,, \tag{3.34}$$

where D_e is the exact "electron" propagator. We note that for connected diagrams in this theory $C_G=1$ so that we are indeed counting diagrams, as can be seen from Fig. 2. Many of these diagrams would vanish in QED because of Furry's theorem. To disentangle the "electron" loop structure of amplitudes we do the integral over the ϕ fields and write the generating function as

$$\mathfrak{z}=\exp\left[-\ln\left(1-g\frac{d}{dJ}\right)+\eta^*\eta\left(1-g\frac{d}{dJ}\right)^{-1}\right]\exp(\tfrac{1}{2}J^2)\,, \tag{3.35}$$

which may be recognized as the "electron loop expansion." Examples of results which may be obtained for this theory with a fixed number of "electron" loops are, for no loops

$$\mathfrak{z}_{\text{no loops}}=\exp\left[\eta^*\eta\left(1-g\frac{d}{dJ}\right)^{-1}\right]\exp(\tfrac{1}{2}J^2), \tag{3.36}$$

$$(D_e)_k=(k-1)!!\,,\quad k\text{ even} \tag{3.37}$$

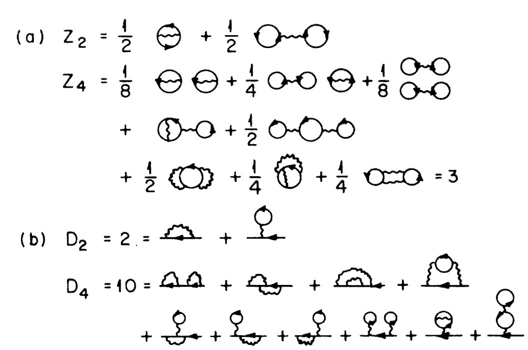

FIG. 2. Combinatoric weights for $\phi^*A\phi$ diagrams; (a) vacuum bubbles and (b) exact "electron" propagator.

or for one loop the proper self-energy for the "photon"

$$\pi_k = (k-1)!!\ ,\quad k \text{ even.} \tag{3.38}$$

C. QED[11]

Furry's theorem states that all diagrams with electron loops attached to an odd number of photons vanish. Recalling (3.35) we may eliminate such loops by the replacement

$$\ln(1-gA) \to \tfrac{1}{2}\ln(1-gA) + \tfrac{1}{2}\ln(1+gA). \tag{3.39}$$

Thus the number of QED diagrams has the generating function

$$\mathfrak{z} = \left(1 - g^2 \frac{d^2}{dJ^2}\right)^{-1/2} \exp\left[\eta^*\eta\left(1 - g\frac{d}{dJ}\right)^{-1}\right]\exp(\tfrac{1}{2}J^2). \tag{3.40}$$

With the expansion

$$(1-g^2A^2)^{-1/2}(1-gA)^{-e} = (1+gA)^e(1-g^2A^2)^{-e-1/2} \times \sum_{s=0}^{\infty}\binom{e}{s}\sum_{r=0}^{\infty}\binom{e+r-\frac{1}{2}}{r}A^{s+2r} \tag{3.41}$$

we obtain for full Green's functions

$$Z_k^{ep} = \frac{(k+p-1)!!\,e!}{k!}\sum_{\substack{e\geq s\geq 0\\ r\geq 0,\ 2r+s=k}}\binom{e}{s}\binom{e+r-\frac{1}{2}}{e-\frac{1}{2}}. \tag{3.42}$$

Again e is the number of electron lines crossing the diagram, and p the number of photon lines entering the diagram. In the interesting cases this expression simplifies considerably

$$Z_k = (k-1)!!^2/k!!\ ,\quad k \text{ even,} \tag{3.43}$$

$$Z_k^{01} = Z_k^{10} = (k+1)!!\,(k-1)!!/k!!,\quad k \text{ even}, \tag{3.44}$$

$$Z_k^{1,1} = k!!^2/(k-1)!!,\quad k \text{ odd.} \tag{3.45}$$

To study connected diagrams we again turn to the Dyson-Schwinger equations. From (3.40) by scaling $gA \to A$ we obtain for the insertion of two photons

$$\frac{d^2}{dJ^2}\mathfrak{z} = \left(1 + JA + g\frac{d}{dg}\right)\mathfrak{z}\,. \tag{3.46}$$

For the insertion of an electron line through the diagram we have

$$\frac{d^2}{d\eta\, d\bar{\eta}}\mathfrak{z} = \left(1 + \eta\frac{d}{d\eta} + g\frac{d}{dg} + \frac{gd/dJ}{1-g^2d^2/dJ^2}\right)\mathfrak{z}\,. \tag{3.47}$$

These two equations give relations for the full Green's functions

$$Z^{e,p+2} = \left(1 + p + g\frac{d}{dg}\right)Z^{ep}, \tag{3.48}$$

$$Z^{e+1,p} = \left(1 + e + g\frac{d}{dg}\right)Z^{ep} + g\sum_{k=0} g^{2k}Z^{e,p+2k+1}\,. \tag{3.49}$$

They also give differential equations for the vacuum bubbles

$$g\frac{dZ}{dg} = g^2\left(1 + g\frac{d}{dg}\right)^2 Z, \tag{3.50}$$

$$g\frac{dW}{dg} = g^2\left\{1 + 3g\frac{dW}{dg} + g^2\left[\frac{d^2W}{dg^2} + \left(\frac{dW}{dg}\right)^2\right]\right\}. \tag{3.51}$$

The exact electron and photon propagators, whose numbers turn out to be the same, are related to W by

$$D = 1 + g\frac{dW}{dg} \tag{3.52}$$

and satisfy the differential equation

$$D = 1 + g^3D' + g^2D^2\,, \tag{3.53}$$

which gives the series

$$D = 1 + g^2 + 4g^4 + 25g^6 + 208g^8 + 2146g^{10} + 26\,368g^{12} + \cdots\,. \tag{3.54}$$

The proper self-energies obey the differential equation

$$\pi = g^2 + g^3\pi' + \pi^2 \tag{3.55}$$

giving the series

$$\pi = g^2 + 3g^4 + 18g^6 + 153g^8 + 1638g^{10} + \cdots\,; \tag{3.56}$$

the low order terms can be checked with Figs. 2 and 3. The proper vertex is related to the proper self-energy by the linear equation

$$\frac{1}{g}\Gamma = 1 + g^2 + \frac{1}{2}g^3\frac{d}{dg}\left(\frac{1}{g^4} + \frac{1}{g}\frac{d}{dg}\right)g^2\pi\,, \tag{3.57}$$

which gives the series known from the magnetic moment calculations[7]

$$\Gamma = g + g^3 + 7g^5 + 72g^7 + 891g^9 + 12\,672g^{11} + \cdots\,. \tag{3.58}$$

An equation for the number of skeleton diagrams

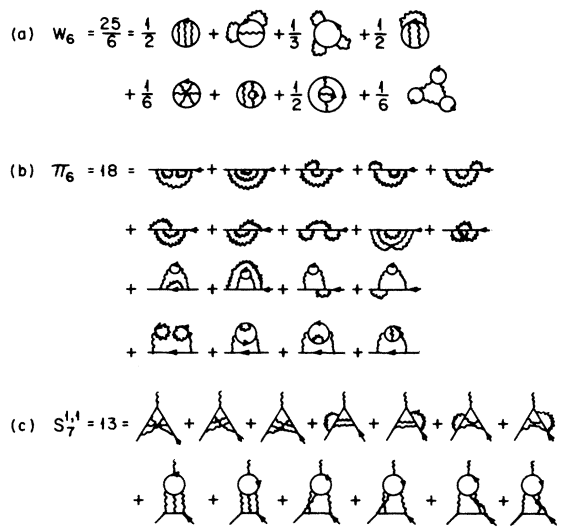

FIG. 3. QED diagrams: (a) connected vacuum bubbles, (b) proper self-energy, and (c) vertex skeletons.

for the vertex may be obtained by rewriting (3.53) in terms of the renormalized coupling (2.26). This yields a differential equation for z_1 analogous to (3.28) for ϕ^3 theory:

$$
\begin{aligned}
(1-z_1) &= g^2+g^4+2g^6 \\
&+\Big[-3g^2+g^4\Big(-1+\tfrac{1}{2}g\frac{d}{dg}\Big) \\
&\quad -g^6\Big(8+\tfrac{1}{2}g\frac{d}{dg}\Big)\Big](1-z_1) \\
&+\Big[\Big(1+\tfrac{1}{2}g\frac{d}{dg}\Big)+(g^2-g^4)\Big(3+\tfrac{3}{4}g\frac{d}{dg}\Big) \\
&\quad +g^6\Big(12+\tfrac{3}{4}g\frac{d}{dg}\Big)\Big](1-z_1)^2 \\
&+\Big[-g^2\Big(1+\tfrac{1}{2}g\frac{d}{dg}\Big)+g^4\Big(5+\tfrac{1}{2}g\frac{d}{dg}\Big) \\
&\quad -g^6\Big(8+\tfrac{1}{2}g\frac{d}{dg}\Big)\Big](1-z_1)^3 \\
&+(-g^4+g^6)\Big(2+\tfrac{1}{8}g\frac{d}{dg}\Big)(1-z_1)^4
\end{aligned}
\tag{3.59}
$$

which yields the series

$$1-z_1=g^2+g^4+13g^6+93g^8+\dots . \tag{3.60}$$

The sixth-order term is illustrated by Fig. 3(c).

IV. EXACT COUNTING WITH GLOBAL SYMMETRIES

In this section we generalize the preceding discussion to the case of fields which transform under some symmetry group. Instead of just counting the number of diagrams we count the number of diagrams weighted by the associated group-theoretic weights. For an action we can take any polynomial in the fields invariant under the action of the group. The methods already developed apply and no new difficulties are encountered.

A. $(\phi_i\phi_i)^N$ theory

As our first example we consider the action

$$S=-\tfrac{1}{2}(\phi_i\phi_i)+\frac{g}{(2N)!}(\phi_i\phi_i)^N+J_i\phi_i,\quad i=1,\dots,n, \tag{4.1}$$

which is invariant under the group of orthogonal transformations $O(n)$. Since the only invariant tensor is δ_{ij}, Green's functions have the tensor structure

$$
\begin{aligned}
G^m_{ij\cdots kl} &= \frac{1}{(m-1)!!}\,\delta_{(ij}\delta\cdots\delta_{kl)_+}G^m, \quad m \text{ even},\\
G^m &= \frac{(n-2)!!\,(m-1)!!}{(m+n-2)!!}\,G^m_{iijj\cdots kk}.
\end{aligned}
\tag{4.2}
$$

Here the normalization has been adjusted so that when $n=1$, we recover the ϕ^{2N} theory considered before. By (2.24) the full Green's functions are

$$Z^m_k=\frac{1}{k!}\,\frac{1}{(2N)!^k}\,\frac{(2Nk+m+n-2)!!\,(m-1)!!}{(m+n-2)!!},\quad \text{m even} \tag{4.3}$$

and obey the recurrence relation

$$Z^{m+2}=\frac{m+1}{m+n}\Big(m+n+2Ng\frac{d}{dg}\Big)Z^m. \tag{4.4}$$

Again for simplicity we will consider a specific case, $N=2$ or ϕ^4 theory. In this case the differential equation for the connected vacuum bubbles is

$$W'=\frac{1}{4!}[n(n+2)+8(n+3)gW'+16g^2(W''+W'^2)]. \tag{4.5}$$

From

$$D=1+\frac{4g}{n}W' \tag{4.6}$$

the exact propagator and proper self-energy are

$$
\begin{aligned}
D&=\frac{g}{3!}(2D+4gD'+nD^2) &(4.7)\\
&=1+\frac{1}{2}\frac{n+2}{3}g+\frac{2}{3}\frac{(n+2)}{3}\frac{(n+3)}{4}g^2\\
&\quad+\frac{33}{8}\frac{(n+2)}{3}\frac{(5n^2+34n+60)}{99}g^3+\cdots, &(4.8)
\end{aligned}
$$

$$
\begin{aligned}
\pi&=\frac{1}{2}\frac{n+2}{3}g+\frac{g}{3}\Big(2g\frac{d}{dg}-1\Big)\pi+\pi^2 &(4.9)\\
&=\frac{1}{2}\frac{(n+2)}{3}g+\frac{5}{12}\frac{(n+2)}{3}\frac{(n+4)}{5}g^2\\
&\quad+\frac{5}{6}\frac{(n+2)}{3}\frac{(n+4)}{5}\frac{(n+5)}{6}g^3+\cdots. &(4.10)
\end{aligned}
$$

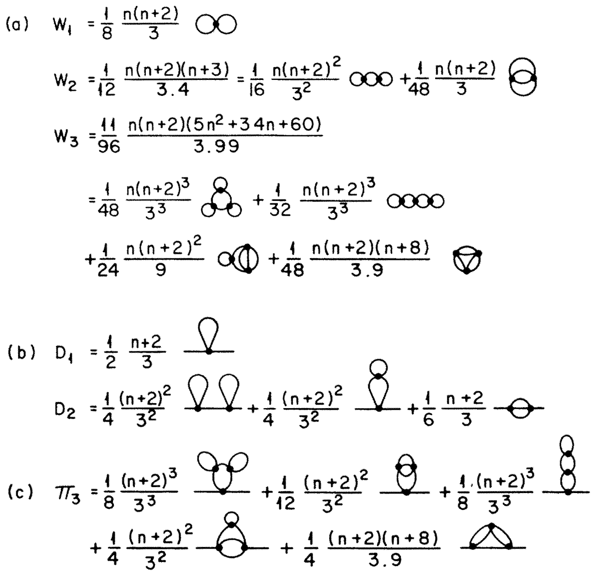

FIG. 4. Combinatoric and group-theoretic weights for $(\phi_i\phi_i)^2$ diagrams: (a) connected vacuum bubbles, (b) exact propagator, and (c) proper self-energy.

The first few terms are shown graphically in Fig. 4.

B. Traces of QED γ matrices

Instead of the *ad hoc* treatment of Sec. III we may consider the action

$$S = -\psi^\dagger\psi - \tfrac{1}{2}A^2 + gA^\mu\psi^\dagger\gamma^\mu\psi + \eta^\dagger\psi + \psi^\dagger\eta + J^\mu A^\mu , \tag{4.11}$$

where ψ_a, $a=1,2,\ldots,n$ is an anticommuting n-dimensional spinor, A^μ, $\mu=1,2,\ldots,d$ transforms under the vector representation of SO(d), and $(\gamma^\mu)^a_b$ are Hermitian Dirac matrices which satisfy

$$\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}, \quad \mathrm{tr}(\gamma^\mu) = 0.$$

Defining $\mathfrak{A} = gA^\mu\gamma^\mu$ we have

$$\mathfrak{A}^2 = g^2A^2 1, \quad \mathrm{tr}\,\mathfrak{A} = 0; \tag{4.12}$$

thus

$$\det(1-\mathfrak{A}) = \exp[\tfrac{1}{2}n \ln(1-g^2A^2)] \tag{4.13}$$

and the generating function is

$$\mathfrak{z} = \exp\left(\tfrac{1}{2}n\ln(1-g^2A^2) + \eta^\dagger \frac{1+\mathfrak{A}}{1-g^2A^2}\eta\right)\exp(\tfrac{1}{2}J^2). \tag{4.14}$$

This incorporates Furry's thoerem and sums all the γ traces for the momentum-independent part of diagrams as well. The results of Sec. IIIC are recovered if we set $d=1$, and $n=-1$. Fermi versus Bose statistics for the electrons amounts to the choice n or $-n$ in the above. For the vacuum bubbles we have

$$gZ' = g^2\left(d + g\frac{d}{dg}\right)\left(-n + g\frac{d}{dg}\right)Z , \tag{4.15}$$

$$gW' = g^2[-nd + (d-n+1)gW' + g^2(W'' + W'^2)]. \tag{4.16}$$

The exact photon and electron propagators are given in terms of W as

$$D_\gamma = \frac{1}{1-\pi} = 1 + \frac{g}{d}W', \tag{4.17}$$

$$D_e = \frac{1}{1-\Sigma} = 1 - \frac{g}{n}W', \tag{4.18}$$

where

$$W^{0,2}_{\mu\nu} = g_{\mu\nu}D_\gamma , \qquad W^{1,0}_{\alpha\beta} = \delta_{\alpha\beta}D_e. \tag{4.19}$$

Since they are related by $d \leftrightarrow -n$, we give just the photon propagators

$$\begin{aligned} D_\gamma &= 1 + g^2[-(d+n)D_\gamma + gD_\gamma' + dD_\gamma^2], \\ \pi &= g^2[-n + (d+n)\pi + g\pi'] + \pi^2 \\ &= -n[g^2 + (2+d)g^4 + (2+d)(4+d-n)g^6 + \cdots]. \end{aligned} \tag{4.20}$$

The exact vertex

$$\gamma^{\mu a}_b\langle\psi\gamma^\mu\psi^\dagger\rangle^b_a = -ndW^{1,1} \tag{4.21}$$

is given by

$$W^{1,1} = -\frac{1}{nd}W' \tag{4.22}$$

and the proper vertex is given in terms of the proper self-energies by

$$\Gamma = \frac{W^{1,1}}{D_e^2D_\gamma} = \frac{n}{(d+n)^2}\,\frac{\Sigma-\pi}{g} + \frac{g}{d+n}[d-(d+n)\Sigma + g\Sigma']. \tag{4.23}$$

These results are illustrated in Fig. 5 for some low order diagrams.

C. QCD group-theoretic weights

The ease with which QED γ-matrix traces can be summed suggests that QCD group-theoretic weights might be similarly summable. We show that this is indeed the case by considering a generalization of the action of the preceding section.

$$S = -\phi^\dagger\phi - \tfrac{1}{2}A^2 + \phi^\dagger\mathfrak{A}\phi + \text{sources}. \tag{4.24}$$

Here ϕ_a, $a=1,2,\ldots,n$ transforms as a complex

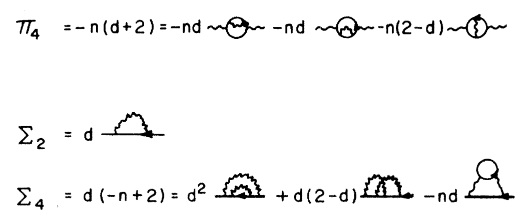

FIG. 5. Sums of traces of QED γ matrices (times −1 for each electron loop): (a) photon self-energy and (b) electron self-energy.

n-dimensional representation of some group G, A_i, $i=1,2,\ldots,n$ is a real n-dimensional representation contained in $n\otimes n$, $(T_i)^a_b$ are the Clebsch-Gordan coefficients for the $n\otimes n\to N$ projection, and $\mathfrak{A}\equiv gT_iA_i$. We have omitted three-gluon couplings because they vanish by the antisymmetry of C_{ijk} unless the group has structure $G=G_c\times G_s$. Any number of such models can be constructed, but they are not physically interesting unless they exhibit a local gauge invariance, a problem beyond the scope of the present paper. For our purposes it will be sufficient to show that one can sum the weights for diagrams without three-gluon couplings. From

$$g\frac{d}{dg}\mathfrak{z}=\phi^\dagger\mathfrak{A}\phi\mathfrak{z} \tag{4.25}$$

and the Dyson-Schwinger equations

$$\begin{aligned}\phi_a\mathfrak{z}&=\eta^\dagger(1+\mathfrak{A})_a\mathfrak{z},\\ A_i\mathfrak{z}&=(J_i+g\phi^\dagger T_i\phi)\mathfrak{z}\end{aligned} \tag{4.26}$$

one obtains

$$\phi^\dagger\phi\mathfrak{z}=\left(n+\eta^\dagger\phi+g\frac{d}{dg}\right)\mathfrak{z}, \tag{4.27}$$

$$\begin{aligned}gZ'&=g\phi^\dagger T_i\phi A_i\mathfrak{z}\,\big|_{J=\eta=0}\\ &=g^2(\phi^\dagger T_i\phi)(\phi^\dagger T_i\phi)\mathfrak{z}\,\big|_{J=\eta=0}\,.\end{aligned} \tag{4.28}$$

The form of the projection operator $(T_i)^a_b(T_i)^c_d$ depends on the choice of the representations n and N, and the symmetry group G. In QCD n is typically a lowest-dimensional representation of some simple Lie group and N is always its adjoint representation. All such projectors (except for E_8) are given in Ref. 8. For example, for SU(n)

$$(T_i)^a_b(T_i)^c_d=\delta^a_d\delta^c_b-\frac{1}{n}\delta^a_b\delta^c_d \tag{4.29}$$

and (4.28) becomes

$$gZ'=g^2\frac{n-1}{n}\left(n+1+g\frac{d}{dg}\right)\left(n+g\frac{d}{dg}\right)Z\,. \tag{4.30}$$

As usual, this gives us an equation for counting connected vacuum bubbles (see Fig. 6):

SU(n):

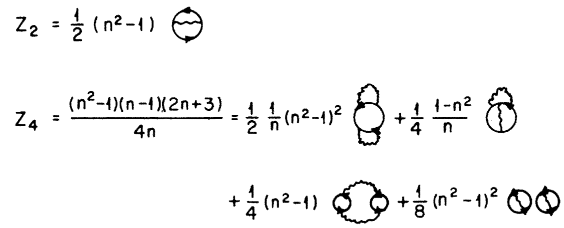

FIG. 6. Combinatoric and group-theoretic weights for vacuum bubbles in SU(n) theory.

$$\begin{aligned}gW'&=g^2\left(n^2-1+2\frac{n^2-1}{n}gW'\right.\\ &\qquad\left.+\frac{n-1}{n}g^2[W''+(W')^2]\right),\end{aligned} \tag{4.31}$$

$$W=\frac{n^2-1}{2}g^2+\frac{(n^2-1)(n-1)(2n+3)}{4n}g^4+\cdots\,.$$

Other Green's functions can be evaluated as in the preceding sections.

V. ASYMPTOTIC ESTIMATES

In the previous sections we have discussed the computation of exact sums for finite orders. In this section we give the behavior for large orders as an asymptotic expansion. All of the Green's functions that we consider can be expressed in the standard form

$$A_k\cong(ak)!A^kk^BC\left(1+\frac{d_1}{k}+\frac{d_2}{k^2}+\cdots\right). \tag{5.1}$$

If a restriction to subset of diagrams affects only $d_1, d_2, \ldots$, we shall refer to it as *insignificant*; otherwise we shall call the change *significant*. Paralleling the previous discussions, we will treat the specific example of ϕ^3 in detail.

There are several different ways to extract the asymptotic behavior. We can begin from the exact answers for the full Green's functions and compute from them the estimates for other Green's functions. Alternatively, we may use the steepest descent method to study the "functional integral." The differential equations of Secs. III and IV offer still another method of arriving at the estimates. The exact result for Z^m_k (3.3) has the Stirling's expansion for large k

$$\begin{aligned}Z^m_k&=\left(\frac{k}{2}\right)!\left(\frac{3}{2}\right)^{k/2}k^{m/2-1}\frac{3^{m/2}}{\pi}\\ &\quad\times\left(1+\frac{m^2/12-5/18}{k}+\cdots\right),\end{aligned} \tag{5.2}$$

The expectation values (2.4) are obtained by dividing out Z. This sort of operation is easily performed for asymptotic series if one remembers that only the terms containing the biggest factorials are important. Removal of vacuum bubbles affects only terms of $1/k$ or smaller:

$$d_1 = -\tfrac{29}{36}, \quad m=1,$$
$$d_1 = \frac{m^2}{12} - \frac{5}{9}, \quad m \geq 2. \tag{5.3}$$

The connected Green's functions (2.7) are related to the expectation values through the cummulant expansion.

$$W^m = \langle \phi^m \rangle - m\langle \phi \rangle \langle \phi^{m-1} \rangle - \cdots \tag{5.4}$$

(which is valid for large enough m). In the present case we obtain for W^m

$$d_1 = \frac{3m^2 - 12m - 20}{36}, \quad m \geq 2. \tag{5.5}$$

The 1PI Green's functions are obtained by amputating external legs and subtracting one-particle reducible graphs. This gives

$$d_1 = \left(-\frac{32}{9}, -\frac{245}{36}, -\frac{89}{9}, \frac{3m^2 - 60m - 20}{36}\right),$$
$$m = (2,3,4,\geq 5), \tag{5.6}$$

An alternative approach which gives some of these results more simply is to begin with the recursion relation satisfied by a Green's function. For example (3.13) can be written for large k as

$$\Phi_{k+2} = \frac{3k+1}{4}\Phi_k + \Phi_1\Phi_k + \Phi_3\Phi_{k-2} + \Phi_5\Phi_{k-3} + \cdots, \tag{5.7}$$

which is a linear equation for Φ_k when $\Phi_1, \Phi_3, \ldots,$ values are substituted. This gives Φ_k up to a constant which must be found independently, as above. Higher m-point functions may then be found from (3.11). This method is very economical for computing higher terms in the asymptotic expansion. For example, we have for the tadpoles

$$\Phi_k \cong \left(\frac{k}{2}\right)! \left(\frac{3}{2}\right)^{k/2} k^{-1/2} \frac{\sqrt{3}}{\pi}$$
$$\times \left(1 - \frac{29}{36k} - \frac{877}{648k^2} - \cdots\right). \tag{5.8}$$

A completely different starting point for obtaining the above results is the steepest descent method. The analysis has been given in many places,[9,10] so we only sketch it here. From

$$Z^m = \frac{1}{\sqrt{2\pi}} \int d\phi\, \phi^m \exp\left(\frac{-\phi^2}{2} + \frac{g\phi^3}{6}\right) \tag{5.9}$$

we have

$$Z_k^m = \frac{-i}{(2\pi)^{3/2}} \oint_C \frac{dg}{g^{k+1}} \int d\phi\, \phi^m \exp\left(\frac{-\phi^2}{2} + \frac{g\phi^3}{6}\right), \tag{5.10}$$

where C is a closed contour enclosing $g=0$. Writing g^{-k} in the exponent gives an effective action for Z_k^m

$$S_{\text{eff}} = -\tfrac{1}{2}\phi^2 + \tfrac{1}{6}g\phi^3 - k\ln g, \tag{5.11}$$

In the combined variables ϕ and g, S_{eff} has a pair of critical points at

$$g_c = \pm 2\sqrt{3k}, \quad \phi_c = \pm\sqrt{3k},$$
$$S_c = S_{\text{eff}}(g_c, \phi_c) = \tfrac{1}{2}k + \tfrac{1}{2}k\ln(3k/4). \tag{5.12}$$

The integral is evaluated by choosing a contour for both the g and ϕ integrals which passes througl both saddle points and expanding S_{eff} to quadratic order about S_c. Higher orders may be computed perturbatively. The result, of course, agrees with (5.2). Since restrictions due to connectivity and reducibility only effect the $O(1/k)$ terms in the asymptotic expansion, this method is useful for giving the leading term. It is also useful in computing the leading terms of the asymptotic expansions of renormalized models or skeletons. Here this is the easiest way to get the constant C in (5.1). For ϕ^3 skeletons we have

$$Z^m = \left(\frac{z_3}{2\pi}\right)^{1/2} \int d\phi \exp\left(-\frac{z_3\phi^2}{2} + \frac{gz_1\phi^3}{6} - J\phi\right)(\sqrt{z_3}\phi)^m, \tag{5.13}$$

where we let z_1, z_2, and J be functions of $g=g_R$ and impose the normalization conditions

$$\Phi = 0,$$
$$D = 1, \tag{5.14}$$
$$\Gamma = g.$$

To lowest order we have

$$\Phi = \tfrac{1}{2}g - J + O(g^2),$$
$$D = z_3^{-1} + \tfrac{1}{2}g^2 + O(g^4), \tag{5.15}$$
$$\Gamma = z_1 g + g^3 + O(g^5),$$

so that

$$z_1 = 1 - g^2 + O(g^4),$$
$$z_2 = 1 + \tfrac{1}{2}g^2 + O(g^4), \tag{5.16}$$
$$J = \tfrac{1}{2}g + O(g^3);$$

comparing with (5.12) the saddle points are now at

$$\phi_c = \pm\sqrt{3k}\,[1 + 7/3k + O(1/k^2)],$$
$$g_c = \pm 2\sqrt{3k}\,[1 + O(1/k^2)], \tag{5.17}$$

which gives

$$S_c = -\tfrac{1}{2}k + \tfrac{1}{2}k\ln(3k/4) - \tfrac{10}{3} + O(1/k). \tag{5.18}$$

It is important to note that the effect of the one-loop counterterms is to change S_c by $-10/3$, to order k^0, and that higher-order corrections change S_c to order $1/k$. Thus the ratio of skeletons to all graphs in leading order is

$$S_k^m / Z_k^m \xrightarrow[k\to\infty]{} e^{-10/3}. \tag{5.19}$$

TABLE I. This table contains the numbers of several interesting types of QED diagrams. A comparison of the first two columns illustrates the effect of Furry's theorem.

Order	Exact electron propagators without Furry's theorem	Exact propagators	Proper self-energies	$\frac{1}{g}\Gamma$, proper vertices	$\frac{1}{g}S^3$, vertex skeletons
2	2	1	1	1	1
4	10	4	3	7	1
6	74	25	18	72	13
8	706	208	153	891	93
10	8 162	2 146	1 638	12 672	1 245
12	110 410	26 368	20 898	202 770	18 093
14	1 708 394	375 733	307 908	3 602 880	308 605
16	29 752 066	6 092 032	5 134 293	70 425 747	5 887 453
18	576 037 442	110 769 550	95 518 278	1 503 484 416	124 221 373
20	12 277 827 850	2 232 792 064	1 967 333 838	34 845 294 582	2 864 305 277
Asymptotic: $\approx (ak)!A^k k^B C(1+d_1/k+d_2/k^2+\cdots)$, where					
a	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
A	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
B	$\frac{1}{2}$	0	0	1	1
C	$\sqrt{2}/\pi$	$2/\pi$	$2/\pi$	$2/\pi$	$2e^{-5/2}/\pi$
d_1	$-\frac{5}{4}$	-1	-3	-5	$-\frac{9}{2}$
d_2	$-\frac{87}{32}$	$-\frac{3}{2}$	$-\frac{9}{2}$	$-\frac{3}{2}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	

Similar analysis yields analogous results for QED, listed in Table I of the Summary.

VI. SUMMARY

In this paper we have shown how to count the number of Feynman diagrams or sum group-theoretic weights by the functional methods. The results fall into two categories. For finite orders we are able to give recursion relations which are very efficient compared to the direct computation from the definitions. The former typically requires k^2 or k^3 operations to find all the Green's functions up to order k while direct computation requires order e^k operations. For large orders we give asymptotic expansions in $1/k$ for the numbers of diagrams. Unlike finite dimensional field theories where determining the higher-order terms can be difficult, here on may easily compute as many terms as needed.

Since they are of some practical interest, we tabulate the numbers of diagrams for QED Green's functions. In each case we give the result for several finite orders and the asymptotic expansion for large orders. Similar results may easily be obtained for the other models discussed here.

Note added in proof. C. Itzykson and J. B. Zuber have also applied a zero-dimensional field theory to QED diagram counting (to be published in their forthcoming book on field theory). We are grateful to Prof. C. Itzykson for informing us of their results.

ACKNOWLEDGMENTS

We are indebted to J. C. Collins and J. Zinn-Justin for suggestions seminal to the above work. The work of P. Cvitanović and R. Pearson was supported by the Department of Energy under Grant No. EY-76-S-02-2220.

*Current address: Niels Bohr Institute, Copenhagen, Denmark.

[1]C. M. Bender and T. T. Wu, Phys. Rev. 184, 1231 (1969).
[2]B. E. Lautrup, Phys. Lett. 69B, 109 (1977).
[3]G. 't Hooft, Erice lectures, 1977 (unpublished).
[4]P. Cvitanović, Nucl. Phys. B127, 176 (1977).
[5]E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).
[6]C. A. Hurst, Proc. R. Soc. London A214, 44 (1952).
[7]B. E. Lautrup, Phys. Lett. 38B, 408 (1972).
[8]P. Cvitanović, Phys. Rev. D 14, 1536 (1976).
[9]L. N. Lipatov, Zh. Eksp. Teor. Fiz. 72, 411 (1977) [Sov. Phys.—JETP 45, 216 (1977)].
[10]E. Brezin, J. G. Le Guillou, and J. Zinn-Justin, Phys. Rev. D 15, 1544 (1977).
[11]See note added in proof.

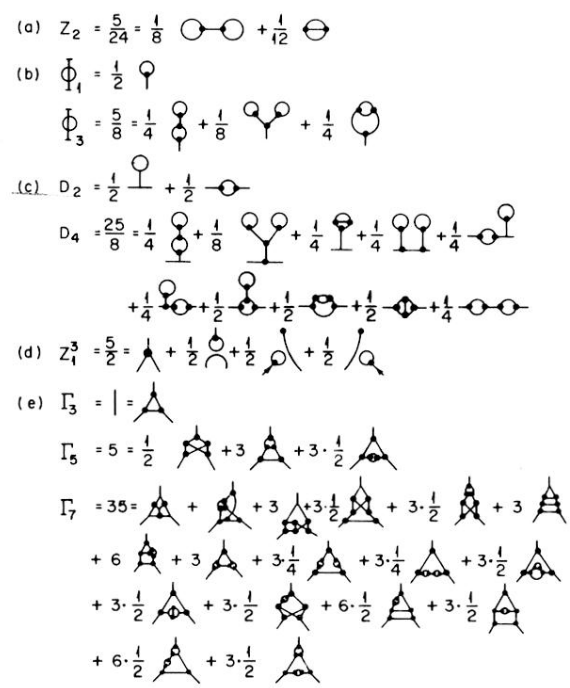

FIG. 1. Combinatoric weights for ϕ^3 diagrams (a) vacuum bubbles, (b) tadpoles, (c) exact propagator, (d) full vertex, and (e) proper vertex (the first factor is the number of diagrams with the same structure).

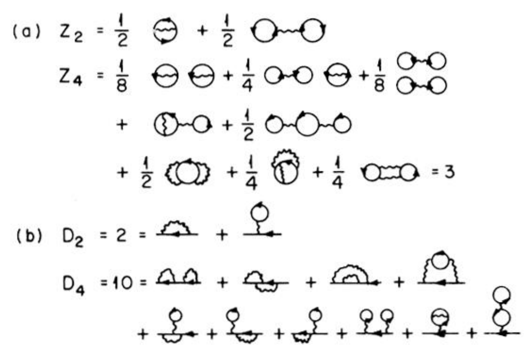

FIG. 2. Combinatoric weights for $\phi^* A \phi$ diagrams; (a) vacuum bubbles and (b) exact "electron" propagator.

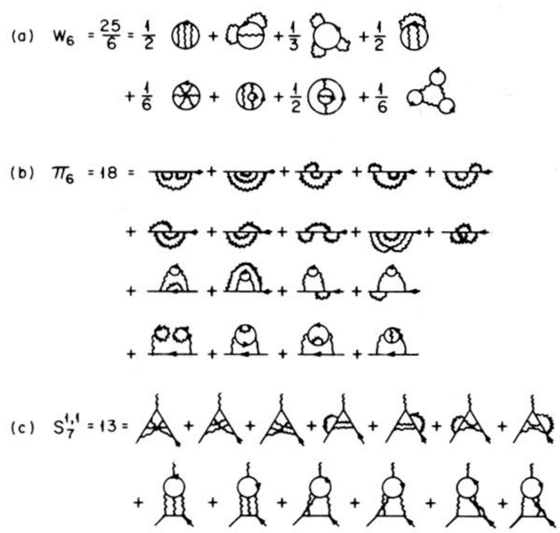

FIG. 3. QED diagrams: (a) connected vacuum bubbles, (b) proper self-energy, and (c) vertex skeletons.

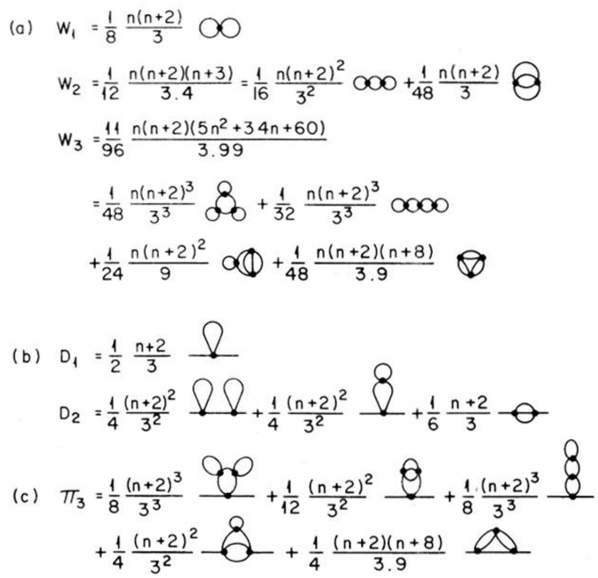

FIG. 4. Combinatoric and group-theoretic weights for $(\phi_i\phi_i)^2$ diagrams: (a) connected vacuum bubbles, (b) exact propagator, and (c) proper self-energy.

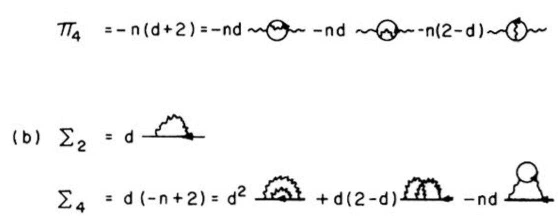

FIG. 5. Sums of traces of QED γ matrices (times -1 for each electron loop): (a) photon self-energy and (b) electron self-energy.

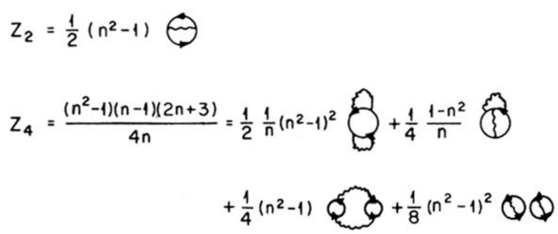

FIG. 6. Combinatoric and group-theoretic weights for vacuum bubbles in SU(n) theory.