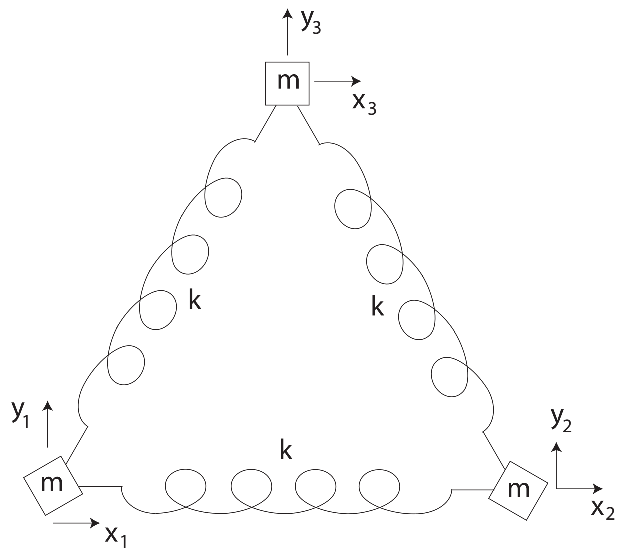

Figure 3.2: The three springs example, showing the coordinate system. Each coordinate pair has its origin at the center of its respective mass in the equilibrium position.

Finally, we use the first orthogonality relation to isolate a particular coefficient, obtaining,

$$a_j = \frac{1}{h} \sum_{k=1}^{n_c} \chi^{(j)*}(C_k)\chi(C_k)N_k. \tag{3.91}$$

3.10 Example Application

For our first example of a physical application, we consider an arrangement of springs and masses which have a particular symmetry in the equilibrium position. We'll consider here the case of an equilateral triangle, expanding on the example in Mathews & Walker chapter 14.

Suppose that we have a system of three equal masses, m, located (in equilibrium) at the vertices of an equilateral triangle. The three masses are connected by three identical springs of strength k. See Fig. 3.2. The question we wish to answer is: If the system is constrained to move in a plane, what are the normal modes? We'll use group theory to analyze what happens when a normal mode is excited, potentially breaking the equilateral triangular symmetry to some lower symmetry.

Let the coordinates of each mass, relative to the equilibrium position, be $x_i, y_i, i = 1, 2, 3$. The state of the system is given by the 6-dimensional vector:

$\eta = (x_1, y_1, x_2, y_2, x_3, y_3)$, as a function of time. The kinetic energy is:

$$T = \frac{m}{2}\sum_{i=1}^{6} \dot{\eta}_i^2. \tag{3.92}$$

Likewise, the potential energy, for small perturbations about equilibrium, is given by:

$$V = \frac{k}{2}\left\{(x_2 - x_1)^2 + \left[-\frac{1}{2}(x_3 - x_2) + \frac{\sqrt{3}}{2}(y_3 - y_2)\right]^2 + \left[\frac{1}{2}(x_1 - x_3) + \frac{\sqrt{3}}{2}(y_1 - y_3)\right]^2\right\}. \tag{3.93}$$

Or, we may write:

$$V = \frac{k}{2}\sum_{i,j=1}^{6} U_{ij}\eta_i\eta_j, \tag{3.94}$$

where

$$U = \frac{1}{4}\begin{pmatrix} 5 & \sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ -4 & 0 & 5 & -\sqrt{3} & -1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 6 \end{pmatrix}. \tag{3.95}$$

The equations of motion ($F = ma$) are:

$$m\ddot{\eta}_i = -\frac{\partial V}{\partial \eta_i} = -k\sum_{j=1}^{6} U_{ij}\eta_j. \tag{3.96}$$

In a normal mode,

$$\eta = \mathbf{A}e^{i\omega t}, \tag{3.97}$$

where $\mathbf{A}$ is a constant 6-vector, and hence,

$$-m\omega^2\eta_i = -k\sum_{j=1}^{6} U_{ij}\eta_j, \tag{3.98}$$

or,

$$\sum_{j=1}^{6} U_{ij}\eta_j = \lambda\eta_i, \quad \text{where } \lambda = \frac{m\omega^2}{k}. \tag{3.99}$$

That is, the normal modes are the eigenvectors of U, with frequencies given in terms of the eigenvalues. In principle, we need to solve the secular equation $|U - \lambda I| = 0$, a sixth-order polynomial equation, in order to get the eigenvalues. Let's see how group theory can help make this tractable, by incorporating the symmetry of the system.

Each eigenvector "generates" an irreducible representation when acted upon by elements of the symmetry group. Consider a coordinate system in which U is diagonal (such a coordinate system must exist, since U is Hermitian):

$$U = \begin{pmatrix} \lambda_a & & & & & & \\ & \ddots & & & & & \\ & & \lambda_a & & & & \\ & & & \lambda_b & & & \\ & & & & \ddots & & \\ & & & & & \lambda_b & \\ & & & & & & \ddots \end{pmatrix}, \tag{3.100}$$

where the first n_a coordinate vectors in this basis belong to eigenvalue λ_a, and transform among themselves according to irreducible representation $D^{(a)}$, and so forth.

What is the appropriate symmetry group? Well, it must be the group, D_3, of operations which leaves an equilateral triangle invariant. This group is generated by taking products of a rotation by $2\pi/3$, which we will call R, and a reflection about the y-axis, which we will call P. The entire group is then given by the 6 elements $\{e, R, R^2, P, PR, PR^2\}$. Note that this group is isomorphic with the group of permutations of three objects, S_3. The classes are:

$$\{e\}, \{R, R^2\}, \{P, PR, PR^2\}. \tag{3.101}$$

As there are three classes, there must be three irreducible representations, and hence their dimensions must be 1, 1, and 2. Thus, we can easily construct the character table in Table 3.3.

The first row is given by the dimensions of the irreps, since these are the traces of the identity matrices in those irreps. The first column is all ones, since this is the trivial irrep where every element of D_3 is represented by the number 1. The second and third row of the second column may be obtained by orthogonality with the first row (remembering the N_k weights), noticing that in a one-dimensional representation the traces are the same as the representation. In particular, the representation of R must be a cube root of one, and the representation of P must be a square root of one. Finally, the second and third rows of the final column are readily determined using the orthogonality relations. Note that in this example, we don't actually need to construct the non-trivial representations to determine the character table. In general, it may be necessary to construct a few of the matrices explicitly.

There is a 6-dimensional representation of D_3 which acts on our 6-dimensional coordinate space. We wish to decompose this representation into irreducible representations (why? because that will provide a breakdown of the normal modes by their symmetry under D_3). It is sufficient to know the characters, which we obtain by explicitly considering the action of one element from each class.

Clearly, $\eta = D(e)\eta$, hence $D(e)$ is the 6×6 identity matrix. Its character is $\chi(C_1) = 6$.

Table 3.3: Character table for D_3.

N_k	$\ell_i \rightarrow$ class $\downarrow$; irrep $\rightarrow$	$\ell_i = 1$ $\chi^{(1)}$	$\ell_2 = 1$ $\chi^{(2)}$	$\ell_3 = 2$ $\chi^{(3)}$
1	$\{e\}$	1	1	2
2	$\{R, R^2\}$	1	1	-1
3	$\{P, PR, PR^2\}$	1	-1	0

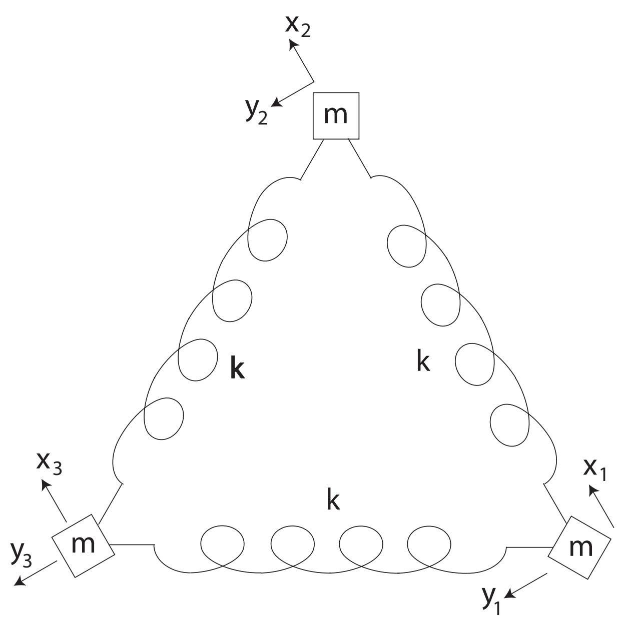

Figure 3.3: The three springs example, showing result of a rotation by $2\pi/3$.

Now consider a rotation by $2\pi/3$, see Fig. 3.3. The 6×6 matrix representing this rotation is:

$$D(R) = \begin{pmatrix} 0 & 0 & r \\ r & 0 & 0 \\ 0 & r & 0 \end{pmatrix}, \tag{3.102}$$

where r is the 2×2 rotation matrix:

$$r = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}. \tag{3.103}$$

We see that the trace is zero, that is $\chi(C_2) = 0$.

The action of P is to interchange masses 1 and 2, and reflect the x coordinates:

$$D(P) = \begin{pmatrix} 0 & p & 0 \\ p & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \tag{3.104}$$

where p is the 2×2 reflection matrix:

$$p = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.105}$$

We see that the trace is again zero, that is $\chi(C_3) = 0$.

With these characters, we are now ready to decompose D into the irreps of D_3. We wish to find the coefficients a_1, a_2, a_3 in:

$$D = a_1 D^{(1)} \oplus a_2 D^{(2)} \oplus a_3 D^{(3)}. \tag{3.106}$$

They are given by:

$$a_j = \frac{1}{h} \sum_{k=1}^{n_c} N_k \chi^{(j)*}(C_k) \chi(C_k). \tag{3.107}$$

The result is:

$$\begin{aligned} a_1 &= \frac{1}{6}(1 \cdot 1 \cdot 6 + 2 \cdot 1 \cdot 0 + 3 \cdot 1 \cdot 0) = 1 \\ a_2 &= \frac{1}{6}(1 \cdot 1 \cdot 6 + 2 \cdot 1 \cdot 0 + 3 \cdot -1 \cdot 0) = 1 \\ a_3 &= \frac{1}{6}(1 \cdot 2 \cdot 6 + 2 \cdot -1 \cdot 0 + 3 \cdot 0 \cdot 0) = 2. \end{aligned} \tag{3.108}$$

That is,

$$D = D^{(1)} \oplus D^{(2)} \oplus 2D^{(3)}. \tag{3.109}$$

In the basis corresponding to the eigenvalues we thus have:

$$U = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & 0 & \\ & & \lambda_{31} & & & \\ & & & \lambda_{31} & & \\ & 0 & & & \lambda_{32} & \\ & & & & & \lambda_{32} \end{pmatrix}, \tag{3.110}$$

where λ_1 corresponds to $D^{(1)}$, λ_2 to $D^{(2)}$, and $\lambda_{31}, \lambda_{32}$ to two instances of $D^{(3)}$. Thus, we already know that there are no more than four distinct eigenvalues, that is, some of the six modes have the same frequency.

Let's see that we can find the actual frequencies without too much further work. Consider $D(g)U$ in this diagonal coordinate system. In this basis we must have:

$$D(g) = \begin{pmatrix} D^{(1)}(g) & & & \\ & D^{(2)}(g) & 0 & \\ & 0 & D^{(3)}(g) & \\ & & & D^{(3)}(g) \end{pmatrix}, \tag{3.111}$$

and hence,

$$D(g)U = \begin{pmatrix} \lambda_1 D^{(1)}(g) & & & \\ & \lambda_2 D^{(2)}(g) & & 0 \\ & & \lambda_{31} D^{(3)}(g) & \\ & 0 & & \lambda_{32} D^{(3)}(g) \end{pmatrix}. \tag{3.112}$$

We don't know what this coordinate system is, but we may consider quantities which are independent of coordinate system, such as the trace:

$$\mathrm{Tr}\,[D(g)U] = \lambda_1 \chi^{(1)}(g) + \lambda_2 \chi^{(2)}(g) + (\lambda_{31} + \lambda_{32})\chi^{(3)}(g). \tag{3.113}$$

Referring to Eqn. 3.95 we find, for $g = e$:

$$\mathrm{Tr}\,[D(e)U] = \mathrm{Tr}\,U = \frac{1}{4}(5 + 3 + 5 + 3 + 2 + 6) = 6. \tag{3.114}$$

For $g = R$:

$$\begin{aligned} \mathrm{Tr}\,[D(R)U] &= \mathrm{Tr}\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & \sqrt{3} & -1 \\ -1 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ \sqrt{3} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & -1 & 0 & 0 \end{pmatrix} \times \\ & \quad \frac{1}{4}\begin{pmatrix} 5 & \sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ -4 & 0 & 5 & -\sqrt{3} & -1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 6 \end{pmatrix} \\ &= \frac{1}{8}(1 + 3 - 3 + 3 + 4 + 0 + 1 - 3 + 3 + 3) = \frac{3}{2} \end{aligned} \tag{3.115}$$

For $g = P$:

$$\begin{aligned} \mathrm{Tr}\,[D(P)U] &= \mathrm{Tr}\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{4}\begin{pmatrix} 5 & \sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ -4 & 0 & 5 & -\sqrt{3} & -1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 6 \end{pmatrix} \\ &= \frac{1}{4}(4 + 0 + 4 + 0 - 2 + 6) = 3. \end{aligned} \tag{3.116}$$

This gives us the three equations:

$$\begin{aligned} 6 &= \lambda_1 + \lambda_2 + 2(\lambda_{31} + \lambda_{32}) \\ \frac{3}{2} &= \lambda_1 + \lambda_2 - (\lambda_{31} + \lambda_{32}) \\ 3 &= \lambda_1 - \lambda_2. \end{aligned} \tag{3.117}$$

Hence,

$$\lambda_1 = 3 \tag{3.118}$$
$$\lambda_2 = 0 \tag{3.119}$$
$$\lambda_{31} + \lambda_{32} = \frac{3}{2}. \tag{3.120}$$

To determine λ_{31} and λ_{32}, we could consider another invariant, such as

$$\mathrm{Tr}U^2 = \lambda_1^2 + \lambda_2^2 + 2\left(\lambda_{31}^2 + \lambda_{32}^2\right). \tag{3.121}$$

Alternatively, we may use some physical insight: There must be three degrees of freedom with eigenvalue 0, corresponding to an overall rotation of the system and overall translation of the system in two directions. Thus, choose $\lambda_{31} = 0$ and then $\lambda_{32} = 3/2$.

The frequencies are $\omega = \sqrt{\lambda k/m}$. The highest frequency is $\omega = \sqrt{3k/m}$, corresponding to the "breathing mode" in which the springs all expand or contract in unison. Note that this is the mode corresponding to the identity representation; the symmetry of the triangle is not broken in this mode.

3.11 Another example

Let us consider another simple example (again an expanded discussion of an example in Mathews & Walker, chapter 16), to try to get a more intuitive picture of the connection between eigenfunctions and irreducible representations:

Consider a square "drumhead", and the connection of its vibrational modes with representations of the symmetry group of the square. We note that two eigenfunctions which are related by a symmetry of the square must have the same eigenvalue – otherwise this would not be a symmetry. The symmetry group of the square (see Fig. 3.4) is generated by a 4-fold axis, plus mirror planes joining the sides and vertices.

This group has the elements:

$$\{e, M_a, M_b, M_\alpha, M_\beta, R_{\pm\pi/2}, R_\pi\}. \tag{3.122}$$

Thus, the order is $h = 8$. The classes are readily seen to be:

$$\begin{aligned} C_1 &= \{e\} \\ C_2 &= \{M_a, M_b\} \\ C_3 &= \{M_\alpha, M_\beta\} \\ C_4 &= \{R_\pi\} \\ C_5 &= \{R_{\pi/2}, R_{-\pi/2}\} \end{aligned} \tag{3.123}$$

We must have $\sum_{i=1}^{n_r} \ell_i^2 = 8$, but $n_r = 5$, and therefore $\ell_1 = \ell_2 = \ell_3 = \ell_4 = 1$, and $\ell_5 = 2$ are the dimensions of the irreducible representations.