mathematical methods - week 11

Group theory

Georgia Tech PHYS-6124

Homework HW #11 due Thursday, October 7, 2014

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

Exercise 11.1 *Decompose a representation of* S_3
(a) 2; (b) 2; (c) 3; and (d) 3 points
(e) 2 and (f) 3 points bonus points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2014-10-28 Predrag Lecture 20 Finite groups
Character orthogonality theorem.

2014-10-30 Predrag Lecture 21 Characters
Reading: Tinkham.

Exercises

11.1. **Decompose a representation of S_3.** Consider a reducible representation $D(g)$, i.e., a representation of group element g that after a suitable similarity transformation takes form

$$D(g) = \begin{pmatrix} D^{(a)}(g) & 0 & 0 & 0 \\ 0 & D^{(b)}(g) & 0 & 0 \\ 0 & 0 & D^{(c)}(g) & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix},$$

with character for class $\mathcal{C}$ given by

$$\chi(\mathcal{C}) = c_a \, \chi^{(a)}(\mathcal{C}) + c_b \, \chi^{(b)}(\mathcal{C}) + c_c \, \chi^{(c)}(\mathcal{C}) + \cdots ,$$

where c_a, the multiplicity of the ath irreducible representation (colloquially called "irrep"), is determined by the character orthonormality relations,

$$c_a = \overline{\chi^{(a)*} \, \chi} = \frac{1}{h} \sum_k^{class} N_k \chi^{(a)}(\mathcal{C}_k^{-1}) \, \chi(\mathcal{C}_k) . \qquad (11.1)$$

Knowing characters is all that is needed to figure out what any reducible representation decomposes into!

As an example, let's work out the reduction of the matrix representation of S_3 permutations. The identity element acting on three objects $[a \, b \, c]$ is a 3×3 identity matrix,

$$D(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transposing the first and second object yields $[b \; a \; c]$, represented by the matrix

$$D(A) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

since

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a \\ c \end{pmatrix}$$

(a) Find all six matrices for this representation.

(b) Split this representation into its conjugacy classes.

(c) Evaluate the characters χ_i for this representation.

(d) Determine multiplicities c_a of irreps contained in this representation.

(e) (bonus) Construct explicitly all irreps.

(f) (bonus) Explain whether any irreps are missing in this decomposition, and why.