Georgia Tech PHYS 6124

Mathematical Methods of Physics I

Instructor: Predrag Cvitanović
Fall semester 2012

Homework Set #3

due Tue, Sept 18 2012

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

[problems from Stone and Goldbart]

Exercise 2.20 Test functions and distributions

Let $f(x)$ be a continuous function. Show that $f(x)\delta(x) = f(0)\delta(x)$. Deduce that

$$\frac{d}{dx}[f(x)\delta(x)] = f(0)\delta'(x)\,.$$

If $f(x)$ were differentiable we might also have used the product rule to conclude that

$$\frac{d}{dx}[f(x)\delta(x)] = f'(0)\delta(x) + f(x)\delta'(x)\,.$$

Show, by integrating both against a test function, that the two expressions for the derivative of $f(x)\delta(x)$ are equivalent.

Exercise 2.21 Let $\phi(x)$ be a test function...

Using the definition of the principal part integrals, show that

$$\frac{d}{dt}\left\{P\int_{-\infty}^{\infty}\frac{\phi(x)}{(x-t)}dx\right\} = P\int_{-\infty}^{\infty}\frac{\phi(x)-\phi(t)}{(x-t)^2}dx$$

in two different ways:

1. Fix the value of the cutoff ϵ. Differentiate the resulting ϵ-regulated integral, taking care to include the terms arising from the t dependence of the limits at $x = t \pm \epsilon$.

2. First make a change of variables $y = x - t$, so that the singularity is fixed at $y = 0$. Now differentiate with respect to t. Next integrate by parts to take the derivative off ϕ and onto the singular factor. (Take care to include the boundary contributions.) Finally change back to the original x, t variables.

Both methods should give the same result!

ChaosBook Exercise 16.1 (a) Integrating over Dirac delta functions

Check the delta function integrals in 1 dimension,

$$\int dx\, \delta(h(x)) = \sum_{\{x:h(x)=0\}} \frac{1}{|h'(x)|}\,, \tag{1}$$

and in d dimensions, $h : \mathbb{R}^d \to \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} dx\, \delta(h(x)) \;=\; \sum_j \int_{\mathcal{M}_j} dx\, \delta(h(x)) \;=\; \sum_{\{x:h(x)=0\}} \frac{1}{\left|\det \frac{\partial h(x)}{\partial x}\right|}\,. \tag{2}$$

where $\mathcal{M}_j$ are arbitrarily small regions enclosing the zeros x_j (with x_j not on the boundary $\partial\mathcal{M}_j$). For a refresher on Jacobian determinants, read, for example, Stone and Goldbart Sect. 12.2.2.

Optional problems

ChaosBook Exercise 16.1 (b) Integrating over $\delta(x^2)$

The delta function can be approximated by a sequence of Gaussians

$$\int dx\,\delta(x)f(x) = \lim_{\sigma\to 0}\int dx\,\frac{e^{-\frac{x^2}{2\sigma}}}{\sqrt{2\pi\sigma}}f(x)\,.$$

Use this approximation to see whether the formal expression

$$\int_{\mathbb{R}} dx\,\delta(x^2)$$

makes sense.

ChaosBook Exercise 16.2 Derivatives of Dirac delta functions

Consider $\delta^{(k)}(x) = \frac{\partial^k}{\partial x^k}\delta(x)\,.$
Using integration by parts, determine the value of

$$\int_{\mathbb{R}} dx\,\delta'(y) \quad , \qquad \text{where } y = f(x) - x \tag{3}$$

$$\int dx\,\delta^{(2)}(y) = \sum_{\{x:y(x)=0\}} \frac{1}{|y'|}\left\{3\frac{(y'')^2}{(y')^4} - \frac{y'''}{(y')^3}\right\} \tag{4}$$

$$\begin{aligned}\int dx\,b(x)\delta^{(2)}(y) &= \sum_{\{x:y(x)=0\}} \frac{1}{|y'|}\left\{\frac{b''}{(y')^2} - \frac{b'y''}{(y')^3}\right. \\ &\qquad \left. + b\left(3\frac{(y'')^2}{(y')^4} - \frac{y'''}{(y')^3}\right)\right\}\,.\end{aligned} \tag{5}$$