Rescaling in Perturbation Theory

Example: $\varepsilon x^2 - 1 = 0$

Unperturbed equation: $1 = 0$ – no solutions

$\Rightarrow$ Don't know the leading order of both solutions

$x = \delta y + \dots$: $x = \delta y \Rightarrow \varepsilon \delta^2 y^2 - 1 = 0$

Dominant balance: $\varepsilon \delta^2 = 1 \Rightarrow \delta = \frac{1}{\sqrt{\varepsilon}} \Rightarrow y^2 - 1 = 0$

$\Rightarrow y = \pm 1 \Rightarrow x = \delta y = \pm \frac{1}{\sqrt{\varepsilon}}$

Example: $\varepsilon x^2 + x - 1 = 0$

Unperturbed equation: $x - 1 = 0 \Rightarrow$ one solution: $x = 1$

$\Rightarrow$ Don't know the leading order behavior of one of the solutions

$x = \delta y + \dots$: $x = \delta y \Rightarrow \varepsilon \delta^2 y^2 + \delta y - 1 = 0$

Dominant balance:

$$\varepsilon \delta^2 = \delta \Rightarrow \delta = \frac{1}{\varepsilon} \Rightarrow \frac{1}{\varepsilon} y^2 + \frac{1}{\varepsilon} y - 1 = \frac{1}{\varepsilon}(y^2 + y - \varepsilon) = 0 \Rightarrow \begin{cases} y = 0 + \dots \\ y = -1 + \dots \end{cases}$$

$$\Rightarrow x = \delta y = -\frac{1}{\varepsilon} + \dots$$

Example: $-\varepsilon x^2 + x^2 - 2x + 1 = 0$

Unperturbed equation: $x^2 + 2x + 1 = 0 \Rightarrow$ two solutions: $x_{1,2} = +1$

- Degeneracy $\Rightarrow$ expect nontrivial scaling of corrections to leading order. Since we don't know the behavior of subleading terms

$\Rightarrow x = \underset{\substack{\uparrow \\ \text{solution of unperturbed eq.}}}{1} + \delta y + \dots$: $x = \delta y \Rightarrow$

$$-\varepsilon(1 + 2\delta y + \delta^2 y^2) + (\cancel{1} + \cancel{2\delta y} + \delta^2 y^2) - 2(\cancel{1 + \delta y}) + \cancel{1} = \overset{\approx 1}{(1-\varepsilon)}\delta^2 y^2 - 2\varepsilon \delta y - \varepsilon = 0$$

Dominant balance:

$$\delta^2 = \varepsilon \Rightarrow \delta = \sqrt{\varepsilon} \Rightarrow \varepsilon y^2 - \cancel{2\varepsilon^{3/2} y} - \varepsilon = 0 \Rightarrow y = \pm 1$$

$$\Rightarrow x = 1 + \delta y = 1 \pm \sqrt{\varepsilon} + \dots$$

Perturbation Theory for Degenerate Matrices

$$(A+\varepsilon B)\vec{x} = \lambda \vec{x}$$

$$\left.\begin{aligned} \lambda_i &= \lambda_i^{(0)} + \varepsilon \lambda_i^{(1)} + \cdots \\ \vec{x}_i &= \vec{x}_i^{(0)} + \varepsilon \vec{x}_i^{(1)} + \cdots \end{aligned}\right\} \quad (A+\varepsilon B)(\vec{x}_i^{(0)} + \varepsilon \vec{x}_i^{(1)} + \cdots) = (\lambda_i^{(0)} + \varepsilon \lambda_i^{(1)} + \cdots)(\vec{x}_i^{(0)} + \varepsilon \vec{x}_i^{(1)} + \cdots)$$

$\underline{\underline{\varepsilon^0}}$: $A\vec{x}_i^{(0)} = \lambda_i^{(0)} \vec{x}_i^{(0)}$

$\Rightarrow \lambda_i^{(0)} = a_i$ – eigenvalue of A, $\vec{x}_i^{(0)}$ – <u>some</u> eigenvector, which corresponds to a_i

We have <u>not</u> determined $\vec{x}_i^{(0)}$, if $\lambda_i^{(0)}$ is degenerate.

If $\lambda_i^{(0)}$ is n-times degenerate, $\vec{x}_i^{(0)} = \alpha_1^i \vec{e}_1 + \alpha_2^i \vec{e}_2 + \cdots \alpha_n^i \vec{e}_n$

Coefficients α_i are <u>not</u> determined by 0^{th} order equations, they are determined by higher order equations!

$\underline{\underline{\varepsilon^1}}$: $A\vec{x}_i^{(1)} + B\vec{x}_i^{(0)} = \underset{\overset{\parallel}{a_i}}{\lambda_i^{(0)}} \vec{x}_i^{(1)} + \lambda_i^{(1)} \vec{x}_i^{(0)}$

Multiply by $\vec{f}_j$ on the left: $a_j(\vec{f}_j \cdot \vec{x}_i^{(1)}) + (\vec{f}_j \cdot B\vec{x}_i^{(0)}) = a_i(\vec{f}_j \cdot \vec{x}_i^{(1)}) + \lambda_i^{(1)} \vec{x}_i^{(0)}$

a) $j = 1, \ldots, n$ $\rightarrow$ n equations for eigenvalue correction $\lambda_i^{(1)}$

b) $j \neq n+1, \ldots, N$ $\Rightarrow$ $N-n$ equations for eigenvector correction $\vec{x}_i^{(1)}$

a) $(\vec{f}_j \cdot B\vec{x}_i^{(0)}) = \sum_{k=1}^{n} \alpha_k^i \underbrace{(\vec{f}_j \cdot B\vec{e}_k)}_{\tilde{B}_{jk}} = \lambda_i^{(1)} \sum_{k=1}^{n} \alpha_k^i \underbrace{(\vec{f}_j \cdot \vec{e}_k)}_{\delta_{jk}} \Rightarrow \tilde{B}\vec{\alpha}^i = \lambda_i^{(1)} \vec{\alpha}^i$

If $\tilde{B}$ is non-degenerate, this eigenproblem determines both $\lambda_i^{(1)}$ (1^{st} order term in eigenvalue expansion) and $\vec{\alpha}^i = (\alpha_1^i, \ldots, \alpha_n^i)$ (0^{th} order term in eigenvector expansion)

If $\tilde{B}$ is also degenerate, α_k^i might (or might not be) determined by higher orders of pert. theory

Example:
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\vec{f}_1 = \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{f}_2 = \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \lambda_{1,2}^{(0)} = 1$$

$$\vec{x}_i^{(0)} = \alpha_1^i \vec{e}_1 + \alpha_2^i \vec{e}_2 ; \quad \tilde{B}_{jk} = \vec{f}_j \circ B \vec{e}_k = \delta_{jk} = B \text{ - degenerate!}$$

$$\Rightarrow \lambda_i^{(1)} = 1, \quad \alpha_{1,2}^i \text{ - arbitrary}$$

(Solve exactly: $\lambda_{1,2} = 1 + \varepsilon, \quad \vec{x}_i = \alpha_1^i \vec{e}_1 + \alpha_2^i \vec{e}_2, \quad \forall \alpha_{1,2}^i$)

Example:
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\tilde{B}_{jk} = \vec{f}_j \cdot B \vec{e}_k = B_{jk} \Rightarrow \lambda_i^{(1)^2} - 1 = 0 \Rightarrow \lambda_i^{(1)} = \pm 1 \text{ - non-degen.!}$$

$$\lambda_i^{(1)} = +1 : \quad \vec{\alpha} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_i^{(0)} = \vec{e}_1 + \vec{e}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_i^{(1)} = -1 : \quad \vec{\alpha} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \vec{x}_i^{(0)} = \vec{e}_1 - \vec{e}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$