Georgia Tech PHYS 6124
Mathematical Methods of Physics I

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Homework Set #4

due Sept 20 2011, in class

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

[All problems in this set are from Goldbart, go bug him]

Problem 1) Unique solutions – a matrix caricature

The purpose of this problem is to provide a matrix caricature of the notion in differential equations that to obtain a unique solution one must search for functions from the appropriate collection. (In other words, one must choose appropriate boundary conditions for differential equations.)

Consider the following simultaneous equations for the three unknown vector components x, y and z:

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\alpha \\ \beta + \gamma \\ \beta - \gamma \end{pmatrix},$$

where we have, without loss of generality, parametrised the arbitrary components of the *source* vector (or *inhomogeneous*) term in terms of the three independent real variables α, β and γ.

a) For arbitrary α, β and γ, can a solution (x, y, z) be found? Explain why?

b) For arbitrary α and β, but $\gamma = 0$, explain if a solution (x, y, z) be found? Is it unique?

c) For arbitrary α and β, but $\gamma = 0$, can a solution (x, y, z) be found from the restricted collection of vectors which are orthogonal to $(0, 1, -1)$? If so, is it unique?

Problem 2) Cauchy-Lipschitz solution to an ordinary differential equation

Consider the second-order ordinary differential equation

$$\frac{d^2}{dt^2}u(t) = -\kappa^2 u(t)\,,$$

the general solution of which you already know to be

$$u(t) = A\cos\kappa t + B\sin\kappa t\,.$$

Confirm that this is indeed a solution by using the Cauchy construction and the initial conditions $u(0) = A$, $du/dt|_{t=0} = B\kappa$.
[Note: the Cauchy construction entails constructing a Taylor series for the solution of an ordinary differential equation by using the initial conditions, the equation itself, and derivatives of the equation, thereby to build the coefficients of the Taylor series.]

Problem 5) Diffusion equation with initial conditions

Consider the diffusion equation in three space dimensions,

$$\frac{\partial}{\partial t}u(\mathbf{r},t) = D\,\nabla^2 u(\mathbf{r},t).$$

By making a Laplace transform with respect to the time variable t (or otherwise) and a Fourier transform with respect to the spatial variable $\mathbf{r}$, find an integral expression for the solution $u(\mathbf{r},t)$ of the diffusion equation subject to the initial condition $u(\mathbf{r},t)|_{t=0} = \bar{u}(\mathbf{r})$.

Problem 6) Integrating factors

Consider the general linear inhomogeneous first order ordinary differential equation $y' + r(x)\,y = s(x)$.

a) By multiplying the entire ordinary differential equation by a suitable function show that it can be written in the form $d\big(t(x)y\big)/dx = u(x)$. Hence, obtain the general solution.

b) Apply this method to obtain the explicit solution when $r(x) = -1/x$ and $s(x) = 1$, with boundary condition $y(\mathrm{e}) = \mathrm{e}$.

Optional problems

Problem 3) Wronskian determinant

(Professor Goldbart is incredulous at my lack of appreciation for Wronskians, so you'll make him happy if you do this one.)

Given a set of n functions $\{z_j(x)\}_{j=1}^{n}$ of a single variable x, the *Wronskian determinant* (or simply Wronskian) $W(x)$ is defined to be

$$W(x) = \begin{vmatrix} z_1 & z_2 & \dots & z_n \\ z_1^{(1)} & z_2^{(1)} & \dots & z_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{(n-1)} & z_2^{(n-1)} & \dots & z_n^{(n-1)} \end{vmatrix},$$

where $z^{(j)}$ denotes $d^j z/dx^j$, $z^{(0)} \equiv z$ and the vertical bars $|\cdots|$ indicate the determinant of the enclosed matrix. If $W(x)$ does not vanish for any x, or vanishes only at an isolated set of values of x, then the set of functions $\{z_j\}$ is linearly independent.

Consider the general n^{th}-order, linear, homogeneous ordinary differential equation,

$$y^{(n)} + p_{n-1}\, y^{(n-1)} + \cdots + p_1\, y^{(1)} + p_0\, y^{(0)} = 0,$$

where the coefficients p_j may depend on x, and suppose that you have constructed a set of n solutions $\{y_j(x)\}_{j=1}^{n}$. By using the following expression for the determinant of an $n \times n$ matrix $A_{i,j}$:

$$\det A = \epsilon_{i_1 i_2 \dots i_n} A_{1,i_1} A_{2,i_2} \dots A_{n,i_n}$$

(in which summation convention is implied, and $\epsilon_{i_1 i_2 \dots i_n}$ is the n-dimensional generalisation of ϵ_{ijk}) show that $W(x)$ for the set of solutions $\{y_j\}$ satisfies the *first-order* ordinary differential equation

$$\frac{d}{dx} W(x) = -p_{n-1}(x)\, W(x),$$

and find the general solution of this ordinary differential equation. Note the striking fact that the Wronskian is simply calculable even when the equation is not readily solvable!

Problem 4) Eliminating constants of integration

Find a differential equation having the general solution $y = c_1(x + c_2)^n$, where c_1 and c_2 are arbitrary constants (but n is presumed fixed).

Problem 7) Classification and characteristics

Classify according to type, and determine the characteristic curves of the following partial differential equations. For parts (a)–(c) discuss the ability of the given boundary information to produce a unique stable solution.

a) $2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0$ with Cauchy boundary conditions on the entire line $x + y = 0$.

b) $4u_{xx} + 12u_{xy} + 9u_{yy} - 2u_x + u = 0$ with Dirichlet boundary conditions on the lines $2y = 3x$ (for $0 \le x \le 2$), $3y = -2x$ (for $x \le 0$) and $3y = -2x + 13$ (for $x \le 2$).

c) $x^{-3}\, u_{xx} - x^3\, u_{yy} = 0$ with Cauchy boundary conditions on the curve $y = 2x^2$ (for $-1 \le x \le 3$).

d) $\mathrm{e}^{2x} u_{xx} + 2\mathrm{e}^{x+y} u_{xy} + \mathrm{e}^{2y} u_{yy} = 0$.

Problem 8) Tricomi's partial differential equation

Tricomi's equation for $u(x, y)$ is a second-order, linear homogeneous partial differential equation having the dimensionless form

$$y\,\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,.$$

a) Sketch the xy–plane indicating regions in which the equation is elliptic, parabolic and hyperbolic.

b) Deduce and sketch the characteristic curves for $y < 0$.

c) For $y < 0$, introduce new independent variables, the so-called *natural variables* ξ and η, which are constant along a characteristic curves. Derive Tricomi's equation in terms of these natural variables.

(For an application of Tricomi's equation to physics, see, *e.g.*, Landau and Lifshitz, *Fluid Mechanics*, §118.)

Problem 9) Mediæval Behaviour

A man stands atop a mountain whose altitude is given by $z = \exp\big(-x^2 - 4y^2\big)$ and pours boiling oil upon the climbers below him. What paths do the rivulets of oil follow? [Assume that the paths are orthogonal to the contour lines of the mountain.]