3/10

Properties of Irreducible Representations (irreps)

As we have discussed a representation of a finite (or compact) group is

- either irreducible
- or it can be cast by a suitable similarity transformation into block-diagonal form with irreducible blocks

There are two important results involving irreps, known as Schur's lemmas

i Any matrix that commutes with all the matrices of an irrep must be proportional to the identity matrix

$$\text{i.e., if } \quad BD(g) = D(g)B \quad \forall g\in G \quad \text{then} \quad B = \underset{\text{proportionality constant}}{\lambda}\overset{\text{identity matrix}}{I}$$

ii If, for two inequivalent reps $\underset{n\times n}{D}$ and $\underset{m\times m}{D'}$ the matrix $\overset{m\times n}{B}$ obeys

$$\overbrace{B\,D(g)}^{(m\times n)\ (n\times n)\ \to\ m\times n} = \underbrace{D'(g)\,B}_{\underbrace{(m\times m)\ (m\times n)}_{(m\times n)}} \quad \forall g\in G \quad \text{then} \quad B = \overset{\text{the zero matrix}}{\underset{m\times n}{0}}.$$

We shall see powerful and useful ideas, especially in quantum mechanics, following from these lemmas.

Fundamental (aka Grand) Orthogonality Theorem for Group Representations

3/20

Take 2 (unitary) irreps $D^{(\mu)}$ and $D^{(\nu)}$.

Form the "scalar product (or group average) of threads": order of group (move to l.h.s)

$$\sum_{g\in G} D^{(\mu)}_{mm'}(g)\; D^{(\nu)}_{nn'}(g)^{*} = \frac{[g]}{n^{(\mu)}}\,\delta^{\mu\nu}\,\delta_{mn}\,\delta_{m'n'}\,.$$

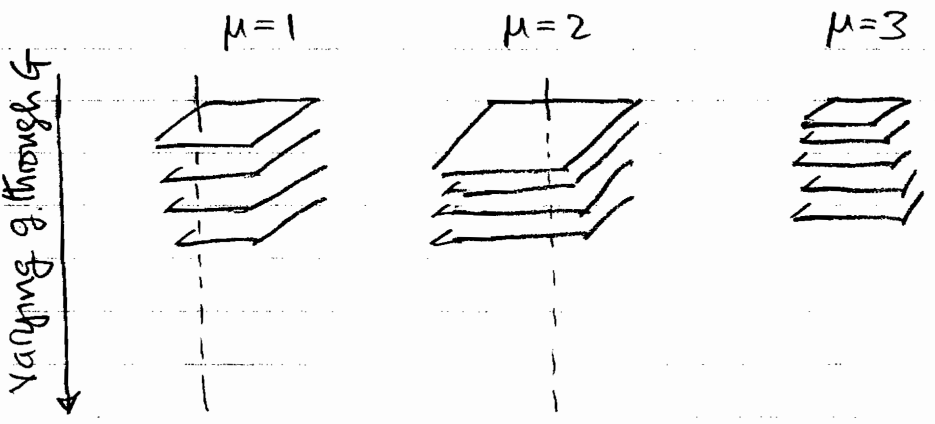
dimension of irrep μ

same irrep.

same thread

Can write this because (by unitarity) it is the same as $D^{(\nu)}_{n'n}(g^{-1})$ ← this is what arrives from the proof

Pictorial interpretation — stack the matrices in each irrep:



Thread through two stacks (ie a site mm' in irrep μ
--- nn' --- ν)

Complex conjugate one string and form the scalar product with the other string

- get 0 if the strings are distinct
- get $\frac{[g]}{n^{(\mu)}}$ if the strings are the same.

Fundamental (aka Grand) Orthogonality Theorem for Group Characters

3/30

Recall that the character of a representation is the collection of traces, one for each element of the group, of the matrices representing the group

$$\chi \equiv \{ \chi(g) \mid g \in G \} \quad \text{where} \quad \chi(g) = \Sigma_j \, D_{jj}(g).$$

Recall, also, that χ does not change when D is replaced by and equivalent representation $S D S^{-1}$.

Let us assume that the representation is unitary; in this case we have

$$\circledast \rightarrow \chi(g^{-1}) = \Sigma_j \, D_{jj}(g^{-1}) = \Sigma_j \, D(g)^{-1}|_{jj}$$

$$\xrightarrow{\text{by the unitarity of } D \text{, i.e., } D^{+} = D^{-1}} = \Sigma_j \, (D(g)|_{jj})^{*} = \chi(g)^{*}$$

$\circledast$ What arrives from the proof

Then we take the trace of the Orthogonality Theorem, ie, we contract with $\delta_{mm'} \, \delta_{nn'}$ to arrive at

$$\frac{1}{[g]} \sum_{g \in G} \chi^{(\mu)}(g) \, \chi^{(\nu)}(g)^{*} = \delta^{\mu\nu}$$

ie the characters coming from inequivalent irreps are orthogonal to one another; each has squared length $[g]$.

Sometimes we write $\langle \chi^{(\nu)}, \chi^{(\mu)} \rangle$ for $\frac{1}{[g]} \sum_{g \in G} \chi^{(\mu)}(g) \, \chi^{(\nu)}(g)^{*}$,

$\nwarrow$ a scalar product between characters of reps

in which case we can say that inequivalent irreps are orthonormal

One simplification (and computational convenience) emerges from the observation that the character $\chi(g)$ of a group element g is unchanged if g is replaced by any element drawn from its conjugacy class (g) — in other words all "like" elements have the same character.

To see this, recall that if a and b are conjugate elements of G then for some $g \in G$ we have $b = g a g^{-1}$. Thus, in the representation D we must have

$$D(b) = D(g a g^{-1}) = D(g)\, D(a)\, D(g^{-1})$$

Now, taking the trace we see that

$$\chi(b) = \Sigma_j D_{jj}(b) = \Sigma_{jk\ell}\, D_{jk}(g)\, D_{k\ell}(a)\, D_{\ell j}(g^{-1})$$

$$= \Sigma_{k\ell}\, D_{k\ell}(a) \underbrace{\Sigma_j\, D_{\ell j}(g^{-1})\, D_{jk}(g)}_{\delta_{\ell k}} = \chi(a).$$

Thus, in our orthogonality theorem for characters we can collect terms in the sum over elements into the sum over conjugacy classes and a sum (of identical terms) within conjugacy classes.

Letting $\alpha = 1, 2, \ldots K$ index the conjugacy classes,
letting k_α be the number of group elements in class α, and
letting $\chi^{(\mu)}_\alpha$ denote the character, in irrep μ, of group elements in class α we arrive at

# of CC's

$$\frac{1}{[g]} \sum_{\kappa=1}^{K} \chi^{(\mu)}_\alpha\, \chi^{(\nu)*}_\alpha\, k_\alpha = \delta^{\mu\nu}.$$

- K is the number of conjugacy classes
- r is the number of irreps

↓ rep(μ)	$\alpha =$ class → 1	2	---	K
1	•	•	•	•
2	•	•	•	•
⋮	⋮	⋮	⋮	⋮
r	•	•	•	•

# of irreps

← Called a character table

This theorem is "orthogonality of rows"
But the number of entries in each row is K,
so the maximum number of orthogonal rows is K,# i.e., $r \leq K$.

But (see Hamermesh for a proof) we also have "orthogonality of columns", ie, orthogonality on the class index:

$$\frac{1}{[g]} \sum_{\mu=1}^{r} \chi_{\alpha}^{(\mu)} \chi_{\beta}^{(\mu)*} = \frac{1}{k_\alpha} \delta_{\alpha\beta}$$

But the number of entries in each column is r,
so the maximum number of orthogonal columns is r,# i.e., $K \leq r$.

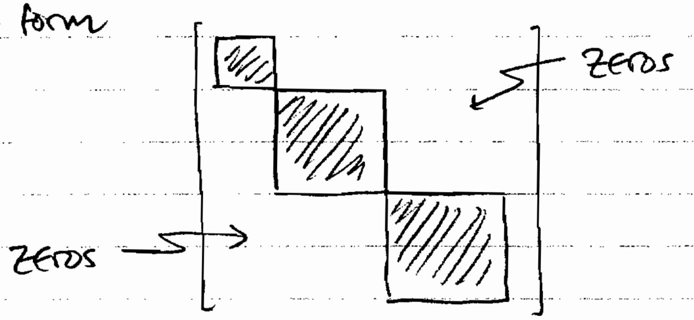
Thus we arrive at the result that the character table is square, ie the number of irreps is the same as the number of conjugacy classes.

Can find at most λ orthogonal vectors in a dimension-λ space

How to decompose completely reducible representations into their constituent irreps?

As we have discussed, for a finite or a compact group, any reducible representation is completely reducible (ie decomposable) into a direct sum of irreps. Said another way, by a suitable similarity transformation (ie change of basis) all the matrices of the representation can be cast into block-diagonal form in which the (non-zero, diagonal) blocks are the matrices of irreducible representations.

Example: We may have a 5x5 representation of some group for which, by a suitable similarity transformation, all matrices acquire the form



This transformation would reveal the fact that the 5x5 rep is composed of three reps: one 1x1, and two 2x2 reps

We write $D = D_1 \oplus D_2 \oplus D_3$ to indicate this

and we say that the reducible rep D is the direct sum (indicated by $\oplus$) of the three irreps D_1, D_2 and D_3

Remarks:
- D_3 may be a repetition of D_2
- That D is decomposible may well not be at all apparent from the original (ie unsimilarity-transformed) representation
- characters and their orthogonality provide us with a powerful tool for decomposing representations.

Even though it may not be apparent, a compound representation is a direct sum of irreps

$$\underset{\substack{\uparrow\\ \text{Compound}\\ \text{rep}}}{D} = \underbrace{D^{(1)} \oplus D^{(1)} \oplus \cdots \oplus}_{\substack{\text{irrep } D^{(1)} \text{ appears}\\ a_1 \text{ times}}} \underbrace{D^{(2)} \oplus}_{\substack{\text{irrep } D^{(2)}\\ \text{appears } a_2\\ \text{times}}} \cdots\cdots = \Sigma^{\oplus} \underset{\substack{\uparrow\\ \text{number of times}\\ (0,1,2,--)\text{ irrep}\\ \mu \text{ appears.}}}{a_\mu} D^{(\mu)} .$$

To compute $\{a_\mu\}_{\mu=1}^{K}$ without having to find the similarity transformation that block-diagonalises the compound rep we first imagine that we have block-diagonalised and taken the trace of the elements of the rep $\{D(g) \mid g \in G\}$ to learn that

$$\underset{\substack{\uparrow\\ \text{Compound}\\ \text{character}}}{\chi(g)} = \sum_{\mu=1}^{K} a_\mu \underset{\substack{\uparrow\\ \text{"simple" characters}}}{\chi^{(\mu)}}(g) . \qquad (\#)$$

Next, we apply the operation $\frac{1}{[g]} \sum_{g \in G} \chi^{(\nu)}(g)^* ----$ to both sides of (#) to obtain

$$\frac{1}{[g]} \sum_{g \in G} \chi(g)\chi^{(\nu)}(g)^* = \sum_{\mu=1}^{K} a_\mu \, {}_{1/[g]}\underbrace{\sum_{g\in G} \chi^{(\mu)}(g)\chi^{(\nu)}(g)^*}_{\delta^{\mu\nu}}$$

ie $a_\nu = \frac{1}{[g]} \sum_{g \in G} \chi(g)\chi^{(\nu)}(g)^*$, which, owing to the invariance of characters across conjugacy classes, can be rewritten as

$$\boxed{a_\nu = \frac{1}{[g]} \underset{\substack{\uparrow\\ \text{sum over conjugacy classes}}}{\sum_{\alpha}} \chi_\alpha \chi_\alpha^{(\nu)*} k_\alpha} \longleftarrow \substack{\#\text{ of elements of } G\\ \text{in conjugacy class } \alpha}$$

The Regular Representation

3/80

Take any group G of finite order $[g]$.
Then we have the regular representation :

$$g \longrightarrow D_{jk}(g) = \delta_{g_j, g g_k} \qquad (\#)$$

a $[g] \times [g]$ matrix — each row and column has exactly 1 one and $[g]-1$ zeros

slight extension of the Kronecker delta symbol $\begin{cases} 1 & \text{if } g_j = g g_k \\ 0 & \text{otherwise} \end{cases}$

Algebraic interpretation :

$$\sum_k D_{jk}(g)\, D_{k\ell}(g') = \sum_k \delta_{g_j, g g_k}\, \delta_{g_k, g' g_\ell}$$

$$= \Sigma_k\, \delta_{g^{-1} g_j, g_k}\, \delta_{g_k, g' g_\ell}$$

$$= \delta_{g^{-1} g_j, g' g_\ell} = \delta_{g_j, g g' g_\ell} = D_{j\ell}(g g')$$

Matrix point of view: The matrix that takes the string $(g_1, g_2, \cdots)$ and replaces it by the string $(g g_1, g g_2, \cdots)$, i.e.,

$$(g g_1, g g_2, \cdots) = (g_1, g_2, \cdots) \begin{pmatrix} D(g) \end{pmatrix}$$

$[g] \times [g]$ of zeros and ones

The matrix (#) accomplishes this :

$$g g_k = \Sigma_j\, g_j\, D_{jk}(g) = \Sigma_j\, g_j\, \delta_{g_j, g g_k} = g g_k \quad \checkmark$$

The regular rep embodies the idea that every row of the group multiplication table is a permutation of the elements of the group (although typically not all permutations appear). In other words, we have Cayley's Theorem: all finite groups of order $[g]$ are subgroups of the permutation group $S_{[g]}$.

How does matrix multiplication realise group multiplication here?

$$g_k \rightarrow g g_k = \sum_j g_j D_{jk}(g) \qquad \forall g \in G$$

[illegible] $\nearrow$ ~~[illegible]~~ $= g'\left(\sum_j g_j D_{jk}(g)\right)$

$\circledast$

$$= \sum_j g g_j D_{jk}(g)$$

$$= \sum_j \left(\sum_\ell g_\ell D_{\ell j}(g')\right) D_{jk}(g)$$

$$= \sum_\ell g_\ell \left(\sum_j D_{\ell j}(g') D_{jk}(g)\right)$$

$$= \sum_\ell g_\ell D_{\ell k}(g'g), \text{ as required.}$$

So, $(g'g) g_k = g'(g g_k) = \circledast$

Example: $G = C_3 = \{e, c, c^2\}$ with $c^3 = e$

Regular rep: (built using $D_{jk} = \delta_{g_j, g g_k}$)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & \boxed{1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$D(e)$ $\qquad$ $D(c)$ $\qquad$ $D(c^2)$

↑

	e	c	c^2
e	e	c	c^2
c	c	c^2	e
c^2	c^2	e	c

↑

$g_j = g g_k$

Ex $j=1, k=3$; $e = c c^2$ ✓

Decomposition of the regular representation into irreps using characters

For $g = e$ (the identity) we have $D_{jk}(e) = \delta_{g_j, g_k} = \delta_{jk}$

Hence $\chi(e) = \Sigma_j D_{jj}(e) = \Sigma_j \delta_{jj} = [g]$.

On the other hand, for $g \neq e$ we have $D_{jj}(g) = 0$ and hence $\chi(g) = 0$.

(no sum)

Now if $D = \Sigma^{\oplus}_{\mu} a_\mu D^{(\mu)}$ (#) then, by the character orthogonality theorem we have

$$a_\mu = \frac{1}{[g]} \sum_{g \in G} \chi(g)\, \chi^{(\mu)}(g)^*$$

0 unless $g = e$; then $[g]$

$n^{(\mu)}$ if $g = e$ ← the dimension of irrep μ

(all elts of the Ds are 0 or 1)

Thus we find that $a_\mu = n^{(\mu)}$ – each irrep features in the regular rep as many times as it has dimensions –

$$D = \Sigma^{\oplus}_{\mu} n^{(\mu)} D^{(\mu)}.$$

This leads to the following useful result: put $g = e$ (in #) and take the trace to find

$$\chi(e) = \Sigma_\mu n^{(\mu)} \chi^{(\mu)}(e)$$

$[g]$ $\qquad$ $n^{(\mu)}$

and hence $\boxed{[g] = \sum_\mu (n^{(\mu)})^2}$ which sets a condition on the dimensions of the irreps.

Irreps for abelian groups

3/1/20

- Abelian groups G have exactly [g] irreps, all of which are one-dimensional

Why? Route I: G has [g] conjugacy classes (ie every element is in its own class);
therefore G has [g] irreps;
thus the dimensions $n^{(\mu)}$ of the irreps obey

$$[g] = \sum_{\mu=1}^{[g]} \underset{\substack{\uparrow \\ \text{only solution is } n^{(\mu)}=1 \\ \forall \mu = 1,2,..,[g]}}{(\overset{\nwarrow 1,2,3,\ldots}{n^{(\mu)}})^2}$$

Route II: Via Schur's lemma: Suppose $D^{(\mu)}$ is an irrep of G.

Then $\underset{\substack{\uparrow \\ \text{fixed elt. of } G}}{D^{(\mu)}(\gamma)}$ commutes with all $D^{(\mu)}(g)$ $\forall g \in G$

By Schur, this means that $D^{(\mu)}(\gamma) \propto$ (identity matrix).
But this is true for all $\gamma \in G$ and, hence, if $D^{(\mu)}$ is not one-dimensional it is reducible, which contradicts the original assumption of irreducibility.

Character Tables (again)

3/130

One can determine the characters of the irreps of the finite groups most commonly encountered in physical settings and present the information in the form of a character table

group	Conj class #1	#2	#3	----
irrep 1	1	1	1	
irrep 2				
⋮				

↯ exactly as many irreps as conjugacy classes

Tools:

- table is square (1 irrep for each conjugacy class)
- orthogonality of rows, normalisation too
- orthogonality of columns, normalisation too
- any other information (eg 1-dimensional reps have characters that obey the group multiplication table)
- always have the trivial rep

Nomenclature in Character Tables for Groups of Rotations

Example: The dihedral group D_3

identity class

group name ↓

	E	$2C_3$	$3C_2$		← name of conjugacy class
	(e)	(c)	(b)		class itself
D_3	$\{e\}$	$\{c, c^2\}$	$\{b, bc, bc^2\}$		elements in class
A_1	1	1	1		
A_2	1	1	-1	z	tells us where the basis
E	2	-1	0	x, y	functions of the vector rep

reside – ie take the

(always relevant for rotations) vector rep; decompose it into irreps; what are the invariant subspaces and how do they transform (z-like A_2; x, y-like E)

The names (eg $2C_3$) given to the conjugacy classes reflect the number and nature of its members

← after all, conjugate elements are rotations through the same angle (although the converse is not true)

For example, take $2C_3$ ← means each element generates a C_3 subgroup of D_3

means there are 2 elements in the class (c, c^2)

$c: \{c, c^2, e\}$

$c^2: \{c^2, c^4(=c), c^6(=e)\}$

Or take $3C_2$ ⇐ 3 subgroups C_2 –

$b: \{b, b^2(=e)\}$

$bc: \{bc, (bc)^2(=e)\}$

$bc^2: \{bc^2, (bc^2)^2(=e)\}$

3 elements $\{b, bc, bc^2\}$

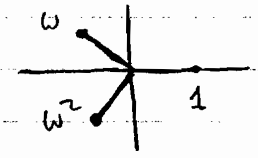
Example : Character Table for the Abelian Group C_3

- $C_3 = \{e, c, c^2\}$ $(c^3 = e)$
- Conjugacy classes $(e), (c), (c^2)$ (three of them)
- # of irreps : three
- dimensionality of irreps : all one-dimensional

simple ↓

	C_3	(e)	(c)	(c^2)	remarks
simple →	$D^{(1)}$	1	1	1	trivial, called A, always present
	$D^{(2)}$	1	ω	ω^2	faithful / isomorphism } E
	$D^{(3)}$	1	ω^2	ω	" "
			↑ ⊛	↑ (#)	

- In 1D reps $\chi(c^3) = \chi(c)^3 = \chi(e) = 1$ ⊛
  ⇒ cube roots of unity $1, \omega, \omega^2$ $(\omega = \exp 2\pi i/3)$
- Similarly $\chi(c^2) = \chi(c)^2$ (#)
  Recall that $1 + \omega + \omega^2 = 0$



(geometrically),
or algebraically:
$\omega^3 - 1 = 0$, so
$(\omega - 1)(\omega^2 + \omega + 1) = 0$, but $\omega - 1 \neq 0$ so $1 + \omega + \omega^2 = 0$.

- So $D^{(2)}$ and $D^{(3)}$ obey the orthogonality requirements on both the rows and the columns.