This homework is **optional** (meaning that you will get bonus points for problems worked through), but you are encouraged to read through this set of problems, alert me if there are topics you feel need a review. Homework assignments will be posted on the web latest on Tuesday and will be due next Tuesday in class.

1) R. P. Agnew's snow plow problem (after *Bender and Orszag*, as are probs. 2 and 3 and quite a few more, this semester): One day it started snowing at a heavy and steady rate. A snow plow started out at noon, going two miles the first hour and one mile the second hour. What time did it start snowing? [Hint: The speed of the plow is inversely proportional to the depth of the snow.]

You may also wish to try the more sophisticated version due to M. S. Klamkin, known as *The great snow plow chase*: One day it started snowing at a heavy and steady rate. Three identical snow plows started out at noon, 1 p.m., and 2 p.m., from the same place, and all collided at the same time. What time did it start snowing? When did the snow plows collide?

2) Four caterpillars: Four caterpillars, initially at rest at the four corners of a square centered at the origin, start walking counter-clockwise, each caterpillar walking directly towards the one in front of him. If each caterpillar walks with unit velocity, show that the trajectories satisfy the differential equation $y' = (y - x)/(y + x)$. By making the substitution $x = r\cos\theta$, $y = r\sin\theta$ show that the trajectories are logarithmic spirals.

3) Farmer and the pig: At $t = 0$, a pig, initially at the origin, runs along the x-axis with constant speed v. At $t = 0$, a farmer, initially 20 m north of the origin, also runs with constant speed v. If the farmer's instantaneous velocity is always directed towards the instantaneous position of the pig, show that the farmer never gets closer than 10 m from the pig.

4) The needle of Georges Louis Leclerc, Comte de Buffon (1707-1788): A grid of parallel lines, equally spaced, is drawn on the ground. A needle, of length equal to the line spacing, is dropped at random. Show that the probability that the needle does not intersect a line is given by $2/\pi$. (This provides an experimental method for determining π.) Generalize to the case in which the needle length need not equal the line spacing.
Note: A computational perspective on this first example of *stochastic geometry* is given by W. Krauth in *Algorithms and Computations* (Oxford University Press, 2006), pp. 9-15.

5) The method of similarity: Consider a particle of mass m, which moves along a certain trajectory $\mathbf{r}(t)$ according to the equation of motion $m\ddot{\mathbf{r}} = -\partial U/\partial\mathbf{r}$. [Note the alternative notation: $\partial/\partial\mathbf{r} \equiv \nabla$.] Suppose (as is usually the case) that the potential energy U is independent of m.

a) Show that a particle of mass αm, moving in the same potential U, can follow the trajectory $\mathbf{R}(t) = \mathbf{r}(\beta t)$, and find the appropriate value of the constant β in terms of the constant α.

b) By using the result of part (a) show that if the mass of a particle is decreased by a factor of 4 then the particle can travel the same orbit in the same force field twice as fast.

Suppose that the potential energy of a central force field is a homogeneous function of degree ν. This means that $U(\mathbf{r}) = w(r)$ and $w(\gamma r) = \gamma^\nu w(r)$, where $r \equiv |\mathbf{r}|$.

c) Show that if the curve $\mathbf{r}(t)$ is a classical trajectory then the (rescheduled, and inflated or deflated) curve $\gamma \mathbf{r}(\epsilon t)$ is also a classical trajectory, albeit with different initial conditions, for some suitable choice of ϵ. If these trajectories are closed orbits, determine the ratio of their periods.

d) Hence deduce that, in the linear regime, pendulum oscillations of any amplitude all have the same period. State the value of ν to which this corresponds?

e) Similarly, deduce Kepler's third law (see, *e.g.*, *Marion and Thornton*, eq. 7.48). To which value of ν does this correspond?

f) A desert animal has to cover great distances between sources of water. How does the maximal time the animal can run depend on the size of the animal?

g) How does the running speed of an animal on level ground and uphill depend on the size L of the animal?

h) How does the height of an animal's jump depend on its size?
[Source for (f-h): V. I. Arnol'd, *Mathematical Methods in Classical Mechanics*, pp. 51-52, who cites J. M. Smith, *Mathematical Ideas in Biology* (Cambridge, 1968).]

6) Separable ordinary differential equations: An ordinary differential equation is said to be *separable* if it can be written in the form

$$y' \equiv \frac{dy}{dx} = a(x)\, b(y).$$

a) Show that the solution is given by

$$\int^{y(x)} \frac{dt}{b(t)} = \int^{x} ds\, a(s) + c\,,$$

where c is a constant of integration.

b-1) Solve $y' = \exp(x + y)$.

b-2) Solve $y' = xy + x + y + 1$.

c) Discuss briefly the problem of determining whether the right hand side of $y' = a(x)\, b(y)$ does indeed have the separable form $a(x)\, b(y)$.

7) Cartesian vectors: The aim of this question is to familiarize you with some of the notational conventions that I shall be using throughout the course, and also to give you some practice with summation convention.

Consider a *Cartesian basis*, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, for 3-dimensional vectors $\mathbf{x}$. Suppose that the basis vectors are normalized to unity and are mutually orthogonal (*i.e.*, they are orthonormal); then they possess the scalar products $\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \delta_{\mu\nu}$, where μ and ν take the values 1, 2, or 3 (or x, y or z). Here, $\delta_{\mu\nu}$ is the Kronecker symbol, which equals 1 when $\mu = \nu$, and equals 0 otherwise. You may think of this as the identity matrix

$$\begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

An arbitrary vector $\mathbf{x}$ is a linear combination of basis vectors, $\mathbf{x} = \sum_{\mu=1}^{3} x_\mu \mathbf{e}_\mu$, with the set of coefficients (called components) $\{x_\mu\}_{\mu=1}^{3}$. Notice that we can extract a component by taking the scalar product of a vector with the appropriate basis vector,

$$\mathbf{e}_\mu \cdot \mathbf{x} = \mathbf{e}_\mu \cdot \sum_{\nu=1}^{3} x_\nu \mathbf{e}_\nu = \sum_{\nu=1}^{3} \mathbf{e}_\mu \cdot \mathbf{e}_\nu x_\nu = \sum_{\nu=1}^{3} \delta_{\mu\nu} x_\nu = x_\mu.$$

It is very useful to adopt a convention, called summation convention, in which summation is implied over any twice-repeated indices; for example

$$\mathbf{x} = \sum_{\mu=1}^{3} x_\mu \mathbf{e}_\mu \equiv x_\mu \mathbf{e}_\mu.$$

In true tensorial equations a given index, say μ, never need occur more than twice. Singly occurring indices are called effective indices, whilst repeated indices are called dummy indices and may be replaced by another index: *e.g.*, $\mathbf{x} = x_\nu \mathbf{e}_\nu = x_\mu \mathbf{e}_\mu$. Dummy indices are rather like dummy variables in integrals.

If two vectors $\mathbf{a}$ and $\mathbf{b}$ are equal then their components are equal, *i.e.*, $a_\mu = b_\mu$. This follows from taking the scalar product of both sides of the equation $\mathbf{a} = \mathbf{b}$ with the basis vector $\mathbf{e}_\mu$. Notice that unrepeated indices balance throughout all terms of an equation. For example, if $\mathbf{a} + \mathbf{b} = \mathbf{c}$ then $a_\mu + b_\mu = c_\mu$. Indices are only considered repeated if they occur in the *same term*. For example, the equation $a_\mu = b_\mu$ contains one effective index and no repeated indices. Using $\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \delta_{\mu\nu}$ and $\mathbf{e}_\mu \cdot \mathbf{x} = x_\mu$, and also the definition of $\delta_{\mu\nu}$, verify the following statements:

a-1) $\mathbf{x} \cdot \mathbf{x} = x_\mu x_\mu$

a-2) $\mathbf{x} \cdot \mathbf{y} = x_\mu y_\mu$

a-3) $\delta_{\mu\nu}\, \delta_{\nu\rho} = \delta_{\mu\rho}$

a-4) $a_\mu = a_\nu \delta_{\nu\mu}$

a-5) $\delta_{\mu\mu} = 3$.

Now consider scalar and vector fields, *i.e.*, scalar-valued functions, $f(\mathbf{x})$, and vector-valued functions, $\mathbf{g}(\mathbf{x}) = \mathbf{e}_\mu \, g_\mu(\mathbf{x})$, of a position vector, $\mathbf{x}$. For Cartesian coordinates, the gradient operator ∇ is defined by

$$\nabla \equiv \sum_{\mu=1}^{3} \mathbf{e}_\mu \, \frac{\partial}{\partial x_\mu} = \mathbf{e}_\mu \, \frac{\partial}{\partial x_\mu} = \mathbf{e}_\mu \, \partial_\mu$$

where, for convenience, we have written ∂_μ for $\partial/\partial x_\mu$.
Verify the following results:

b-1) $\nabla \cdot \mathbf{g}(\mathbf{x}) = \partial_\mu g_\mu(\mathbf{x})$

b-2) $\nabla f(\mathbf{x}) = \mathbf{e}_\mu \partial_\mu f(\mathbf{x})$

b-3) $(\mathbf{x} \cdot \nabla)\, f(\mathbf{x}) = x_\mu \, \partial_\mu f(\mathbf{x})$

b-4) $\nabla \cdot \big(\nabla f(\mathbf{x})\big) = \nabla^2 f(\mathbf{x})$ where $\nabla^2 \equiv \partial_x^2 + \partial_y^2 + \partial_z^2 = \partial_\mu \partial_\mu$

b-5) $\nabla \cdot \mathbf{x} = \partial_\mu \, x_\mu = \delta_{\mu\mu} = 3$

b-6) $\nabla(\mathbf{x} \cdot \mathbf{x}) = 2\,\mathbf{x}$

b-7) $\nabla^2 \, (\mathbf{x} \cdot \mathbf{x}) = 6$

b-8) $\nabla|\mathbf{x}| = \mathbf{x}/|\mathbf{x}|$

b-9) $\partial_\mu \big(x_\nu/|\mathbf{x}|\big) = \big(x^2 \delta_{\mu\nu} - x_\mu \, x_\nu\big)/x^3$.

b-10) $\nabla^2 \, (1/|\mathbf{x}|) = -4\pi \, \delta(\mathbf{x})$ [Hint: apply the divergence theorem]

b-11) $\nabla \, \big(\mathbf{x} \cdot \mathbf{g}(\mathbf{x})\big) = \mathbf{g}(\mathbf{x}) + \mathbf{e}_\mu x_\nu \partial_\mu g_\nu(\mathbf{x})$

b-12) $\nabla \cdot \big(\mathbf{x}\, f(\mathbf{x})\big) = 3\, f(\mathbf{x}) + (\mathbf{x} \cdot \nabla)\, f(\mathbf{x})$

b-13) For constant $\mathbf{h}$, $\oint_\Gamma d\mathbf{x} \cdot \big(\tfrac{1}{2}\mathbf{h}\times\mathbf{x}\big) = \pi \mathbf{h} \cdot \mathbf{n}$, where Γ is a any circle of unit radius, and the unit vector $\mathbf{n}$ specifies the axis of the circle and the sense in which it is traversed [Hint: apply Stokes's theorem]

b-14) $\nabla \, \exp(i\mathbf{k} \cdot \mathbf{x}) = i\mathbf{k} \, \exp(i\mathbf{k} \cdot \mathbf{x})$

b-15) $f(\mathbf{x} + \mathbf{a}) = f(\mathbf{x}) + (\mathbf{a} \cdot \nabla) f(\mathbf{x}) + \cdots = \mathrm{e}^{\mathbf{a}\cdot\nabla} f(\mathbf{x})$.

8) More on Cartesian vectors: The purpose of this question is two-fold. Firstly, we will investigate some of the properties of the vector product, denoted $\times$, and the related differential operator, curl, denoted $\nabla\times$. Secondly, we will solve the problems using summation convention so that we get some more practice with it. As with the previous problem, we consider an orthonormal basis, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (*i.e.*, $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$) for 3-dimensional Cartesian vectors, $\mathbf{x}$. The basis is said to be right-handed because

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$$
$$\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$$
$$\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2.$$

We can express these relationships much more compactly using the symbol $\epsilon_{\mu\nu\rho}$, known as the Levi-Civita symbol, or the antisymmetric third-rank tensor. This tensor takes on the following values in all Cartesian coordinate systems:

$$\epsilon_{\mu\nu\rho} = \begin{cases} +1, & \text{if } \mu\nu\rho = 123, 231, \text{ or } 312; \\ -1, & \text{if } \mu\nu\rho = 132, 213, \text{ or } 321; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $\epsilon_{\mu\nu\rho}$ is totally antisymmetric, *i.e.*, its value changes sign whenever any pair of indices are exchanged, e.g., $\epsilon_{123} = -\epsilon_{213} = 1$. This requirement forces $\epsilon_{\mu\nu\rho}$ to vanish whenever two or more of its indices are the same, e.g. $\epsilon_{113} = 0$. This property is extremely useful, as we shall see, when it comes to proving certain results involving vector products and the curl operator.

In terms of $\epsilon_{\mu\nu\rho}$, the vector products between basis vectors become

$$\mathbf{e}_\mu \times \mathbf{e}_\nu = \epsilon_{\mu\nu\rho}\, \mathbf{e}_\rho,$$

where the implied summation on ρ recovers the previously-given results for the cases $\mu \neq \nu$, and also includes results when $\mu = \nu$.

Starting with these definitions, and the results from the previous problem, if necessary, verify the following statements using summation convention:

a-1) $\epsilon_{\mu\nu\rho}\, \epsilon_{\mu\sigma\tau} = \delta_{\nu\sigma}\, \delta_{\rho\tau} - \delta_{\nu\tau}\, \delta_{\rho\sigma}$

a-2) $\epsilon_{\mu\nu\rho}\, \epsilon_{\mu\nu\tau} = 2\, \delta_{\rho\tau}$

a-3) $\epsilon_{\mu\nu\rho}\, \epsilon_{\mu\nu\rho} = 6$

a-4) $\mathbf{A}\times\mathbf{B} = A_\mu\, B_\nu\, \epsilon_{\mu\nu\rho}\, \mathbf{e}_\rho$

a-5) $(\mathbf{A}\times\mathbf{B})_\rho = A_\mu\, B_\nu\, \epsilon_{\mu\nu\rho}$

a-6) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \epsilon_{\mu\nu\rho}\, A_\mu\, B_\nu\, C_\rho$

a-7) $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = (\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}$

a-8) $\mathbf{A}\times\mathbf{A} = \mathbf{0}$

a-9) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$

Now consider scalar and vector fields, *i.e.*, scalar-valued functions, $f(\mathbf{x})$, and vector-valued functions, $\mathbf{g}(\mathbf{x}) = \mathbf{e}_\mu\, g_\mu(\mathbf{x})$, of a position vector, $\mathbf{x}$. The curl operator, $\nabla\times$, operates on a vector field, $\mathbf{g}(\mathbf{x})$ to produce new vector field, denoted $\nabla\times\mathbf{g}(\mathbf{x})$. It is defined in the following way:

$$\nabla\times\mathbf{g}(\mathbf{x}) \equiv \sum_{\mu,\nu,\rho=1}^{3} \mathbf{e}_\mu\, \epsilon_{\mu\nu\rho}\, \frac{\partial}{\partial x_\nu} g_\rho(\mathbf{x}) = \mathbf{e}_\mu\, \epsilon_{\mu\nu\rho}\, \frac{\partial}{\partial x_\nu}\, g_\rho(\mathbf{x}) = \mathbf{e}_\mu\, \epsilon_{\mu\nu\rho}\, \partial_\nu\, g_\rho(\mathbf{x}).$$

Using these definitions, verify the following statements:

b-1) $\nabla\times\mathbf{x} = \mathbf{0}$

b-2) $\nabla\times(\mathbf{H}\times\mathbf{x}) = 2\,\mathbf{H}$, for constant $\mathbf{H}$

b-3) $\nabla\cdot\big(\nabla\times\mathbf{g}(\mathbf{x})\big) = 0$

b-4) $\nabla\times\big(\nabla f(\mathbf{x})\big) = \mathbf{0}$

b-5) $\nabla\times\big(f(\mathbf{x})\,\mathbf{g}(\mathbf{x})\big) = f\,\nabla\times\mathbf{g} + (\nabla f)\times\mathbf{g}$

b-6) $\nabla\times\big(\mathbf{g}(\mathbf{x})\times\mathbf{h}(\mathbf{x})\big) = \mathbf{g}\,\nabla\cdot\mathbf{h} - \mathbf{h}\,\nabla\cdot\mathbf{g} + (\mathbf{h}\cdot\nabla)\,\mathbf{g} - (\mathbf{g}\cdot\nabla)\,\mathbf{h}$