C

Curvilinear coordinates

The distance between two points Euclidean space takes the simplest form (2-4) in Cartesian coordinates. The geometry of concrete physical problems may make non-Cartesian coordinates more suitable as a basis for analysis, even if the distance becomes more complicated in the new coordinates. Since the new coordinates are non-linear functions of the Cartesian coordinates, they define three sets of intersecting curves, and are for this reason called *curvilinear coordinates*.

At a deeper level, it is often the *symmetry* of a physical problem that points to the most convenient choice of coordinates. Cartesian coordinates are well suited to problems with translational invariance, cylindrical coordinates for problems that are invariant under rotations around a fixed axis, and spherical coordinates for problems that are invariant or partially invariant under arbitrary rotations. Elliptic and hyperbolic coordinates are also of importance but will not be discussed here (see [18, p. 455]).

C.1 Cylindrical coordinates

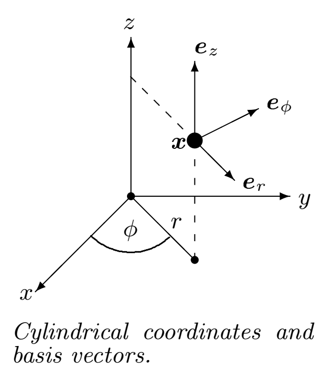
The relation between Cartesian coordinates x, y, z and cylindrical coordinates r, ϕ, z is given by

$$x = r\cos\phi \ , \tag{C-1a}$$

$$y = r\sin\phi \ , \tag{C-1b}$$

$$z = z \tag{C-1c}$$

with the range of variation $0 \leq r < \infty$ and $0 \leq \phi < 2\pi$. The two first equations



Cylindrical coordinates and basis vectors.

simply define polar coordinates in the xy-plane[1]. The last is rather trivial but included to emphasize that this is a transformation in 3-dimensional space.

Curvilinear basis

The *curvilinear basis vectors* are defined from the tangent vectors, obtained by differentiating the Cartesian position after the cylindrical coordinates,

$$\boldsymbol{e}_r = \frac{\partial \boldsymbol{x}}{\partial r} = (\cos\phi, \sin\phi, 0) \ , \tag{C-2a}$$

$$\boldsymbol{e}_\phi = \frac{1}{r}\frac{\partial \boldsymbol{x}}{\partial \phi} = (-\sin\phi, \cos\phi, 0) \ , \tag{C-2b}$$

$$\boldsymbol{e}_z = \frac{\partial \boldsymbol{x}}{\partial z} = (0, 0, 1) \ . \tag{C-2c}$$

As may be directly verified, they are orthogonal and normalized everywhere, and thus define a local curvilinear basis with an orientation that changes from place to place. An arbitrary vector field may therefore be resolved in this basis

$$\boldsymbol{V} = \boldsymbol{e}_r V_r + \boldsymbol{e}_\phi V_\phi + \boldsymbol{e}_z V_z \ , \tag{C-3}$$

where the vector coordinates

$$V_r = \boldsymbol{V}\cdot\boldsymbol{e}_r \ , \quad V_\phi = \boldsymbol{V}\cdot\boldsymbol{e}_\phi \ , \quad V_z = \boldsymbol{V}\cdot\boldsymbol{e}_z \ , \tag{C-4}$$

are the projections of $\boldsymbol{V}$ on the local basis vectors.

Resolution of the gradient

The derivatives after the cylindrical coordinates are found by differentiation through the Cartesian coordinates

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y} \ ,$$
$$\frac{\partial}{\partial \phi} = \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} = -r\sin\phi\frac{\partial}{\partial x} + r\cos\phi\frac{\partial}{\partial y} \ .$$

From these relations we may calculate the projections of the gradient operator $\boldsymbol{\nabla} = (\partial_x, \partial_y, \partial_z)$ on the cylindrical basis, and we obtain

$$\nabla_r = \boldsymbol{e}_r \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial r} \ , \tag{C-5a}$$

$$\nabla_\phi = \boldsymbol{e}_\phi \cdot \boldsymbol{\nabla} = \frac{1}{r}\frac{\partial}{\partial \phi} \ , \tag{C-5b}$$

$$\nabla_z = \boldsymbol{e}_z \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial z} \ . \tag{C-5c}$$

[1]Some texts use Θ instead of ϕ as the conventional name for the polar angle in the plane. Various arguments can be given one way or the other by comparing with spherical coordinates. But what's in a name? A polar angle by any name still works as sweet.

Conversely, the gradient may be resolved on the basis

$$\boxed{\boldsymbol{\nabla} = \boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\phi \nabla_\phi + \boldsymbol{e}_z \nabla_z = \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} + \boldsymbol{e}_z \frac{\partial}{\partial z}} \ . \tag{C-6}$$

Together with the non-vanishing derivatives of the basis vectors

$$\frac{\partial \boldsymbol{e}_r}{\partial \phi} = \boldsymbol{e}_\phi \ , \tag{C-7a}$$

$$\frac{\partial \boldsymbol{e}_\phi}{\partial \phi} = -\boldsymbol{e}_r \ , \tag{C-7b}$$

we are now in possession of all the necessary tools for calculating in cylindric coordinates.

The Laplacian

An operator which often occurs in differential equations is the *Laplace operator* or *Laplacian*,

$$\boldsymbol{\nabla}^2 = \nabla_x^2 + \nabla_y^2 + \nabla_z^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \ . \tag{C-8}$$

In cylindrical coordinates this operator takes a different form, which may be found by squaring the resolution of the gradient (C-6). Keeping track of the order of the operators and basis vectors we get

$$\begin{aligned}
\boldsymbol{\nabla}^2 &= (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\phi \nabla_\phi + \boldsymbol{e}_z \nabla_z) \cdot (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\phi \nabla_\phi + \boldsymbol{e}_z \nabla_z) \\
&= (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\phi \nabla_\phi + \boldsymbol{e}_z \nabla_z) \cdot \boldsymbol{e}_r \nabla_r \\
&+ (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\phi \nabla_\phi + \boldsymbol{e}_z \nabla_z) \cdot \boldsymbol{e}_\phi \nabla_\phi \\
&+ (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\phi \nabla_\phi + \boldsymbol{e}_z \nabla_z) \cdot \boldsymbol{e}_z \nabla_z \\
&= \nabla_r^2 + \frac{1}{r}\nabla_r + \nabla_\phi^2 + \nabla_z^2 \ .
\end{aligned}$$

In the second line we have distributed the first factor on the terms of the second, and in going to the last line we have furthermore distributed the terms of the first factor, using the orthogonality of the basis and taking into account that differentiation after ϕ may change the basis vectors according to (C-7).

Finally, using (C-5) we arrive the cylindrical Laplacian,

$$\boxed{\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}} \ , \tag{C-9}$$

expressed in terms of the usual partial derivatives.

C.2 Spherical coordinates

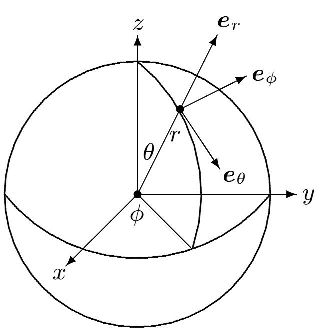
The treatment of spherical coordinates follows much the same pattern as cylindrical coordinates. Spherical or polar coordinates consist of the radial distance r, the polar angle θ and the azimuthal angle ϕ. If the z-axis is chosen as polar axis and the x-axis as the origin for the azimuthal angle, the transformation from spherical to Cartesian coordinates becomes,

$$x = r\sin\theta\cos\phi \ , \tag{C-10a}$$
$$y = r\sin\theta\sin\phi \ , \tag{C-10b}$$
$$z = r\cos\theta \ . \tag{C-10c}$$

The domain of variation for the spherical coordinates is $0 \le r < \infty$, $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$.

Curvilinear basis

The normalized tangent vectors along the directions of the spherical coordinate are,



Spherical coordinates and their basis vectors.

$$\boldsymbol{e}_r = \frac{\partial \boldsymbol{x}}{\partial r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \ , \tag{C-11a}$$
$$\boldsymbol{e}_\theta = \frac{1}{r}\frac{\partial \boldsymbol{x}}{\partial \theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \ , \tag{C-11b}$$
$$\boldsymbol{e}_\phi = \frac{1}{r\sin\theta}\frac{\partial \boldsymbol{x}}{\partial \phi} = (-\sin\phi, \cos\phi, 0) \ . \tag{C-11c}$$

They are orthogonal, so that an arbitrary vector field may be resolved after these directions,

$$\boldsymbol{V} = \boldsymbol{e}_r V_r + \boldsymbol{e}_\theta V_\theta + \boldsymbol{e}_\phi V_\phi \tag{C-12}$$

with $V_a = \boldsymbol{e}_a \cdot \boldsymbol{V}$ for $a = r, \theta, \phi$.

Resolution of the gradient

The gradient operator may also be resolved on the basis,

$$\boxed{\boldsymbol{\nabla} = \boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi} \ , \tag{C-13}$$

where

$$\nabla_r = \boldsymbol{e}_r \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial r} \ , \tag{C-14a}$$
$$\nabla_\theta = \boldsymbol{e}_\theta \cdot \boldsymbol{\nabla} = \frac{1}{r}\frac{\partial}{\partial \theta} \ , \tag{C-14b}$$
$$\nabla_\phi = \boldsymbol{e}_\phi \cdot \boldsymbol{\nabla} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \ . \tag{C-14c}$$

The non-vanishing derivatives of the basis vectors are

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \boldsymbol{e}_\theta, \qquad \frac{\partial \boldsymbol{e}_r}{\partial \phi} = \sin\theta\, \boldsymbol{e}_\phi \ , \tag{C-15a}$$

$$\frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_r, \qquad \frac{\partial \boldsymbol{e}_\theta}{\partial \phi} = \cos\theta \boldsymbol{e}_\phi \ , \tag{C-15b}$$

$$\frac{\partial \boldsymbol{e}_\phi}{\partial \phi} = -\sin\theta\, \boldsymbol{e}_r - \cos\theta\, \boldsymbol{e}_\theta \ . \tag{C-15c}$$

These are all the relations necessary for calculations in spherical coordinates.

The Laplacian

The Laplacian (C-8) becomes in this case

$$\begin{aligned}
\boldsymbol{\nabla}^2 &= (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \cdot (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \\
&= (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \cdot \boldsymbol{e}_r \nabla_r \\
&+ (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \cdot \boldsymbol{e}_\theta \nabla_\theta \\
&+ (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \cdot \boldsymbol{e}_\phi \nabla_\phi \\
&= \left(\nabla_r^2 + \frac{1}{r} \nabla_r + \frac{\sin\theta}{r \sin\theta} \nabla_r \right) + \left(\nabla_\theta^2 + \frac{\cos\theta}{r\sin\theta} \nabla_\theta \right) + \nabla_\phi^2 \ .
\end{aligned}$$

In the last step we have used the orthogonality and the derivatives (C-15). Finally, using (C-14) this becomes

$$\boxed{\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}} \ . \tag{C-16}$$

in standard notation.

