15

Nearly ideal flow

The most important fluids of our daily life, air and water, are lively and easily set into irregular motion. Getting out of a bathtub creates visible turbulence in the soapy water, whereas we have to imagine the unruly air behind us when we jog. Internal friction, or *viscosity*, seems to play only a minor role in these fluids. Other fluids, like honey and grease, are highly viscous, do not easily become turbulent, and would certainly be very hard to swim in. Being sluggish or lively is, however, not an absolute property of a fluid, but rather a condition of the circumstances under which it flows. Lava may be very sluggish in small amounts, but when it streams down a mountainside it appears to be quite lively. We shall later see that there is a way of characterizing fluid flow by means of a real number, called the Reynolds number, which is typically large for lively and small for sluggish flow.

The earliest quantitative model of fluid behavior goes back about 250 years and did not include viscosity. Although Newton introduced the concept, viscosity first entered fluid mechanics in its modern formulation almost a century later. Fluids with no viscosity have been called *ideal* or perfect, and sometimes "dry", because they do not hang on to containing surfaces. An ideal fluid is able to slip along container walls with finite velocity, whereas a real fluid has to adjust its velocity field so that it matches the speed of the container walls and the surfaces of moving objects.

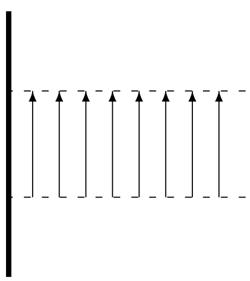
Although ideal fluids do not really exist, except for a component of superfluid helium close to zero kelvin, viscous fluids may nevertheless flow with such high Reynolds number that they behave as nearly ideal. Although never perfectly so. Around solid obstacles and near the walls of fluid conduits there will always be boundary layers in which viscosity becomes dominant. In this chapter we focus on nearly ideal flow, and postpone the discussion of viscosity to chapter 17.

15.1 The Euler equation

Leonhard Euler (1707–83). *Swiss mathematician who made fundamental contributions to calculus, geometry, number theory, and to practical ways of solving mathematical problems. His books on differential calculus (1755) and integral calculus (1768–70) have been especially useful for physics.*

In 1755 Euler was the first to write down Newton's second law of motion for fluids without viscosity. In such an *ideal* or *perfect* fluid the only forces at play are pressure and gravity, but in distinction to hydrostatics, these two forces are no more in balance, but give rise to an effective density of force $\boldsymbol{f}^* = \rho \boldsymbol{g} - \boldsymbol{\nabla} p$. Inserting this into the dynamic equation (14-35) and dividing by the density ρ, we obtain *Euler's equation* for ideal fluids,

$$\boxed{\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{v} = -\frac{1}{\rho} \boldsymbol{\nabla} p + \boldsymbol{g}} \ . \tag{15-1}$$



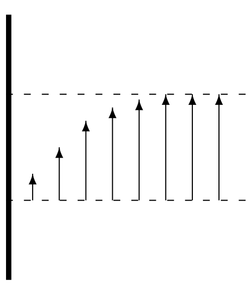
In an ideal fluid the tangential component of the velocity field is nonvanishing all the way to the boundary.

Together with the equation of continuity which we repeat here in the form (14-27),

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\rho = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{v} \ , \tag{15-2}$$

and a barotropic equation of state $p = p(\rho)$, we have obtained a closed set of five equations for the five fields, p, ρ, and $\boldsymbol{v}$ (assuming that $\boldsymbol{g}$ is known). If the equation of state also depends on temperature, $p = p(\rho, T)$, an equation for the temperature field must be added to the set (see chapter 27).

Partial differential equations require boundary conditions. As in hydrostatics, Newton's third law demands that the pressure must be continuous across any material interface (in the absence of surface tension). Furthermore, since moving fluids are normally contained in tubes, pipes, or other kinds of conduits that are impenetrable to the fluid, it follows that the velocity component normal to a containing surface must vanish, *i.e.* $\boldsymbol{n} \cdot \boldsymbol{v} = 0$. There are, however, no conditions on the tangential velocity in ideal flow.



In a real fluid the tangential component of the velocity field rises linearly at a boundary and joins smoothly with the general flow.

It is precisely here that the greatest difference between ideal and real viscous fluids come in. Ideal fluids are able to slip along container surfaces with finite tangential velocity, whereas the omnipresent viscosity of real fluids demands that also the tangential velocity must vanish on containing surfaces at rest. Viscosity thereby causes boundary layers to arise even in almost ideal fluids. Such boundary layers will in general soften any sharp transitions in tangential velocity (see chapter 25).

Exploring the extremes

Although we are now in possession of the fundamental equations for ideal fluids, solving them is another matter. The bad news about nonlinear partial differential equations is that they are very hard to solve and that makes it imperative to explore their usually simpler extreme limits. One such limit is the *linear approximation* in which the nonlinearities are dropped, another is *incompressible flow* in which the density is taken to be a constant, and still another is *steady flow* where all fields become time independent. In the following sections the various limits will be discussed in detail.

15.2 Small-amplitude sound waves

When you clap your hands together, you create momentarily a small disturbance in the air which propagates to your ear and tells you that something happened. The diaphragm of the loudspeaker in your radio vibrates in tune with the music carried by the radio waves and transfers its vibrations to the air where they continue as *sound*. No significant bulk movement of air takes place over longer distances, but locally the air oscillates rapidly back and forth with small spatial amplitude, and the velocity, density and pressure fields oscillate along with it.

Wave equation

Before the sound starts, the fluid is assumed to be in hydrostatic equilibrium with constant density ρ_0 and constant pressure p_0. For simplicity we assume that there is no gravity (see however problem 15.13). We now disturb the equilibrium by setting fluid into motion with a tiny velocity field, $\boldsymbol{v}(\boldsymbol{x}, t)$. The disturbance generates a small change in the density, $\rho = \rho_0 + \Delta\rho$, and in the pressure $p = p_0 + \Delta p$. Inserting this into the Euler equations we obtain to first order in the small quantities, $\boldsymbol{v}$, Δp, and $\Delta\rho$,

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\rho_0}\boldsymbol{\nabla}\Delta p \ , \qquad\qquad \frac{\partial \Delta\rho}{\partial t} = -\rho_0 \boldsymbol{\nabla}\cdot\boldsymbol{v} \ . \tag{15-3}$$

Differentiating the second equation and inserting the first, we obtain

$$\frac{\partial^2 \Delta\rho}{\partial t^2} = \boldsymbol{\nabla}^2 \Delta p \ . \tag{15-4}$$

Assuming that the fluid obeys a barotropic equation of state $p = p(\rho)$ we get a relation between the pressure and density corrections. From the definition of the bulk modulus (4-49) on page 76 we get to first order,

$$\Delta p = \frac{dp}{d\rho}\Delta\rho = \frac{K}{\rho}\Delta\rho \approx \frac{K_0}{\rho_0}\Delta\rho \ , \tag{15-5}$$

where K_0 is the bulk modulus in hydrostatic equilibrium. Inserting this into (15-4) we get a linear second order *wave equation* for the density correction,

$$\boxed{\frac{\partial^2 \Delta\rho}{\partial t^2} = c_0^2 \boldsymbol{\nabla}^2 \Delta\rho \ ,} \tag{15-6}$$

where we for convenience (and with foresight) have introduced the constant,

$$c_0 = \sqrt{\frac{K_0}{\rho_0}} \ . \tag{15-7}$$

It has the dimension of a velocity, and may as we shall see below be identified with the *speed of sound*. For water with $K_0 \approx 2.3$ GPa and $\rho_0 \approx 10^3$ kg/m^3 the sound speed comes to about $c_0 \approx 1500$ m/s ≈ 5500 km/h.

Isentropic sound speed in an ideal gas

Sound vibrations in air are normally so rapid that temperature equilibrium is never established, allowing us to assume that the oscillations take place without heat conduction, *i.e.* adiabatically. From the bulk modulus (4-49) for an isentropic ideal gas is $K_0 = \gamma p_0$ where γ is the adiabatic index, we obtain

$$\boxed{c_0 = \sqrt{\frac{\gamma p_0}{\rho_0}} = \sqrt{\gamma \frac{RT_0}{M_{\mathsf{mol}}}}\ .} \tag{15-8}$$

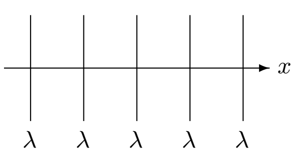
In the last step we have used the ideal gas equation $p_0 = \rho_0 RT_0/M_{\mathsf{mol}}$.

Example 15.2.1: For air at 20°C with $\gamma = 7/5$ and $M_{\mathsf{mol}} = 29$ g/mol, this comes to $c_0 \approx 343$ m/s ≈ 1235 km/h. Since the temperature of the homentropic atmosphere falls linearly with height according to (4-57), the speed of sound varies with height z above the ground as

$$c = c_0\sqrt{1 - \frac{z}{h_2}}\ , \tag{15-9}$$

where c_0 is the sound speed at sea level and $h_2 \approx 30$ km is the homentropic scale height (4-58). At the flying altitude of modern jet aircraft, $z \approx 10$ km, the sound speed has dropped to $c \approx 280$ m/s ≈ 1000 km/h. At greater heights this expression begins to fail because the the homentropic model of the atmosphere fails. Above $z = h_2$ it is of course meaningless.

Plane wave solution



Plane density wave propagating along the x-axis with wave length λ. There is constant density and pressure in all planes orthogonal to the direction of propagation.

An elementary plane density wave moving along the x-axis with wavelength λ, period τ, and amplitude $\rho_1 > 0$ is described by a density correction of the form,

$$\Delta\rho = \rho_1 \cos(kx - \omega t)\ , \tag{15-10}$$

where $k = 2\pi/\lambda$ is the wave number and $\omega = 2\pi/\tau$ the circular frequency. Inserting the plane wave into the wave equation (15-6), we obtain $\omega^2 = c_0^2 k^2$ or $c_0 = \omega/k = \lambda/\tau$. The surfaces of constant density (and pressure) are planes orthogonal to the direction of propagation, satisfying $kx - \omega t = \text{const}$. Differentiating this equation after time, we see that the planes of constant density move with velocity $dx/dt = \omega/k = c_0$, also called the *phase velocity* of the wave. This shows that c_0 given by (15-7) may indeed be identified with the speed of sound in the material.

Inserting the plane density wave into (15-5) we obtain,

$$\Delta p = p_1 \cos(kx - \omega t)\ , \qquad p_1 = c_0^2 \rho_1\ . \tag{15-11}$$

Inserting this result into the x-component of the Euler equation (15-3), and solving for the velocity field, we find

$$v_x = v_1 \cos(kx - \omega t)\ , \qquad v_1 = \frac{p_1}{\rho_0 c_0} = \frac{\rho_1}{\rho_0} c_0\ . \tag{15-12}$$

Since $v_y = v_z = 0$, the velocity field of a sound wave is always *longitudinal*, *i.e.* parallel to the direction of wave propagation. The corresponding spatial displacement u_x satisfying $v_x = \partial u_x/\partial t$ becomes,

$$u_x = -a_1 \sin(kx - \omega t) , \qquad a_1 = \frac{v_1}{\omega} . \tag{15-13}$$

The displacement is 90° out of phase with the density, pressure and velocity.

Validity of the approximation

It only remains to check whether the approximation of disregarding the inertial term is valid. The actual ratio between the magnitudes of the inertial term $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ and the pressure term $\boldsymbol{\nabla}p/\rho_0$ is,

$$\frac{|(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}|}{|\boldsymbol{\nabla}p/\rho_0|} \approx \frac{kv_1^2}{kp_1/\rho_0} = \frac{v_1}{c_0} = \frac{\rho_1}{\rho_0} = \frac{p_1}{K_0} = \frac{2\pi a_1}{\lambda} . \tag{15-14}$$

The condition for the approximation is thus that *the amplitude of the velocity oscillations should be much smaller than the speed of sound*, $v_1 \ll c_0$, or equivalently that $\rho_1 \ll \rho_0$, or $p_1 \ll K_0$, or $a_1 \ll \lambda/2\pi$.

Example 15.2.2 (Loudspeaker): A certain loudspeaker transmits sound to air at a frequency $\omega/2\pi = 1000$ Hz with diaphragm displacement amplitude of about $a_1 = 1$ mm. The velocity amplitude becomes $v_1 = a_1\omega \approx 6$ m/s, and since $v_1/c_0 \approx 1/57$ the approximation of leaving out the non-linear terms is well justified.

15.3 Steady incompressible flow

In many practical uses of fluids, the flow does not change with time, and is said to be *steady* or *stationary*. In this section we shall for simplicity also assume that the fluid is incompressible with constant density. Steady flow in compressible fluids will be analyzed in section 15.4 where we shall learn that *a fluid in steady flow is effectively incompressible when the flow velocity is everywhere much smaller than the speed of sound.*

Taking $\partial\boldsymbol{v}/\partial t = \boldsymbol{0}$ and $\rho = \rho_0$ in Eulers equations we now find,

$$\boxed{(\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} = -\frac{1}{\rho_0}\boldsymbol{\nabla}p + \boldsymbol{g} , \qquad \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 .} \tag{15-15}$$

Truly steady flow is like true incompressibility an idealization, only valid to a certain approximation. A river may flow steadily for days and weeks, while over several months seasonal changes in rainfall makes its water level rise and subside again. If one empties a cistern filled with water through a narrow pipe, the flow is almost steady in the pipe for a time, except that the level of water in the cistern slowly goes down and thereby reduces the flow speed in the pipe. In such cases, the flow should rather be called *nearly steady* or *quasistationary.*

This explanation begs the question of how a steady flow is established, even when boundary conditions are kept strictly constant. Since all flows start out being time dependent, there must be friction forces capable of removing the surplus energy from a lively flow to calm it down. Such forces are not included in the Euler equation, but even if viscosity is included (as it will in chapter 17), there is no guarantee that the flow will become steady after sufficiently long time. You just have to open the water faucet wide and watch the persistent turbulence in the kitchen sink to realize that steady flow will not always arise.

Bernoulli's theorem for incompressible fluid

The negative sign of the pressure gradient in Euler's steady-flow equation shows that in the absence of gravity a flow accelerating in a certain direction must be accompanied by a drop in pressure in the same direction, so that regions of high fluid velocity must generally have a lower pressure than regions of low velocity.

Daniel Bernoulli (1700–82). *Dutch born mathematician who made major contributions to the theory of elasticity, fluid mechanics, and the mechanics of musical instruments. Pointed out the relation between pressure and velocity in the world's first book on hydrodynamics,* Hydrodynamica*, which he published in 1738. (It was actually not Bernoulli who formulated the quantitative theorem which now bears his name, but rather Lagrange in his famous book on analytic mechanics from 1788 [7].)*

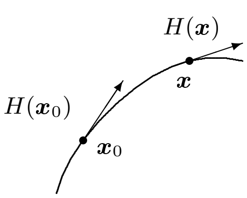
For an incompressible fluid with constant density $\rho = \rho_0$, Bernoulli's theorem states that the field,

$$\boxed{H = \frac{1}{2}\boldsymbol{v}^2 + \Phi + \frac{p}{\rho_0}\ ,} \tag{15-16}$$

is *constant along streamlines*. In the absence of gravity, the constancy of $H(\boldsymbol{x})$ implies as we foresaw that an increase in velocity squared $|\boldsymbol{v}|$ along a streamline must be compensated by a decrease in pressure p, and conversely. Notice that the first two terms in the Bernoulli function H make up the total mechanical (*i.e.* kinetic plus potential) energy of a unit mass particle, also called the *specific mechanical energy*. We shall discuss how the Bernoulli function is related to energy in section 16.11.

The proof of the theorem is straightforward. The comoving rate of change of H along a particle orbit is given by the material derivative (14-30), and since all fields are time independent in steady flow so that $D/Dt = \boldsymbol{v} \cdot \boldsymbol{\nabla}$, we get

$$\begin{aligned} \frac{DH}{Dt} &= \boldsymbol{v} \cdot \frac{D\boldsymbol{v}}{Dt} + \frac{D\Phi}{Dt} + \frac{1}{\rho_0}\frac{Dp}{Dt} \\ &= \boldsymbol{v} \cdot (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \frac{1}{\rho_0}\boldsymbol{v} \cdot \boldsymbol{\nabla} p + \boldsymbol{v} \cdot \boldsymbol{\nabla}\Phi \\ &= \boldsymbol{v} \cdot \left((\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \frac{1}{\rho_0}\boldsymbol{\nabla} p - \boldsymbol{g} \right) \\ &= 0\ , \end{aligned}$$



H is constant along a streamline in steady flow, i.e. $H(\boldsymbol{x}) = H(\boldsymbol{x}_0)$.

where we in the last step used the steady flow Euler equation (15-15). This shows that H is constant along any particle orbit, and thus along any streamline since streamlines coincide with particle orbits in steady flow. The modifications of Bernoulli's theorem necessary for compressible fluids will be discussed in section 15.4.

Terminology

The importance of Bernoulli's theorem for many practical hydrodynamical applications has led to several different terminologies. Since $\rho_0 H$ has the dimension of pressure, the term $\frac{1}{2}\rho_0 \boldsymbol{v}^2$ is often called the *dynamic pressure* as opposed to the *static pressure* p. The combination $p + \rho_0 \Phi$ is called the *effective pressure*. A point where the fluid has zero velocity, $\boldsymbol{v} = \boldsymbol{0}$, is called a *stagnation point* for the flow. In the absence of gravity, the pressure at a stagnation point is $p_0 = \rho_0 H$, also called the *stagnation pressure*.

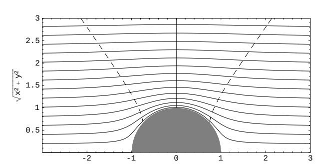
An often encountered engineering terminology is used in constant gravity, $\boldsymbol{g} = (0, 0, -g_0)$, where the Bernoulli field becomes (in flat-earth coordinates),

$$H = \frac{1}{2}\boldsymbol{v}^2 + g_0 z + \frac{p}{\rho_0} \ . \tag{15-17}$$

The vertical height z of a point on the streamline above some fixed reference level $z = 0$ is called the *static head*. Similarly $p/\rho_0 g_0$ is called the *pressure head*, and $v^2/2g_0$ is called the *velocity head*. The total head, H/g_0, is by Bernoulli's theorem the same everywhere along a streamline.

Applications of Bernoulli's theorem

Bernoulli's theorem is highly useful because most of the flows that we deal with in our daily lives are nearly ideal and nearly incompressible. Bernoulli's theorem often provides us with a first idea about the behavior of a flow in a given geometry. The drop in pressure accompanying an increase in flow velocity lies, for example, at the root of lift generation for both animals and machines, whether they swim or fly.



The flow has to quicken around an obstacle (here a half sphere) on the bottom of a stream and by Bernoulli's theorem, there will be a lower pressure above the obstacle, i.e. a lift.

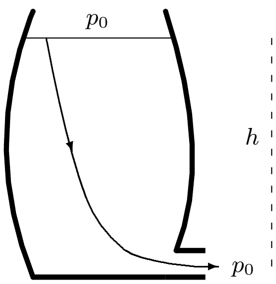
Example 15.3.1: Lift is usually thought of as beneficial, but that may not always be the case. Some fish hide by burrowing superficially into the sandy bottom of a stream. The fish's curved upper surface forces the passing water to speed up, leading to a pressure drop above the fish that grows with the square of the flow velocity. If the stream velocity increases, the fish may be lifted out of the sand, whether it wants to or not. That is probably why flatfish are indeed — flat.

A warning is in place at this point. Viscosity is never completely absent, but mostly it is negligible well away from the boundaries of the ducts and containers that we use to handle fluids. Exploiting the constancy of H along a streamline is always an approximation, and in any realistic problem there will be what the engineers call "head loss" due to viscosity, to secondary flow or turbulence generated by irregularities and sharp corners in bounding surfaces.

Bernoulli's theorem only furnishes a partial solution of Euler's equation (15-15). Although it relates the pressure and the magnitude of the velocity along a streamline, nothing is said about the direction of the velocity or how different streamlines relate to each other. To determine that one needs to solve the Euler equation with the boundary conditions imposed by the problem at hand.

Torricelli's Law

A barrel of wine with a little spout close to the bottom is a prototypical example of a fluid container. If the plug in the spout is suddenly removed, gravity makes wine emerge with considerable speed. Provided the spout is narrow compared to size of the barrel, a nearly steady flow will soon establish itself. This is a case where Bernoulli's theorem readily gives us the answer.



Wine running out of a barrel. The wine emerges with the same speed as it would have obtained by falling freely through the height h of the fluid in the barrel.

Consider a streamline starting near the top of the barrel and running with the flow down through the middle of the spout. Near the top at a height $z = h$ over the position of the spout, the fluid is almost at rest, *i.e.* $v \approx 0$. The pressure is atmospheric, $p = p_0$, and the gravitational potential may be taken to be $g_0 h$ so that

$$H_{\text{top}} = g_0 h + \frac{p_0}{\rho_0} \ . \tag{15-18}$$

Just outside the spout the fluid has some horizontal velocity v, and the pressure is also atmospheric, $p = p_0$, with no contribution from gravity, because the potential has been chosen to vanish here. Hence

$$H_{\text{bottom}} = \frac{1}{2}v^2 + \frac{p_0}{\rho_0} \ . \tag{15-19}$$

Equating the values of H at the top and the bottom we find

$$\frac{1}{2}v^2 + \frac{p_0}{\rho_0} = \frac{p_0}{\rho_0} + g_0 h \ ,$$

which has the solution

$$\boxed{v = \sqrt{2 g_0 h}} \ . \tag{15-20}$$

Surprisingly, this is exactly the same velocity as a drop of wine would have obtained by falling freely from the top of the barrel to the spout.

Evangelista Torricelli (1608 – 1647). *Italian physicist. Constructed the first mercury barometer and noticed that the barometric pressure varied from day to day. Served as companion and secretary for Galileo in the last months of Galileo's life.*

In a sense the barrel acts as a device for diverting the vertical momentum of the falling liquid into the horizontal direction (see chapter 16). The same result holds if the spout is replaced by a pipe that may not even be horizontal, but may turn and twist. As long as friction can be ignored in the pipe, the exit velocity will be equal to the free-fall velocity from the fluid surface at the top of the barrel to the exit from the pipe. This result is called *Torricelli's Law*, and was in its time a major step forward in the understanding of fluid mechanics.

Example 15.3.2: A large cylindrical wine barrel has diameter 1 m and height 2 m. According to Torricelli's law the wine will emerge from the spout with the free-fall speed of about 6.3 m/s. If the spout opening has diameter 5 cm, about 12.3 liters of wine will be spilled on the floor per second. At this rate it would take 2 minutes to empty the barrel, but we shall see below that it actually takes the double because the level sinks.

We have not specified precisely which streamline to take. Other streamlines will start out in different places at the top of the barrel, but all will begin with essentially zero velocity, be subject to the same gravitational potential, and end up in roughly the same place. The calculation of the velocity must therefore lead to the same result, except for streamlines running very near to the walls of the barrel and spout, where the unavoidable viscosity slows down the flow of wine. The wine emerges from the spout with the same velocity all over the opening, a flow pattern that is sometimes called *plug flow*.

Time to empty a wine barrel

If the barrel has constant cross section A_0, Leonardo's law (14-12) tells us that the average vertical flow velocity in the barrel is $v_0 = vA/A_0$ where A is the cross section of the spout and $v = \sqrt{2g_0 z}$ the average horizontal flow velocity through it when the water level is z. Since $dz/dt = -v_0$, we obtain the following differential equation for quasistationary emptying of the barrel,

$$\frac{dz}{dt} = -\frac{A}{A_0}\sqrt{2g_0 z}\ . \tag{15-21}$$

Integrating this equation we obtain the time it takes to empty the barrel from the original height $z = h$

$$T = \int_h^0 \frac{dt}{dz}\,dz = -\int_h^0 \frac{A_0}{A}\frac{dz}{\sqrt{2g_0 z}} = \frac{A_0}{A}\sqrt{\frac{2h}{g_0}}\ . \tag{15-22}$$

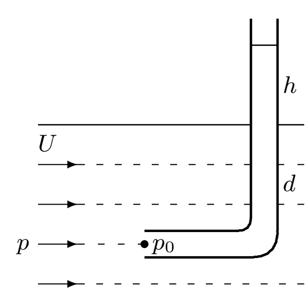
This time equals the free-fall time from height h multiplied with the usually huge ratio of the barrel and spout cross sections. For the cylindrical wine barrel of example 15.3.2 the free fall time is 0.64 seconds and it takes about 400 times longer, about 4 minutes, to empty the barrel.

Pitot's tube

Henri Pitot (1695–1771). *French mathematician, astronomer, and hydraulic engineer. Invented the Pitot tube around 1732.*

Fast aircraft often have a sharply pointed nose which on closer inspection is seen to end in a little open tube. On other aircraft the tube mostly sticks orthogonally out from the side and is then bent forward into the air stream. This device is called a *Pitot tube*, and in today's technology it is used in many variant forms to measure flow speeds in gases and liquids. In its simplest and original form, the Pitot tube is just an open glass tube bent through a right angle. The tube is lowered into a steadily streaming river with velocity U, one end turned horizontally towards the current and the other vertically in the air above. The flow will stem water up into the vertical part of the tube, until the hydrostatic pressure of the water column balances the dynamic pressure from the flow. After the flow has steadied, the water in the tube rises to a height z above the river surface.

Since the water speed must be zero at the entrance to the horizontal part of the tube, a horizontal streamline arriving here from afar must come to an end



The principle of the Pitot tube. The pressure increase along the stagnating streamline must equal the weight of the raised water column.

in a *stagnation point* where the velocity vanishes. The gravitational potential is constant everywhere along the horizontal streamline, and can be disregarded, so that Bernoulli's theorem yields,

$$\frac{p_0}{\rho_0} = \frac{1}{2}U^2 + \frac{p}{\rho_0} , \tag{15-23}$$

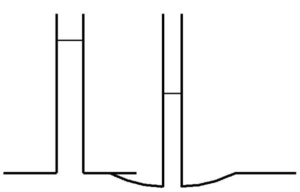
where p is the pressure at infinity, and p_0 is the stagnation pressure. The excess stagnation pressure $\Delta p = p_0 - p = \frac{1}{2}\rho_0 U^2$ must also equal the extra hydrostatic pressure from the raised water column, $\Delta p = \rho_0 g_0 h$, and we get

$$U = \sqrt{2g_0 h} . \tag{15-24}$$

Again we find the simple and surprising result that the speed of the water is exactly the same as it would be after a free fall from the height h. At a typical flow speed of 1 m/s the water in the tube is raised 5 cm above the river level.

Example 15.3.3: Farmers know better than to leave the barn door open towards the wind in a storm, even if they are not familiar with Bernoulli's theorem. A gust of wind will not only decrease the pressure above the barn roof because the wind has to move faster above the roof than at the ground, but the Pitot effect will also increase the pressure inside, giving the roof a double reason to blow off.

Example 15.3.4 (Water scoop): Forest fires are sometimes combated by aircraft dropping large amounts of sea or lake water. To avoid excessive landing and take-off, the aircraft collects water by lowering a scoop into the water while flying slowly at very low altitude. If the scoop turns directly forward and the aircraft velocity is U, it can like the Pitot tube raise the water to a maximal height, $h = U^2/2g_0$. Even for a speed as low as $U = 120$ km/h ≈ 33 m/s, this comes to $h = 55$ m. In practice, the height of the water tank over the lake surface is much smaller, so that the water ideally could arrive in the tank with nearly maximal speed U. A scoop with an opening area of just $A \approx 300$ cm^2 can deliver water at a rate of $Q = UA \approx 1$ m^3/s. Turbulence and friction lowers this somewhat, but typically such an aircraft can collect 6 m^3 in just 12 s.



Sketch of a Venturi tube experiment. Water streams from the left through a constriction in the horizontal tube where the pressure is lower than in the tube outside the constriction because of the Venturi effect, as shown by the lower water level in the vertical tube.

Giovanni Batista Venturi (1746-1822). *Italian physicist and engineer. Studied how fluids behave in a duct with a constriction.*

The Venturi effect

A simple duct with slowly varying cross section carries a constant volume flux Q of incompressible fluid. For simplicity we assume the duct is horizontal, such that gravity can be disregarded. Taking a streamline running horizontally through the duct, we obtain from Bernoulli's theorem that $H = \frac{1}{2}\boldsymbol{v}^2 + p/\rho_0$ takes the same value everywhere in the duct. Approximating the velocity with its average $U = Q/A$ over the cross section A, the pressure becomes,

$$p = \rho_0 \left(H - \frac{Q^2}{2A^2} \right) . \tag{15-25}$$

This demonstrates the *Venturi effect*: The pressure decreases when the duct cross section decreases (and conversely).

15.4 Steady compressible flow

In steady compressible flow, the velocity, pressure, and density are independent of time, and Euler's equation and the continuity equation take the form,

$$\boxed{(\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \boldsymbol{g}\ , \qquad \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0\ .} \tag{15-26}$$

We shall for simplicity assume that the equation of state is barotropic, $p = p(\rho)$ or $\rho = \rho(p)$, such that we have a closed set of five field equations for the five fields, $\boldsymbol{v}$, ρ, and p. In this section we shall mostly ignore gravity.

Effective incompressibility

We shall now demonstrate the claim made in the preceding section that *all fluids in steady flow are effectively incompressible when the flow speed is everywhere much smaller than the local speed of sound.*

Writing the divergence condition in the form $\boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = \rho \boldsymbol{\nabla} \cdot \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\rho = 0$, and using the relation $\boldsymbol{\nabla} p = (dp/d\rho)\boldsymbol{\nabla}\rho = c^2 \boldsymbol{\nabla}\rho$ together with Euler's equation without gravity, we find the exact result,

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = -\frac{1}{\rho}(\boldsymbol{v} \cdot \boldsymbol{\nabla})\rho = -\frac{1}{\rho c^2}(\boldsymbol{v} \cdot \boldsymbol{\nabla})p = \frac{\boldsymbol{v} \cdot (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}}{c^2}\ . \tag{15-27}$$

Applying the Schwarz inequality to the numerator (see problem 15.6) we get

$$|\boldsymbol{\nabla} \cdot \boldsymbol{v}| \leq \frac{|\boldsymbol{v}|^2}{c^2}\, |\boldsymbol{\nabla} \boldsymbol{v}|\ , \tag{15-28}$$

where $|\boldsymbol{\nabla} \boldsymbol{v}| = \sqrt{\sum_{ij} (\nabla_j v_i)^2}$ is the norm of the velocity gradient matrix. This clearly demonstrates that for $|\boldsymbol{v}| \ll c$ the divergence is much smaller than the general velocity gradients, making the divergence condition, $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, a good approximation.

The Mach number

The local speed of sound, $c = \sqrt{K/\rho} = \sqrt{dp/d\rho}$, plays an important role for compressible fluids. The ratio of the local flow speed $\boldsymbol{v}$ (relative to the a static solid object or boundary wall) and the local sound speed c is denoted by,

$$\boxed{\mathsf{Ma} = \frac{|\boldsymbol{v}|}{c}\ ,} \tag{15-29}$$

called the *Mach number*. Whereas a fluid in steady flow is effectively incompressible when the local Mach number is everywhere small, $\mathsf{Ma} \ll 1$, a fluid is truly compressible if the Mach number anywhere is comparable to unity or larger.

Ernst Mach (1838-1916). *Austrian positivist philosopher and physicist. Made early advances in psychophysics, the physics of sensations. His rejection of Newton's absolute space and time prepared the way for Einstein's theory of relativity. Proposed the principle that inertia results from the interaction between a body and all other matter in the universe.*

Example 15.4.1: Waving your hands in the air, you generate flow velocities at most of the order of meters per second, corresponding to $\mathsf{Ma} \approx 0.01$. Driving a car at 120 km/h $\approx$ 33 m/s corresponds to $\mathsf{Ma} \approx 0.12$. A passenger jet flying at a height of 10 km with velocity about 900 km/h $\approx$ 250 m/s has $\mathsf{Ma} \approx 0.9$. Even if this speed is subsonic, considerable compression of the air must occur especially at the front of the wings and body of the aircraft. The Concorde and modern fighter aircraft operate at supersonic speeds at Mach 2-3, whereas the Space Shuttle enters the atmosphere at the hypersonic speed of Mach 25. The strong compression of the air at the frontal parts of such aircraft create shock waves that appear to us as sonic booms.

Bernoulli's theorem for barotropic fluids

For compressible fluids, Bernoulli's theorem is still valid in a slightly modified form. If the fluid obeys a barotropic equation of state, $\rho = \rho(p)$, the Bernoulli function becomes,

$$H = \frac{1}{2}\boldsymbol{v}^2 + \Phi + w(p) \ , \tag{15-30}$$

where

$$w(p) = \int \frac{dp}{\rho(p)} \tag{15-31}$$

is the *pressure function*, previously defined in (**??**). The proof of the modified Bernoulli theorem is elementary and follows the same lines as before, using that $Dw/Dt = (dw/dp)Dp/Dt = (1/\rho)Dp/Dt$.

The most interesting barotropic fluid is an ideal gas with adiabatic index γ. Writing the isentropic condition (4-48) as $\rho = Cp^{-1/\gamma}$, and carrying out the integral, we find after a little rearrangement,

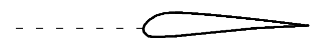
$$w = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = c_p T \qquad\qquad c_p = \frac{\gamma}{\gamma - 1}\frac{R}{M_{\mathsf{mol}}} \ . \tag{15-32}$$

In the last step we have used the ideal gas law (4-24), and the definition (4-51) of the specific heat at constant pressure. Thus, in the absence of gravity, a drop in velocity along a streamline in isentropic flow is accompanied by a rise in temperature (as well as a rise in pressure and density).

There is a conceptual subtlety in understanding isentropic steady flow because of the unavoidable heat conduction that takes place in all real fluids. Since truly steady flow lasts "forever", one might think that there would be ample time for a local temperature change to spread to all of the fluid, independently of how badly it conducts heat. But remember that steady flow is not static, and any heat added to a stream will be whisked away by the steady flow. So provided the flow is sufficiently fast, heat conduction will have little effect. The physics of heat and flow is discussed in chapter 27.

Stagnation point temperature rise

An object moving through a fluid has at least one stagnation point in the front where the fluid is at rest relative to the object. There is also at least one stagnation point at the rear of a body, but turbulence will generally disturb the flow in this region and tangle the streamlines, so that we cannot use Bernoulli's theorem to calculate the physical properties near the rearwards stagnation point.



A static airfoil in an airstream coming in horizontally from the left. The shown streamline (dashed) ends in the forward stagnation point.

At the forward stagnation point the gas is compressed and the temperature will always be higher than in the fluid at large. In the frame of reference where the object is at rest and the fluid asymptotically moves with constant speed U and temperature T the flow is steady, and we find from (15-30) and (15-32) in the absence of gravity,

$$\frac{1}{2}U^2 + c_p T = c_p T_0 \ ,$$

where T_0 is the stagnation point temperature. Accordingly the temperature rise due to adiabatic compression becomes,

$$\boxed{\Delta T = T_0 - T = \frac{U^2}{2c_p} = \frac{1}{2}U^2 \frac{\gamma - 1}{\gamma} \frac{M_{\text{mol}}}{R}} \ . \qquad (15\text{-}33)$$

Notice that the stagnation temperature rise depends only on the velocity difference between the body and the fluid far from the body, and not on the pressure or density of the gas.

> **Example 15.4.2:** Taking $\gamma = 7/5$ we obtain for a car moving at 100 km/h a stagnation point temperature rise of merely 0.4 K. At the front of a passenger jet travelling at 900 km/h the stagnation point rises a moderate 31 K, whereas a supersonic aircraft travelling at 2300 km/h suffers a stagnation point temperature rise of about 200 K. When a reentry vehicle hits the dense atmosphere with a speed of 3 km/s the stagnation point temperature rise becomes nearly 4500 K. At the full free fall speed from outer space, 11.2 km/s, the stagnation point temperature would be 63,000 K, but the density of the air in the outer reaches of the atmosphere would probably be so small that the continuum assumption is no more fulfilled.

Whether the tip of a moving object actually attains the stagnation point temperature depends on primarily on how efficiently heat is conducted away from this region by the material of the object. The moving object is usually solid with a vastly greater heat capacity than the air near the stagnation point. In addition to adiabatic compression viscous friction also produces heat, and for extreme aircraft such as the Space Shuttle it has been necessary to use special ceramic materials to withstand temperatures that are otherwise capable of melting and burning metals. Freely falling meteorites appear as shooting stars in the sky because of viscous friction.

Stagnation point properties

It is often convenient to express the ratio of the stagnation point temperature and the local temperature in terms of the local Mach number $\mathsf{Ma} = |\boldsymbol{v}|/c$ where $c = \sqrt{\gamma RT/M_{\mathsf{mol}}}$ is the local sound velocity. From (15-33) we obtain

$$\frac{T_0}{T} = 1 + \frac{1}{2}(\gamma - 1)\mathsf{Ma}^2 \ . \tag{15-34}$$

Even if the streamline does not actually end in a stagnation point, T_0 will be a constant for the streamline because it represents the value of the Bernoulli function, $H = c_p T_0$, which constant along the streamline. The stagnation temperature may be thought of as the temperature that would be obtained if a tiny object was inserted into the flow far downstream from the observation point.

The stagnation density ρ_0 and pressure ρ_0 are obtained from the isentropic conditions,

$$T\rho^{1-\gamma} = T_0\rho_0^{1-\gamma} \ , \qquad\qquad T^\gamma p^{1-\gamma} = T_0^\gamma p_0^{1-\gamma} \ ,$$

which may be written,

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{\gamma-1}} \ , \qquad\qquad \frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} \ . \tag{15-35}$$

Again it follows that ρ_0 and p_0 are constants for any given streamline.

Sonic point properties

A point where the velocity of a steady flow passes through the local velocity of sound is called a *sonic point*. The collection of sonic points typically form a sonic surface, for example in the duct flow to be discussed below. The sonic point temperature may be calculated from the stagnation point temperature (15-34) by setting $\mathsf{Ma} = 1$,

$$\frac{T_1}{T_0} = \frac{2}{\gamma + 1} \ . \tag{15-36}$$

For $\gamma = 7/5$, the ratio is $T_1/T_0 = 5/6$. Multiplying (15-34) by T_1/T_0 and rearranging the expression, we obtain the sonic temperature in terms of the local temperature and Mach number,

$$\frac{T_1}{T} = 1 + \frac{\gamma - 1}{\gamma + 1}\left(\mathsf{Ma}^2 - 1\right) \ . \tag{15-37}$$

The sonic temperature T_1 is, like the stagnation temperature T_0, a constant for any streamline, independently of whether the flow actually becomes sonic on this streamline. The sonic density ρ_1 and pressure p_1 may similarly be obtained from stagnation values and related to the local Mach number by relations like (15-35).

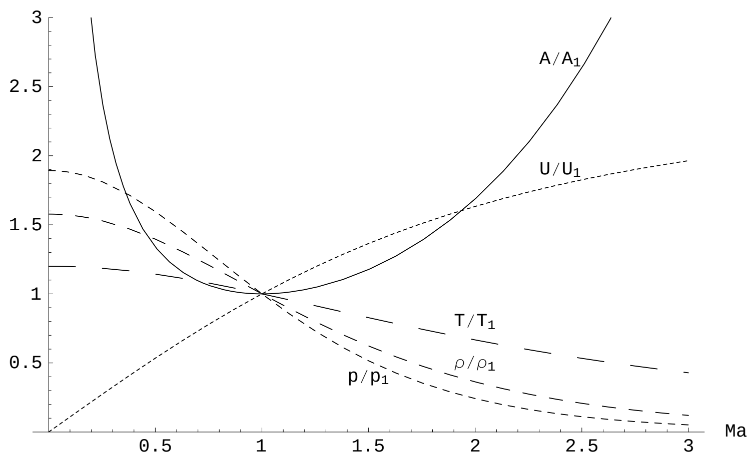
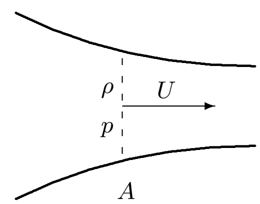


Figure 15.1: *Flow in a slowly varying duct. Plot of the ratio of local to sonic values as a function of the Mach number. The ratio A/A_1 is fully drawn, T/T_1 has large dashes, ρ/ρ_1 medium dashes , p/p_1 small dashes, and U/U_1 dotted.*

Ideal gas flow in duct with slowly varying cross section



Flow in a slowly converging duct. All parameters are assumed constant on every planar cross section A.

Consider now an ideal gas flowing through a duct with so slowly varying cross section that the temperature T, density ρ, pressure p, and normal velocity U may be assumed to be constant over any given (but otherwise arbitrary) planar duct cross section of area A. We are interested in determining the conditions under which the flow may become sonic in the duct.

The constancy of the mass flux along the duct,

$$Q = \rho A U \ , \tag{15-38}$$

provides us with a relation between the duct area and the local Mach number. At the sonic point we have $\rho A U = \rho_1 A_1 U_1$, and using that $U = \mathsf{Ma}\, c$ and $U_1 = c_1$ where c and c_1 are the local and sonic sound velocities, we find

$$\frac{A}{A_1} = \frac{\rho_1 U_1}{\rho U} = \frac{1}{\mathsf{Ma}} \frac{\rho_1}{\rho} \frac{c_1}{c} = \frac{1}{\mathsf{Ma}} \frac{\rho_1}{\rho} \sqrt{\frac{T_1}{T}} = \frac{1}{\mathsf{Ma}} \left(\frac{T_1}{T}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} .$$

In the last step we also used that $\rho_1/\rho = (T_1/T)^{1/(\gamma-1)}$. Inserting T_1/T from (15-37), we arrive at the sought for relation,

$$\boxed{\frac{A}{A_1} = \frac{1}{\mathsf{Ma}} \left(1 + \frac{\gamma - 1}{\gamma + 1} \left(\mathsf{Ma}^2 - 1\right)\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}} \ . \tag{15-39}$$

This function is shown (for $\gamma = 7/5$) as the fully drawn curve in fig. 15.1 together with the corresponding temperature, density, pressure, and velocity ratios.

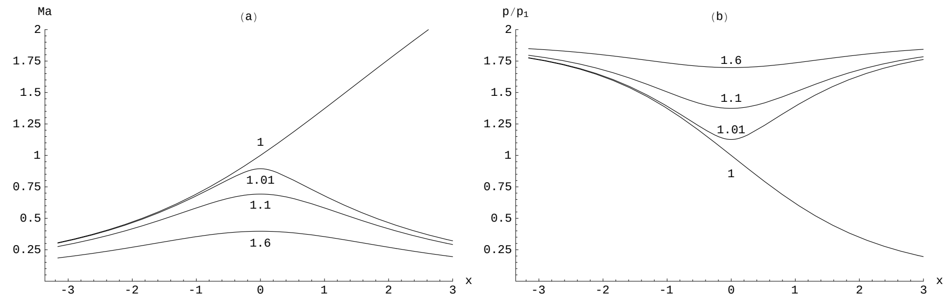
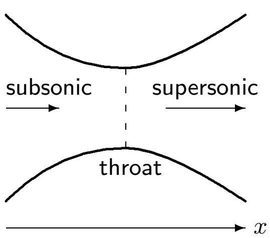


Figure 15.2: *Simple model of a constricted duct, $A(x) = A_{\mathsf{throat}} + kx^2$, with $A_{\mathsf{throat}} = 1$ and $k = 0.1$ (and $\gamma = 7/5$).* **(a)** *Plot of the Mach number $\mathsf{Ma}(x)$ as a function of the duct coordinate x. The different curves are labelled by the ratio A_{throat}/A_1.* **(b)** *The pressure ratio $p(x)/p_1$ under the same conditions. Notice that the pressure is lowest in the throat (the Venturi effect) for all $A_1 < A_{\mathsf{throat}}$, but drops to much lower values for $A_1 = A_{\mathsf{throat}}$.*



A subsonic flow may become supersonic in a duct with a constriction where it changes from converging to diverging. The transition happens at the narrowest point, the throat, where the cross section $A(x)$ is minimal.

The main observation to make from fig. 15.1 is that the local area A has a minimum at $\mathsf{Ma} = 1$. This shows that it is only possible to make a *smooth* transition from subsonic to supersonic flow by sending the gas through a constriction in which the duct first converges and then diverges, forming a *throat* at the point where its area is minimal. If the flow parameters are set up such that the sonic area A_1 precisely equals the physical area A_{throat} of the throat, the subsonic flow entering the converging part of the duct will travel down the left hand branch of the fully drawn curve in fig. 15.1, increasing its velocity until the Mach number reaches unity exactly at the throat. Having passed the throat, the flow is now supersonic and travels up the right hand branch of the fully drawn curve while increasing its velocity further in the diverging part of the constriction.

The sonic point is not always reached. After all, flutes and other musical instruments, including our voices, have constrictions in the airflow that do not give rise to supersonic flow (which would surely destroy the music). If the physical throat area is larger than the sonic area, $A_{\mathsf{throat}} > A_1$, the Mach number does not reach unity at the throat. The maximal Mach number at the throat, $\mathsf{Ma}_{\mathsf{throat}}$, is determined from (15-39) by setting $A = A_{\mathsf{throat}}$. Graphically, it may be read off from the left hand branch of fully drawn curve in fig. 15.1 at the point it crosses through A_{throat}/A_1. In fig. 15.2 this is illustrated in a simple model.

Suppose now that the duct is carrying a subsonic flow through the throat with $A_{\mathsf{throat}} > A_1$, and that we begin to *throttle* the throat by diminishing its area A_{throat} without changing the flow parameters at the entry. Since the sonic area A_1 is determined by the entry values, it will not change, so the throttling may continue until $A_{\mathsf{throat}} = A_1$. At this point the flow becomes sonic at the throat. If we now continue throttling, the entry parameters are at least forced to change in such a way that the the sonic area follows the diminishing throat area, $A_1 = A_{\mathsf{throat}}$, and the flow stays sonic at the throat. A duct with a throat operating at the sonic point is said to be *choked* because there is no way you can increase the mass flow through the throat by varying the entry parameters (for further details see [16, 37]).

15.5 Vorticity

The value of the Bernoulli field $H(\boldsymbol{x})$ in a point $\boldsymbol{x}$ is only a function of the streamline going through this point. Different streamlines will in general have different values of H. This can be illustrated by considering the case of Newton's rotating bucket which was discussed before on page 121.

Bernoulli field in Newton's bucket

In the corotating (bucket) coordinate system, the fluid is at rest and subject to both the force of gravity and the centrifugal force. Since the velocity vanishes, the Bernoulli function becomes

$$H_0 = \frac{p}{\rho_0} + g_0 z - \frac{1}{2}\Omega^2 r^2 \ , \qquad \text{(15-40)}$$

where the last term is the centrifugal potential and $r = \sqrt{x^2 + y^2}$ the distance from the axis of rotation. Hydrostatic balance ensures that $\boldsymbol{\nabla} H_0 = 0$ so that H_0 is a true constant, independent of both r and z. Solving for the pressure we get,

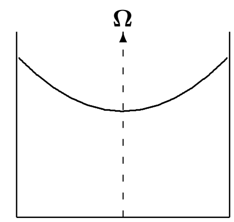
$$p = \rho_0 H_0 - g_0 z + \frac{1}{2}\Omega^2 r^2 \ . \qquad \text{(15-41)}$$

The constancy of the pressure on the open surface, determines its parabolic shape $z = \Omega^2 r^2/2g_0 + \text{const}$.

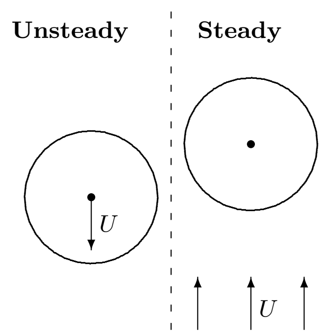
In the non-rotating (laboratory) system, there is no centrifugal force, but the fluid moves steadily with velocity $v = r\Omega$, and the streamlines are concentric circles. The Bernoulli function becomes in this system,

$$H(r) = \frac{1}{2}\Omega^2 r^2 + g_0 z + \frac{p}{\rho_0} = \Omega^2 r^2 + H_0 \ . \qquad \text{(15-42)}$$

where in the last step we have made use of (15-40). The Bernoulli function depends in this case on the radial distance from the axis, but is of course constant on the streamlines as Bernoulli's theorem ensures us. H is furthermore independent of z, and we shall see below that this also follows from general rules.



The water surface is parabolic in a bucket rotating with angular velocity Ω.



A body moving at constant speed through a fluid is physically equivalent to the same body being at rest in a steady flow which is asymptotically uniform.

Asymptotically uniform flow

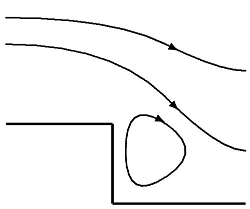
It is often possible to relate the values of H for different streamlines. A general and frequently occurring example is a body moving with constant velocity through a fluid otherwise at rest, were it not for the disturbance created by the body. The relativity of motion in Newtonian mechanics tells us that this unsteady flow is physically equivalent to a steady flow around a stationary body in a fluid which far away from the body moves with constant velocity under constant pressure. We tacitly used this equivalence in the discussion of the water scoop above.

At great distances from the body the flow will have the same velocity $\boldsymbol{U}$ and in the absence of gravity the same pressure P for all streamlines. Consequently, the Bernoulli field must take the same value

$$H_0 = \frac{1}{2}\boldsymbol{U}^2 + \frac{P}{\rho_0} \tag{15-43}$$

for all streamlines coming in from afar. If one can be sure that all streamlines around the body originate infinitely far away, then H must take the same value all over space, *i.e.* $H(\boldsymbol{x}) = H_0$.

Vorticity field



Closed streamlines may appear when a fluid flows past an edge.

The simple result that H is spatially constant for asymptotically uniform flow, is spoiled by the possibility that there may be streamlines forming closed curves unconnected with the flow at infinity. This was the case for Newton's bucket in the non-rotating coordinate system where the streamlines were concentric circles, but common experience indicates that such circulating flow may occur in the wake of the disturbance created by the shape of a container or a moving body. We are thus naturally led to the study of local circulation in a fluid, and we shall see that H *is* in fact globally constant for flow completely without local circulation anywhere.

The general conditions for H being constant may be calculated from its gradient, using Euler's equation for steady incompressible flow (15-15). Employing index notation, we find

$$\begin{aligned}\nabla_i H &= \frac{1}{2}\nabla_i \boldsymbol{v}^2 + \frac{1}{\rho_0}\nabla_i p + \nabla_i \Phi \\ &= \boldsymbol{v}\cdot\nabla_i \boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{\nabla} v_i \\ &= (\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{v}))_i \ .\end{aligned}$$

This result, which is also valid for barotropic compressible fluids, allows us to write

$$\boxed{\boldsymbol{\nabla} H = \boldsymbol{v}\times\boldsymbol{\omega}} \ , \tag{15-44}$$

where we have defined a new field, the *vorticity field*

$$\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v} \ . \tag{15-45}$$

The vorticity field also goes back to Cauchy (1841) and is a quantitative measure of the local circulation in the fluid.

Example 15.5.1: The curl of the field $\boldsymbol{v} = (x^2, 2xy, 0)$ is $\boldsymbol{\omega} = (0, 0, 2y)$, whereas the curl of $\boldsymbol{v} = (y^2, 2xy, 0)$ vanishes.

In any region where the vorticity field vanishes, we have $\boldsymbol{\nabla} H = 0$, so that the Bernoulli field must take one value only in that region, *i.e.* $H(\boldsymbol{x}) = H_0$. Flow completely free of vorticity everywhere is called *irrotational* flow and leads to a particularly simple formalism that we shall investigate in the following section.

Vorticity and local rotation

The prime example of a flow with vorticity is a steadily rotating rigid body, for example Newton's bucket in the laboratory system. If the rotation vector of the body is $\mathbf{\Omega}$, the velocity field becomes $\boldsymbol{v} = \mathbf{\Omega} \times \boldsymbol{x}$, and the vorticity

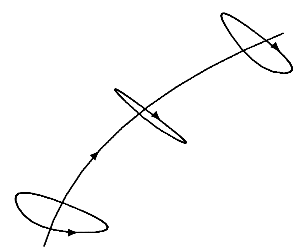
$$\begin{aligned}\boldsymbol{\omega} &= \boldsymbol{\nabla} \times (\mathbf{\Omega} \times \boldsymbol{x}) \\ &= \mathbf{\Omega}(\boldsymbol{\nabla} \cdot \boldsymbol{x}) - (\mathbf{\Omega} \cdot \boldsymbol{\nabla})\boldsymbol{x} \\ &= 3\mathbf{\Omega} - \mathbf{\Omega} = 2\mathbf{\Omega} \ .\end{aligned}$$

The vorticity is in this case constant and equal to twice the rotation vector. The factor of 2 seems a bit strange but is a general result which may be verified by calculating the gradient of H for Newton's bucket. We find using (15-44)

$$\boldsymbol{\nabla} H = (\mathbf{\Omega} \times \boldsymbol{x}) \times 2\mathbf{\Omega} = 2\Omega^2(x, y, 0) \ ,$$

which agrees with the gradient of the Bernoulli field calculated from (15-42).

Vortex lines



Around a vortex line there is local circulation of fluid.

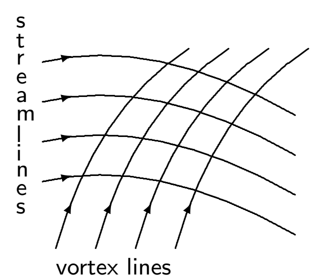
The field lines of the vorticity field are called *vortex lines*, and are defined as curves that are everywhere tangent to the vorticity field. They are solutions to the ordinary differential equation

$$\frac{d\boldsymbol{x}}{ds} = \boldsymbol{\omega}(\boldsymbol{x}, t_0) \ . \tag{15-46}$$

In a steady flow these lines are fixed curves in space, just like the streamlines.

Bernoulli's theorem, $(\boldsymbol{v} \cdot \boldsymbol{\nabla})H = 0$, follows immediately from (15-44) by multiplying with $\boldsymbol{v}$. Similarly, by multiplying with $\boldsymbol{\omega}$ we obtain

$$\boxed{(\boldsymbol{\omega} \cdot \boldsymbol{\nabla})H = 0} \ . \tag{15-47}$$



The Bernoulli field is constant on surfaces made from vortex lines and streamlines.

The Bernoulli field is therefore also constant along vortex lines.

Together these results imply that *the Bernoulli field is constant on the two-dimensional surfaces formed by combining vortex lines and streamlines.* Since streamlines for circulating fluid tend to form closed curves, these surfaces will be long tubes, called *vortex tubes*. For Newton's bucket the vortex tubes are cylinders concentric with the axis of rotation, and this explains that the Bernoulli field cannot depend on z, as we noticed earlier.

Equation of motion for vorticity

The vorticity field is derived from the velocity field, so the equation of motion for the vorticity field must follow from the equation of motion for the velocity field, (15-1). Eliminating the pressure by means of the Bernoulli field (15-16) and retracing the steps leading to (15-44), Euler's equation for an incompressible fluid may be written,

$$\frac{\partial \boldsymbol{v}}{\partial t} = -(\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{v} - \frac{1}{\rho_0}\boldsymbol{\nabla} p + \boldsymbol{g} = \boldsymbol{v} \times \boldsymbol{\omega} - \boldsymbol{\nabla} H \ . \tag{15-48}$$

The end result is also valid for compressible barotropic fluids. For steady flow where $\partial \boldsymbol{v}/\partial t = \boldsymbol{0}$, we recover of course (15-44). The equation of motion for vorticity is obtained by calculating the curl of both sides of this equation, using that the curl of a gradient vanishes, $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times H) = \boldsymbol{0}$, arriving at

$$\boxed{\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\omega}) \ .} \tag{15-49}$$

There is a major lesson to draw from this equation. If the vorticity vanishes identically, $\boldsymbol{\omega}(\boldsymbol{x}, t) = \boldsymbol{0}$, everywhere inside a region V of space at one instant of time t, then we have $\partial \boldsymbol{\omega}/\partial t = \boldsymbol{0}$ in V at t, and the vorticity field will not change in the next instant. Continuing this argument, we conclude that if the vorticity field vanishes in V at t, it will vanish forever after.

In other words: in the absence of viscosity, *vorticity cannot be generated by the flow of the fluid but must be present from the outset.* Thus if you accelerate a body from rest in a truly ideal fluid, the flow will forever remain without vorticity, because $\boldsymbol{\omega} = \boldsymbol{0}$ in the beginning. The whirling air that trails a speeding car or an airplane must for this reason somehow be caused by viscous forces, independently of how tiny the viscosity is.

15.6 Steady, incompressible potential flow

Flow with a vanishing vorticity field $\boldsymbol{\omega} = \boldsymbol{0}$ is called *irrotational* and obeys a much simpler formalism than flow with vorticity, in particular when it is also incompressible. The results to be derived below for incompressible flow can be generalized to compressible as well as unsteady flow (see problem 15.8), although much of the simplicity is lost. But even if the equations for steady, incompressible, irrotational flow can be solved analytically in many geometries, the solutions are unfortunately not as useful as might be thought at first, because vorticity is always generated in the unavoidable viscous boundary layers of the bodies and containers that define the geometry, and may from there spread to large parts of the fluid.

The velocity potential

The vanishing of the vorticity, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v} = 0$, everywhere in the fluid implies (see problem 15.7) that the velocity has to be the gradient of a scalar field Ψ, called the *flow potential* or the *velocity potential*[1],

$$\boxed{\boldsymbol{v} = \boldsymbol{\nabla}\Psi \ .} \tag{15-50}$$

For an incompressible fluid, the vanishing of the divergence of the velocity field implies that the flow potential must satisfy Laplace's equation

$$\boxed{\boldsymbol{\nabla}^2\Psi = 0 \ .} \tag{15-51}$$

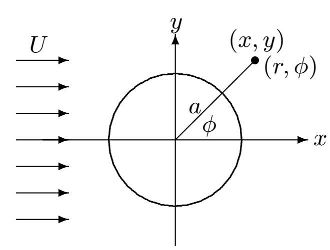
Typically, the boundary conditions consist in requiring the normal velocity $\boldsymbol{n}\cdot\boldsymbol{v} = \boldsymbol{n}\cdot\boldsymbol{\nabla}\Psi$ to vanish at all impermeable container walls, and that the flow potential asymptotically must approach the potential, $\Psi = \boldsymbol{U}\cdot\boldsymbol{x}$, of a uniform flow with velocity $\boldsymbol{U}$.

From the gradient of the Bernoulli field (15-44), it immediately follows that it is spatially constant, $H(\boldsymbol{x}) = H_0$, when $\boldsymbol{\omega} = \boldsymbol{0}$. Solving for the pressure we obtain from (15-16)

$$\boxed{p = p_0 - \rho_0\left(\frac{1}{2}\boldsymbol{v}^2 + \Phi\right) \ ,} \tag{15-52}$$

where $p_0 = \rho_0 H_0$ is a constant. Thus, in steady, incompressible potential flow where $\boldsymbol{v} = \boldsymbol{\nabla}\Psi$, the pressure is simply derived from the velocity potential found by solving the linear Laplace equation (15-51) with suitable boundary conditions. All the original nonlinearity of the Euler equation is thus being relegated to the expression for the pressure.

Potential flow around a cylinder in cross-wind



Cylinder of radius a in asymptotically uniform cross-wind U.

A circular cylinder of radius a is an infinitely extended three-dimensional object which is invariant under translations along as well as rotations around its axis. We choose as usual a coordinate system with the z-axis coinciding with the cylinder (see page 30. An asymptotically uniform "cross-wind" U along the x-axis does not break the longitudinal symmetry, which makes it natural to look for a velocity potential, $\Psi = \Psi(x, y)$, that is independent of z. Alternatively, the potential may be expressed in plane polar coordinates $\Psi = \Psi(r, \phi)$.

Asymptotically, for $r \to \infty$, the potential must approach the field $\Psi \to Ux = Ur\cos\phi$ of the uniform cross-wind. The linearity of the Laplace equation (15-51) demands that the potential everywhere is linear in the asymptotic field,

$$\Psi = U\cos\phi f(r) \ , \tag{15-53}$$

where $f(r)$ is a so far unknown function of r only which behaves as $f(r) \to r$ for $r \to \infty$. Using the Laplacian in cylindrical coordinates (2-68), we obtain

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{f}{r^2} = 0 \ . \tag{15-54}$$

[1]The preferred symbol for the velocity potential seems to be ϕ, but we shall in this book use Ψ which does not clash with other uses.

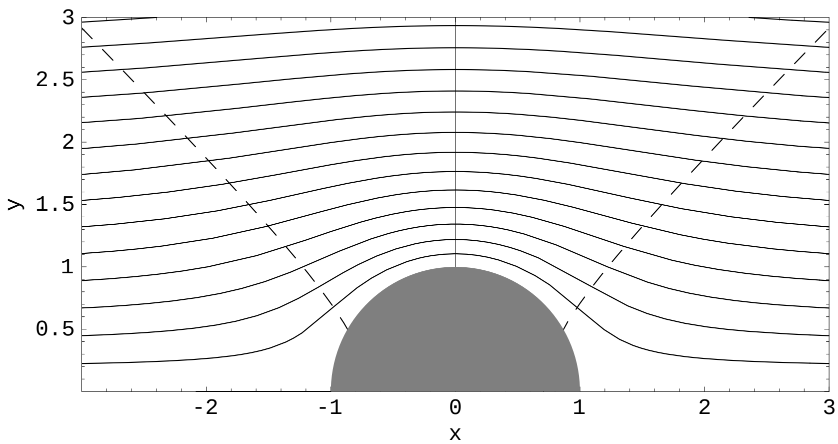


Figure 15.3: *Potential flow around cylinder with $a = 1$ and $U = 1$. Only the upper half shown here (the lower half is the mirror image). The pressure vanishes on the dashed lines. The streamlines have been obtained by numeric integration of the differential equation for streamlines (14-15) with the velocity field given by the solution (15-56) converted to Cartesian coordinates. The streamlines are equidistantly spaced by $\Delta y = 0.1$ for $x = -20$.*

Since all three terms are of order $1/r^2$, we should look for power law solutions of the form, $f \sim r^\alpha$. Inserting this into the equation we find $\alpha = \pm 1$ so that the most general solution is of the form $f = Ar + B/r$, where A and B are arbitrary constants. The asymptotic condition implies $A = 1$, and B is determined by requiring the radial field $v_r = \nabla_r \Psi$ to vanish at the surface of the cylinder, $r = a$. This leads to $B = a^2$, so that the solution is

$$\boxed{\Psi = Ur\cos\phi\left(1 + \frac{a^2}{r^2}\right) .} \tag{15-55}$$

Calculating the gradient by means of (2-64) we finally obtain the velocity field

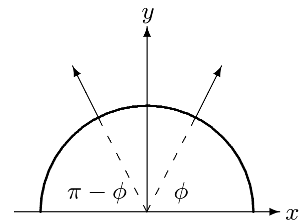
$$v_r = \nabla_r \Psi = \quad U\cos\phi\left(1 - \frac{a^2}{r^2}\right) , \tag{15-56a}$$

$$v_\phi = \nabla_\phi \Psi = -U\sin\phi\left(1 + \frac{a^2}{r^2}\right) . \tag{15-56b}$$

The flow is plotted in fig. 15.3. The radial flow $v_r|_{r=a} = 0$ vanishes at the surface of the cylinder as it should, whereas the tangential flow, $v_\phi|_{r=a} = -2U\sin\phi$, only vanishes at the stagnation points $\phi = 0, \pi$.

The pressure is calculated from (15-52). In the absence of gravity and normalized to vanish at infinity, it becomes

$$p = \frac{1}{2}\rho_0\left(U^2 - \boldsymbol{v}^2\right) = \frac{1}{2}\rho_0 U^2 \frac{a^2}{r^2}\left(4\cos^2\phi - 2 - \frac{a^2}{r^2}\right) , \tag{15-57}$$



The projection of the pressure force on the x-axis is equal and opposite for ϕ and $\pi - \phi$.

which on the surface of the cylinder simplifies to,

$$p|_{r=a} = \frac{1}{2}\rho_0 U^2 \left(4\cos^2\phi - 3\right) . \qquad (15\text{-}58)$$

It is negative for $30° < \phi < 150°$. It is clear from the symmetry of the problem that there can be no total force or *lift* in the y-direction; there is simply nothing in the flow geometry to fix its sign. This is also confirmed explicitly by the up/down invariance of the pressure under $\phi \to 2\pi - \phi$. What is more surprising is that due to the forwards/backwards invariance of the pressure (under $\phi \to \pi - \phi$) the total force or *drag* along the x-direction must also vanish, even for the upper half of the cylinder, although in this case the incoming flow would fix its sign (to be positive).

The lift acting on the half cylinder over a length L in the z-direction is obtained by projecting the surface element $dS = La\, d\phi$ on the y-direction,

$$\mathcal{L} = -\int_{y\geq 0} p\, dS_y = -\int_0^\pi p|_{r=a} \sin\phi\, Lad\phi = \frac{5}{3}\rho_0 U^2 La . \qquad (15\text{-}59)$$

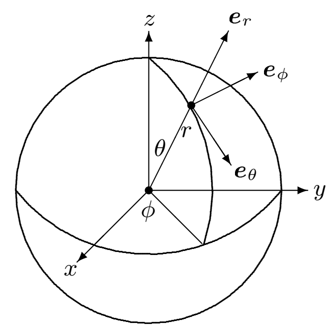
There is of course an equal and opposite "lift" acting on the lower half.

Example 15.6.1: A cylindrical pipe of radius $a = 10$ cm and average density of $\rho_1 = 1.6\rho_0$ lies half buried in the sand at the bottom of a stream flowing with $U = 1$ m/s. Its mass compensated for buoyancy is $M = \pi a^2 L(\rho_1 - \rho_0)$, so that the ratio of lift (on the upper half) to weight becomes

$$\frac{\mathcal{L}}{Mg_0} = \frac{5}{3\pi}\frac{U^2}{ag_0}\frac{\rho_0}{\rho_1 - \rho_0} \approx 0.9 . \qquad (15\text{-}60)$$

The pipe is nearly weightless in this flow.

Potential flow around a sphere



Spherical coordinates and their basis vectors.

The natural coordinates for a sphere of radius a inserted into an asymptotically uniform flow are of course spherical (page 32) with the z-axis along the asymptotic velocity U. The solution follows along exactly the same lines as for the cylinder above. Asymptotically, for $r \to \infty$, the velocity potential has to approach the uniform flow $\Psi \to Ur\cos\theta$. The symmetry of the problem implies that Ψ cannot depend on the azimuthal angle ϕ, and the linearity of the Laplace equation (15-51) requires the velocity potential to be linear in the asymptotic flow, or $\Psi = U\cos\theta f(r)$ where $f(r)$ is a function of the radial distance r only. Inserting this into the spherical Laplacian (2-75) one obtains an ordinary differential equation leading to $f = Ar + B/r^2$. The asymptotic boundary condition implies that $A = 1$, and the vanishing of the radial field $v_r = \nabla_r\Psi$ at the surface of the sphere requires $f'(a) = 0$, leading to $B = a^3/2$.

The velocity potential around a sphere is thus

$$\boxed{\Psi = Ur\cos\theta\left(1 + \frac{a^3}{2r^3}\right)} , \qquad (15\text{-}61)$$

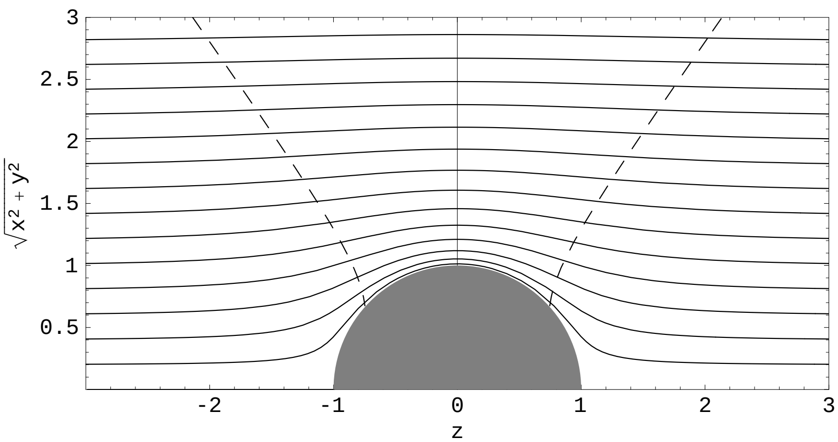


Figure 15.4: *Potential flow around the sphere with $a = 1$ and $U = 1$. Only the upper half shown here (the lower half is the mirror image). The streamlines have been obtained by numeric integration of the differential equation for streamlines (14-15) with the velocity field given by the solution (15-62) converted to Cartesian coordinates ($z, s = \sqrt{x^2+y^2}$). The streamlines are equidistantly spaced by $\Delta s = 0.1$ for $z = -20$. The field appears qualitatively different from the flow around a cylinder in fig. 15.3 and hugs much closer to the surface of the sphere. The pressure vanishes on the dashed lines.*

and the velocity field is calculated from the spherical representation of the gradient (2-72),

$$v_r = \nabla_r \Psi = \quad U \cos\theta \left(1 - \frac{a^3}{r^3}\right) \;, \tag{15-62a}$$

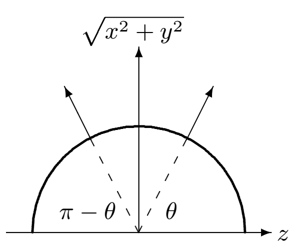
$$v_\theta = \nabla_\theta \Psi = -U \sin\theta \left(1 + \frac{a^3}{2r^3}\right) \;, \tag{15-62b}$$

$$v_\phi = \nabla_\phi \Psi = 0 \;. \tag{15-62c}$$

The streamlines are shown in fig. 15.4.

The pressure is obtained from (15-52),

$$p = \frac{1}{2}\rho_0 U^2 \frac{a^3}{r^3}\left(3\cos^2\theta - 1 - \frac{1}{4}(1 + 3\cos^2\theta)\frac{a^3}{r^3}\right) \;. \tag{15-63}$$



There is no drag because the projection of the pressure force on the z-axis is equal and opposite for θ and $\pi - \theta$.

On the surface of the sphere the pressure is

$$p|_{r=a} = \frac{1}{2}\rho_0 U^2 \frac{9\cos^2\theta - 5}{4} \;. \tag{15-64}$$

It is negative for $42^\circ \lesssim \theta \lesssim 138^\circ$. Again the symmetry $\theta \to \pi - \theta$ shows that there is no drag on a sphere.

d'Alembert's paradox: No drag in steady potential flow

The absence of drag in steady potential flow which we have explicitly verified for the cylinder and sphere may be formally shown to be true for any body shape (see page 566). But since everyday experience tells us that a moving object is subject to drag from the fluid that surrounds it, even if the viscosity is vanishingly small, we have exposed a problem called *d'Alembert's paradox*.

The resolution of the paradox elucidates the danger in assuming potential flow. Although a tiny viscosity may not give rise to an appreciable friction force between body and fluid, it will generate vorticity close to the surface of the body. The vorticity will then spread into the fluid and produce a *trailing wake* behind the moving body, carrying a non-vanishing kinetic energy. The constant loss of kinetic energy produces a drag on the body. In potential flow around a cylinder or sphere, the fluid does not create a wake but returns to its original state with no kinetic energy, implying that there is no resultant drag. Potential flow may be a mathematically correct solution, but it misses in this case important aspects of the physics of real flow. We shall see in chapter 26 that d'Alembert's paradox may in fact be viewed as a theorem about the smallness of drag compared to lift for streamlined bodies in nearly ideal fluids; a theorem with important consequences for the emergence of powered flight.

The paradox elucidates that the solutions to Euler's equation are not unique. Besides the potential flow solution which is unique, there may be other solutions containing vorticity that also satisfy the correct boundary conditions. Apparently, we can conclude that the Euler equation does not in itself constitute a complete theory of steady ideal flow. The problem is, however, more general. Even in the presence of viscosity, the equations of fluid mechanics have the inherent weakness that there may be more than one solution to a given flow problem. This is most clearly exposed by turbulent flow, where the precise flow pattern at any given moment depends on so minute details in the past that it becomes virtually unpredictable.

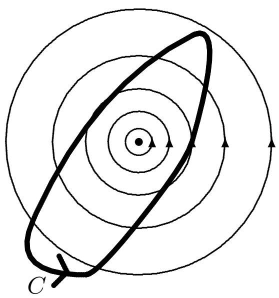
∗ Effective mass in unsteady potential flow

Even if no kinetic energy is lost from the fluid in steady potential flow around a sphere, there will be kinetic energy in the flow around the body. In the reference frame where the fluid is asymptotically at rest and the sphere moves with velocity $-\boldsymbol{U}$, the total kinetic energy of the sphere and fluid becomes (problem 15.10)

$$\mathcal{T} = \frac{1}{2}MU^2 + \int_{r\geq a} \frac{1}{2}\rho_0(\boldsymbol{v}-\boldsymbol{U})^2\,dV = \frac{1}{2}\left(M + \frac{2\pi}{3}a^3\rho_0\right)U^2\,, \qquad (15\text{-}65)$$

where M is the mass of the sphere. This shows that the *effective mass* of the sphere plus fluid is $M+m/2$ where $m = \frac{4\pi}{3}a^3\rho_0$ is the mass of the fluid displaced by the sphere. Apart from the factor of $1/2$, Archimedes would have liked this result!

* 15.7 Circulation



A closed curve C encircling a whirl.

The vorticity field is a measure of *local* circulation in the fluid, and we shall now see that there also is a *global* measure of circulation, related to the flux of vorticity.

Streamlines may form closed curves like the circles of Newton's bucket but often they are much more complicated. To avoid the problem of streamlines we shall instead consider an arbitrary closed curve C. If it encircles a region of whirling fluid, the projection of the velocity field onto the curve will tend to be of the same sign all the way around. Formally, the *circulation* of the velocity field around a closed curve C is defined to be the integrated projection of the velocity field on the line elements of the curve,

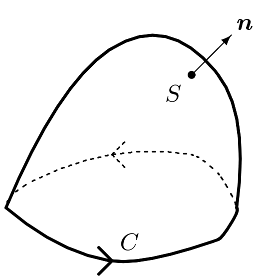
$$\boxed{\Gamma(C,t) = \oint_C \boldsymbol{v}(\boldsymbol{x},t) \cdot d\boldsymbol{\ell}} \ . \qquad (15\text{-}66)$$

Whether it is positive or negative depends on whether the curve runs with the whirling flow or against it. We emphasize that the circulation may be calculated for any curve, not just a streamline encircling a whirl, although that may be the natural thing to do.

Stokes' theorem

George Gabriel Stokes (1819–1903). *British mathematician and physicist.*

The most important theorem about circulation is due to Stokes. It is completely general and states that the circulation of the velocity field around a closed curve is equal to the flux of the vorticity through any surface bounded by the curve,



A surface S with perimeter C. Notice how the normal to the surface is consistent with the orientation of C (here using a right hand rule).

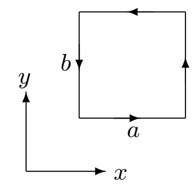
$$\boxed{\oint_C \boldsymbol{v} \cdot d\boldsymbol{\ell} = \int_S \boldsymbol{\nabla} \times \boldsymbol{v} \cdot d\boldsymbol{S}} \ . \qquad (15\text{-}67)$$

It does not matter which surface S the flux is calculated for, as long as it has C as boundary, lies entirely within the fluid, and is oriented consistently with the orientation of C. Stokes' theorem is like Gauss' theorem (6-3) valid for any vector field.

Example 15.7.1: A fluid rotates like a solid body with velocity $v = \Omega r$. The circulation around a circle (also a streamline) with radius r is (in the non-rotating laboratory system) obtained by multiplying the constant velocity with the circumference of the circle,

$$\Gamma(r) = 2\pi r\ \Omega r = 2\pi\Omega r^2 = 2\Omega\ \pi r^2 \ . \qquad (15\text{-}68)$$

The last expression shows that the circulation is also the product of the constant vorticity 2Ω with the area of the circle, thus confirming Stokes' theorem.



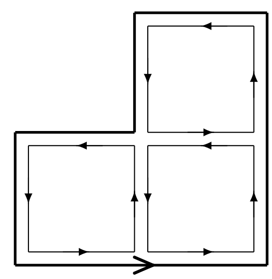
Circulation around a small rectangle of dimensions $a \times b$.

Proof of Stokes' theorem: As before the relation between global and local quantities is established by calculating the global quantity for an infinitesimal geometric figure, in this case a tiny rectangle in the xy-plane with sides a and b. To first order in the sides we find the circulation (suppressing z and t)

$$\begin{aligned}\oint_{a\times b} \boldsymbol{v}\cdot d\boldsymbol{\ell} = & \int_x^{x+a} v_x(x',y)\,dx' + \int_y^{y+b} v_y(x+a,y')\,dy' \\ & - \int_x^{x+a} v_x(x',y+b)\,dx' - \int_y^{y+b} v_y(x,y')\,dy' \\ & \approx -\int_x^{x+a} b\,\nabla_y v_x(x',y)\,dx' + \int_y^{y+b} a\,\nabla_x v_y(x,y')\,dy' \\ & \approx ab\big(\nabla_x v_y(x,y) - \nabla_y v_x(x,y)\big) \\ & = ab\,(\boldsymbol{\nabla}\times\boldsymbol{v})_z\end{aligned}$$

The last expression is the projection $(\boldsymbol{\nabla}\times\boldsymbol{v})\cdot d\boldsymbol{S}$ of the vorticity field on the small vector surface element of the rectangle, $d\boldsymbol{S} = (0,0,ab)$. A similar result would of course have been obtained for any other orientation of the little rectangle.

Consider now a surface built up from little rectangles of this kind. Adding together the circulation for each rectangle, the contributions from the inner common edges cancel and one is left only with the circulation around the outer perimeter of the surface. Since an arbitrary surface may be built up from infinitesimal rectangles, we conclude that the circulation around the perimeter is equal to the flux of the vorticity field through the surface, which is Stokes theorem. Since the normal to each little rectangle is oriented consistently with the way the perimeter is followed, this must also be required for the surface S.



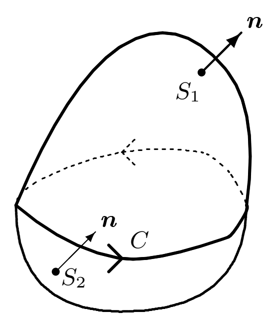
Adding rectangles together, the circulation cancels along the edges where two rectangles meet. This is also valid if the rectangles bend into the other coordinate directions.

> Stokes theorem is like Gauss' theorem a multi-dimensional version of the trivial result $\int_a^b f'(x)\,dx = f(b) - f(a)$. Where Gauss theorem was a relation between a three-dimensional volume and its surface, Stokes theorem is a relation between a two-dimensional surface and its boundary curve. Along this line one might even view the gradient integral $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\nabla} p(\boldsymbol{x})\cdot d\boldsymbol{\ell} = p(\boldsymbol{b}) - p(\boldsymbol{a})$ as a relation between a one-dimensional line segment and its endpoints.

The proof implies that the shape of the surface S does not matter. This may be explicitly verified by calculating the difference between two surfaces S_1 and S_2, both having the curve C as perimeter and oriented consistently,

$$\int_{S_1} \boldsymbol{\nabla}\times\boldsymbol{v}\cdot d\boldsymbol{S} - \int_{S_2} \boldsymbol{\nabla}\times\boldsymbol{v}\cdot d\boldsymbol{S} = \oint_S \boldsymbol{\nabla}\times\boldsymbol{v}\cdot d\boldsymbol{S} = \int_V \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})\,dV = 0\ .$$

Here $S = S_1 - S_2$ is the closed surface formed by the two open surfaces with an extra minus sign because of the requirement that the closed surface should have an outwardly oriented normal. Gauss' theorem has been used to convert the integral over S to an integral over the volume V contained in S.



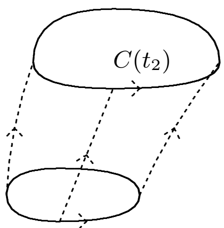
The flux of vorticity is the same for two surfaces with the same bounding curve. The normals to the surfaces are both consistent with the orientation of the curve.

Kelvin's circulation theorem

William Thomson , alias lord Kelvin (1824–1907). *British physicist.*

Kelvin's famous circulation theorem from 1868 states that in an *ideal* flow the circulation around a *comoving* closed curve (also called a closed material curve) is independent of time. In other words, if $C(t)$ is a comoving closed curve, then

$$\boxed{\frac{D\Gamma}{Dt} = \frac{d\Gamma(C(t),t)}{dt} = 0} \ . \tag{15-69}$$



In an ideal flow the circulation around the comoving curve $C(t)$ is the same at t_1 as at t_2.

A comoving closed curve is washed along with the fluid and may thus change shape dramatically without change in circulation. In steady flow the circulation around any fixed closed curve is like all other quantities independent of time, but the theorem concerns a curve following the material of the fluid whether the motion is steady or not.

Kelvins theorem applies only to ideal or nearly ideal flow. For a viscous fluid, the circulation will change at a rate proportional to the viscosity. Viscosity will act both as dissipator and generator of vorticity. It is as mentioned before virtually impossible to generate vorticity — or get rid of it — without the aid of viscosity.

Proof of Kelvin's theorem: The proof of the theorem is straightforward. We shall carry it through for an incompressible fluid in the absence of gravity, but it is equally valid for compressible fluids and including gravity. Let $C(t)$ be the comoving closed curve. In a small time interval dt the circulation along this curve changes by

$$\begin{aligned}
\delta\Gamma(C(t),t) &= \Gamma(C(t+\delta t), t+\delta t) - \Gamma(C(t),t) \\
&= \oint_{C(t+\delta t)} \boldsymbol{v}(\boldsymbol{x}, t+\delta t)\cdot d\boldsymbol{\ell} - \oint_{C(t)} \boldsymbol{v}(\boldsymbol{x},t)\cdot d\boldsymbol{\ell} \\
&= \oint_{C(t)} [\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{v}(\boldsymbol{x},t)\delta t, t+\delta t) - \boldsymbol{v}(\boldsymbol{x},t)]\cdot d\boldsymbol{\ell} \\
&= \delta t \oint_{C(t)} \left[\frac{\partial \boldsymbol{v}(\boldsymbol{x},t)}{\partial t} + (\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla})\boldsymbol{v}(\boldsymbol{x},t)\right]\cdot d\boldsymbol{\ell} \\
&= -\delta t \oint_{C(t)} \boldsymbol{\nabla}\left(\frac{p(\boldsymbol{x},t)}{\rho_0} + \Phi(\boldsymbol{x},t)\right)\cdot d\boldsymbol{\ell} \\
&= 0 \ .
\end{aligned}$$

In going to the third line we have used that the point $\boldsymbol{x}$ at time t of a comoving curve is found at $\boldsymbol{x}+\boldsymbol{v}(\boldsymbol{x},t)dt$ at time $t+dt$, and this generates the comoving derivative in the fourth line. Euler's equation (15-1) with constant density is then used in going to the fifth line (for compressible fluids the theorem may be proven by means of the pressure function (**??**)). Finally, in the last line we have used that the pressure and the gravitational potential are single-valued functions of space so that the integral of the gradient around a closed curve must vanish.

Problems

15.1 There is a small correction to the flow from the wine barrel (page 272) because the velocity of the flow does not vanish exactly on the top of the barrel. Estimate this correction from the ratio of the barrel cross section A_0 and the spout cross section A.

15.2 A wine barrel has two spouts with different cross sections A_1 and A_2 at the same horizontal level. Show that under steady flow conditions the wine emerges with the same speed from the two spouts.

15.3 Consider the quasistationary emptying of the wine barrel. Determine how the actual height z varies as a function of time.

15.4 Assume that air is an ideal gas with constant temperature T_0. a) Calculate the relation between pressure and air velocity for a Pitot tube which is closed in one end. b) Estimate the pressure increase relative to outside pressure in the Pitot tube of a passenger jet flying at a height of 10 km with speed 250 m/s.

15.5 Incompressible fluid flows along x in an open channel with a weir without any dependence on y. Show that if the horizontal flow $v_x(x)$ is independent of z, the vertical flow will be

$$v_z = v_x \frac{(h-z)b' + (z-b)h'}{h-b} \tag{15-70}$$

where $b(x)$ and $h(x)$ are the bottom the surface heights.

15.6 Use the Schwarz inequality

$$\left| \sum_n A_n B_n \right|^2 \leq \sum_n A_n^2 \sum_m B_m^2 \tag{15-71}$$

to derive (15-28).

15.6 Writing out the double sum,

$$\boldsymbol{v} \cdot (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \sum_{ij} v_i v_j \nabla_j v_i \tag{15-72}$$

we find

$$|\boldsymbol{v} \cdot (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}|^2 \leq \sum_{ij} (v_i v_j)^2 \sum_{kl} (\nabla_l v_k)^2 = |\boldsymbol{v}|^4 \, |\boldsymbol{\nabla} \boldsymbol{v}|^2 \tag{15-73}$$

* **15.7** Show that when $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$ everywhere then there exists a potential Ψ such that $\boldsymbol{v} = \boldsymbol{\nabla}\Psi$.

∗ **15.8** Show that for unsteady, compressible potential flow the Bernoulli field is only a function of time,

$$\frac{\partial \Psi}{\partial t} + \frac{1}{2}\boldsymbol{v}^2 + \Phi + w(p) = H(t) \tag{15-74}$$

where $w(p)$ is the pressure function (15-31).

15.9 Calculate the lift on a sphere half buried at the bottom of a stream with asymptotic velocity U.

15.10 Show that the kinetic energy of the fluid surrounding a sphere moving with velocity U in steady potential flow is $\frac{\pi}{3}\rho_0 a^3 U^2$.

15.11 Show that the vector area of a surface bounded by a closed curve C only depends on the boundary.

15.12 Show that Kelvin's theorem is valid for a compressible ideal fluid.

15.13 Consider a fluid with "large" bulk modulus in a constant gravitational field $\boldsymbol{g} = (0, 0, -g_0)$. a) Show that in the hydrostatic limit $\rho \approx \rho_0$ and $p \approx p_0 - \rho_0 g_0 z$ where ρ_0 and p_0 are constants. b) Show that the first Euler equation for small-amplitude oscillations (15-3) becomes,

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\rho_0}\boldsymbol{\nabla}\Delta p + \frac{\Delta\rho}{\rho_0}\boldsymbol{g} \ . \tag{15-75}$$

c) Show that the wave equation becomes

$$\frac{\partial^2 \Delta p}{\partial t^2} = \frac{K_0}{\rho_0}\boldsymbol{\nabla}^2\Delta p + g_0\frac{\partial \Delta p}{\partial z} \tag{15-76}$$

d) Estimate under which conditions the last term can be disregarded.