7 Strain

All materials deform when subjected to external forces, but different materials react in different ways. Elastic materials bounce back again to the original configuration when the forces cease to act. Others are plastic and retain their shape after deformation. Viscoelastic materials behave like elastic solids under rapid deformation, but creep like viscous liquid over longer periods of time. Elasticity is itself an idealization, limited to a certain range of forces. If the external forces become excessive, all materials become plastic and undergo permanent deformation or may even fracture.

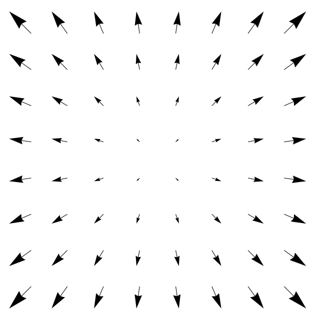
When a body is deformed, its material is displaced away from its original position. Small deformations are mathematically much easier to handle than large deformations where parts of a body become greatly and non-uniformly displaced relative to other parts, as for example when you crumple a piece of paper. A rectilinear coordinate system embedded in the original body and deformed along with the material of the body becomes a curvilinear coordinate system after the deformation. It can therefore come as no surprise that the general theory of finite deformation is mathematically at the same level of difficulty as general curvilinear coordinate systems. Luckily, our buildings and machines are rarely subjected to such violent treatment, and in most cases the deformation may be assumed to be tiny.

Although displacement is naturally described by a vector field, the description of deformation inevitably leads to the introduction of a new tensor quantity, the *strain tensor* which characterizes the state of *local deformation* or *strain* in a material. It can come as no surprise that material strain causes tension or stress—as do strained relations among people. In this chapter we shall focus exclusively on the description of strain, and postpone the discussion of the stress-strain relationship for elastic materials to chapter 8.

7.1 Displacement

The prime example of deformation is a *uniform scaling* in which the coordinates of all material particles in a body are multiplied with the scale factor κ. A material particle originally situated in the point $\boldsymbol{X}$ is thus displaced to the point,

$$\boldsymbol{x} = \kappa \boldsymbol{X}. \tag{7.1}$$



Uniform dilatation. The arrows indicate how material particles are displaced.

It is emphasized that *both $\boldsymbol{X}$ and $\boldsymbol{x}$ refer to the same coordinate system*. Uniform scaling with $\kappa > 1$ is also called uniform *dilatation* whereas scaling with $0 < \kappa < 1$ is called uniform *compression*. Negative scaling with $\kappa < 0$ is physically impossible.

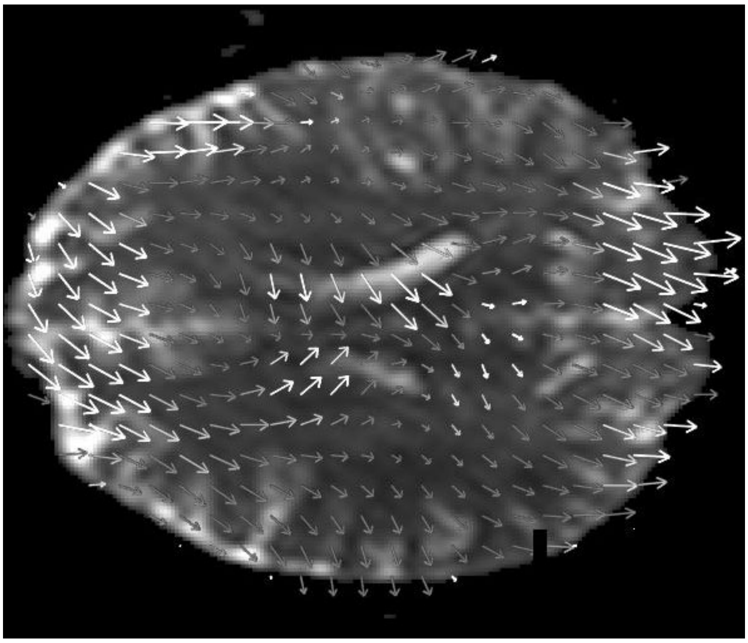
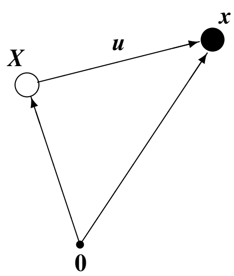


Figure 7.1. In statistical brain image analysis it is necessary to align brain images from different individuals so that structurally similar regions are brought to overlap. This process, called registration, may be viewed as a deformation of one brain into another. In the above image the small arrows picture the displacement field connecting two brain images. Image courtesy Hauge Bartsch (permission to be obtained).

The only point which does not change place during uniform scaling is the origin of the coordinate system. Although it superficially looks as if the origin of the coordinate system plays a special role, this is not really the case. All relative positions of material particles scale in the same way, because $\boldsymbol{x} - \boldsymbol{y} = \kappa(\boldsymbol{X} - \boldsymbol{Y})$, independent of the origin of the coordinate system. There is no special center for a uniform scaling, either geometrically or physically. The origin of the coordinate system is simply an *anchor point* for the mathematical description of scaling.



Geometry of displacement. The particle that originally was located at $\boldsymbol{X}$ has been displaced to $\boldsymbol{x}$ by the displacement vector $\boldsymbol{u}$.

Linear displacements

Under a displacement the center-of-mass of a material particle is moved from its original position $\boldsymbol{X}$ to its actual position $\boldsymbol{x}$. The *displacement vector* is always defined as the difference between the *actual* and the *original* coordinates,

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}. \tag{7.2}$$

For the case of uniform scaling, the displacement vector becomes

$$\boldsymbol{u} = (\kappa - 1)\,\boldsymbol{X} = \left(1 - \frac{1}{\kappa}\right)\boldsymbol{x} \tag{7.3}$$

Mathematically we are completely free to express the displacement in terms of the original or actual position of the material particle. For scaling, the displacement is in both cases a linear function of the coordinates.

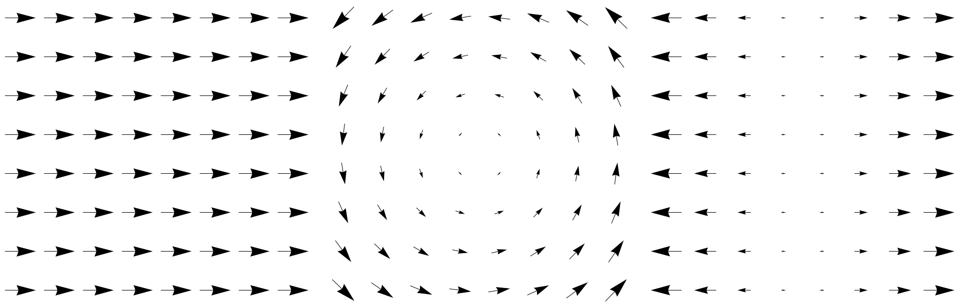


Figure 7.2. Arrow plots of the displacement fields for simple translation, simple rotation and simple dilatation.

More generally a linear displacement (and its inverse) takes the form,

$$\boldsymbol{x} = \boldsymbol{A} \cdot \boldsymbol{X} + \boldsymbol{b}, \qquad \boldsymbol{X} = \boldsymbol{A}^{-1} \cdot (\boldsymbol{x} - \boldsymbol{b}). \tag{7.4}$$

where $\boldsymbol{A}$ is a non-singular constant matrix and $\boldsymbol{b}$ is a constant vector. As for scaling, the displacement vector may be expressed as a function of either the original or the actual positions,

$$\boldsymbol{u} = (\boldsymbol{A} - \mathbf{1}) \cdot \boldsymbol{X} + \boldsymbol{b} = \left(\mathbf{1} - \boldsymbol{A}^{-1}\right) \cdot \boldsymbol{x} + \boldsymbol{A}^{-1} \cdot \boldsymbol{b}. \tag{7.5}$$

There is strong similarity between the general linear displacements and the transformations of Cartesian coordinates (seen appendix B), but the class of linear displacements is larger, because the matrix $\boldsymbol{A}$ is not restricted to be orthogonal.

The general linear displacement may like coordinate transformations be resolved into simpler types, namely translation along a coordinate axis, rotation by a fixed angle around a coordinate axis, and scaling by a fixed factor along a coordinate axis (see figure 7.2). The physically impossible reflections in a coordinate axis are excluded. We shall not prove here that the general linear displacement may be resolved in this way, but instead rely on geometric intuition.

Simple translation

A rigid body translation of the material through a distance b along the x-axis is described by $x = X + b$, $y = Y$, and $z = Z$. The displacement vector becomes,

$$u_x = b, \qquad u_y = 0, \qquad u_z = 0. \tag{7.6}$$

The geometric relationships in a body are evidently unchanged under any translation, so this is not a deformation.

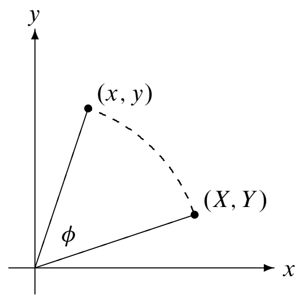
Simple rotation

A rigid body rotation through the angle ϕ around the z-axis takes the form,

$$x = X\cos\phi - Y\sin\phi, \qquad X = x\cos\phi + y\sin\phi \tag{7.7a}$$

$$y = X\sin\phi + Y\cos\phi, \qquad Y = -x\sin\phi + y\cos\phi \tag{7.7b}$$

$$z = Z, \qquad Z = z. \tag{7.7c}$$



A rigid body rotation through an angle ϕ moves the material particle at (X, Y) to (x, y).

The corresponding displacement vector components are

$$u_x = -X\,(1 - \cos\phi) - Y\,\sin\phi \qquad = x(1 - \cos\phi) - y\sin\phi, \tag{7.8a}$$

$$u_y = X\,\sin\phi - Y\,(1 - \cos\phi) \qquad = x\sin\phi - y(1 - \cos\phi), \tag{7.8b}$$

$$u_z = 0. \tag{7.8c}$$

Since all distances in the body are unchanged, this is not a deformation.

Simple scaling

Multiplying all x-coordinates by the factor κ, we get $x = \kappa X$, $y = Y$, and $z = Z$. The displacement vector becomes,

$$u_x = (\kappa - 1)X = kx, \tag{7.9a}$$
$$u_y = 0, \tag{7.9b}$$
$$u_z = 0, \tag{7.9c}$$

where $k = 1 - 1/\kappa$. Simple dilatation corresponds to $k > 0$ and simple compression to $k < 0$. Uniform scaling (7.1) is a combination of three such scalings by the same factor along the three coordinate axes. Scaling is a true deformation.

7.2 The displacement field

In this book we have systematically adopted a "materialistic" attitude towards the description of continuous matter. Field values, such as the density $\rho(\boldsymbol{x})$ and gravity $\boldsymbol{g}(\boldsymbol{x})$, represent the physical properties in the immediate neighborhood of the point $\boldsymbol{x}$. When a macroscopic material body is deformed, all its material particles are in general simultaneously displaced. In keeping with the materialistic attitude, we let the field $\boldsymbol{X}(\boldsymbol{x})$ denote the original position of the material particle now situated at $\boldsymbol{x}$, so that the *displacement field* becomes a function of the actual position,

$$\boxed{\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{X}(\boldsymbol{x}).} \tag{7.10}$$

This (material) representation of displacement is also called the *Euler representation*.

Mathematically, we could—as we did for the linear displacements—solve the equation, $\boldsymbol{X}(\boldsymbol{x}) = \boldsymbol{X}$ for $\boldsymbol{x}$, to obtain the actual position in terms of the original, $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X})$. In that case the displacement, $\boldsymbol{u} = \boldsymbol{x}(\boldsymbol{X}) - \boldsymbol{X}$, becomes a function of the original position. Although it seems physically awkward, this *Lagrange representation* of displacement is conceptually convenient in many situations and has played a great role in the long history of continuum physics. Here we shall mainly deal with slowly varying displacement fields, for which there is essentially no difference between the Euler and the Lagrange representations. A bit of the general theory of arbitrary displacements is presented in section 7.5.

Local deformation

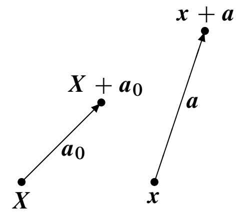
A general displacement field also includes all kinds of ordinary rigid body translations and rotations, and it would be wrong to classify all displacement fields as deformations. Sailing a submarine at the surface of the water will only translate or rotate it horizontally, not deform it, whereas taking it to the bottom of the sea will also compress it. *A true deformation must involve changes in geometric relationships*, i.e. lengths and angles, in the body.

Large scale deformation can be very complex. Think of all the loops and knots that weavers make from a roll of yarn. We should for this reason not expect to find a simple formalism for global deformation. Weaving, knitting, folding, winding, writhing, wringing and squashing may bring particles that were originally far apart into close proximity. Even the wildest weave consists, however, locally of small pieces of straight yarn that have only been translated, rotated, stretched or contracted, but not folded, spindled or mutilated. We may therefore expect to find a much simpler description of deformation for very small pieces of matter.

Displacement of an infinitesimal "needle"

Consider a tiny elongated piece of matter, a "needle" or *material vector* $\boldsymbol{a}$, now actually connecting the points $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{a}$ in the displaced material. Before the displacement this needle connected the points $\boldsymbol{X}=\boldsymbol{X}(\boldsymbol{x})$ and $\boldsymbol{X}+\boldsymbol{a}_0=X(\boldsymbol{x}+\boldsymbol{a})$. Subtracting these equations and using that $\boldsymbol{X}(\boldsymbol{x})=\boldsymbol{x}-\boldsymbol{u}(\boldsymbol{x})$, we find

$$\boldsymbol{a}_0=\boldsymbol{X}(\boldsymbol{x}+\boldsymbol{a})-\boldsymbol{X}(\boldsymbol{x})=\boldsymbol{a}-\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{a})+\boldsymbol{u}(\boldsymbol{x}). \tag{7.11}$$



Displacement of a tiny material needle from $\boldsymbol{a}_0$ to $\boldsymbol{a}$. It may be translated, rotated, and scaled. Only the latter corresponds to a true deformation.

Expanding to the displacement field to first order in $\boldsymbol{a}$, we get

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{a})&=\boldsymbol{u}(\boldsymbol{x})+a_x\frac{\partial\boldsymbol{u}(\boldsymbol{x})}{\partial x}+a_y\frac{\partial\boldsymbol{u}(\boldsymbol{x})}{\partial y}+a_z\frac{\partial\boldsymbol{u}(\boldsymbol{x})}{\partial z}+\mathcal{O}\left(\boldsymbol{a}^2\right)\\&=\boldsymbol{u}(\boldsymbol{x})+(\boldsymbol{a}\cdot\nabla)\,\boldsymbol{u}(\boldsymbol{x})+\mathcal{O}\left(\boldsymbol{a}^2\right).\end{aligned} \tag{7.12}$$

This shows that the displacement changes an infinitesimal needle vector by

$$\boxed{\delta\boldsymbol{a}\equiv\boldsymbol{a}-\boldsymbol{a}_0=(\boldsymbol{a}\cdot\nabla)\,\boldsymbol{u}(\boldsymbol{x}).} \tag{7.13}$$

Since it is a relation between infinitesimal quantities, this transformation is of course *linear* in $\boldsymbol{a}$. In index notation, it may be written as,

$$\delta a_i=\sum_j a_j\,\nabla_j u_i. \tag{7.14}$$

The coefficients of the linear transformation of $\boldsymbol{a}$ are computed from the set of derivatives of the displacement field, $\{\nabla_j u_i\}$, also called the *displacement gradients*. For a general linear displacement (7.5) we find $\nabla_j u_i=\delta_{ij}-(\boldsymbol{A}^{-1})_{ij}$.

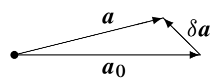
Example 7.1 [Simple rotation]: The displacement gradient matrix of a simple rotation (7.7) is,

$$\{\nabla_j u_i\}=\begin{pmatrix}1-\cos\phi & -\sin\phi & 0\\ \sin\phi & -1+\cos\phi & 0\\ 0 & 0 & 0\end{pmatrix} \tag{7.15}$$

where the index i enumerates the rows and j the columns.

Slowly varying displacement field

Displacements and coordinates have dimension of length, so that *the displacement gradients are dimensionless*, i.e. pure numbers. This makes it meaningful to speak of small displacement gradients in an absolute way. A displacement field is said to be *slowly varying* if all the displacement gradients are small everywhere,



The change in a needle vector is small compared to its length when the displacement gradients are small.

$$\left|\nabla_j u_i(\boldsymbol{x})\right|\ll 1, \tag{7.16}$$

for all i,j and $\boldsymbol{x}$. If we define the norm of a matrix as $|\boldsymbol{A}|=\sum_{ij}\left|A_{ij}\right|^2$, this may also be written $|\nabla\boldsymbol{u}|\ll 1$. Except for section 7.5, where a few aspects of the theory of finite deformations are presented, we shall from now on assume that the displacement field is slowly varying, so that the change in a material needle is much smaller than its length, $|\delta\boldsymbol{a}|\ll|\boldsymbol{a}|$.

Small displacement gradients do not automatically guarantee that the displacement field itself is small compared to the size of the body, because the displacement could include a rigid body translation to the other end of the universe, and that would not affect its gradient. But relative to a fixed anchor point in the body, a slowly varying field will always be small compared to the size L of the body and thus fulfill,

$$|\boldsymbol{u}(\boldsymbol{x})| \ll L. \tag{7.17}$$

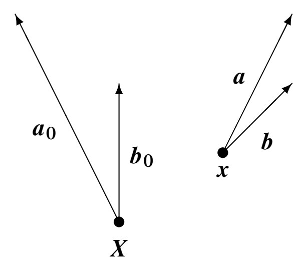
A displacement field, satisfying this condition everywhere, will in general also be slowly varying, though there are notable exceptions. If you, for example, make a small crease in your shirt when you iron it, the displacement gradients will be almost infinitely large in the crease although none of the shirt's material is greatly displaced compared to its size.

Example 7.2 [Small rotations]: For small rotation angle $\phi \ll 1$, the displacement field of a simple rotation becomes $\boldsymbol{u} = (-y, x, 0)\,\phi$, and the gradient matrix (7.15) becomes,

$$\{\nabla_j u_i\} = \begin{pmatrix} 0 & -\phi & 0 \\ \phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.18}$$

to lowest order in ϕ. The change in $\boldsymbol{a}$ becomes $\delta a_x = -\phi a_y$ and $\delta a_y = \phi a_x$. This may also be written as a cross product, $\delta\boldsymbol{a} = \boldsymbol{\phi} \times \boldsymbol{a}$, where $\boldsymbol{\phi} = \phi\,\hat{\boldsymbol{e}}_z$.

Cauchy's strain tensor



Displacement of a pair of infinitesimal material needles may affect their lengths as well as the angle between them.

The scalar product of two needles $\boldsymbol{a} \cdot \boldsymbol{b}$ is unchanged by translation and rotation, so it ought to be a useful indicator for a change in geometry. Using (7.13) , we calculate the change in the scalar product $\delta(\boldsymbol{a} \cdot \boldsymbol{b}) \equiv \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{a}_0 \cdot \boldsymbol{b}_0$ to first order in the small displacement gradients,

$$\begin{aligned}\delta(\boldsymbol{a} \cdot \boldsymbol{b}) &= \delta\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \delta\boldsymbol{b} \\ &= (\boldsymbol{a} \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{a} \\ &= \sum_{ij} \left(\nabla_i u_j + \nabla_j u_i\right) a_i b_j.\end{aligned}$$

or

$$\delta(\boldsymbol{a} \cdot \boldsymbol{b}) = 2 \sum_{ij} u_{ij} a_i b_j = 2\, \boldsymbol{a} \cdot \boldsymbol{\mathsf{u}} \cdot \boldsymbol{b}, \tag{7.19}$$

where $\boldsymbol{\mathsf{u}} = \{u_{ij}\}$ is the symmetrized displacement gradient tensor,

$$\boxed{u_{ij} = \tfrac{1}{2}\left(\nabla_i u_j + \nabla_j u_i\right).} \tag{7.20}$$

This tensor is called *Cauchy's (infinitesimal) strain tensor*, or just the *strain tensor* when that is unambiguous. It is sometimes convenient to write this relation in matrix form,

$$\boldsymbol{\mathsf{u}} = \tfrac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top\right) \tag{7.21}$$

where $(\nabla\boldsymbol{u})_{ij} = \nabla_i u_j$ and $(\nabla\boldsymbol{u}^\top)_{ij} = \nabla_j u_i$ is its transposed.

The *strain tensor contains all the information about geometric changes caused by the displacement* and is accordingly a good measure of deformation. All bodily translations and rotations have been automatically taken out, and any displacement which is a combination of translations and rotations must consequently yield a vanishing strain tensor. It should, however, be emphasized that *Cauchy's expression is only valid for small displacement gradients*. When that is not the case, a more complicated expression must be used, involving the square of the displacement gradients (see section 7.5). The relative error committed by using Cauchy's approximation rather than the true strain tensor is therefore of the same magnitude as the displacement gradients.

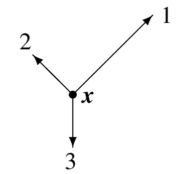
It is instructive and useful for practical calculations to write out all the components of the strain tensor explicitly, once and for all. The six independent components are,

$$u_{xx} = \nabla_x u_x, \qquad u_{yz} = u_{zy} = \tfrac{1}{2}\left(\nabla_y u_z + \nabla_z u_y\right), \tag{7.22a}$$

$$u_{yy} = \nabla_y u_y, \qquad u_{zx} = u_{xz} = \tfrac{1}{2}\left(\nabla_z u_x + \nabla_x u_z\right), \tag{7.22b}$$

$$u_{zz} = \nabla_z u_z, \qquad u_{xy} = u_{yx} = \tfrac{1}{2}\left(\nabla_x u_y + \nabla_y u_x\right). \tag{7.22c}$$

Had we not assumed that the displacement was slowly varying, there would as mentioned above also have been quadratic terms in the displacement gradients, and the strain tensor might take large values. But with our assumption of small displacement gradients (7.16) , the strain tensor field is likewise small, $\left|u_{ij}(\boldsymbol{x})\right| \ll 1$ for all i, j and $\boldsymbol{x}$. Contrariwise, a small strain tensor does not imply that the displacement gradients are small.



Principal strain basis in a point $\boldsymbol{x}$. The deformation consists entirely of scale changes along the principal axes, often shown by the lengths of the basis vectors.

Example 7.3 [Simple linear displacements]: The matrix of displacement gradients of a simple translation, $\boldsymbol{u}(\boldsymbol{x}) = (b, 0, 0)$, vanishes trivially, and so does the strain tensor. This confirms that a translation is not a deformation. For small angles of rotation, $|\phi| \ll 1$, the displacement gradient matrix (7.18) is antisymmetric. Cauchy's symmetric strain tensor therefore vanishes, confirming that a small rotation is not a deformation. For a simple scaling $\boldsymbol{u} = k(x, 0, 0)$ the displacement gradient matrix is symmetric and equals therefore strain tensor

$$\{u_{ij}\} = k \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.23}$$

Evidently this is a true deformation.

* Principal axes of strain

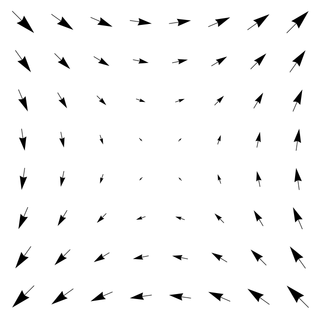
According to its definition (7.20) the strain tensor is born *symmetric* under exchange of its indices

$$\boxed{u_{ij} = u_{ji}.} \tag{7.24}$$

It differs in this respect from the stress tensor, for which symmetry is not self-evident and must be viewed as a constitutive equation (see page 104).

A symmetric tensor can always be *diagonalized*. The eigenvectors of the strain tensor at a given point are called the *principal axes of strain*, and form a *principal basis* at every point of the body. In the principal basis for any given point, the strain tensor is diagonal, and the angles between the principal axes are unchanged under the displacement. The signs and magnitudes of the eigenvalues determine how much the material is being stretched or contracted along the principal axes. It should, however, be remembered that the principal basis of the strain tensor varies from point to point in space, and defines three orthogonal unit vector fields and three scalar eigenvalue fields (see figure 7.3).

A symmetric tensor has six independent component whereas the displacement field has only three independent components. Every strain tensor must consequently satisfy consistency or compatibility conditions that remove three degrees of freedom. These conditions are formulated in problem 7.11 and will not be further discussed here.



Arrow plot of the two-dimensional Lagrangian linear displacement field $\boldsymbol{u} = (y, x, 0)$ in the square $-1 < x < 1$ and $-1 < y < 1$. The material is dilated along one diagonal and contracted along the other. These are the principal directions of strain everywhere (see problem 7.5).

Figure 7.3. Principal strain axis distribution for ground displacements in Japan, determined by GPS over two years. Only the two horizontal axes are shown with lengths proportional to the magnitude of the eigenvalues. The black axes (running mainly southeast-to-northwest) indicate contraction and the gray extension. Geography & Crustal Dynamics Research Center (Permission to be obtained).

7.3 Geometrical meaning of the strain tensor

The strain tensor contains all the relevant information about local changes in geometric relationships, such as lengths of material needles and the angles between them. Other local geometric quantities, for example curve, surface and volume elements, are also changed under a deformation.

Lengths and angles

It is useful for the following discussion to define the *projection* u_{ab} of a tensor u_{ij} on the directions of two arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$u_{ab} = \hat{\boldsymbol{a}} \cdot \boldsymbol{u} \cdot \hat{\boldsymbol{b}} = \frac{\boldsymbol{a} \cdot \boldsymbol{u} \cdot \boldsymbol{b}}{|\boldsymbol{a}|\ |\boldsymbol{b}|}. \tag{7.25}$$

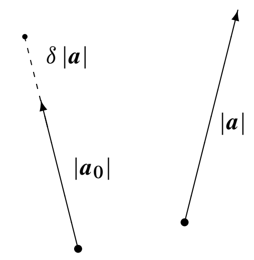
Then we may simply write,

$$\delta(\boldsymbol{a} \cdot \boldsymbol{b}) = 2\,|\boldsymbol{a}|\,|\boldsymbol{b}|\,u_{ab}, \tag{7.26}$$

for the change in a scalar product (7.19) .

The change in the length of a needle, $\delta\,|\boldsymbol{a}| \equiv |\boldsymbol{a}| - |\boldsymbol{a}_0|$, is obtained by setting $\boldsymbol{b} = \boldsymbol{a}$ in (7.26) , and using that $2u_{aa}\,|\boldsymbol{a}|^2 = \delta(\boldsymbol{a}\cdot\boldsymbol{a}) = \delta(|\boldsymbol{a}|^2) = 2\,|\boldsymbol{a}|\,\delta\,|\boldsymbol{a}|$ we get,

$$\boxed{\frac{\delta\,|\boldsymbol{a}|}{|\boldsymbol{a}|} = u_{aa}.} \tag{7.27}$$



The length of the original needle is $|\boldsymbol{a}_0| = |\boldsymbol{a}| - \delta\,|\boldsymbol{a}|$ where $\delta\,|\boldsymbol{a}| = u_{aa}\,|\boldsymbol{a}|$ is determined by the diagonal projection of the strain tensor.

The diagonal strain projection u_{aa} thus equals the *fractional change of lengths* in the direction $\boldsymbol{a}$. Obtaining this relation is part of the reason behind the conventional factor 2 in the definition (7.20) of the strain tensor. Another reason is given in problem 7.8.

Introducing the angle ϕ between two needles we have $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\phi$, and thus

$$\delta(\boldsymbol{a}\cdot\boldsymbol{b}) = \delta\,|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\phi + |\boldsymbol{a}|\,\delta\,|\boldsymbol{b}|\cos\phi - |\boldsymbol{a}|\,|\boldsymbol{b}|\sin\phi\,\delta\phi.$$

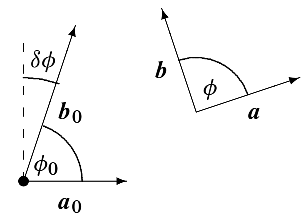
Solving for $\delta\phi$ and using(7.26) and (7.27) , we get

$$\delta\phi \equiv \phi - \phi_0 = \frac{(u_{aa} + u_{bb})\cos\phi - 2u_{ab}}{\sin\phi}. \tag{7.28}$$

For actually orthogonal vectors, such as the coordinate axes, we have $\phi = 90°$, and the change in angle simplifies to

$$\boxed{\delta\phi = -2u_{ab}.} \tag{7.29}$$

The off-diagonal projections of the strain tensor thus determine the change in angle between actually orthogonal needles.



The original angle between two vectors is $\phi_0 = \phi - \delta\phi$ where in this case the actual angle is $\phi = 90°$. The Euler representation makes the drawing a bit awkward because it is based on the actual geometry and not the original.

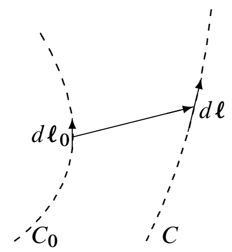
Infinitesimal elements

Curve, surface, and volume integrals appear everywhere in continuum physics, and the mathematics of these integrals is discussed in appendix C. When material is displaced, the infinitesimal elements of the integrals also change, and the formalism of displacement with small gradients introduced in this chapter allows us to calculate how they transform.

Curve element

A curve element is nothing but a small needle. Under a displacement, the curve element changes from $d\boldsymbol{\ell}_0$ to $d\boldsymbol{\ell}$, and simply transforms like the needle,

$$\boxed{\delta(d\boldsymbol{\ell}) \equiv d\boldsymbol{\ell} - d\boldsymbol{\ell}_0 = d\boldsymbol{\ell}\cdot\nabla\boldsymbol{u} = \nabla\boldsymbol{u}^{\top}\cdot d\boldsymbol{\ell}.} \tag{7.30}$$



A line element is stretched and rotated by the displacement that changes the curve from C_0 to C.

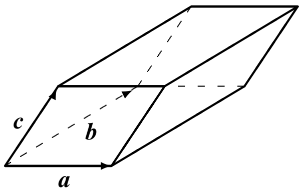
Here we have as before used a compact matrix notation for the displacement gradient tensor, $(\nabla\boldsymbol{u})_{ij} = \nabla_i u_j$. The transposed displacement gradient matrix (with rows and columns interchanged) then becomes $(\nabla\boldsymbol{u})^{\top}_{ij} = \nabla_j u_i$.

Example 7.4: Let $\boldsymbol{g}(\boldsymbol{x})$ be a vector field, and consider the integral $\int_C \boldsymbol{g}\cdot d\boldsymbol{\ell}$ along a curve C. Assuming that the end points are not displaced, the change in the integral becomes,

$$\begin{aligned}\delta\int_C \boldsymbol{g}\cdot d\boldsymbol{\ell} &= \int_C \delta\boldsymbol{g}\cdot d\boldsymbol{\ell} + \int_C \boldsymbol{g}\cdot\delta(d\boldsymbol{\ell})\\ &= \int_C (\boldsymbol{u}\cdot\nabla)\boldsymbol{g}\cdot d\boldsymbol{\ell} + \int_C \boldsymbol{g}\cdot(\nabla\boldsymbol{u})^{\top}\cdot d\boldsymbol{\ell}.\end{aligned} \tag{7.31}$$

The first term is due to the change in the field $\boldsymbol{g}$ under the displacement, $\delta\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}-\boldsymbol{u}) \approx (\boldsymbol{u}\cdot\nabla)\boldsymbol{g}$. This requires the displacement to be small compared to the length scales for changes in the $\boldsymbol{g}$-field. The second term is due to the change in the curve elements, and only requires the displacement gradients to be small (see also problem 7.17).

Volume element



Three infinitesimal needles span a parallelepiped with volume $dV \boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}$.

Consider a tiny material volume, $dV = \boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}$, of a parallelepiped spanned by three linearly independent infinitesimal needles. Under the displacement the volume changes by,

$$\begin{aligned}\delta(dV) &= \delta(\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}) \\ &= \delta\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{a} \times \delta\boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{a} \times \boldsymbol{b} \cdot \delta\boldsymbol{c} \\ &= (\boldsymbol{a} \cdot \nabla)\, \boldsymbol{u} \times \boldsymbol{b} \cdot \boldsymbol{c} + (\boldsymbol{b} \cdot \nabla)\, \boldsymbol{a} \times \boldsymbol{u} \cdot \boldsymbol{c} + (\boldsymbol{c} \cdot \nabla)\, \boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{u} \\ &= \left[\boldsymbol{b} \times \boldsymbol{c}\, (\boldsymbol{a} \cdot \nabla) + \boldsymbol{c} \times \boldsymbol{a}\, (\boldsymbol{b} \cdot \nabla) + \boldsymbol{a} \times \boldsymbol{b}\, (\boldsymbol{c} \cdot \nabla)\right] \cdot \boldsymbol{u}.\end{aligned}$$

In the last step we used that $\boldsymbol{u} \times \boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{b} \times \boldsymbol{c} \cdot \boldsymbol{u}$, and $\boldsymbol{a} \times \boldsymbol{u} \cdot \boldsymbol{c} = \boldsymbol{c} \times \boldsymbol{a} \cdot \boldsymbol{u}$ to pull the only $\boldsymbol{x}$-dependent factor $\boldsymbol{u}(\boldsymbol{x})$ out to the right.

We now use an identity between four arbitrary vectors,

$$\boldsymbol{a} \times \boldsymbol{b}\, (\boldsymbol{c} \cdot \boldsymbol{d}) + \boldsymbol{b} \times \boldsymbol{c}\, (\boldsymbol{a} \cdot \boldsymbol{d}) + \boldsymbol{c} \times \boldsymbol{a}\, (\boldsymbol{b} \cdot \boldsymbol{d}) = (\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c})\, \boldsymbol{d}. \tag{7.32}$$

It expresses the simple fact that in a three-dimensional space, four vectors will always be linearly dependent. It may be verified by dotting from left with $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ (see also problem B.10 on page 610). Replacing $\boldsymbol{d}$ by ∇, it follows immediately that

$$\boxed{\delta(dV) = \nabla \cdot \boldsymbol{u}\, dV,} \tag{7.33}$$

We have thus shown that the for small displacement gradients, the divergence of the displacement field, $\nabla \cdot \boldsymbol{u} = \sum_i \nabla_i u_i$, determines the fractional change $\delta(dV)/dV$ in the local volume. There are several other ways of deriving this relation, a couple of which are explored in problems 7.15 and 7.16.

Example 7.5 [Simple linear displacements]: A translation $\boldsymbol{u} = \boldsymbol{b}$ does not change the density because $\nabla \cdot \boldsymbol{u} = 0$. Likewise for an infinitesimal rotation around the z-axis through a small angle ϕ, the displacement field $\boldsymbol{u} = (-y, x, 0)\phi$ has vanishing divergence, so that the density is unchanged. A uniform scaling $\boldsymbol{u} = k\boldsymbol{x}$ has $\nabla \cdot \boldsymbol{u} = 3k$. If $k > 0$ the volume increases while the density diminishes with a contribution k from each dimension.

The change in volume of a material particle induces a change in the local density of the displaced matter. Using that the mass of the particle is unchanged during the displacement,

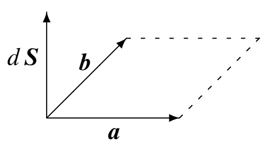
$$\delta(dM) = \delta(\rho dV) = \rho\, \delta(dV) + \delta\rho\, dV = 0,$$

and making use of (7.33) , the change in density becomes,

$$\boxed{\delta\rho = -\rho\, \nabla \cdot \boldsymbol{u}.} \tag{7.34}$$

Thus, if the divergence vanishes, there is no change of volume or density.

Surface element



Two infinitesimal needles span a parallelogram with area $d\boldsymbol{S} = \boldsymbol{a} \times \boldsymbol{b}$.

An tiny material surface element, $d\boldsymbol{S} = \boldsymbol{a} \times \boldsymbol{b}$, also changes under a displacement. Using that the volume element equals $dV = \boldsymbol{c} \cdot d\boldsymbol{S}$ we find from (7.33) ,

$$\boldsymbol{c} \cdot \delta(d\boldsymbol{S}) = \delta(\boldsymbol{c} \cdot d\boldsymbol{S}) - \delta\boldsymbol{c} \cdot d\boldsymbol{S} = \nabla \cdot \boldsymbol{u}(\boldsymbol{c} \cdot d\boldsymbol{S}) - (\boldsymbol{c} \cdot \nabla\boldsymbol{u}) \cdot d\boldsymbol{S}.$$

Since $\boldsymbol{c}$ is an arbitrary vector, it can be "divided out", and we get (using matrix notation)

$$\boxed{\delta(d\boldsymbol{S}) = \left(\nabla \cdot \boldsymbol{u}\, \mathbf{1} - \nabla\boldsymbol{u}\right) \cdot d\boldsymbol{S}.} \tag{7.35}$$

Both the magnitude and direction of the vector surface element are changed by the displacement, but the rule is quite different from that of the vector curve element (7.30) .

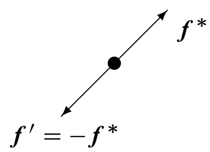
7.4 Work and energy

Deforming a body takes work, a fact known to everyone who has ever kneaded clay or dough. In these cases, the work you perform seems to get lost inside the material, but in other cases, as for example when you squeeze an elastic rubber ball, the material appears to store the work and release it again when you relinquish your grip. Many ball games like ping-pong or tennis rely entirely on the elasticity of the ball. No material is, however, perfectly elastic. Some work is always lost to internal friction. A hard steel ball may jump many times on a hard floor, but eventually it loses all its energy and comes to rest, partly due to air resistance, partly due to losses in the ball and, perhaps more importantly, in the floor. But even when your work seems to disappear into the dough, this is not really the case. The energy you have put into the dough has in the end been converted into heat which, however, cannot easily be recovered. We shall analyze the interplay of mechanics and heat in chapter 22.

In continuum physics it can be quite subtle to derive the correct energy relations. The simplest way to proceed is to *follow the work*. This is quite analogous to the admonition, “follow the money”, often used with success to uncover economic or political fraud.

Virtual displacement work

A volume of an arbitrary material which is not in mechanical equilibrium, will left to itself seek towards equilibrium. If the effective force $d\mathcal{F} = \boldsymbol{f}^* \, dV$, acting on a material particle of volume dV, does not vanish everywhere, the particle is (literally) forced to move until the effective force vanishes. If we wish to keep all material particles in their non-equilibrium positions, we must act on the body with an external volume distribution of so-called *virtual forces*, $\boldsymbol{f}' = -\boldsymbol{f}^*$, to compensate the effective internal forces. Even if such forces may be impossible to realize in practice, they will — in a thought experiment — freeze all the material particles in their positions for as long as we wish.



Every material particle can be kept in place by acting on it with an additional external force that balances the already existing effective body force on the particle.

Imagine now that in this frozen state all the material particles of the body are displaced infinitesimally by $\delta\boldsymbol{u}(\boldsymbol{x})$. To prevent the constant external forces on the surface of the body from performing work, we shall choose to keep the surface S of its volume V unchanged, so that the infinitesimal displacement must vanish at the surface, $\delta\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0}$ for $\boldsymbol{x} \in S$. The work of the virtual forces under this displacement then becomes,

$$\delta W = \int_V \boldsymbol{f}' \cdot \delta\boldsymbol{u} \, dV = -\int_V \boldsymbol{f}^* \cdot \delta\boldsymbol{u} \, dV. \tag{7.36}$$

Inserting $\boldsymbol{f}^* = \boldsymbol{f} + \nabla \cdot \boldsymbol{\sigma}^\top$ we find for the stress term (using index notation for clarity),

$$\sum_{ij} (\nabla_j \sigma_{ij}) \, \delta u_i = \sum_{ij} \nabla_j (\sigma_{ij} \delta u_i) - \sum_{ij} \sigma_{ij} \nabla_j \delta u_i \, .$$

Integrating over V, the first term is converted into a surface integral which vanishes because the displacement field $\delta\boldsymbol{u}$ vanishes at the surface. We now introduce a convenient notation for the trace of the product of two matrices, $\boldsymbol{A} \colon \boldsymbol{B} = \mathrm{Tr}(\boldsymbol{A} \cdot \boldsymbol{B}) = \sum_{ij} A_{ij} B_{ji}$. The work of the virtual forces may then be written,

$$\delta W = -\int_V \boldsymbol{f} \cdot \delta\boldsymbol{u} \, dV + \int_V \boldsymbol{\sigma} \colon \nabla \delta\boldsymbol{u} \, dV, \tag{7.37}$$

where $\boldsymbol{f}$ is the true force density due to long range forces and $\boldsymbol{\sigma} \colon \nabla\delta\boldsymbol{u} = \sum_{ij} \sigma_{ij} \nabla_j \delta u_i$.

The first term represents the part of the displacement work that is spent by the virtual forces *against* the true body forces, for example gravity. In that case the work contributes to the gravitational energy of the body. The second term represents the part of the work of the virtual forces that is spent against the *internal stresses* in the body,

$$\boxed{\delta W_{\text{deform}} = \int_V \boldsymbol{\sigma} : \nabla \delta \boldsymbol{u} \, dV.} \tag{7.38}$$

If the stress tensor is symmetric (which it normally is), $\sigma_{ij} = \sigma_{ji}$, the integrand may be written,

$$\boldsymbol{\sigma} : \nabla \delta \boldsymbol{u} = \sum_{ij} \sigma_{ij} \nabla_j \delta u_i = \sum_{ij} \sigma_{ij} \delta u_{ij} = \boldsymbol{\sigma} : \delta \boldsymbol{u}$$

where $\delta u_{ij} = \frac{1}{2}(\nabla_i \delta u_j + \nabla_j \delta u_i)$ is the infinitesimal change in the strain tensor. Evidently the work (7.38) work is associated with deformation of the material and contributes to the *deformation energy* of the body. In chapter 8 we shall derive an explicit expression for the deformation energy.

Example 7.6 [Thermodynamic work]: Suppose the stresses are only due to pressure, $\sigma_{ij} = -p\,\delta_{ij}$. Then the deformation work becomes

$$\delta W_{\text{deform}} = -\int_V p \, \nabla \cdot \delta \boldsymbol{u} \, dV. \tag{7.39}$$

Since $\delta(dV) = (\nabla \cdot \delta \boldsymbol{u})\, dV$ is the change in volume of a material particle, we see that the deformation work is identical to the thermodynamic work $-p\, \delta(dV)$ summed over all material particles.

∗ 7.5 Large deformations

When the condition (7.16) for slowly varying displacement is not fulfilled, we can no longer use the simple Cauchy strain tensor (7.20) . The local description of large deformation is essentially equivalent to the formalism of general curvilinear coordinate systems, but because space is Euclidean the description is not quite as complicated as that of truly non-Euclidean spaces [Green and Zerna 1992]. Although many aspects of the theory of large deformations were developed in the nineteenth century, the subject was not fully established until the mid-twentieth century through Rivlin's work on nonlinear materials. Here we shall only touch briefly on the most general aspects of large deformation theory which is a mathematically rather challenging subject [Green and Adkins 1960, Doghri 2000].

Ronald Samuel Rivlin (1915–2005). British born mathematician and physicist. Contributed to the understanding of nonlinear materials during the 1940s and 1950s. Discovered exact nonlinear solutions for isotropic materials.

The Euler representation

When there are no restrictions on the magnitude of the displacement field or the displacement gradients, the transformation $\boldsymbol{x} \to \boldsymbol{X}(\boldsymbol{x})$ becomes a completely general non-singular differentiable point-to-point map between two regions of space representing the actual body and its original situation. There is then no particular reason to split off the displacement, except to make contact with the description of small deformations in the preceding part of this chapter. The local properties of the displacement field will still be of importance, because the map is nearly linear in the neighborhood of any point in the body.

Consider again an infinitesimal material "needle" (in the actual body) described by the vector $d\boldsymbol{x}$. It originated in a material needle with coordinates,

$$dX_i = \sum_j \frac{\partial X_i}{\partial x_j} dx_j. \tag{7.40}$$

The scalar product of the infinitesimal vectors $d\boldsymbol{X}$ and $d\boldsymbol{Y}$ then becomes

$$d\boldsymbol{X} \cdot d\boldsymbol{Y} = \sum_{ij} g_{ij}(\boldsymbol{x}) dx_i dx_j, \tag{7.41}$$

where the tensor field

$$g_{ij} = \sum_k \frac{\partial X_k}{\partial x_i} \frac{\partial X_k}{\partial x_j} \tag{7.42}$$

is called the *Eulerian deformation tensor*. It contains all information about geometric changes taking place under the displacement. Writing

$$g_{ij} = \delta_{ij} - 2u_{ij}, \tag{7.43}$$

the change in the scalar product can be written in the same way as the expression (7.19) for slowly varying displacement,

$$d\boldsymbol{x} \cdot d\boldsymbol{y} - d\boldsymbol{X} \cdot d\boldsymbol{Y} = 2 \sum_{ij} u_{ij} dx_i dy_j. \tag{7.44}$$

Finally, we insert $\boldsymbol{X}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{u}(\boldsymbol{x})$, and arrive at the strain tensor in the Euler representation,

$$\boxed{u_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \sum_i \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right).} \tag{7.45}$$

It only differs from the infinitesimal Cauchy strain tensor (7.20) by the last non-linear term. This tensor was first introduced by Emilio Almansi in 1911 and Georg Hamel in 1912 (see [Chandrasekharaiah and Debnath 1994]).

Emilio Almansi (1869–1948). Italian mathematical physicist. Worked on nonlinear elasticity theory, electrostatics and celestial mechanics.

Georg Hamel (1877–1954). German mathematician. Solved one of the famous Hilbert problems in his doctoral thesis under Hilbert (1901).

Example 7.7 [Uniform scaling]: For uniform scaling $\boldsymbol{x} = \kappa \boldsymbol{X}$, we have $\boldsymbol{X} = \kappa^{-1}\boldsymbol{x}$, so that $\partial X_i/\partial x_j = \kappa^{-1}\delta_{ij}$. Consequently the displacement gradient tensor becomes,

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j} - \frac{\partial X_i}{\partial x_j} = \left(1 - \kappa^{-1}\right)\delta_{ij}, \tag{7.46}$$

from which we get the Euler-Almansi strain tensor,

$$u_{ij} = \frac{1}{2}\left[\left(1-\kappa^{-1}\right)\delta_{ij} + \left(1-\kappa^{-1}\right)\delta_{ij} - \left(1-\kappa^{-1}\right)^2\delta_{ij}\right] = \frac{1}{2}\left(1-\kappa^{-2}\right)\delta_{ij}, \tag{7.47}$$

valid for any value of κ. For $\kappa = 1 + k$ with $|k| \ll 1$, this expression becomes $u_{ij} = k\delta_{ij}$ to leading order.

The Lagrange representation

Even if the Lagrange representation of large deformation goes against the "materialistic" attitude of this book—in which the properties of material particles are functions of their actual positions—it is sometimes useful, for exampel in numerical computations. The basic relation between the two representation is simply that the Lagrangian displacement field by $\boldsymbol{U}(\boldsymbol{X})$ must take the same value as the Eulerian one at corresponding points, so that

$$\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{U}(\boldsymbol{X}) \qquad \text{with } \boldsymbol{U}(\boldsymbol{X}) = \boldsymbol{u}(\boldsymbol{x}). \tag{7.48}$$

Given the Euler displacement field, $\boldsymbol{u}(\boldsymbol{x})$, this nonlinear equation may be solved for $\boldsymbol{U}(\boldsymbol{X})$.

The local analysis now proceeds as before via the infinitesimal line element,

$$d\boldsymbol{x} = \sum_j \frac{\partial x_i}{\partial X_j} dX_j. \tag{7.49}$$

The scalar product becomes

$$d\boldsymbol{x} \cdot d\boldsymbol{y} = \sum_{ij} G_{ij} dX_i dY_j, \qquad G_{ij} = \sum_k \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}, \tag{7.50}$$

where G_{ij} is the *Lagrangian deformation tensor*. Writing

$$G_{ij} = \delta_{ij} + 2U_{ij} \tag{7.51}$$

it follows from (7.48) that the *Lagrangian strain tensor* is,

George Green (1793–1841). Largely self-taught English mathematician and mathematical physicist. Contributed to hydrodynamics, electricity and magnetism and partial differential equations.

$$\boxed{U_{ij} = \frac{1}{2}\left(\frac{\partial U_j}{\partial X_i} + \frac{\partial U_i}{\partial X_j} + \sum_k \frac{\partial U_k}{\partial X_i}\frac{\partial U_k}{\partial X_j}\right).} \tag{7.52}$$

It is called the *Lagrange-Green strain tensor*.

Example 7.8 [Uniform scaling]: For a uniform scaling $\boldsymbol{x} = \kappa \boldsymbol{X}$, the Lagrangian displacement field becomes

$$\boldsymbol{U}(\boldsymbol{X}) = \boldsymbol{x} - \boldsymbol{X} = (\kappa - 1)\boldsymbol{X}. \tag{7.53}$$

The Lagrange-Green strain tensor becomes,

$$U_{ij} = \tfrac{1}{2}\left[(\kappa - 1)\delta_{ij} + (\kappa - 1)\delta_{ij} + (\kappa - 1)^2\delta_{ij}\right] = \tfrac{1}{2}(\kappa^2 - 1)\delta_{ij}, \tag{7.54}$$

for any value of κ. It vanishes for $\kappa = \pm 1$, i.e. for no displacement and for a (physically impossible) pure reflection in the origin.

The scalar product of two infinitesimal needles then becomes,

$$d\boldsymbol{x} \cdot d\boldsymbol{y} = \kappa^2 d\boldsymbol{X} \cdot d\boldsymbol{Y}, \tag{7.55}$$

and just reflects that all vectors are scaled by the same amount κ.

Problems

7.1 Prove that eq. (7.7) is the correct transformation for a simple rotation.

7.2 Calculate displacement gradients and the strain tensor for the transformation,

$$u_x = \alpha(5x - y + 3z),$$
$$u_y = \alpha(x + 8y),$$
$$u_z = \alpha(-3x + 4y + 5z),$$

where α is small.

7.3 A displacement field is given by

$$u_x = \alpha(x + 2y) + \beta x^2,$$
$$u_y = \alpha(y + 2z) + \beta y^2,$$
$$u_z = \alpha(z + 2x) + \beta z^2,$$

where α and β are 'small'. Calculate the divergence and curl of this field. Calculate Cauchy's strain tensor.

7.4 Calculate the strain tensor for the displacement field $\boldsymbol{u} = (Ax + Cy, Cx - By, 0)$ where A, B, C are small constants. Under what condition will the volume be unchanged?

7.5 Calculate the strain tensor for $\boldsymbol{u} = \alpha(y, x, 0)$ where $0 < \alpha \ll 1$. Determine the principal directions of strain and the change in length scales along these.

7.6 **(a)** Calculate the displacement gradients and the strain tensor for the displacement field $\boldsymbol{u} = \alpha(y^2, xy, 0)$ with $|\alpha| \ll 1/L$, where L is the size of the body. **(b)** Calculate the principal directions of strain and the scaling factors.

7.7 Show that the change in a scalar product under a deformation is derivable from changes in length, i.e. from the diagonal projections u_{aa} of the strain tensor.

7.8 Show that the general displacement rule for a an infinitesimal needle (7.13) may be written

$$\boldsymbol{a}' = \boldsymbol{a} + \boldsymbol{\phi} \times \boldsymbol{a} + \mathsf{u} \cdot \boldsymbol{a} \qquad (7.56)$$

where $\boldsymbol{\phi} = \frac{1}{2}\nabla \times \boldsymbol{u}$ and $\mathsf{u} = \{u_{ij}\}$ is Cauchy's strain tensor (7.20) . What does the second term mean?

7.9 Show that the most general solution, for which Cauchy's strain tensor (7.20) vanishes, is

$$u_x = A + Dy + Ez$$
$$u_y = B - Dx + Fz$$
$$u_z = C - Ex - Fy$$

where A, B, C are arbitrary constants and D, E, F are small.

7.10 A deformable material undergoes two successive displacements, $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{x}'' = \boldsymbol{x}' + \boldsymbol{u}'(\boldsymbol{x}')$, both having small strain. Calculate the final strain tensor for the total deformation u''_{ij} relative to the original reference state.

∗ **7.11** Show that Cauchy's strain tensor satisfies the relation (going back to Saint-Venant)

$$\nabla_i \nabla_j u_{kl} + \nabla_k \nabla_l u_{ij} = \nabla_i \nabla_l u_{kj} + \nabla_k \nabla_j u_{il}. \qquad (7.57)$$

[Conversely, if this relation is fulfilled for a symmetric tensor field u_{ij} then there is a displacement field such that the strain tensor is given by (7.20) .]

∗ **7.12** Show that for finite deformations

$$\delta_{ij} + 2u_{ij} = \sum_k (\delta_{ik} + \nabla_i u_k)(\delta_{jk} + \nabla_j u_k), \tag{7.58}$$

and use this to prove that the matrix $\{\delta_{ij} + 2u_{ij}\}$ is positive definite. Show that

$$\det\{\delta_{ij} + \nabla_i u_j\} = \sqrt{\det\{\delta_{ij} + 2u_{ij}\}}. \tag{7.59}$$

∗ **7.13** Show that the only finite displacements with vanishing strain tensor are the rigid body translations and rotations.

∗ **7.14** Consider a shear deformation of a slab of elastic material in the xz-plane by a force in the x-direction. Assume that the sides of the slab are kept free to move, so that the only non-vanishing components of the strain tensor are $u_{xy} = u_{yx} = \frac{1}{2}\alpha$. Show that the displacement becomes

$$u_x = \alpha y, \tag{7.60}$$

$$u_y = -\left(1 - \sqrt{1-\alpha^2}\right) y. \tag{7.61}$$

for a deformation which is not assumed to be small. Describe what happens for $\alpha \to 1$.

7.15 Show that the Jacobian determinant of an arbitrary transformation $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$ represents the ratio of the infinitesimal volumes

$$\frac{dV'}{dV} = \det|\mathbf{1} + \boldsymbol{\nabla}\boldsymbol{u}|. \tag{7.62}$$

Show that for small displacement gradients, the right hand side becomes $1 + \boldsymbol{\nabla}\cdot\boldsymbol{u}$ and compare with eq. (7.33) .

7.16 Show that a small volume change can be represented by a surface integral

$$\delta V = V' - V = \oint_S \boldsymbol{u}\cdot d\boldsymbol{S}. \tag{7.63}$$

and use this result to derive (7.33) .

7.17 Consider the gravitational field $\boldsymbol{g}(\boldsymbol{x}) = -\boldsymbol{\nabla}\Phi(\boldsymbol{x})$ and calculate the change due to the displacement in the integral $\int_C \boldsymbol{g}\cdot d\boldsymbol{\ell}$ along a piece C of a curve. Show that eq. (7.31) yields the same result.