Linear theory of Faraday instability in viscous liquids

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The linear stability of the plane free surface of a viscous liquid on a horizontal plate under vertical sinusoidal oscillation is analysed theoretically. The free surface of a laterally unbounded liquid of any depth h may always be excited to standing waves if the external acceleration is raised above a critical value a_c. For a fixed external frequency ω, solutions are possible only within certain bands of wave numbers k for a given forcing amplitude above a_c, that is, within tongue-like stability zones in the a–k plane. The analysis for a shallow layer of viscous fluids shows new qualitative behaviours compared to the nearly inviscid theory. It predicts a series of bicritical points, where both harmonic and subharmonic solutions exist for the same forcing amplitude and forcing frequency. This makes harmonic solutions possible at the onset in a laterally large container, which is qualitatively different from the results of nearly inviscid theory. For a low viscosity fluid of small depths, the damping coefficient may be considered proportional to $(\nu\omega)^{1/2}/h$ in contrast to νk^2 predicted by the nearly inviscid theory. An approximate analytic expression is derived for the lower part of the lowest marginal curve in cases when the depth of the liquid is much larger than the thickness of the viscous boundary layer formed at the bottom plate. This approximate threshold agrees well with that of recent experiments with viscous liquids.

1. Introduction

The generation of standing waves at the plane free surface of a liquid subjected to vertical oscillation has been known since the observations of Faraday (1831). Faraday also noted that the resulting waves had fundamental frequency half the excitation frequency, i.e. the response was subharmonic. Lord Rayleigh (1883) also performed his own experiments and confirmed the experimental observations of Faraday. Recent experiments with viscous liquids show stripes (Edwards & Fauve 1993), regular triangular pattern (Müller 1993), competing hexagons and equilateral triangles (Kumar & Bajaj 1994) at the free surface in contrast to the earlier experiments with low viscosity fluids (see, for instance, Ezerskii *et al.* 1986; Tuffilaro *et al.* 1989; Ciliberto *al.* 1991) showing square patterns at the onset. These experimental observations suggest the importance of viscosity in pattern formation under parametric excitation.

On the theoretical front (see for a review Miles & Henderson 1990), Benjamin &

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Proc. R. Soc. Lond. A (1996) **452**, 1113–1126
Printed in Great Britain

TEX Paper

Ursell (1954) analysed the linear problem for ideal liquids enclosed in a container, vibrating sinusoidally in the vertical plane, and showed that the fluid dynamical equations can be reduced to a system of Mathieu equations, which allow harmonic as well as subharmonic responses. The free surface can be destabilized only in the tongue-like zones in the plane of the forcing amplitude a and the selected wave number k. The tongues correspond alternately to subharmonic and harmonic responses. For ideal liquids, the lowest points of all these tongues occur for vanishing forcing amplitude a and, therefore, both kind of responses are possible at the onset. However, in presence of small viscous dissipation, all tongues move away from the k-axis leading to finite a for the onset of standing waves. It turns out that the first tongue, which is subharmonic, moves least. Therefore, the first instability is always subharmonic in a container of large lateral dimensions according to the nearly inviscid theory. Harmonic responses cannot occur at onset in this situation. This result may or may not apply to shallow layers of viscous fluids. In addition, the experimental results for the critical acceleration a_c even for small viscosities do not compare well with the prediction of the nearly inviscid theory. This further emphasizes the need to understand the role of viscosity.

This work is motivated to provide a quantitative linear theory of viscous liquids, which is a necessay step to understand the basic mechanisms of pattern selection. The purposes of this paper are twofold. The first is to provide a linear theory for the stability of the plane free surface of a liquid of arbitrary viscosity and depth under vertical vibration. It will be shown that the stability problem can be reduced to finding eigenvalues and eigenvectors of a banded square matrix with non-zero elements only in two subdiagonals. For liquids of depth h much larger than the size of viscous boundary layers $b(=\sqrt{2\nu/\omega})$ due to the bottom plate, the lower part of the lowest marginal curve $a(k)$ can be predicted analytically. It is then sufficient to compute the stability threshold a_c and the critical wave number k_c by minimizing $a(k)$ with respect to k.

The second purpose is to explore the possibility of harmonic response at the onset. For the situations when the size of the viscous boundary layer b becomes comparable with the depth h of the liquid layer, the harmonic as well as subharmonic responses can coexist (see § 4) at the onset leading to a series of bicritical points.

2. Hydrodynamic system

We consider an incompressible liquid of uniform density ρ and dynamic viscosity η resting on a horizontal plate, which is subjected to a vertical sinusoidal oscillation of amplitude a and frequency ω. In a frame of reference fixed with the oscillating plate, the free surface is initially flat and stationary and the oscillation is equivalent to a temporally modulated gravitational acceleration,

$$G(t) = g - a\cos(\omega t). \tag{2.1}$$

Linearizing about the basic state of rest, which has time dependent pressure field $P(t) = P_0 - \rho G(t) z$, the equations for the perturbation fields $\boldsymbol{u}$, p in the the liquid read:

$$\rho \partial_t \boldsymbol{u} = -\nabla p + \eta \nabla^2 \boldsymbol{u}, \tag{2.2}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{2.3}$$

Taking curl twice of equation (2.2), we obtain for the vertical velocity w,

$$(\partial_t - \nu\nabla^2)\nabla^2 w = 0, \tag{2.4}$$

where ν $(= \eta/\rho)$ is the kinematic viscosity of the liquid.

The fluid rests on a rigid plate at which all three components of the velocity field must vanish.

$$w = 0 \quad \text{at} \quad z = -h, \tag{2.5}$$

$$\boldsymbol{u}_H = 0 \Longrightarrow -\nabla_H \cdot \boldsymbol{u}_H = \partial_z w = 0 \quad \text{at} \quad z = -h, \tag{2.6}$$

and h is the depth of the liquid. In derivation of (2.6) the equation of continuity (2.3) is used.

The free surface is initially flat, stationary, and coincident with the $z = 0$ plane by choice of the coordinate system. As soon as the instability sets in, the free surface is located at $z = \zeta(\boldsymbol{x}, t)$, where $\boldsymbol{x} \equiv (x, y)$, and obeys the kinematic boundary condition (Lamb 1932, § 9),

$$[\partial_t + (\boldsymbol{u} \cdot \nabla)]\zeta = w\,|_{z=\zeta}, \tag{2.7}$$

which states that the free surface is advected by the fluid motion.

Since we are interested in the linear stability of a free and flat surface, we may Taylor-expand the fields and their z-derivatives around $z = 0$ and retain only the lowest-order terms. It is then sufficient to compute the fields and their vertical derivatives at $z = 0$. The kinematic condition (2.7), after linearization, simplifies to

$$\partial_t \zeta = w\,|_{z=0}\,. \tag{2.8}$$

The boundary conditions at the free surface are determined by considering the stress tensor given by

$$\pi_{jk} = -p\delta_{jk} + \eta(\partial_j u_k + \partial_k u_j) + \rho G(t)\zeta\delta_{jz}\delta_{kz}. \tag{2.9}$$

The last term above is the stress due to the surface deformation in the effective gravitational acceleration. As there are no tangential stress components at the free surface, we have

$$\pi_{xz} = \pi_{yz} = 0. \tag{2.10}$$

Since these stress components vanish everywhere on the free surface, we may write

$$\eta(\partial_x \pi_{xz}) = \eta(\partial_y \pi_{yz}) = 0. \tag{2.11}$$

Inserting the definition (2.9) in (2.11) and using the equation of continuity (2.3) lead to

$$\eta(\nabla_H^2 - \partial_{zz})w = 0. \tag{2.12}$$

On the other hand the normal component of the stress tensor at the free surface must be equated to the surface tension σ times the curvature of the free surface. For small curvature, which is the case for the linear theory, this leads to

$$\pi_{zz}\,|_{z=0} = \sigma\nabla_H^2\zeta. \tag{2.13}$$

Substituting the definition (2.9) of the stress tensor into (2.13), we obtain the expression for the pressure at the free surface,

$$p|_{z=\zeta} = 2\eta(\partial_z w)_{z=0} + \rho G(t)\zeta - \sigma\nabla_H^2\zeta. \tag{2.14}$$

Another expression for the pressure can be derived by taking the horizontal divergence of the equations (2.2) and applying (2.3):

$$\begin{aligned}\nabla_H^2 p &= (\eta\nabla^2 - \rho\partial_t)\nabla_H \cdot \boldsymbol{u}_H \\ &= (\rho\partial_t - \eta\nabla^2)\partial_z w.\end{aligned} \tag{2.15}$$

Eliminating p from (2.14) and (2.15), we obtain at the free surface

$$[(\rho\partial_t - \eta\nabla^2)\partial_z w]_{z=0} = 2\eta(\nabla_H^2\partial_z w)_{z=0} + \rho G(t)\nabla_H^2\zeta - \sigma\nabla_H^4\zeta, \tag{2.16}$$

which serves as an additional boundary condition for (2.4), and is the only equation in which the external forcing $G(t)$ appears explicitly.

Because of the infinite extension of the fluid in the horizontal direction we can express the fields in the normal modes of the horizontal plane, i.e. $\sin(\boldsymbol{k}\cdot\boldsymbol{x})$, where the horizontal wave number k ($k^2 = k_x^2 + k_y^2$) can take any real and positive value. We now replace $w(\boldsymbol{x}, z, t)$ by $w(z,t)\sin(\boldsymbol{k}\cdot\boldsymbol{x})$, $\zeta(\boldsymbol{x},t)$ by $\zeta(t)\sin(\boldsymbol{k}\cdot\boldsymbol{x})$, and the differential operator ∇_H^2 by a number $-k^2$. The relevant equations then read:

$$[\partial_t - \nu(\partial_{zz} - k^2)](\partial_{zz} - k^2)w = 0, \tag{2.17}$$

$$(\partial_{zz}w + k^2 w)_{z=0} = 0, \tag{2.18}$$

$$w|_{z=-h} = 0, \tag{2.19}$$

$$(\partial_z w)_{z=-h} = 0, \tag{2.20}$$

$$[(\rho\partial_t - \eta\partial_{zz} + 3\eta k^2)\partial_z w]_{z=0} = -[\rho G(t) + \sigma k^2]k^2\zeta, \tag{2.21}$$

$$\partial_t\zeta = w|_{z=0}. \tag{2.22}$$

The above set of equations (2.17)–(2.22) constitute the complete linear stability problem for laterally unbounded viscous liquid, of arbitrary depth, under parametric oscillation.

3. Linear stability analysis

(a) *Liquids of finite depth*

The stability problem (2.17)–(2.22) is analysed by applying Floquet theory. Because $G(t)$, the effective gravitational acceleration in the moving frame, is a periodic function, the solutions to (2.17)–(2.22) are assumed of Floquet form (see also Kumar & Tuckerman 1994). The surface deformation ζ is then expressed as

$$\zeta(t) = \mathrm{e}^{\mu t} Z(t \bmod 2\pi/\omega). \tag{3.1}$$

The Floquet exponent μ can be expressed as

$$\mu = s + \mathrm{i}\alpha\omega, \tag{3.2}$$

where s and α can take any real and finite value. The function Z is periodic in time with period $2\pi/\omega$, and may therefore be expanded in the Fourier series

$$Z(t \bmod 2\pi/\omega) = \sum_{n=-\infty}^{\infty} \zeta_n \mathrm{e}^{\mathrm{i}n\omega t}. \tag{3.3}$$

The reality condition for the field ζ implies that

$$\zeta_{-n} = \zeta_n^* \quad \text{for} \quad \alpha = 0 \tag{3.4}$$

and

$$\zeta_{-n} = \zeta^*_{n-1} \quad \text{for} \quad \alpha = 1/2. \tag{3.5}$$

For $0 < \alpha < 1/2$, the complex conjugate of the right-hand side in equation (3.1) must be added to form a real $\zeta(t)$. However, all non-zero integer values of α can be absorbed into the periodic function Z. In addition, for any α between 0 and 1/2, the complex conjugate terms are equivalent to considering α between 1/2 and 1. Therefore, it is sufficient to consider only the range $0 \leqslant \alpha \leqslant 1/2$. The solutions $\zeta(t)$ corresponding to $\alpha = 0$ and $\alpha = 1/2$ are referred to as harmonic and subharmonic solutions respectively. Now from (2.22)

$$w(z=0,t) = \mathrm{e}^{\mu t} \sum_{n=-\infty}^{\infty} w_n(z=0)\mathrm{e}^{\mathrm{i}n\omega t}, \tag{3.6}$$

where

$$w_n(z=0) = (\mu + \mathrm{i}n\omega)\zeta_n. \tag{3.7}$$

The vertical velocity must reduce to this form at $z=0$ for all t. We therefore assume that $w(z,t)$ may be expressed as

$$w(z,t) = \mathrm{e}^{\mu t} \sum_{n=-\infty}^{\infty} w_n(z)\mathrm{e}^{\mathrm{i}n\omega t} \tag{3.8}$$

with the same reality condition on $w_n(z)$ as for ζ_n. The choice of trial functions $\zeta(t)$ as in (3.1)–(3.3) and for $w(z,t)$ as in (3.8) reproduces the results for unforced ($a=0$) surface waves in viscous fluids (Chandrasekhar 1961) by setting $n=0$.

Inserting (3.8) into (2.17) we obtain for each Fourier component n, a fourth-order ordinary differential equation in z:

$$(\partial_{zz} - k^2)(\partial_{zz} - q_n^2)w_n(z) = 0, \tag{3.9}$$

where

$$q_n^2 \equiv k^2 + \frac{[s + \mathrm{i}(\alpha+n)\omega]}{\nu} \tag{3.10}$$

with the convention that q_n is the root of (3.10) with positive real part.

The general solution of (3.9) is

$$w_n(z) = P_n \cosh(kz) + Q_n \sinh(kz) + R_n \cosh(q_n z) + S_n \sinh(q_n z). \tag{3.11}$$

Inserting (3.11) in (2.18)–(2.20) and using (2.22) leads to

$$P_n = \nu(q_n^2 + k^2)\zeta_n, \tag{3.12}$$

$$R_n = -2\nu k^2 \zeta_n, \tag{3.13}$$

$$S_n = -\frac{[kP_n + R_n\{k\cosh(q_n h)\cosh(kh) - q_n \sinh(q_n h)\sinh(kh)\}]}{[q_n \cosh(q_n h)\sinh(kh) - k\sinh(q_n h)\cosh(kh)]}, \tag{3.14}$$

$$\begin{aligned} Q_n = {} & R_n[q_n \sinh(q_n h)\cosh(kh) - k\cosh(q_n h)\cosh(kh)] \\ & -S_n[q_n \cosh(q_n h)\cosh(kh) - k\sinh(q_n h)\sinh(kh)]. \end{aligned} \tag{3.15}$$

Inserting (3.11) into (2.21) and using (3.12)–(3.15) we arrive at the following recursion relation:

$$A_n\zeta_n = a(\zeta_{n-1} + \zeta_{n+1}), \tag{3.16}$$

where

$$A_n = \frac{2}{k}\left[gk + \frac{\sigma}{\rho}k^3 - \nu^2\left(\frac{4q_n k^2(q_n^2+k^2) - C_n\cosh(q_n h)\cosh(kh) + D_n\sinh(q_n h)\sinh(kh)}{q_n\cosh(q_n h)\sinh(kh) - k\sinh(q_n h)\cosh(kh)}\right)\right] \tag{3.17}$$

with

$$C_n = q_n(q_n^4 + 2q_n^2k^2 + 5k^4), \tag{3.18}$$

$$D_n = k(q_n^4 + 6q_n^2k^2 + k^4). \tag{3.19}$$

The condition (3.7) implies that $w_0(z=0) = 0$ when $[\mu + in\omega] = 0$, which, together with the boundary and continuity conditions, insures that $w_0(z) = 0$ for all z. Therefore,

$$A_0(\mu + in\omega = 0) \equiv A_0^h = \frac{2}{k}\left(gk + \frac{\sigma}{\rho}k^3\right). \tag{3.20}$$

In absence of the external forcing (i.e. $a = 0$), there is no coupling of temporal modes. We can then set $n = 0$ in (3.16)–(3.17) and the resulting equation,

$$gk + \frac{\sigma}{\rho}k^3 - \nu^2\left(\frac{[4q_0k^2(q_0^2+k^2) - C_0\cosh(q_0h)\cosh(kh) + D_0\sinh(q_0h)\sinh(kh)]}{[q_0\cosh(q_0h)\sinh(kh) - k\sinh(q_0h)\cosh(kh)]}\right) = 0, \tag{3.21}$$

is the dispersion relation for the surface waves in viscous liquids of finite depth (for a historical reference, see Basset 1888, § 523). The exponent $\mu(= s + \mathrm{i}\alpha)$ is then simply the complex decay rate of the surface waves.

The external forcing couples different temporal modes making the exact stability analysis for arbitrary viscosities difficult. However, the stability of the free surface can be determined with any preassigned accuracy by converting the recursion relation (3.16) to a matrix equation:

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & A_{-2} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & A_{-1} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & A_0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & A_1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & A_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \vdots \\ \zeta_{-2} \\ \zeta_{-1} \\ \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \end{pmatrix} = a \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \vdots \\ \zeta_{-2} \\ \zeta_{-1} \\ \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \end{pmatrix}, \tag{3.22}$$

which can be written symbolically as

$$\mathcal{A}\zeta = a\mathcal{B}\zeta. \tag{3.23}$$

$\mathcal{A}$ is a diagonal square matrix with complex elements and $\mathcal{B}$ is a banded square matrix with with two sub-diagonals. An ordinary eigenvalue problem can easily be constructed from (3.23) by inverting $\mathcal{A}$:

$$(\mathcal{A}^{-1}\mathcal{B})\zeta = \frac{1}{a}\zeta. \tag{3.24}$$

The eigenvalues of $\mathcal{A}^{-1}\mathcal{B}$ are inverse of the forcing amplitudes. For a usual stability analysis, the wave number k and the forcing amplitude a are fixed for given fluid parameters, and the Floquet exponent $\mu = s + i\alpha\omega$ is computed such that the corresponding growth rate s is the largest. The marginal stability boundaries are defined by the curves in a–k plane on which $s(a, k) = 0$. In the present method, μ is fixed instead, usually at $s = 0$ and at $\alpha = 0$ or $\alpha = 1/2$. We then solve for the eigenvalues $1/a$. Only real and positive values of $1/a$ are meaningful as complex values of a do not corespond to a real forcing for given parameters. The largest, or several largest, real positive eigenvalues of $1/a$ is selected for $s = 0$ and $\alpha = 1/2$ or $\alpha = 0$. These give the marginal stability curves $a(k, s = 0, \alpha = 1/2)$ for subharmonic and $a(k, s = 0, \alpha = 0)$ for harmonic instability without any interpolation. Any non-zero real eigenvalue of $\mathcal{A}^{-1}\mathcal{B}$ implies a finite forcing amplitude a. The largest eigenvalues of this matrix are the inverse of the lowest forcing amplitudes a for a preassigned k and given fluid parameters. The matrix $\mathcal{A}^{-1}\mathcal{B}$ is a banded matrix and its structure is similar to that of the matrix $\mathcal{B}$ but with, in general, complex elements. For the harmonic case it is given by

$$\mathcal{A}^{-1}\mathcal{B} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 1/A_2^* & 0 & 0 & 0 & \cdots \\ \cdots & 1/A_1^* & 0 & 1/A_1^* & 0 & 0 & \cdots \\ \cdots & 0 & 1/A_0^h & 0 & 1/A_0^h & 0 & \cdots \\ \cdots & 0 & 0 & 1/A_1 & 0 & 1/A_1 & \cdots \\ \cdots & 0 & 0 & 0 & 1/A_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{3.25}$$

and for the subharmonic case it reads

$$\mathcal{A}^{-1}\mathcal{B} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 1/A_1^* & 0 & 0 & \cdots \\ \cdots & 1/A_0^* & 0 & 1/A_0^* & 0 & \cdots \\ \cdots & 0 & 1/A_0 & 0 & 1/A_0 & \cdots \\ \cdots & 0 & 0 & 1/A_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{3.26}$$

Note that the complex conjugate of matrix $\mathcal{A}^{-1}\mathcal{B}$, for harmonic and subharmonic cases, can be constructed by interchanging its rows and columns even number of times. Therefore, possible eigenvalues for these two cases must either be *real* or

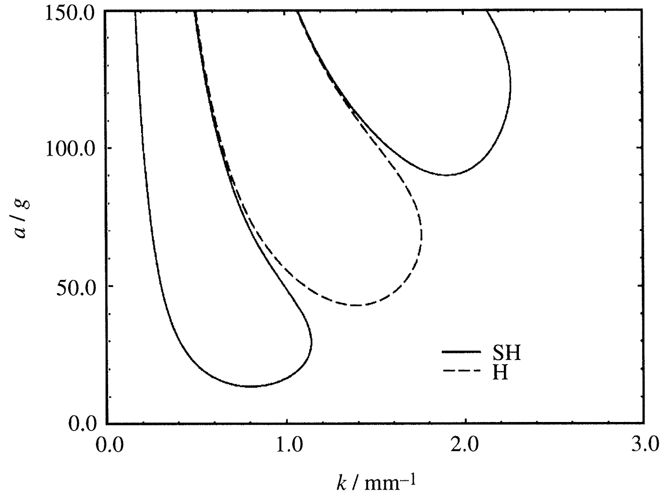

Figure 1. Stability boundaries of glycerine–water mixture ($\nu = 1.02 \times 10^{-4}\ \mathrm{m^2\ S^{-1}}$, $\sigma = 67.6 \times 10^{-3}\ \mathrm{N\ m^{-1}}$) of thickness 2.0×10^{-3} m at the external frequency $\omega/2\pi = 60$ Hz. Subharmonic (SH) and harmonic (H) tongues alternate.

complex conjugate pairs. For the range $0 < \alpha < 1/2$, the matrix $\mathcal{A}^{-1}\mathcal{B}$ is still a banded matrix of similar structure, but none of the elements is complex conjugate of other. It is then not obvious that there might be a real eigenvalue. Numerically we see either zero or infinitesimally small real eigenvalues in this case when the real part s of the Floquet exponent μ is set to zero. The corresponding forcing amplitude are infinitely large. This suggests that the solutions corresponding to these Floquet exponents may never be made unstable at any realizable forcing amplitudes. These cases then seem to be irrelevant for the linear theory. For an exact statement about the solutions in these cases, one needs to get the complete spectrum of all possible eigenvalues. We shall consider in this paper only the harmonic and the subharmonic cases, which are the most relevant ones. Eigenvalues of the matrix $\mathcal{A}^{-1}\mathcal{B}$ can be obtained numerically. Thus one can predict the stability boundary as well as the stability threshold. If the growth rate s is set to a negative (positive) value, the corresponding smallest values of a fall on curves below (above) the marginal stability curves for the same fluid parameters. So the meaning of these stability zones are precisely the same as in usual stability analysis.

In figure 1, we show the inverse of the first two largest eigenvalues (for $\alpha = 1/2$ as well as for $\alpha = 0$) of the matrix $\mathcal{A}^{-1}\mathcal{B}$ with $s = 0$ for glycerine–water mixture, subjected to vertical vibration at a fixed frequency, as a function of the wave number k. There exist tongue-like zones in the a–k plane, within which the solutions of the hydrodynamic system (3.9) with (3.10) are unstable, i.e. s is positive. As a is raised above a critical value a_c – the lowest value of a for the lowest tongue – the plane free surface of the liquid becomes unstable to standing waves. For a liquid of depth h much larger than the typical size b $(= \sqrt{2\nu/\omega})$ of the viscous boundary layer, the lowest tongue corresponds to the subharmonic solutions ($\alpha = 1/2$). Therefore, in these circumstances the onset of the instability is always subharmonic.

As long as $h \gg b$, the first tongue, which also corresponds to the subharmonic response, is the lowest. To compute the stability threshold a_c and the critical wave number k_c, one need only know the lower part of the first tongue. That is, only the largest eigenvalue of the matrix $\mathcal{A}^{-1}\mathcal{B}$ for the subharmonic case is required. It is then sufficient to truncate the matrix at $n = 1$, which means considering a 4×4 subdiagonal matrix only. The eigenvalues are then

$$a(k, \omega, \nu, \sigma, h) = \pm\left(\frac{|A_1|^2 + (A_0 A_1 + \text{c.c.})}{2|A_0|^2|A_1|^2} \pm \sqrt{\left[\frac{|A_1|^2 + (A_0 A_1 + \text{c.c.})}{2|A_0|^2|A_1|^2}\right]^2 - \frac{1}{|A_0|^2|A_1|^2}}\right)^{-1/2}, \tag{3.27}$$

where A_ns are given by (3.17). Note that, because of the severe truncation of the Fourier expansion, the above formula is valid only for smaller ks. As the wave number k becomes larger, the second minimum value of a diverges. This indicates that the lowest value of a is no longer accurate and one needs to include more terms in the expansion. If $h \gg b$, which is the case in most experiments, the above formula gives the first tongue, up to the first turning point after the first minimum, within 1% of the exact numerical result. Minimizing $a(k, \omega)$ with respect to k yields critical value $k_c(\omega)$ and, therefore, $a_c(\omega)$ for given values of the fluid parameters. Figure 2 shows a comparison of the prediction of the above formula to experimental results obtained with a glycerine–water mixture (Edwards, personal communication 1993) of thickness $h = 2.9 \times 10^{-3}$ m. The mixure had kinematic viscosity $\nu = 10^{-4}\ \text{m}^2\ \text{s}^{-1}$ and surface tension $\sigma = 65 \times 10^{-3}\ \text{N m}^{-1}$. In this experiment, the free surface was pinned using brimful technique to avoid the effect due to variation of contact angles. The fluid container was covered to minimize evaporation of water as well as contamination of the free surface. The stability threshold a_c as well as the critical wavelength λ_c are in good agreement for entire range of forcing frequency. The dispersion relation for shallow layers of ideal fluid is given by

$$\tfrac{1}{4}\omega^2 = \omega_0^2 = (gk + (\sigma/\rho)k^3)\tanh(kh), \tag{3.28}$$

and its prediction does not agree at all with the experimental results (see dashed lines in figure 2).

Once the critical amplitude a_c and critical wave number k_c are known, the critical modes can easily be computed. Again for cases when $h \gg b$, we can express the deformation ζ from the flat surface approximately by two temporal modes ζ_0, ζ_1 and their complex conjugates,

$$\zeta_c(t) = (\zeta_0 e^{i\omega/2} + \zeta_1 e^{3i\omega/2} + \text{c.c.}), \tag{3.29}$$

where ζ_1 is related to ζ_0 by

$$\zeta_1^* = \left(\frac{1}{A_0^* A_1^*}\right)\left(\frac{1}{a_c^2} - \frac{1}{A_0^* A_1^*}\right)^{-1} \zeta_0. \tag{3.30}$$

A_0^* and A_1^* in the above expression are to be computed at the critical point. The vertical velocity $w_c(z, t)$ can be evaluated by inserting (3.29)–(3.30) into (3.11)–(3.15).

(*b*) *Liquids of infinite depth*

For the case of liquids of infinite depth, the velocity field $w(z, t)$ and its z-derivative must vanish at $z = -\infty$. The solution of (3.9)–(3.10) can be expressed as

$$w_n^\infty(z) = P_n^\infty \exp(kz) + R_n^\infty \exp(q_n z). \tag{3.31}$$

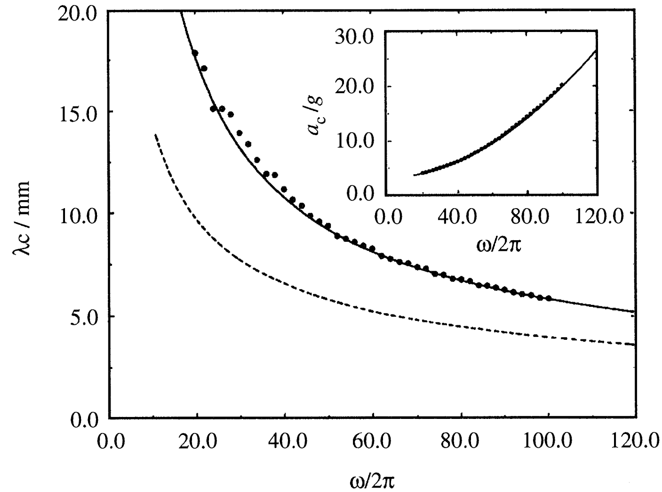

Figure 2. Comparison of the theoretical prediction (solid line) for the dispersion relation with the experimental results (filled circles) with glycerine–water mixture of thickness $h = 2.9 \times 10^{-3}$ m. The ideal fluid results for finite depth (dashed line) do not agree with the experimental observations. Inset: Stability threshold a_c as a function of the external frequency $\omega/2\pi$. The fluid parameters for the theoretical curve are as in figure 2.

Following the same procedure as before, we have

$$P_n^\infty = \nu(q_n^2 + k^2)\zeta_n, \tag{3.32}$$

$$R_n^\infty = -2\nu k^2\zeta_n. \tag{3.33}$$

The equation (2.21) for the pressure at the free surface leads to the recursion relation

$$A_n^\infty \zeta_n = a(\zeta_{n+1} + \zeta_{n-1}), \tag{3.34}$$

where A_n^∞ is defined as:

$$A_n^\infty = \frac{2}{k}\left[gk + \frac{\sigma}{\rho}k^3 + \nu^2(q_n^4 + 2q_n^2k^2 - 4q_nk^3 + k^4)\right]. \tag{3.35}$$

In absence of external forcing, there is no coupling of the temporal modes and (3.34)–(3.35) lead to the known dispersion relation (Chandrasekhar 1961, § 94e) for the surface waves in viscous liquids of infinite depth:

$$gk + \frac{\sigma}{\rho}k^3 + \nu^2(q_0^4 + 2q_0^2k^2 - 4q_0k^3 + k^4) = 0. \tag{3.36}$$

In the case of a liquid of infinite depth, the first tongue, which always corresponds to the subharmonic response, is the lowest. The first instability is therefore always subharmonic and the stability threshold can be evaluated, within less than 1%, by minimizing the expression (3.27) with respect to k. In the above A_ns are defined by (3.35).

4. Bicritical points

As the depth h of the liquid layer becomes comparable with the size b of the boundary layer formed due to the presence of the rigid plate, one expects distortion

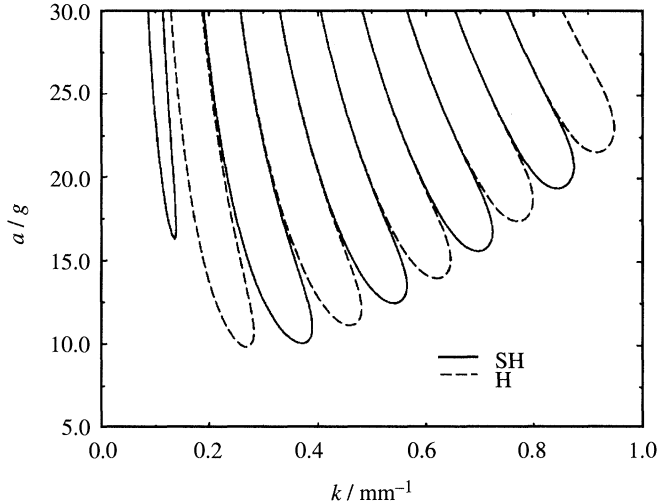

Figure 3. Stability boundaries of glycerine–water for $\omega/2\pi = 6$ Hz. All other parameters are same as in figure 2.

of the critical modes. The approximate expression for a in (3.27) is no longer accurate and one needs to retain more terms in the Fourier expansion of the fields. We then find out the eigenvalues of the matrix $(\mathcal{A}^{-1}\mathcal{B})$ numerically. Figure 3 shows the first few tongues for the case when the thickness of the liquid layer is 2 mm and the forcing frequency is 6 Hz. Comparing it with figure 1, we see that the allowed bands of the wave number k are much narrower and all the tongues are shifted towards lower k as the excitation frequency is decreased. The most important difference between the two cases are at lower ks. The first tongue, which used to be the lowest is no longer so as the frequency is lowered because the size of the boundary layer is now roughly the same as that of the thickness of the liquid layer. The first tongue is pushed up while all other tongues are pushed down. This leads to bicritical points, where the subharmonic as well as the harmonic responses are possible for the same value of the critical forcing amplitude a_c. The first two bicritical points are shown in figure 4. As the excitation frequency is lowered, the lowest points of all the tongues (see figure 4a, for the first tongue this happens at larger excitation frequencies and is not shown here) as well as the corresponding wave numbers ks (figure 4b) decrease first. When the selected wavelength λ_c $(= 2\pi/k_c)$ becomes comparable to the depth h of the liquid layer, the effect of the bottom plate becomes strong and, consequently, the the stability threshold rises. The lowest points of other tongues at relatively higher ks still decrease because the excitation frequency decreases. This leads to a bicritical point. As frequency is further lowered, the effect of the bottom plate becomes stronger. The second tongue is pushed up while the third and higher tongues are still coming down. Between the first and the second bicritical points, the first instabilty is harmonic. In principle, there could be a series of bicritical points. This effect is purely due to the viscosity of the liquid.

A qualitatively similar phenomenon occurs as the thickness h of the liquid is decreased or the viscosity of the liquid is increased while other parameters are kept constant. Interestingly, Lord Rayleigh (1883) conducted his experiments with thin fluid layers and noted the strong influence of the lower plate. He used low-viscosity fluid, for which the first instability is always subharmonic as he observed.

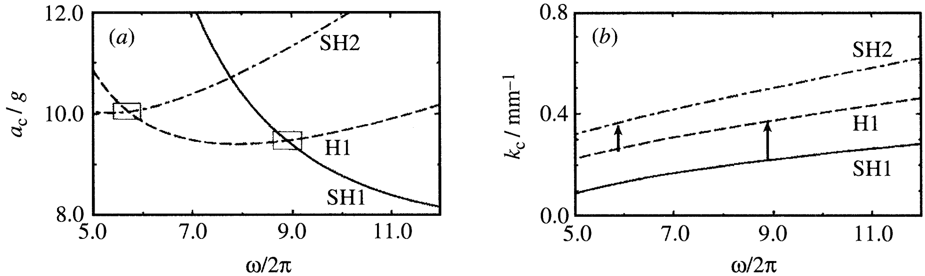

Figure 4. The first two bicritical points (enclosed by boxes in figure 4*a*). As the frequency is decreased, the first subharmonic (SH1) and the first harmonic (H1) response occur at the same value a_c. At these points the critical wave number k_c jumps (figure 4*b*) to higher values.

5. Limit of small viscous dissipation

For $\nu k^2 \ll \omega$, the expression for A_n in (3.16) can be expanded in the powers of k/q_n to compute the effect of viscosity perturbatively. The recursion relation (3.16) then can, up to second order in k/q_n, be written as

$$\frac{2}{k\tanh(kh)}\left[\left(g+\frac{\sigma}{\rho}k^2\right)k\tanh(kh)+\nu^2q_n^4\left\{1+\frac{2}{\sinh(2kh)}\left(\frac{k}{q_n}\right)\right.\right.$$
$$\left.\left.+\ (1+\coth^2(kh))\left(\frac{k}{q_n}\right)^2\right\}\right]\zeta_n=a(\zeta_{n-1}+\zeta_{n+1}), \qquad (5.1)$$

which can be further simplified to

$$\frac{2}{k\tanh(kh)}\left[\left(g+\frac{\sigma}{\rho}k^2\right)k\tanh(kh)+[s+\mathrm{i}(\alpha+n)\omega]^2\right.$$
$$\left.+\ \frac{2\nu^{1/2}k}{\sinh(2kh)}[s+\mathrm{i}(\alpha+n)\omega]^{3/2}+\nu k^2[3+\coth^2(kh)][s+\mathrm{i}(\alpha+n)\omega]\right]\zeta_n$$
$$=a(\zeta_{n-1}+\zeta_{n+1}). \qquad (5.2)$$

In the limit of $\nu \to 0$, the above recursion relation is equivalent to ideal fluid results (see Appendix A). However, the lowest correction to the ideal fluid result is proportional to $(\omega\nu)^{1/2}k/\sin(2kh)$ rather than νk^2, which is generally used in the nearly inviscid models to take account of the viscous damping. For small k as well as h, the correction is proportional to $(\nu\omega)^{1/2}/h$. Since $[s+\mathrm{i}(n+\alpha)]$ is equivalent to ∂_t and the first correction involves an exponent equal to 3/2, one finds fractional derivative of ζ with respect to time. The recursion relation (5.2), therefore, is no longer equivalent to the Mathieu equation with a traditional linear damping term. Only in the limit of infinite depth ($kh \gg 1$), does the first viscous correction become proportional to νk^2 and the resulting equation may be considered equivalent to a traditional linear viscous damping term ($= 4\nu k^2\dot{\zeta}$) in the Mathieu equation for ζ for a given mode of wave number k.

6. Conclusions

The linear theory of the stability of the plane free surface of a viscous liquid of arbitrary viscosity and depth under parametric excitation is presented. We showed that a viscous liquid layer yields qualitatively different results when its depth becomes comparable with the thickness of the viscous boundary layer created at the bottom plate. Only in the limit of small viscosity and infinite depth, the Mathieu equation with a linear damping term may represent parametric instability in a viscous liquid. For viscous liquids, the theory predicts the possibility of series of bicritical points at the instability onset (figures 3 and 4). For liquids of depth large compared with the size of viscous boundary layer, the lower part of the lowest marginal curve can be predicted analytically within reasonable accuracy and the prediction of this approximate formula compares well quantitatively with the experimental data for the stability thresholds as well as for the wavelengths observed experimentally. The viscous boundary conditions are essential to get correct stability thresholds as well as, in our opinion, to understand the mechanism of pattern formation in parametrically excited viscous fluids from the first principles. The theory can easily be extended for the study of the stability of the interface(s) of two (or more) fluids (Kumar & Tuckerman 1994) under parametric excitation.

The author is thankful to S. Fauve, L. S. Tuckerman, H. W. Müller and W. S. Edwards for having fruitful discussions with them. This work has been supported by the CNES (Centre National d'Etudes Spatiales) under Contract No. 92/0328. The recent experimental observations of Bechhoefer *et al.* (1995) for viscous fluids agree well with the prediction of this theory.

Appendix A. Recursion relation for the Mathieu equation

Benjamin & Ursell (1954) showed that the hydrodynamic equations for inviscid fluids can be written for every wave number k as

$$\ddot{\zeta} + \omega_0^2[1 - \hat{a}\cos(\omega t)]\zeta = 0, \tag{A 1}$$

where

$$\omega_0^2 = gk + \sigma k^3/\rho, \tag{A 2}$$

$$\hat{a} = ak/\omega_0^2. \tag{A 3}$$

In the nearly inviscid theory, a linear damping is added to the Mathieu equation. Following the argument of Landau & Lifshitz (1987, § 25), the damping coefficient is found to be $\gamma = 2\nu k^2$. Therefore, the relevant equation now becomes

$$\ddot{\zeta} + 2\gamma\dot{\zeta} + \omega_0^2[1 - \hat{a}\cos(\omega t)]\zeta = 0. \tag{A 4}$$

Substituting the Floquet form for ζ (3.1) in (A 4), and using (A 2)–(A 3) we find the required recursion relation

$$A_n\zeta_n = a(\zeta_{n-1} + \zeta_{n+1}), \tag{A 5}$$

$$A_n = \frac{2}{k}\left[\{s + \mathrm{i}(n+\alpha)\omega\}^2 + 4\nu k^2\{s + \mathrm{i}(n+\alpha)\omega\} + gk + \frac{\sigma}{\rho}k^3\right]. \tag{A 6}$$

References

Basset, A. B. 1888 *A treatise on hydrodynamics*, 1st edn. Deighton, Bell & Co.

Bechhoefer, J., Ego, V., Manneville, S. & Johnson, B. 1995 *J. Fluid Mech.* **288**, 325.

Benjamin, T. B. & Ursell, F. 1954 *Proc. R. Soc. Lond.* A **225**, 505.

Chandrasekhar, S. 1961 *Hydrodynamic and hydromagnetic stability*, 3rd edn. Oxford: Clarendon Press.

Ciliberto, S., Douady, S. & Fauve, S. 1991 *Europhys. Lett.* **15**, 23.

Edwards, W. S. & Fauve, S. 1993 *Phys. Rev.* E **47**, R788.

Ezerskii, A. B., Korotin, P. I. & Rabinovich, M. I. 1985 *Zh. Eksp. theor. Fiz.* **41**, 129–131 (transl. 1986 *Sov. Phys. JEPT Lett.* **41**, 157–160).

Faraday, M. 1831 *Phil. Trans. R. Soc. Lond.* A **52**, 319.

Kumar, K. & Bajaj, K. M. S. 1995 *Phys. Rev.* E **52**, R4606.

Kumar, K. & Tuckerman, L. S. 1994 *J. Fluid Mech.* **279**, 49.

Lamb, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

Landau, L. & Lifshitz, E. M. 1987 *Fluid mechanics*, 2nd edn. Pergamon Press.

Miles, J. W. & Henderson, D. 1990 *A. Rev. Fluid Dyn.* **22**, 143.

Müller, H. W. 1993 *Phys. Rev. Lett.* **71**, 3287.

Rayleigh, Lord 1883 *Phil. Mag.* **16**(5), 50.

Tufillaro, N., Ramshankar, R. & Gollub, J. P. 1989 *Phys. Rev. Lett.* **62**, 422.

Received 5 August 1993; revised 7 June 1994; accepted 4 January 1995