Chapter 18

Counting

I’m gonna close my eyes
And count to ten
I’m gonna close my eyes
And when I open them again
Everything will make sense to me then
—Tina Dico, ‘Count To Ten’

We are now in a position to apply the periodic orbit theory to the first and
easiest problem in theory of chaotic systems: cycle counting. This is the simplest
illustration of the raison d’etre of periodic orbit theory; we derive a duality transformation
that relates local information - in this case the next admissible symbol in a symbol sequence - to
global averages, in this case the mean rate of growth of the number of cycles with increasing cycle
period. In chapter 17 we have transformed, by means of the transition matrices / graphs, the
topological dynamics of chapter 14 into a multiplicative operation. Here we show
that the $n$th power of a transition matrix counts all itineraries of length $n$. The
asymptotic growth rate of the number of admissible itineraries is therefore given
by the leading eigenvalue of the transition matrix; the leading eigenvalue is in turn
given by the leading zero of the characteristic determinant of the transition matrix,
which is - in this context - called the topological zeta function.

For flows with finite transition graphs this determinant is a finite topological
polynomial which can be read off the graph. However, (a) even something as
humble as the quadratic map generically requires an infinite partition (sect.18.5),
but (b) the finite partition approximants converge exponentially fast.

The method goes well beyond the problem at hand, and forms the core of the
entire treatise, making tangible the abstract notion of “spectral determinants” yet
to come.
18.1 How many ways to get there from here?

In the 3-disk system of example 14.2 the number of admissible trajectories doubles with every iterate: there are $K_n = 3 \cdot 2^n$ distinct itineraries of length $n$. If disks are too close and a subset of trajectories is pruned, this is only an upper bound and explicit formulas might be hard to discover, but we still might be able to establish a lower exponential bound of the form $K_n \geq C e^{n \tilde{h}}$. Bounded exponentially by $3e^{n \ln 2} \geq K_n \geq C e^{n \tilde{h}}$, the number of trajectories must grow exponentially as a function of the itinerary length, with rate given by the topological entropy:

$$h = \lim_{n \to \infty} \frac{1}{n} \ln K_n \quad (18.1)$$

We shall now relate this quantity to the spectrum of the transition matrix, with the growth rate of the number of topologically distinct trajectories given by the leading eigenvalue of the transition matrix.

The transition matrix element $T_{ij} \in \{0, 1\}$ in (17.1) indicates whether the transition from the starting partition $j$ into partition $i$ in one step is allowed or not, and the $(i, j)$ element of the transition matrix iterated $n$ times

$$(T^n)_{ij} = \sum_{k_1, k_2, \ldots, k_{n-1}} T_{ik_1} T_{k_1 k_2} \cdots T_{k_{n-1} j} \quad (18.2)$$

receives a contribution 1 from every admissible sequence of transitions, so $(T^n)_{ij}$ is the number of admissible $n$ symbol itineraries starting with $j$ and ending with $i$.

The total number of admissible itineraries of $n$ symbols is

$$K_n = \sum_{ij} (T^n)_{ij} = \left[1, 1, \ldots, 1\right] T^n \left[1 \ldots 1\right]^T \quad (18.3)$$

A finite $[N \times N]$ matrix $T$ has eigenvalues $\{\lambda_0, \lambda_1, \ldots, \lambda_{m-1}\}$ and (right) eigenvectors $\{\varphi_0, \varphi_1, \ldots, \varphi_{m-1}\}$ satisfying $T \varphi_\alpha = \lambda_\alpha \varphi_\alpha$. Expressing the initial vector in (18.3) in this basis,

$$T^n = T^n \sum_{\alpha=0}^{m-1} b_\alpha \varphi_\alpha = \sum_{\alpha=0}^{m-1} b_\alpha \lambda_\alpha^n \varphi_\alpha$$
and contracting with $[1,1,\ldots,1]$, we obtain

$$K_n = \sum_{\alpha=0}^{m-1} c_{\alpha} \lambda^{n}_{\alpha}.$$ 

The constants $c_{\alpha}$ depend on the choice of initial and final partitions: In this example we are sandwiching $T^n$ between the vector $[1,1,\ldots,1]$ and its transpose, but any other pair of vectors would do, as long as they are not orthogonal to the leading eigenvector $\varphi_0$. In an experiment the vector $[1,1,\ldots,1]$ would be replaced by a description of the initial state, and the right vector would describe the measurement time $n$ later.

As $n$ increases, the sum

$$\frac{1}{n} \ln c_0 \lambda^n_0 \left[ 1 + \frac{c_1}{c_0} \frac{\lambda_1}{\lambda_0} \right]^n + \cdots$$

is dominated by the leading eigenvalue of the transition matrix, $\lambda_0 > |\text{Re} \lambda_\alpha|$, $\alpha = 1,2,\cdots,m-1$, and the topological entropy (18.1) is given by

$$h = \lim_{n \to \infty} \frac{1}{n} \ln c_0 \lambda^n_0 \left[ 1 + \frac{c_1}{c_0} \frac{\lambda_1}{\lambda_0} \right]^n + \cdots$$

$$= \ln \lambda_0 + \lim_{n \to \infty} \left[ \ln c_0 \frac{n}{n} + \frac{1}{n} \frac{c_1}{c_0} \left( \frac{\lambda_1}{\lambda_0} \right)^n + \cdots \right]$$

$$= \ln \lambda_0.$$ (18.4)

What have we learned? The transition matrix $T$ is a one-step, short time operator, advancing the trajectory from one partition to the next admissible partition. Its eigenvalues describe the rate of growth of the total number of trajectories at the asymptotic times. Instead of painstakingly counting $K_1, K_2, K_3,\ldots$ and estimating (18.1) from a slope of a log-linear plot, we have the exact topological entropy if we can compute the leading eigenvalue of the transition matrix $T$.

### 18.2 Topological trace formula

There are two standard ways of computing eigenvalues of a matrix - by evaluating the trace $\text{tr} T^n = \sum \lambda^n_\alpha$, or by evaluating the determinant $\text{det} (1 - zT)$. We start by evaluating the trace of transition matrices. The main lesson will be that the trace receives contributions only from itineraries that return to the initial partition, i.e., periodic orbits.

Consider an $M$-step memory transition matrix, like the 1-step memory example (17.11). The trace of the transition matrix counts the number of partitions that
Table 18.1: Prime cycles for the binary symbolic dynamics up to length 9. The numbers of prime cycles are given in table 18.3.

<table>
<thead>
<tr>
<th>$n_p$</th>
<th>$p$</th>
<th>$n_p$</th>
<th>$p$</th>
<th>$n_p$</th>
<th>$p$</th>
<th>$n_p$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0 1</td>
<td>0 0 1</td>
<td>1</td>
<td>0 1 0</td>
<td>0 0 1</td>
<td>1</td>
<td>0 1 0</td>
</tr>
<tr>
<td>3</td>
<td>0 0 1</td>
<td>0 1 0</td>
<td>0 0 1</td>
<td>0 1 0</td>
<td>0 0 1</td>
<td>0 1 0</td>
<td>0 0 1</td>
</tr>
<tr>
<td>4</td>
<td>0 0 0 1</td>
<td>0 1 0 1</td>
<td>0 0 0 1</td>
<td>0 1 0 1</td>
<td>0 0 0 1</td>
<td>0 1 0 1</td>
<td>0 0 0 1</td>
</tr>
<tr>
<td>5</td>
<td>0 0 0 0 1</td>
<td>0 1 1 1 1</td>
<td>0 0 0 0 1</td>
<td>0 1 1 1 1</td>
<td>0 0 0 0 1</td>
<td>0 1 1 1 1</td>
<td>0 0 0 0 1</td>
</tr>
<tr>
<td>6</td>
<td>0 0 0 0 0 1</td>
<td>0 1 1 1 1 1</td>
<td>0 0 0 0 0 1</td>
<td>0 1 1 1 1 1</td>
<td>0 0 0 0 0 1</td>
<td>0 1 1 1 1 1</td>
<td>0 0 0 0 0 1</td>
</tr>
<tr>
<td>7</td>
<td>0 0 0 0 0 0 1</td>
<td>0 1 1 1 1 1 1</td>
<td>0 0 0 0 0 0 1</td>
<td>0 1 1 1 1 1 1</td>
<td>0 0 0 0 0 0 1</td>
<td>0 1 1 1 1 1 1</td>
<td>0 0 0 0 0 0 1</td>
</tr>
</tbody>
</table>

map into themselves. More generally, each closed walk through $n$ concatenated entries of $T$ contributes to $\text{tr} T^n$ the product (18.2) of the matrix entries along the walk. Each step in such a walk shifts the symbolic string by one symbol; the trace ensures that the walk closes on a periodic string $c$. Define $t_c$ to be the local trace, the product of matrix elements along a cycle $c$, each term being multiplied by a book keeping variable $z$.

The quantity $z^n \text{tr} T^n$ is then the sum of $t_c$ for all cycles of period $n$. The $t_c$ = (product of matrix elements along cycle $c$ is manifestly cyclically invariant, $t_{100} = t_{010} = t_{001}$, so a prime cycle $p$ of period $n_p$ contributes $n_p$ times, once for each periodic point along its orbit. For the purposes of periodic orbit counting, the local trace takes values

$$t_p = \begin{cases} z^{n_p} & \text{if } p \text{ is an admissible cycle} \\ 0 & \text{otherwise} \end{cases}$$

(18.5)

i.e., (setting $z = 1$) the local trace is $t_p = 1$ if the cycle is admissible, and $t_p = 0$ otherwise.
Table 18.2: The total numbers \( N_n \) of periodic points of period \( n \) for binary symbolic dynamics. The numbers of contributing prime cycles illustrates the preponderance of long prime cycles of period \( n \) over the repeats of shorter cycles of periods \( n_p \), where \( n = n_r n_p \). Further enumerations of binary prime cycles are given in tables 18.1 and 18.3. (L. Rondoni)

\[
\begin{array}{cccccccccc}
 n & N_n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
 1 & 2 & 2 & & & & & & & & & \\
 2 & 4 & 2 & 1 & & & & & & & & \\
 3 & 8 & 2 & 1 & 2 & & & & & & & \\
 4 & 16 & 2 & 1 & 3 & & & & & & & \\
 5 & 32 & 2 & & & 6 & & & & & & \\
 6 & 64 & 2 & 1 & 2 & & & & 9 & & & \\
 7 & 128 & 2 & & & & 18 & & & & & \\
 8 & 256 & 2 & 1 & 3 & & & & 30 & & & \\
 9 & 512 & 2 & 1 & 2 & & & & 56 & & & \\
10 & 1024 & 2 & 1 & 6 & & & & 99 & & & \\
\end{array}
\]

Hence \( \text{tr} T^n = N_n \) counts the number of admissible periodic points of period \( n \). The \( n \)th order trace (18.28) picks up contributions from all repeats of prime cycles, with each cycle contributing \( n_p \) periodic points, so \( N_n \), the total number of periodic points of period \( n \) is given by

\[
z^n N_n = z^n \text{tr} T^n = \sum_{n_p \mid n} n_p t_p^{n/n_p} = \sum_p n_p \sum_{r=1}^{\infty} \delta_{n,n_p} t_p^r. \quad (18.6)
\]

Here \( m \mid n \) means that \( m \) is a divisor of \( n \). An example is the periodic orbit counting in table 18.2.

In order to get rid of the awkward divisibility constraint \( n = n_r n_p \) in the above sum, we introduce the generating function for numbers of periodic points

\[
\sum_{n=1}^{\infty} z^n N_n = \text{tr} \frac{z T}{1 - z T}. \quad (18.7)
\]

The right hand side is the geometric series sum of \( N_n = \text{tr} T^n \). Substituting (18.6) into the left hand side, and replacing the right hand side by the eigenvalue sum \( \text{tr} T^n = \sum \lambda_n \), we obtain our first example of a trace formula, the topological trace formula

\[
\sum_{\alpha = 0}^{\infty} \frac{z \lambda_{\alpha}}{1 - z \lambda_{\alpha}} = \sum_p \frac{n_p t_p}{1 - t_p}. \quad (18.8)
\]

A trace formula relates the spectrum of eigenvalues of an operator - here the transition matrix - to the spectrum of periodic orbits of a dynamical system. It is a statement of duality between the short-time, local information - in this case the next admissible symbol in a symbol sequence - to long-time, global averages, in this case the mean rate of growth of the number of cycles with increasing cycle period.
18.3 Determinant of a graph

Our next task is to determine the zeros of the spectral determinant of an \([m \times m]\) transition matrix

\[
\det (1 - z T) = \prod_{a=0}^{m-1} (1 - z \lambda_a) .
\]

(18.9)

We could now proceed to diagonalize \(T\) on a computer, and get this over with. It pays, however, to dissect \(\det (1 - z T)\) with some care; understanding this computation in detail will be the key to understanding the cycle expansion computations of chapter 23 for arbitrary dynamical averages. For \(T\) a finite matrix, (18.9) is just the characteristic polynomial for \(T\). However, we shall be able to compute this object even when the dimension of \(T\) and other such operators becomes infinite, and for that reason we prefer to refer to (18.9) loosely as the “spectral determinant.”

There are various definitions of the determinant of a matrix; we will view the determinant as a sum over all possible permutation cycles composed of the traces \(\text{tr} T^K\), in the spirit of the determinant–trace relation (1.16):

\[
\text{det} \begin{pmatrix} 1 - z T \end{pmatrix} = \exp (\text{tr} \ln(1 - z T)) = \exp \left( - \sum_{n=1} z^n \frac{\text{tr} T^n}{n} \right)
\]

\[= 1 - z \text{tr} T - \frac{z^2}{2} (\text{tr} T^2 - \text{tr} T^2) - \ldots \]  

(18.10)

Formally, the right hand is a Taylor series in \(z\) about \(z = 0\). If \(T\) is an \([m \times m]\) finite matrix, then the characteristic polynomial is at most of order \(m\). In that case the coefficients of \(z^n\) must vanish exactly for \(n > m\).

We now proceed to relate the determinant in (18.10) to the corresponding transition graph of chapter 17: toward this end, we start with the usual textbook expression for a determinant as the sum of products of all permutations

\[
\text{det} M = \sum_{\{\pi\}} (-1)^\pi M_{1, \pi_1} M_{2, \pi_2} \cdots M_{m, \pi_m}
\]

(18.11)

where \(M = 1 - z T\) is a \([m \times m]\) matrix, \(\{\pi\}\) denotes the set of permutations of \(m\) symbols, \(\pi_k\) is the permutation \(\pi\) applied to \(k\), and \((-1)^\pi = \pm 1\) is the parity of permutation \(\pi\). The right hand side of (18.11) yields a polynomial in \(T\) of order \(m\).
in $z$: a contribution of order $n$ in $z$ picks up $m - n$ unit factors along the diagonal, the remaining matrix elements yielding
\[
(-z)^n (-1)^n T_{s_1 \pi s_1} \cdots T_{s_n \pi s_n}
\]  
(18.12)
where $\pi$ is the permutation of the subset of $n$ distinct symbols $s_1 \cdots s_n$ indexing $T$ matrix elements. As in (18.28), we refer to any combination $t_c = T_{s_1 s_2} T_{s_3 s_4} \cdots T_{s_{2k} s_1}$, for a given itinerary $c = s_1 s_2 \cdots s_{2k}$, as the local trace associated with a closed loop $c$ on the transition graph. Each term of the form (18.12) may be factored in terms of local traces $t_{c_1} t_{c_2} \cdots t_{c_k}$, i.e., loops on the transition graph. These loops are non-intersecting, as each node may only be reached by one link, and they are indeed loops, as if a node is reached by a link, it has to be the starting point of another single link, as each $s_j$ must appear exactly once as a row and column index.

So the general structure is clear, a little more thinking is only required to get the sign of a generic contribution. We consider only the case of loops of length 1 and 2, and leave to the reader the task of generalizing the result by induction. Consider first a term in which only loops of unit length appear in (18.12), i.e., only the diagonal elements of $T$ are picked up. We have $k = m$ loops and an even permutation $\pi$ so the sign is given by $(-1)^k$, where $k$ is the number of loops. Now take the case in which we have $i$ single loops and $j$ loops of length $n = 2j + i$. The parity of the permutation gives $(-1)^j$ and the first factor in (18.12) gives $(-1)^n = (-1)^{2j+i}$. So once again these terms combine to $(-1)^k$, where $k = i + j$ is the number of loops. Let $f$ be the maximal number of non-intersecting loops. We may summarize our findings as follows:

The characteristic polynomial of a transition matrix is given by the sum of all possible partitions $\pi$ of the corresponding transition graph into products of $k$ non-intersecting loops, with each loop trace $t_p$ carrying a minus sign:
\[
\det (1 - zT) = \sum_{k=0}^{f} \sum'_{\pi} (-1)^k t_{p_1} \cdots t_{p_k}
\]  
(18.13)
Any self-intersecting loop is shadowed by a product of two loops that share the intersection point. As both the long loop $t_{ab}$ and its shadow $t_{db}$ in the case at hand carry the same weight $z^{n_a+n_b}$, the cancelation is exact, and the loop expansion (18.13) is finite. In the case that the local traces count prime cycles (18.5), $t_p = 0$ or $z^0$, we refer to $\det (1 - zT)$ as the topological polynomial.

We refer to the set of all non-self-intersecting loops $\{t_{p_1}, t_{p_2}, \cdots t_{p_f}\}$ as the fundamental cycles (for an explicit example, see the loop expansion of example 18.6). If the graph has $m$ nodes, no fundamental cycle is of period longer than $m$, as any longer cycle is of necessity self-intersecting.
The above loop expansion of a determinant in terms of traces is most easily grasped by working through a few examples. The complete binary dynamics transition graph of figure 17.5 is a little bit too simple, but let us start humbly and consider it anyway.

Similarly, for the complete symbolic dynamics of \( N \) symbols the transition graph has one node and \( N \) links, yielding

\[
\det (1 - zT) = 1 - Nz, \tag{18.14}
\]

which gives the topological entropy \( h = \ln N \).

18.4 Topological zeta function

What happens if there is no finite-memory transition matrix, if the transition graph is infinite? If we are never sure that looking further into the future will reveal no further forbidden blocks? There is still a way to define the determinant, and this idea is central to the whole treatise: the determinant is then defined by (18.10)

\[
\det (1 - zT) = 1 - \sum_{n=1}^{\infty} \hat{c}_n z^n. \tag{18.15}
\]

For finite dimensional matrices the expansion is a finite polynomial, and (18.15) is an identity; however, for infinite dimensional operators the cumulant expansion coefficients \( \hat{c}_n \) define the determinant.

Let us now evaluate the determinant in terms of traces for an arbitrary transition matrix. In order to obtain an expression for the spectral determinant (18.9) in terms of cycles, substitute (18.6) into (18.15) and sum over the repeats of prime
cycles using \( \ln(1 - x) = -\sum_r x^r / r \),

\[
det (1 - zT) = \exp \left( -\sum_p \sum_{r=1}^{\infty} \frac{t_p^r}{r} \right) = \exp \left( \sum_p \ln(1 - t_p) \right) \\
\prod_\alpha (1 - z \lambda_\alpha) = \prod_p (1 - t_p), \tag{18.16}
\]

where for the topological entropy the weight assigned to a prime cycle \( p \) of period \( n_p \) is \( t_p = z^{n_p} \) if the cycle is admissible, or \( t_p = 0 \) if it is pruned. This determinant is called the topological or the Artin-Mazur zeta function, conventionally denoted by

\[
1/\xi_{\text{top}}(z) = \prod_p (1 - z^{n_p}) = 1 - \sum_{n=1} \hat{c}_n z^n. \tag{18.17}
\]

Counting cycles amounts to giving each admissible prime cycle \( p \) weight \( t_p = z^{n_p} \) and expanding the Euler product (18.17) as a power series in \( z \). The number of prime cycles \( p \) is infinite, but if \( T \) is an \([m \times m]\) finite matrix, then the number of roots \( \lambda_\alpha \) is at most \( m \), the characteristic polynomial is at most of order \( m \), and the coefficients of \( z^n \) vanish for \( n > m \).

The topological entropy \( h \) can now be determined from the leading zero \( z = e^{-h} \) of the topological zeta function. For a finite \([m \times m]\) transition matrix, the number of terms in the characteristic equation (18.13) is finite, and we refer to this expansion as the topological polynomial of order \( \leq m \).
18.6 Shadowing

The topological zeta function is a pretty function, but the infinite product (18.16) should make you pause. For finite transition matrices the left hand side is a determinant of a finite matrix, therefore a finite polynomial; so why is the right hand side an infinite product over the infinitely many prime periodic orbits of all periods?

The way in which this infinite product rearranges itself into a finite polynomial is instructive, and crucial for all that follows. You can already take a peek at the
full cycle expansion (23.8) of chapter 23; all cycles beyond the fundamental $t_0$ and $t_1$ appear in the shadowing combinations such as

$$I_{s_1s_2\ldots s_n} - I_{s_1s_2\ldots s_m t_{m+1} \ldots s_n}.$$ 

For subshifts of finite type such shadowing combinations cancel exactly, if we are counting cycles as we do in (18.29) and (18.36), or if the dynamics is piecewise linear, as in exercise 22.2. As we argue in sect. 1.5.4, for nice hyperbolic flows whose symbolic dynamics is a subshift of finite type, the shadowing combinations almost cancel, and the spectral determinant is dominated by the fundamental cycles from (18.13), with longer cycles contributing only small “curvature” corrections.

These exact or nearly exact cancelations depend on the flow being smooth and the symbolic dynamics being a subshift of finite type. If the dynamics requires an infinite state space partition, with pruning rules for blocks of increasing length, most of the shadowing combinations still cancel, but the few corresponding to new forbidden blocks do not, leading to a finite radius of convergence for the spectral determinant, as depicted in figure 18.2.
18.7 Counting cycles

In what follows, we shall occasionally need to compute all cycles up to topological period $n$, so it is important to know their exact number. The formulas are fun to derive, but a bit technical for plumber on the street, and probably best skipped on the first reading.

18.7.1 Counting periodic points

The number of periodic points of period $n$ is denoted $N_n$. It can be computed from (18.15) and (18.7) as a logarithmic derivative of the topological zeta function

$$\sum_{n=1} N_n z^n = \text{tr} \left( -z \frac{d}{dz} \ln(1-zT) \right) = -z \frac{d}{dz} \ln \det (1-zT)$$

$$= -z \frac{d}{dz} \left( \frac{1}{\zeta_{\text{top}}} \right) \frac{1}{\zeta_{\text{top}}}. \tag{18.24}$$

Observe that the trace formula (18.8) diverges at $z \to e^{-h}$, because the denominator has a simple zero there.

18.7.2 Counting prime cycles
Table 18.3: Number of prime cycles for various alphabets and grammars up to period 10. The first column gives the cycle period, the second gives the formula (18.26) for the number of prime cycles for complete $N$-symbol dynamics, and columns three through five give the numbers of prime cycles for $N = 2, 3$ and 4.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$M_n(N)$</th>
<th>$M_n(2)$</th>
<th>$M_n(3)$</th>
<th>$M_n(4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>$N(N - 1)/2$</td>
<td>1</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>$N(N^2 - 1)/3$</td>
<td>2</td>
<td>8</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>$N^2(N^2 - 1)/4$</td>
<td>3</td>
<td>18</td>
<td>60</td>
</tr>
<tr>
<td>5</td>
<td>$(N^5 - N)/5$</td>
<td>6</td>
<td>48</td>
<td>204</td>
</tr>
<tr>
<td>6</td>
<td>$(N^6 - N^3 - N^2 + N)/6$</td>
<td>9</td>
<td>116</td>
<td>670</td>
</tr>
<tr>
<td>7</td>
<td>$(N^7 - N)/7$</td>
<td>18</td>
<td>312</td>
<td>2340</td>
</tr>
<tr>
<td>8</td>
<td>$N^4(N^4 - 1)/8$</td>
<td>30</td>
<td>810</td>
<td>8160</td>
</tr>
<tr>
<td>9</td>
<td>$N^3(N^6 - 1)/9$</td>
<td>56</td>
<td>2184</td>
<td>29120</td>
</tr>
<tr>
<td>10</td>
<td>$(N^{10} - N^5 - N^2 + N)/10$</td>
<td>99</td>
<td>5880</td>
<td>104754</td>
</tr>
</tbody>
</table>

We list the number of prime cycles up to period 10 for 2-, 3- and 4-letter complete symbolic dynamics in table 18.3, obtained by Möbius inversion (18.26).

Résumé

The main result of this chapter is the cycle expansion (18.17) of the topological zeta function (i.e., the spectral determinant of the transition matrix):

$$1/\zeta_{\text{top}}(z) = 1 - \sum_{k=1}^{\infty} \hat{c}_k z^k.$$
Table 18.4: List of 3-disk prime cycles up to period 10. Here $n$ is the cycle period, $M_n$ is the number of prime cycles, $N_n$ is the number of periodic points, and $S_n$ the number of distinct prime cycles under $D_3$ symmetry (see chapter 25 for further details). Column 3 also indicates the splitting of $N_n$ into contributions from orbits of periods that divide $n$. The prefactors in the fifth column indicate the degeneracy $m_p$ of the cycle; for example, 3·12 stands for the three prime cycles $12$, $13$ and $23$ related by $2\pi/3$ rotations. Among symmetry-related cycles, a representative $\hat{p}$ which is lexically lowest is listed. The cycles of period 9 grouped with parentheses are related by time reversal symmetry, but not by any $D_3$ transformation.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$M_n$</th>
<th>$N_n$</th>
<th>$S_n$</th>
<th>$m_p \cdot \hat{p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>6=3·2</td>
<td>1</td>
<td>3·12</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>6=2·3</td>
<td>1</td>
<td>2·123</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>18=3·2+3·4</td>
<td>1</td>
<td>3·1213</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>30=3·6·5</td>
<td>1</td>
<td>6·12123</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>66=3·2+2·3+9·6</td>
<td>2</td>
<td>6·121213 + 3·121323</td>
</tr>
<tr>
<td>7</td>
<td>18</td>
<td>126=18·7</td>
<td>3</td>
<td>6·1212123 + 6·1212313 + 6·1213123</td>
</tr>
<tr>
<td>8</td>
<td>30</td>
<td>258=3·2+3·4+30·8</td>
<td>6</td>
<td>6·12121213 + 3·12121313 + 6·12121323 + 6·12123123</td>
</tr>
<tr>
<td>9</td>
<td>56</td>
<td>510=2·3+56·9</td>
<td>10</td>
<td>6·121212123 + 6·(121212313 + 121212323) + 6·(121213123 + 121213213) + 6·(121213213 + 121213231) + 6·(121213231 + 121213233) + 6·(121213233 + 121213321)</td>
</tr>
<tr>
<td>10</td>
<td>99</td>
<td>1022</td>
<td>18</td>
<td></td>
</tr>
</tbody>
</table>

Table 18.5: The 4-disk prime cycles up to period 8. The symbols is the same as shown in table 18.4. Orbits related by time reversal symmetry (but no $C_4$ symmetry) already appear at cycle period 5. Cycles of period 7 and 8 have been omitted.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$M_n$</th>
<th>$N_n$</th>
<th>$S_n$</th>
<th>$m_p \cdot \hat{p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>12=6·2</td>
<td>2</td>
<td>4·12 + 2·13</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>24=8·3</td>
<td>1</td>
<td>8·123</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>84=6·2+18·4</td>
<td>4</td>
<td>8·1213 + 4·1214 + 2·1234 + 4·1243</td>
</tr>
<tr>
<td>5</td>
<td>48</td>
<td>240=48·5</td>
<td>6</td>
<td>8·(12123 + 12124) + 8·12313 + 8·(12134 + 12143) + 8·12413</td>
</tr>
<tr>
<td>6</td>
<td>116</td>
<td>732=6·2+8·3+116·6</td>
<td>17</td>
<td>8·121213 + 8·121214 + 8·121234 + 8·121235 + 8·121314 + 4·121323 + 8·(121324 + 121423) + 4·121434 + 8·123124 + 8·123134 + 4·123143 + 4·124213 + 8·124243</td>
</tr>
<tr>
<td>7</td>
<td>312</td>
<td>2184</td>
<td>39</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>810</td>
<td>6564</td>
<td>108</td>
<td></td>
</tr>
</tbody>
</table>
For subshifts of finite type, the transition matrix is finite, and the topological zeta function is a finite polynomial evaluated by the loop expansion (18.13) of \( \det(1 - zT) \). For infinite grammars the topological zeta function is defined by its cycle expansion. The topological entropy \( h \) is given by the leading zero \( z = e^{-h} \). This expression for the entropy is \textit{exact}; in contrast to the initial definition (18.1), no \( n \to \infty \) extrapolations of \( \ln K_n/n \) are required.

What have we accomplished? We have related the number of topologically distinct paths from one state space region to another region to the leading eigenvalue of the transition matrix \( T \). The spectrum of \( T \) is given by topological zeta function, a certain sum over traces \( \text{tr} T^n \), and in this way the periodic orbit theory has entered the arena through the trace formula (18.8), already at the level of the topological dynamics.

Contrary to claims one all too often encounters in the literature, “exponential proliferation of trajectories” is \textit{not} the problem; what limits the convergence of cycle expansions is the proliferation of the grammar rules, or the “algorithmic complexity,” as illustrated by sect. 18.5, and figure 18.2 in particular.
18.8 Examples

Example 18.1 3-disk itinerary counting. The \((T^2)_{13} = T_{12}T_{23} = 1\) element of \(T^2\) for the 3-disk transition matrix (17.9)
\[
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}^2
= \begin{bmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix}.
\] (18.27)

Corresponds to path 3 \(\rightarrow\) 2 \(\rightarrow\) 1, the only 2-step path from 3 to 1, while \((T^2)_{33} = T_{31}T_{13} + T_{32}T_{23} = 2\) counts the two returning, periodic paths \(31\) and \(32\). Note that the trace \(\text{tr}T^2 = (T^2)_{11} + (T^2)_{22} + (T^2)_{33} = 2T_{13}T_{31} + 2T_{21}T_{12} + 2T_{32}T_{23}\) has a contribution from each 2-cycle \(12, 13, 23\) twice, one contribution from each periodic point.
Example 18.2 Traces for binary symbolic dynamics. For example, for the $[8 \times 8]$ transition matrix $T_{8 \times 8}$ version of (17.11), or any refined partition $[2^n \times 2^n]$ transition matrix, $n$ arbitrarily large, the periodic point $100$ contributes $t_{100} = \zeta z^3 T_{0010} T_{0101} T_{1000} T_{0010}$ to $\zeta^3 T^3$. This product is manifestly cyclically invariant, $t_{100} = t_{010} = t_{001}$, so a prime cycle $p = 001$ of period 3 contributes 3 times, once for each periodic point along its orbit.

For the binary labeled non–wandering set the first few traces are given by (consult tables 18.1 and 18.2)

\[
\begin{align*}
\zeta t^0 T &= t_0 + t_1, \\
\zeta^2 t^2 T^2 &= t_0^2 + t_1^2 + 2t_{10}, \\
\zeta^3 t^3 T^3 &= t_0^3 + t_1^3 + 3t_{10} + 3t_{101}, \\
\zeta^4 t^4 T^4 &= t_0^4 + t_1^4 + 2t_{10}^2 + 4t_{100} + 4t_{1001} + 4t_{1011}.
\end{align*}
\]

(18.28)

In the binary case the trace picks up only two contributions on the diagonal, $T_{0 \cdot 0 \cdot 0 \cdot 0} + T_{1 \cdot 1 \cdot 1 \cdot 1}$, no matter how much memory we assume. We can even take infinite memory $M \rightarrow \infty$, in which case the contributing partitions are shrunk to the fixed points, $\text{tr } T = T_{00} + T_{11}$.

If there are no restrictions on symbols, the symbolic dynamics is complete, and all binary sequences are admissible (or allowable) itineraries. As this type of symbolic dynamics pops up frequently, we list the shortest binary prime cycles in Table 12.1.

Example 18.3 Topological polynomial for complete binary dynamics: (continuation of example 17.1) There are only two non-intersecting loops, yielding

\[
\begin{align*}
\det (1 - \zeta T) &= 1 - t_0 - t_1 - (t_{01} - t_{01}) = 1 - 2z \\
&= 1 - 0 - 0 - 0 = 1 - 0 - 0.
\end{align*}
\]

(18.29)

Due to the symmetry under $0 \leftrightarrow 1$ interchange, this is a redundant graph (the 2-cycle $t_{01}$ is exactly shadowed by the 1-cycles). Another way to see is that itineraries are labeled by the $[0, 1]$ links, node labels can be omitted. As both nodes have 2 in-links and 2 out-links, they can be identified, and a more economical presentation is in terms of the $[1 \times 1]$ adjacency matrix (17.12)

\[
\begin{align*}
\det (1 - \zeta A) &= 1 - t_0 - t_1 = 1 - 2z \\
&= 1 - 0 - 0 = 1 - 0.
\end{align*}
\]

(18.30)

The leading (and only) zero of this characteristic polynomial yields the topological entropy $h = 2$. As there are $K_n = 2^n$ binary strings of length $N$, this comes as no surprise.

Example 18.4 Golden mean pruning: The “golden mean” pruning of example 17.5 has one grammar rule: the substring $\_11\_1$ is forbidden. The corresponding transition
graph non-intersecting loops are of length 1 and 2, so the topological polynomial is given by

\[
\det (1 - zT) = 1 - t_0 - t_{01} = 1 - z - z^2 \quad (18.31)
\]

\[
\begin{array}{c}
\includegraphics[width=1cm]{loop_graph.png}
\end{array} = 1 - \begin{array}{c}
\includegraphics[width=1cm]{golden_loop_graph.png}
\end{array}.
\]

The leading root of this polynomial is the golden mean, so the entropy (18.4) is the logarithm of the golden mean, \( h = \ln \frac{1 + \sqrt{5}}{2} \).
Exercise 341

Exercises

18.1. **A transition matrix for 3-disk pinball.**

   a) Draw the transition graph corresponding to the 3-disk ternary symbolic dynamics, and write down the corresponding transition matrix corresponding to the graph. Show that iteration of the transition matrix results in two coupled linear difference equations, one for the diagonal and one for the off diagonal elements. (Hint: relate $\text{tr} T^n$ to $\text{tr} T^{n-1} + \ldots$)

   b) Solve the above difference equation and obtain the number of periodic orbits of length $n$. Compare your result with table 18.4.

   c) Find the eigenvalues of the transition matrix $T$ for the 3-disk system with ternary symbolic dynamics and calculate the topological entropy. Compare this to the topological entropy obtained from the binary symbolic dynamics $\{0, 1\}$.

18.2. **3-disk prime cycle counting.** A prime cycle $p$ of length $n_p$ is a single traversal of the orbit; its label is a non-repeating symbol string of $n_p$ symbols. For example, $1\overline{2}$ is prime, but $1\overline{2}2\overline{1}$ is not, since it is $2\overline{1}$ repeated.

   Verify that a 3-disk pinball has 3, 2, 3, 6, 9, \ldots prime cycles of length 2, 3, 4, 5, 6, \ldots.

18.3. **Sum of $A_{ij}$ is like a trace.** Let $A$ be a matrix with eigenvalues $\lambda_k$. Show that

$$\Gamma_n := \sum_{i,j} [A^n]_{ij} = \sum_k c_k A^n_k .$$

(a) Under what conditions do $\ln |\text{tr} A^n|$ and $\ln |\Gamma_n|$ have the same asymptotic behavior as $n \to \infty$, i.e., their ratio converges to one?

(b) Do eigenvalues $\lambda_k$ need to be distinct, $\lambda_k \neq \lambda_l$ for $k \neq l$? How would a degeneracy $\lambda_k = \lambda_l$ affect your argument for (a)?

18.4. **Loop expansions.** Prove by induction the sign rule in the determinant expansion (18.13):

$$\det (1 - zT) = \sum_{k \geq 0} \sum_{p_n \cdots p_1} (-1)^k t_{p_1} t_{p_2} \cdots t_{p_k} .$$

18.5. **Transition matrix and cycle counting.** Suppose you are given the transition graph
18.6. **Alphabet \{0,1\}, prune 00.** The transition graph example 17.8 implements this pruning rule which implies that “0” must always be bracketed by “1”s; in terms of a new symbol 2 := 10, the dynamics becomes unrestricted symbolic dynamics with with binary alphabet \{1,2\}. The cycle expansion (18.13) becomes

\[
\frac{1}{\zeta} = (1 - t_1)(1 - t_2)(1 - t_{12})(1 - t_{112}) \ldots
\]

\[
= 1 - t_1 - t_2 - (t_{12} - t_1 t_2) - (t_{112} - t_2 t_1) - (t_{122} - t_{12} t_2) - \ldots
\]

(18.41)

In the original binary alphabet this corresponds to:

\[
\frac{1}{\zeta} = 1 - t_1 - t_{10} - (t_{110} - t_1 t_{10}) - (t_{1110} - t_{110} t_1) - (t_{11110} - t_{1110} t_{10}) - \ldots
\]

(18.42)

This symbolic dynamics describes, for example, circle maps with the golden mean winding number. For unimodal maps this symbolic dynamics is realized by the tent map of exercise 14.6.

18.7. **“Golden mean” pruned map.** (continuation of exercise 14.6) Show that the total number of periodic orbits of length \(n\) for the “golden mean” tent map is

\[
\frac{(1 + \sqrt{5})^n + (1 - \sqrt{5})^n}{2^n}.
\]

Continued in exercise 22.1. See also exercise 18.8.

---

18.8. **A unimodal map with golden mean pruning.** Consider the unimodal map

for which the critical point maps into the right hand fixed point in three iterations, \(S^+ = 100\). Show that the admissible itineraries are generated by the above transition graph, with transient neighborhood of \(\overline{0}\) fixed point, and \(\overline{00}\) pruned from the recurrent set. (K.T. Hansen)
References


