Chapter 12

Relativity for cyclists

Physicists like symmetry more than Nature
— Rich Kerswell

WHAT IF THE LAWS OF MOTION retain their form for a family of coordinate frames related by *continuous* symmetries? The finite groups intuition is of little use here.

First of all, why worry about continuous symmetries? In physics, we usually assume isotropy, that is the laws of nature do not depend on where we are. In many cases (single or many body quantum mechanics, statistical physics, field theories etc.), the system studied is defined over an infinite or periodic domain. For linear problems of this kind, one takes care of the spatial dependence via a Fourier expansion (2.16), and solves the problem for each mode separately. In nonlinear theories one might start with a Fourier expansion, but the modes couple nonlinearly, and hence one needs to solve for all of them simultaneously. The two-modes system, which we shall introduce in sect. 12.4.2 and use for illustrations throughout this chapter and the next, is an example of a few modes truncation of such Fourier expansion. Figure 12.1 illustrates the effect continuous symmetry has on dynamics of this particular system. The strange attractor is a mess, and, as we shall demonstrate, what makes it messy is its continuous symmetry.

section 12.4.2

remark 12.2

example 12.5
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We shall refer to the component of the dynamics along the continuous symmetry directions as a 'drift'. In the presence of a continuous symmetry an orbit explores the manifold swept by combined action of the dynamics and the symmetry induced drifts. Further problems arise when we try to determine whether an orbit shadows another orbit (see figure 16.1 for a sketch of a close pass to a periodic orbit), or develop symbolic dynamics (partition the state space, as in chapter 14): here a 1-dimensional trajectory is replaced by a $(N+1)$-dimensional 'sausage', a dimension for each continuous symmetry (N being the total number of parameters

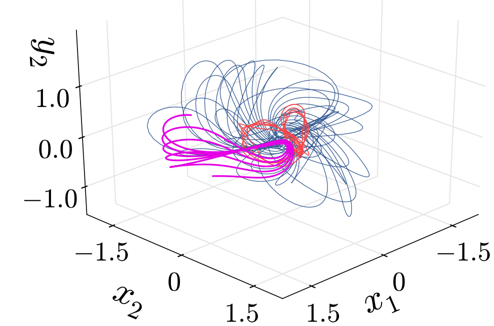

Figure 12.1: Several trajectories of the 4-dimensional two-modes system of example 12.5, a 3-dimensional projection: A long trajectory that originated close to the relative equilibrium TW_1 of the two-modes flow (12.38), with the starting point on its unstable manifold. The initial segment of this trajectory, which follows closely the orbit of TW_1 (see figure 12.4), is colored red; beyond that the trajectory falls onto the strange attractor (colored blue). Superimposed, in magenta, are four repeats of the shortest relative periodic orbit $\overline{1}$ (see figure 12.6 (b)). (N.B. Budanur)

specifying the continuous transformation, and '1' for the time parameter t). How are we to measure distances between such objects? In this chapter and the next one we shall learn how to develop visualizations of such flows, quotient symmetries, and offer computationally straightforward methods of reducing the dynamics to lower-dimensional, reduced state spaces. The methods should also be applicable to high-dimensional flows, such as translationally invariant fluid flows bounded by pipes or planes.

example 12.10

Instead of writing yet another tome on group theory, in what follows we continue to serve group theoretic nuggets on need-to-know basis, through a series of pedestrian examples (but take a slightly higher, cyclist road in the text proper).

12.1 Continuous symmetries

I've always hated the term 'group orbit'
— John F. Zappatista

But first, a lightning review of the theory of Lie groups. The group-theoretical concepts of sect. 10.1 apply to compact continuous groups as well, and will not be repeated here. All the group theory that you need is contained in the *Peter-Weyl theorem*, and its corollaries: A compact Lie group G is fully reducible, its representations are fully reducible, every compact Lie group is a closed subgroup of a unitary group $U(n)$ for some n, and every continuous, unitary, irrep of a compact Lie group is finite dimensional.

example 12.1
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example 12.2
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Let G be a group, and $g\mathcal{M} \longrightarrow \mathcal{M}$ a group action on the state space $\mathcal{M}$. The

$[d\times d]$ matrices g acting on vectors in the d-dimensional state space $\mathcal{M}$ form a linear representation of the group G. If the action of every element g of a group G commutes with the flow

$$gv(x) = v(gx)\,, \qquad g f^\tau(x) = f^\tau(gx)\,, \tag{12.1}$$

G is a symmetry of the dynamics, and, as in (10.4), the dynamics is said to be *G-equivariant*.

In order to explore the implications of equivariance for the solutions of dynamical equations, we start by examining the way a compact Lie group acts on state space $\mathcal{M}$.

Definition: Group orbit For any $x \in \mathcal{M}$, the *group orbit* $\mathcal{M}_x$ of x is the set of all points that x is mapped to under the groups actions,

$$\mathcal{M}_x = \mathrm{Orb}(x) = \{g\,x \mid g \in G\}\,. \tag{12.2}$$

See page 167 and figure 12.2 (a).

Definition: Fixed-point subspace $\mathcal{M}_H$, or a 'centralizer' of a subgroup $H \subset G$, is the set of all state space points that are *H-fixed*, *point-wise* invariant under action of the subgroup

$$\mathcal{M}_H = \mathrm{Fix}\,(H) = \{x \in \mathcal{M} \mid h\,x = x \text{ for all } h \in H\}\,. \tag{12.3}$$

Points in the *fixed-point subspace* $\mathcal{M}_G$ are fixed points of the full group action, i.e., points whose group orbit consists of only the point itself ($\mathcal{M}_x = \{x\}$). They are called *invariant points*,

$$\mathcal{M}_G = \mathrm{Fix}\,(G) = \{x \in \mathcal{M} \mid g\,x = x \text{ for all } g \in G\}\,. \tag{12.4}$$

If a point is an invariant point of the symmetry group, by the definition of equivariance (12.1) the velocity at that point is also in $\mathcal{M}_G$, so the trajectory through that point will remain in $\mathcal{M}_G$. $\mathcal{M}_G$ is disjoint from the rest of the state space since no trajectory can ever enter or leave it.

As we saw in example 12.2, the time evolution itself is a noncompact 1-parameter Lie group. Thus the time evolution and the continuous symmetries can be considered on the same Lie group footing. For a given state space point x a symmetry group of N continuous transformations together with the evolution in time sweeps out, in general, a smooth $(N{+}1)$-dimensional manifold of equivalent solutions (if the solution has a symmetry, the manifold may have a dimension less than $N+1$). For solutions for which the group orbit of x_p is periodic in time T_p, the group orbit sweeps out a *compact* invariant manifold $\mathcal{M}_p$. The simplest example is the $N = 0$, no symmetry case, where the invariant manifold $\mathcal{M}_p$ is the 1-torus traced out by a periodic trajectory p. If $\mathcal{M}$ is a smooth C^∞ manifold, and G is compact and acts smoothly on $\mathcal{M}$, the reduced state space can be realized as

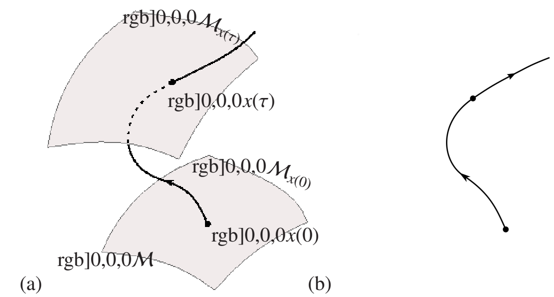

Figure 12.2: (a) The group orbit $\mathcal{M}_{x(0)}$ of state space point $x(0)$, and the group orbit $\mathcal{M}_{x(t)}$ reached by the trajectory $x(t)$ time t later. As any point on the manifold $\mathcal{M}_{x(t)}$ is physically equivalent to any other, the state space is stratified into the union of group orbits. (b) Symmetry reduction $\mathcal{M} \to \hat{\mathcal{M}}$ replaces each full state space group orbit $\mathcal{M}_x$ by a single point $\hat{x} \in \hat{\mathcal{M}}$.

rgb]0,0,0$\hat{x}(\tau)$

rgb]0,0,0$\hat{\mathcal{M}}$ rgb]0,0,0$\hat{x}(0)$

a 'stratified manifold', meaning that each group orbit (a 'stratum') is represented by a point in the reduced state space, see figure 12.2 and sect. 13.2. Generalizing the description of a non–wandering set of sect. 2.1.1, we say that for flows with continuous symmetries the non–wandering set Ω of dynamics (2.3) is the closure of the set of compact invariant manifolds $\mathcal{M}_p$. Without symmetries, we visualize the non–wandering set as a set of points; in presence of a continuous symmetry, each such 'point' is a group orbit.

12.1.1 Lie groups for pedestrians

> [...] which is an expression of consecration of angular momentum.
>
> — Mason A. Porter's student

Definition: A Lie group is a topological group G such that (i) G has the structure of a smooth differential manifold, and (ii) the composition map $G \times G \to G : (g, h) \to gh^{-1}$ is smooth, i.e., $\mathbb{C}^\infty$ differentiable.

Do not be mystified by this definition. Mathematicians also have to make a living. Historically, the theory of compact Lie groups that we will deploy here emerged as a generalization of the theory of SO(2) rotations, i.e., Fourier analysis. By a 'smooth differential manifold' one means objects like the circle of angles that parameterize continuous rotations in a plane, example 12.1, or the manifold swept by the three Euler angles that parameterize SO(3) rotations.

An element of a Lie group continuously connected to identity can be written as

$$g(\phi) = e^{\phi\cdot\mathbf{T}}\,, \qquad \phi\cdot\mathbf{T} = \sum_{a=1}^{N}\phi_a\mathbf{T}_a\,, \tag{12.5}$$

where $\phi\cdot\mathbf{T}$ is a *Lie algebra* element, and ϕ_a are the parameters of the transformation. Repeated indices are summed throughout this chapter, and the dot product refers to a sum over Lie algebra generators. We find it convenient to use bra-ket notation for the Euclidean product of two real vectors $x, y \in \mathcal{M}$, i.e., indicate x-transpose times y by

$$\langle x|y\rangle = x^\top y = \sum_i^d x_i y_i \,. \tag{12.6}$$

Unitary transformations $\exp(\phi \cdot \mathbf{T})$ are generated by sequences of infinitesimal steps of form

$$g(\delta\phi) \simeq 1 + \delta\phi \cdot \mathbf{T}\,, \quad \delta\phi \in \mathbb{R}^N\,, \quad |\delta\phi| \ll 1\,, \tag{12.7}$$

where $\mathbf{T}_a$, the *generators* of infinitesimal transformations, are a set of linearly independent $[d\times d]$ anti-hermitian matrices, $(\mathbf{T}_a)^\dagger = -\mathbf{T}_a$, acting linearly on the d-dimensional state space $\mathcal{M}$. In order to streamline the exposition, we postpone discussion of combining continuous coordinate transformations with the discrete ones to sect. 12.2.1.

exercise 12.1

Unitary and orthogonal groups (as well as their subgroups) are defined as groups that preserve 'length' norms, $\langle gx|gx\rangle = \langle x|x\rangle$, and infinitesimally their generators (12.7) induce no change in the norm, $\langle \mathbf{T}_a x|x\rangle + \langle x|\mathbf{T}_a x\rangle = 0$, hence the Lie algebra generators $\mathbf{T}$ are antisymmetric for orthogonal groups, and antihermitian for unitary ones,

$$\mathbf{T}^\dagger = -\mathbf{T}\,. \tag{12.8}$$

For continuous groups the Lie algebra, i.e., the set of N generators $\mathbf{T}_a$ of infinitesimal transformations, takes the role that the $|G|$ group elements play in the theory of discrete groups. The flow field at the state space point x induced by the action of the group is given by the set of N *tangent fields*

$$t_a(x)_i = (\mathbf{T}_a)_{ij} x_j\,, \tag{12.9}$$

which span the group *tangent space* at state space point x. The antisymmetry (12.8) of generators implies that the action of the group on vector x is locally normal to it,

$$\langle x|t_a(x)\rangle = 0\,. \tag{12.10}$$

A group tangent (12.9) is labelled by a pair of indices, as it is a vector both in the group tangent space and in the state space. We shall indicate by $\langle t_a(x)|t_b(y)\rangle$ the sum over state space inner product only, and by

$$\langle t(x)|t(y)\rangle = \sum_{a=1}^N \langle t_a(x)|t_a(y)\rangle = \langle x|\mathbf{T}^\dagger \cdot \mathbf{T}\, y\rangle \tag{12.11}$$

the sum over both group and spatial dimensions.

Any representation of a compact Lie group G is fully reducible, and invariant tensors constructed by contractions of $\mathbf{T}_a$ are useful for identifying irreps. The simplest such invariant is

$$\mathbf{T}^\top \cdot \mathbf{T} = \sum_\alpha C_2^{(\alpha)}\ \mathbb{1}^{(\alpha)}\,, \tag{12.12}$$

where $C_2^{(\alpha)}$ is a number called the quadratic Casimir for irrep labeled α, and $\mathbb{1}^{(\alpha)}$ is the identity on the α-irreducible subspace, 0 elsewhere. The dot product of two tangent fields is thus a sum weighted by Casimirs,

$$\langle t(x)|t(x')\rangle = \sum_\alpha C_2^{(\alpha)} x_i\, \delta_{ij}^{(\alpha)} x'_j\,. \tag{12.13}$$

example 12.3
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example 12.6
p. 206

The really interesting Lie groups are the non-abelian semisimple ones -but- as we will discuss nothing much more complicated than the abelian special orthogonal group SO(2) of rotations in a plane, we shall not discuss the non-abelian case here.

fast track:
sect. 12.2, p. 197

12.1.2 Equivariance under infinitesimal transformations

A flow $\dot{x} = v(x)$ is G-equivariant (12.1), if symmetry transformations commute with time evolution

$$v(x) = g^{-1}\, v(g\, x)\,, \qquad \text{for all } g \in G\,. \tag{12.14}$$

For an infinitesimal transformation (12.7) the G-equivariance condition becomes

$$v(x) = (1 - \phi \cdot \mathbf{T})\, v(x + \phi \cdot \mathbf{T}x) + \cdots = v(x) - \phi \cdot \mathbf{T}v(x) + \frac{dv}{dx}\, \phi \cdot \mathbf{T}x + \cdots .$$

The $v(x)$ cancel, and ϕ_a are arbitrary. Denote the *group flow tangent field* at x by $t_a(x)_i = (\mathbf{T}_a)_{ij} x_j$. Thus the infinitesimal, Lie algebra G-equivariance condition is

$$t_a(v) - A(x)\, t_a(x) = 0\,, \tag{12.15}$$

where $A = \partial v / \partial x$ is the stability matrix (4.3). A learned remark: The directional derivative along direction ξ is $\lim_{t \to 0}(f(x + t\xi) - f(x))/t$. The left-hand side of (12.15) is the *Lie derivative* of the dynamical flow field v along the direction of the infinitesimal group-rotation induced flow $t_a(x) = \mathbf{T}_a x$,

$$\mathcal{L}_{t_a} v = \left(\mathbf{T}_a - \frac{\partial}{\partial y} (\mathbf{T}_a x) \right) v(y) \Big|_{y=x} . \tag{12.16}$$

The equivariance condition (12.15) states that the two flows, one induced by the dynamical vector field v, and the other by the group tangent field t, commute if their Lie derivatives (or the 'Lie brackets ' or 'Poisson brackets') vanish. exercise 12.9

example 12.7
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example 12.8
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Checking equivariance as a Lie algebra condition (12.15) is easier than checking it for global, finite angle rotations (12.14).

12.2 Symmetries of solutions

Let $v(x)$ be the dynamical flow, and f^τ the trajectory or 'time-τ forward map' of an initial point x_0,

$$\frac{dx}{dt} = v(x) , \qquad x(\tau) = f^\tau(x_0) = x_0 + \int_0^\tau d\tau' \, v(x(\tau')) . \tag{12.17}$$

As discussed in sect. 11.1, solutions $x(\tau)$ of an equivariant system can satisfy all of the system's symmetries, a subgroup of them, or have no symmetry at all. For a given solution $x(\tau)$, the subgroup that contains all symmetries that fix x (that satisfy $gx = x$) is called the isotropy (or stabilizer) subgroup of x. A generic ergodic trajectory $x(\tau)$ has no symmetry beyond the identity, so its isotropy group is $\{e\}$, but recurrent solutions often do. At the other extreme is equilibrium or steady solution (2.8), whose isotropy group is the full symmetry group G.

Definition: Equilibrium $x_{EQ} = \mathcal{M}_{EQ}$ is a fixed, time-invariant solution,

$$\begin{aligned} v(x_{EQ}) &= 0 , \\ x(x_{EQ}, \tau) &= x_{EQ} + \int_0^\tau d\tau' \, v(x(\tau')) = x_{EQ} . \end{aligned} \tag{12.18}$$

An *equilibrium* with full symmetry,

$$g\, x_{EQ} = x_{EQ} \qquad \text{for all } g \in G ,$$

lies, by definition, in Fix (G) subspace, for example the x_3 axis in figure 12.3 (a). The multiplicity of such solution is one. An equilibrium x_{EQ} with symmetry G_{EQ} smaller than the full group G belongs to a group orbit G/G_{EQ}. If G is finite there are $|G|/|G_{EQ}|$ equilibria in the group orbit, and if G is continuous then the group orbit of x is a continuous family of equilibria of dimension $\dim G - \dim G_{EQ}$. For example, if the angular velocity c in figure 12.3 (b) equals zero, the group orbit consists of a circle of (dynamically static) equivalent equilibria.

Definition: Relative equilibrium solution $x_{TW}(\tau) \in \mathcal{M}_{TW}$: the dynamical flow field points along the group tangent field, with constant 'angular' velocity c, and the trajectory stays on the group orbit, see figure 12.3 (a):

$$\begin{aligned} v(x) &= c \cdot t(x) , \qquad x \in \mathcal{M}_{TW} \\ x(\tau) &= g(-\tau\, c)\, x(0) \;=\; e^{-\tau\, c \cdot \mathbf{T}} x(0) . \end{aligned} \tag{12.19}$$

A *traveling wave* remark 13.2

$$x(\tau) = g(-c\tau)\, x_{TW} = x_{TW} - c\,\tau , \qquad c \in \mathbb{R}^d \tag{12.20}$$

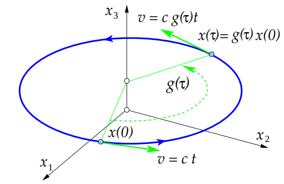


Figure 12.3: (a) A *relative equilibrium orbit* starts out at some point $x(0)$, with the dynamical flow field $v(x) = c \cdot t(x)$ pointing along the group tangent space. For the SO(2) symmetry depicted here, the flow traces out the group orbit of $x(0)$ in time $T = 2\pi/c$. (b) An *equilibrium* lives either in the fixed Fix (G) subspace (x_3 axis in this sketch), or on a group orbit as the one depicted here, but with zero angular velocity c. In that case the circle (in general, N-torus) depicts a continuous family of fixed equilibria, related only by the group action.

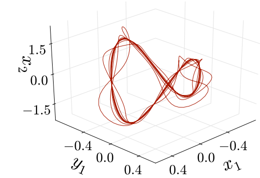


Figure 12.4: $\{x_1, y_1, x_2\}$ plot of the two-modes system with initial point on the unstable manifold of TW_1. In figure 12.1 this trajectory is integrated longer, until it falls on to the strange attractor. (N.B. Budanur)

is a special type of a relative equilibrium of equivariant evolution equations, where the action is given by translation (12.30), $g(y)\, x(0) = x(0) + y$. A *rotating wave* is another special case of relative equilibrium, with the action is given by angular rotation. By equivariance, all points on the group orbit are equivalent, the magnitude of the velocity c is same everywhere along the orbit, so a 'traveling wave' moves at a constant speed. For an $N > 1$ trajectory traces out a line within the group orbit. As the c_a components are generically not in rational ratios, the trajectory explores the N-dimensional group orbit (12.2) quasi-periodically. In other words, the group orbit $g(\tau)\, x(0)$ coincides with the dynamical orbit $x(\tau) \in \mathcal{M}_{TW}$ and is thus flow invariant.

Definition: Periodic orbit. Let x be a periodic point on the periodic orbit p of period T,

$$f^T(x) = x , \qquad x \in \mathcal{M}_p .$$

By equivariance, $g\, x$ is another periodic point, with the orbits of x and gx either identical or disjoint.

If gx lands on the same orbit, g is an element of periodic orbit's symmetry group G_p. If the symmetry group is the full group G, we are back to (12.19),

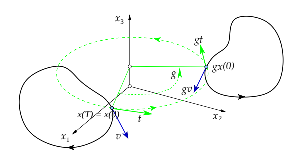


Figure 12.5: A periodic orbit starts out at $x(0)$ with the dynamical v and group tangent t flows pointing in different directions, and returns after time T_p to the initial point $x(0) = x(T_p)$. The group orbit of the temporal orbit of $x(0)$ sweeps out a $(1+N)$-dimensional torus, a continuous family of equivalent periodic orbits, two of which are sketched here. For SO(2) this is topologically a 2-torus.

i.e., the periodic orbit is the group orbit traced out by a relative equilibrium. The other option is that the isotropy group is discrete, the orbit segment $\{x, gx\}$ is pre-periodic (or eventually periodic), $x(0) = g_p x(T_p)$, where T_p is a fraction of the full period, $T_p = T/m$, and thus

$$\begin{aligned} x(0) &= g_p x(T_p)\,, \qquad x \in \mathcal{M}_p\,, \qquad g_p \in G_p \\ x(0) &= g_p^m x(m\,T_p) = x(T) = x(0)\,. \end{aligned} \tag{12.21}$$

If the periodic solutions are disjoint, as in figure 12.5, their multiplicity (if G is finite, see sect. 10.1), or the dimension of the manifold swept under the group action (if G is continuous) can be determined by applications of $g \in G$. They form a family of conjugate solutions (10.6),

$$\mathcal{M}_{g\,p} = g\,\mathcal{M}_p\,g^{-1}\,. \tag{12.22}$$

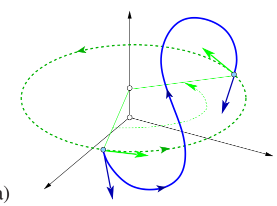
Definition: Relative periodic orbit p is an orbit $\mathcal{M}_p$ in state space $\mathcal{M}$ which exactly recurs

$$x_p(0) = g_p x_p(T_p)\,, \qquad x_p(\tau) \in \mathcal{M}_p\,, \tag{12.23}$$

at a fixed *relative period* T_p, but shifted by a fixed group action g_p which brings the endpoint $x_p(T_p)$ back into the initial point $x_p(0)$, see figure 12.6 (a). The group action g_p parameters $\phi = (\phi_1, \phi_2, \cdots \phi_N)$ are referred to as 'phases', or 'shifts'. In contrast to the pre-periodic (12.21), here the phases are irrational, and the trajectory sweeps out ergodically the group orbit without ever closing into a periodic orbit. For dynamical systems with only continuous (no discrete) symmetries, the parameters $\{t, \phi_1, \cdots, \phi_N\}$ are real numbers, ratios π/ϕ_j are almost never rational, likelihood of finding a periodic orbit for such system is zero, and such relative periodic orbits are almost never eventually periodic.

example 12.11
p. 208

Relative periodic orbits are to periodic solutions what relative equilibria (traveling waves) are to equilibria (steady solutions). Equilibria satisfy $f^\tau(x) - x = 0$ and relative equilibria satisfy $f^\tau(x) - g(\tau)\,x = 0$ for any τ. In a co-moving frame, i.e., frame moving along the group orbit with velocity $v(x) = c \cdot t(x)$, the relative equilibrium appears as an equilibrium. Similarly, a relative periodic orbit is periodic in its mean velocity $c_p = \phi_p/T_p$ co-moving frame (see figure 12.7), but in the stationary frame its trajectory is quasiperiodic. A co-moving frame is helpful in visualizing a single 'relative' orbit, but useless for viewing collections of orbits, as each one drifts with its own angular velocity. Visualization of all relative periodic orbits as periodic orbits we attain only by global symmetry reductions, to be undertaken in sect. 13.2.



(a)

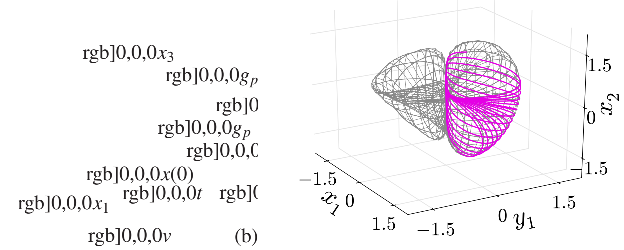


Figure 12.6: (a) A very idealized sketch: a relative periodic orbit starts out at $x(0)$ with the dynamical v and group tangent t flows pointing in different directions, and returns to the group orbit of $x(0)$ after time T_p at $x(T_p) = g_p x(0)$, a rotation of the initial point by g_p. For flows with continuous symmetry a generic relative periodic orbit (not pre-periodic to a periodic orbit) fills out ergodically what is topologically a torus, as in (b); if you are able to draw such a thing, kindly send us the figure. (b) The simplest example, 4-dimensional two-modes system of example 12.5, a 3-dimensional projection: 15 repeats of the shortest relative periodic orbit $\overline{1}$ (magenta) winding around the torus (here visualized as the gray wireframe) it lives on. (N.B. Budanur)

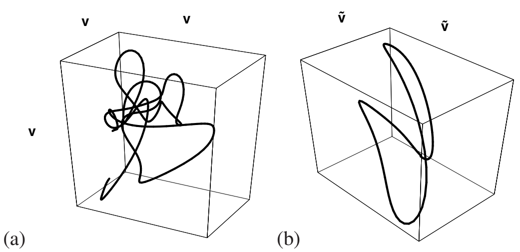


Figure 12.7: A relative periodic orbit of Kuramoto-Sivashinsky flow projected on (a) the stationary state space coordinate frame $\{v_1, v_2, v_3\}$, traced for four periods T_p; (b) the co-moving $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ coordinate frame, moving with the mean angular velocity $c_p = \phi_p/T_p$. (from ref. [A1.84])

12.2.1 Discrete and continuous symmetries together

We expect to see relative periodic orbits because a trajectory that starts on and returns to a given torus of a symmetry equivalent solutions is unlikely to intersect it at the initial point, unless forced to do so by a discrete symmetry. This we will make explicit in sect. 13.2, where relative periodic orbits will be viewed as periodic orbits of the reduced dynamics.

If, in addition to a continuous symmetry, one has a discrete symmetry which is not its subgroup, one does expect equilibria and periodic orbits. However, a relative periodic orbit can be pre-periodic if it is equivariant under a discrete symmetry, as in (12.21): If $g^m = 1$ is of finite order m, then the corresponding orbit is periodic with period mT_p. If g is not of a finite order, a relative periodic orbit is periodic only after a shift by g_p, as in (12.23). Morally, as it will be shown in chapter 25, such orbit is the true 'prime' orbit, i.e., the shortest segment that under action of G tiles the entire invariant submanifold $\mathcal{M}_p$.

Definition: Relative orbit $\mathcal{M}_{Gx}$ in state space $\mathcal{M}$ is the time evolved *group orbit* $\mathcal{M}_x$ of a state space point x, the set of all points that can be reached from x by all symmetry group actions and evolution of each in time.

$$\mathcal{M}_{x(t)} = \{gx(t) \mid t \in \mathbb{R}, g \in G\} \,. \tag{12.24}$$

In presence of symmetry, an equilibrium is the set of all equilibria related by symmetries, an relative periodic orbit is the hyper-surface traced by a trajectory in time T and all group actions, etc..

chapter 25

12.3 Stability

A spatial derivative of the equivariance condition (12.1) yields the matrix equivariance condition satisfied by the stability matrix (stated here both for the finite group actions, and for the infinitesimal, Lie algebra generators):

$$gA(x)g^{-1} = A(gx) \,, \qquad [\mathbf{T}_a, A] = \frac{\partial A}{\partial x} t_a(x) \,. \tag{12.25}$$

For a flow within the fixed $\mathrm{Fix}\,(G)$ subspace, $t(x)$ vanishes, and the symmetry imposes strong conditions on the perturbations out of the $\mathrm{Fix}\,(G)$ subspace. As in this subspace stability matrix A commutes with the Lie algebra generators $\mathbf{T}$, the spectrum of its eigenvalues and eigenvectors is decomposed into irreps of the symmetry group. This we have already observed for the EQ_0 of the Lorenz flow in example 11.5.

A infinitesimal symmetry group transformation maps the initial and the end point of a finite trajectory into a nearby, slightly rotated equivalent points, so we expect the perturbations along to group orbit to be marginal, with unit eigenvalues. The argument is akin to (4.9), the proof of marginality of perturbations along a periodic orbit. Consider two nearby initial points separated by an N-dimensional infinitesimal group transformation (12.7): $\delta x_0 = g(\delta\phi)x_0 - x_0 = \delta\phi \cdot \mathbf{T} x_0 = \delta\phi \cdot t(x_0)$. By the commutativity of the group with the flow, $g(\delta\phi) f^\tau(x_0) = f^\tau(g(\delta\phi)x_0)$. Expanding both sides, keeping the leading term in $\delta\phi$, and using the definition of the Jacobian matrix (4.5), we observe that $J^\tau(x_0)$ transports the N-dimensional group tangent space at $x(0)$ to the rotated tangent space at $x(\tau)$ at time τ:

$$t_a(\tau) = J^\tau(x_0)\, t_a(0) \,, \qquad t_a(\tau) = \mathbf{T}_a\, x(\tau) \,. \tag{12.26}$$

For a relative periodic orbit, $g_p x(T_p) = x(0)$, at any point along cycle p the group tangent vector $t_a(\tau)$ is an eigenvector of the Jacobian matrix with an eigenvalue of unit magnitude,

$$J_p\, t_a(x) = \ t_a(x) \,, \qquad J_p(x) = g_p J^{T_p}(x) \,, \qquad x \in \mathcal{M}_p \,. \tag{12.27}$$

For a relative equilibrium flow and group tangent vectors coincide, $v = c \cdot t(x)$. Dotting by the velocity c (i.e., summing over $c_a t_a$) the equivariance condition (12.15), $t_a(v) - A(x)\, t_a(x) = 0$, we get

$$(c \cdot \mathbf{T} - A) v = 0 \,. \tag{12.28}$$

In other words, in the co-rotating frame the eigenvalues corresponding to group tangent are marginal, and the velocity v is the corresponding right eigenvector.

Two successive points along the cycle separated by $\delta x_0 = \delta\phi \cdot t(\tau)$ have the same separation after a completed period $\delta x(T_p) = g_p \delta x_0$, hence eigenvalue of magnitude 1. In presence of an N-dimensional Lie symmetry group, N eigenvalues equal unity.

Résumé

The message: If a dynamical systems has a symmetry, use it!

We conclude with a few general observations: Higher dimensional dynamics requires study of compact invariant sets of higher dimension than 0-dimensional equilibria and 1-dimensional periodic orbits studied so far. In sect. 2.1.1 we made an attempt to classify 'all possible motions:' (a) equilibria, (b) periodic orbits, (c) everything else. Now one can discern in the fog of dynamics an outline of a more serious classification - long time dynamics takes place on the closure of a set of all invariant compact sets preserved by the dynamics, and those are: (a) 0-dimensional equilibria $\mathcal{M}_{EQ}$, (b) 1-dimensional periodic orbits $\mathcal{M}_p$, (3) global symmetry induced N-dimensional relative equilibria $\mathcal{M}_{TW}$, (c) (N+1)-dimensional relative periodic orbits $\mathcal{M}_p$, (d) terra incognita. We have some inklings of the 'terra incognita:' for example, in symplectic symmetry settings one finds KAM-tori, and in general dynamical settings we encounter *partially hyperbolic invariant M-tori*, isolated tori that are consequences of dynamics, not of a global symmetry. They are harder to compute than anything we have attempted so far, as they cannot be represented by a single relative periodic orbit, but require a numerical computation of full M-dimensional compact invariant sets and their infinite-dimensional linearized Jacobian matrices, marginal in M dimensions, and hyperbolic in the rest. We expect partially hyperbolic invariant tori to play important role in high-dimensional dynamics. In this chapter we have focused on the simplest example of such compact invariant sets, where invariant tori are a robust consequence of a global continuous symmetry of the dynamics. The direct product structure of a global symmetry that commutes with the flow enables us to reduce the dynamics to a desymmetrized ($d-1-N$)-dimensional reduced state space $\mathcal{M}/G$.

Relative equilibria and relative periodic orbits are the hallmark of systems with continuous symmetry. Amusingly, in this extension of 'periodic orbit' theory from unstable 1-dimensional closed periodic orbits to unstable (N+1)-dimensional compact manifolds $\mathcal{M}_p$ invariant under continuous symmetries, there are either no or proportionally few periodic orbits. In presence of a continuous symmetry, likelihood of finding a periodic orbit is *zero*. Relative periodic orbits are almost never eventually periodic, i.e., they almost never lie on periodic trajectories in the full state space, so looking for periodic orbits in systems with continuous symmetries is a fool's errand.

However, dynamical systems are often equivariant under a combination of continuous symmetries and discrete coordinate transformations of chapter 10, for example the orthogonal group O(n). In presence of discrete symmetries relative periodic orbits within discrete symmetry-invariant subspaces are eventually periodic. Atypical as they are (no generic chaotic orbit can ever enter these discrete invariant subspaces) they will be important for periodic orbit theory, as there the shortest orbits dominate, and they tend to be the most symmetric solutions.

chapter 25

Commentary

Remark 12.1 Ideal is not real. (continued from remark 10.1): The literature on symmetries in dynamical systems is immense, most of it deliriously unintelligible. Would it kill them [12.2, 13.66, 12.4, 13.19] to say 'symmetry of orbit p' instead of carrying on about 'isotropies, quotients, factors, normalizers, centralizers and stabilizers?' Group action being 'free, faithful, proper, regular?' Symmetry-reduced state space being 'orbifold?' For the dynamical systems applications at hand we need only the basic Lie group facts, on the level of any standard group theory textbook [26.5]. We found Roger Penrose [12.7] introduction to the subject both enjoyable and understandable. Chapter 2. of ref. [26.7] offers a pedagogical introduction to Lie groups of transformations, and Nakahara [12.9] to Lie derivatives and brackets. The presentation given here is in part based on Siminos thesis [13.27] and ref. [13.28]. The reader is referred to the monographs of Golubitsky and Stewart [12.4], Hoyle [12.2], Olver [13.24], Bredon [13.33], and Krupa [13.60] for more depth and rigor than would be wise to wade into here.

Remark 12.2 Two-modes equations. Dangelmayr [12.15], Armbruster, Guckenheimer and Holmes [12.16], Jones and Proctor [12.17], and Porter and Knobloch [12.18] (see Golubitsky *et al.* [12.19], Sect. XX.1) have investigated bifurcations in 1:2 resonance ODE normal form models to third order in the amplitudes. Budanur *et al.* [13.1] studied (12.35), a particular case of Danglmayr [12.15] and Porter and Knobloch [12.18] 2-Fourier mode SO(2)-equivariant ODEs, as a relatively simple testing ground for the periodic orbit theory on a system with a continuous symmetry. In this chapter and the next, we will use this model, which we will refer to as 'two-modes' system, to illustrate the effects of continuous symmetry on the dynamics and symmetry reduction by the method of slices.

Remark 12.3 Modulated traveling waves. When a 'traveling wave' goes unstable through a Hopf bifurcation, the resulting motion resembles the initial traveling wave weakly periodically 'modulated' in time, hence such relative periodic orbit is often called a *modulated traveling wave* (MTW). These were studied, for instance, by Armbruster *et al.* [12.16], and a detailed computation of numerous bifurcation branches of these solutions was presented by Brown and Kevrekidis [12.21]. They find quasiperiodic secondary Hopf bifurcations. In chaos unstable recurrent motions typically arise come from other, stretching and folding mechanisms, so for our purposes 'MTW' is too narrow a concept, merely a particular case of a relative periodic orbit.

12.4 Examples

12.4.1 Special orthogonal group SO(2)

Example 12.1 ***Special orthogonal group*** $SO(2)$ *(or S^1) is a group of length-preserving rotations in a plane. 'Special' refers to requirement that* $\det g = 1$*, in contradistinction* to the orthogonal group $O(n)$ which allows for length-preserving inversions through the origin, with $\det g = -1$. A group element can be parameterized by angle ϕ, with the group multiplication law $g(\phi')g(\phi) = g(\phi' + \phi)$, and its action on smooth periodic functions $u(\phi + 2\pi) = u(\phi)$ generated by

$$g(\phi') = e^{\phi' \mathbf{T}}\,, \quad \mathbf{T} = \frac{d}{d\phi}\,. \tag{12.29}$$

Expand the exponential, apply it to a differentiable function $u(\phi)$, and you will recognize a Taylor series. So $g(\phi')$ shifts the coordinate by ϕ', $g(\phi')\,u(\phi) = u(\phi' + \phi)$. click to return: p. ??

Example 12.2 ***Translation group:*** Differential operator $\mathbf{T}$ in (12.29) is reminiscent of the generator of spatial translations. The 'constant velocity field' $v(x) = v = c \cdot \mathbf{T}$' acts on x_j by replacing it by the velocity vector c_j. It is easy to verify by Taylor expanding a function $u(x)$ that the time evolution is nothing but a coordinate translation by (time× velocity):

$$e^{-\tau c \cdot \mathbf{T}} u(x) = e^{-\tau c \cdot \frac{d}{dx}} u(x) = u(x - \tau\, c)\,. \tag{12.30}$$

As x is a point in the Euclidean $\mathbb{R}^d$ space, the group is not compact. A sequence of time steps in time evolution always forms an abelian Lie group, albeit never as trivial as this free ballistic motion.

If the group actions consist of N rotations which commute, for example act on an N-dimensional cell with periodic boundary conditions, the group is an abelian group that acts on a torus T^N. click to return: p. ??

Example 12.3 ***SO*(2) *irreps:*** Consider the action (12.29) of the one-parameter rotation group $SO(2)$ on a smooth periodic function $u(\phi + 2\pi) = u(\phi)$ defined on a $1D$-dimensional configuration space domain $x \in [0, 2\pi)$. The state space matrix representation of the $SO(2)$ counter-clockwise (right-handed) rotation $g(\phi')u(\phi) = u(\phi + \phi')$ by angle ϕ' is block-diagonal, acting on the kth Fourier coefficient pair (x_k, y_k) in the Fourier series (2.16),

$$u(\phi) = x_0 + \sum_{k=1}^{\infty} \left(x_k \cos k\phi + y_k \sin k\phi\right)\,. \tag{12.31}$$

by multiplication by

$$g^{(k)}(\phi') = \begin{pmatrix} \cos k\phi' & -\sin k\phi' \\ \sin k\phi' & \cos k\phi' \end{pmatrix}, \qquad \mathbf{T}^{(k)} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \tag{12.32}$$

where $\mathbf{T}^{(k)}$ is the kth Fourier mode Lie group generator. The $SO(2)$ group tangent (12.9) to state space point $u(\phi)$ on the kth invariant subspace is

$$t^{(k)}(u) = k \begin{pmatrix} -y_k \\ x_k \end{pmatrix}. \tag{12.33}$$

The L^2 norm of $t(u)$ is weighted by the $SO(2)$ quadratic Casimir (12.12), $C_2^{(k)} = k^2$,

$$\langle t(u)^\top | t(u) \rangle = \oint \frac{d\phi}{2\pi}\, u(\phi)^\top \mathbf{T}^\top \mathbf{T} u(2\pi - \phi) = \sum_{k=1}^{\infty} k^2 \left(x_k^2 + y_k^2 \right) , \tag{12.34}$$

and converges only for sufficiently smooth $u(\phi)$. What does that mean? We saw in (12.30) that $\mathbf{T}$ generates translations, and by (12.32) the velocity of the kth Fourier mode is k times higher than for the $k = 1$ component. If $|u^{(k)}|$ does not fall off faster the $1/k$, the action of $SO(2)$ is overwhelmed by the high Fourier modes. click to return: p. ??

Example 12.4 ***Invariance under fractional rotations:*** *Consider a velocity field $v(x)$ equivariant (12.1) under discrete cyclic subgroup $C_m = \{e, C^{1/m}, C^{2/m}, \cdots, C^{(m-1)/m}\}$ of $SO(2)$ rotations by $2\pi/m$),* exercise 12.2

$$C^{1/m} v(x) = v(C^{1/m} x) , \qquad (C^{1/m})^m = e .$$

The field $v(x)$ on the fundamental domain $2\pi/m$ is now a tile whose m copies tile the entire domain. It is periodic on the fundamental domain, and thus has Fourier expansion with Fourier modes $\cos(2\pi m j x)$, $\sin(2\pi m j x)$. The Fourier expansion on the full interval $(0, 2\pi)$ cannot have any other modes, as they would violate the C_m symmetry. This means that $SO(2)$ always has an infinity of discrete subgroups $C_2, C_3, \cdots, C_m, \cdots$; for each the non-vanishing coefficients are only for Fourier modes whose wave numbers are multiples of m.

12.4.2 Two-modes SO(2)-equivariant flow

Example 12.5 ***Two-modes flow:*** *Consider the pair of $U(1)$-equivariant complex ODEs*

$$\begin{aligned} \dot z_1 &= (\mu_1 - \mathrm{i}\, e_1)\, z_1 + a_1\, z_1 |z_1|^2 + b_1\, z_1 |z_2|^2 + c_1\, \bar z_1\, z_2 \\ \dot z_2 &= (\mu_2 - \mathrm{i}\, e_2)\, z_2 + a_2\, z_2 |z_1|^2 + b_2\, z_2 |z_2|^2 + c_2\, z_1^2 , \end{aligned} \tag{12.35}$$

with z_1, z_2 complex, and all parameters real valued.

The two-modes system, which we shall introduce in sect. 12.4.2 and use for illustrations throughout this chapter and the next, is an example of a few modes truncation of a Fourier expansion, truncated in such a way that the model exhibits the same symmetry structure as many nonlinear field problems, while being drastically simpler to study.

We shall refer to this toy model as the two-modes system. It belongs to the family of simplest ODE systems that we know that (a) have a continuous $U(1)$ / $SO(2)$, but no discrete symmetry (if at least one of $e_j \neq 0$). (b) models 'weather', in the same sense that Lorenz equation models 'weather', (c) exhibits chaotic dynamics, (d) can be easily visualized, in the dimensionally lowest possible setting required for chaotic dynamics, with the full state space of dimension $d = 4$, and the $SO(2)$-reduced dynamics taking place in 3 dimensions, and (e) for which the method of slices reduces the symmetry by a single global slice hyperplane.

For parameters far from the bifurcation values, this is a merely a toy model with no physical interpretation, just like the iconic Lorenz flow (2.22): We use it to

illustrate the effects of continuous symmetry on chaotic dynamics. We have not found a second order truncation of such models that exhibits interesting dynamics, hence the third order in the amplitudes, and the unreasonably high number of parameters. After some experimentation we fix or set to zero various parameters, and in the numerical examples that follow, we settle for parameters set to

$$\begin{aligned} \mu_1 &= -2.8\, ,\ \mu_2 = 1\, ,\ e_1 = 0\, ,\ e_2 = 1\, , \\ a_1 &= -1\, ,\ a_2 = -2.66\, ,\ b_1 = 0\, ,\ b_2 = 0\, ,\ c_1 = -7.75\, ,\ c_2 = 1\, , \end{aligned} \tag{12.36}$$

unless explicitly stated otherwise. For these parameter values the system exhibits chaotic behavior. Experiment. If you find a more interesting behavior for some other parameter values, please let us know. The simplified system of equations can now be written as a 3-parameter $\{\mu_1, c_1, a_2\}$ two-modes system,

$$\begin{aligned} \dot z_1 &= \mu_1\, z_1 - z_1 |z_1|^2 + c_1\, \bar z_1\, z_2 \\ \dot z_2 &= (1 - \mathrm{i})\, z_2 + a_2\, z_2 |z_1|^2 + z_1^2 . \end{aligned} \tag{12.37}$$

In order to numerically integrate and visualize the flow, we recast the equations in real variables by substitution $z_1 = x_1 + i\, y_1$, $z_2 = x_2 + i\, y_2$. The two-modes system (12.35) is now a set of four coupled ODEs

$$\begin{aligned} \dot x_1 &= (\mu_1 - r^2)\, x_1 + c_1\, (x_1 x_2 + y_1 y_2) , \qquad r^2 = x_1^2 + y_1^2 \\ \dot y_1 &= (\mu_1 - r^2)\, y_1 + c_1\, (x_1 y_2 - x_2 y_1) \\ \dot x_2 &= x_2 + y_2 + x_1^2 - y_1^2 + a_2 x_2 r^2 \\ \dot y_2 &= -x_2 + y_2 + 2\, x_1 y_1 + a_2 y_2 r^2 . \end{aligned} \tag{12.38}$$

Try integrating (12.38) with random initial conditions, for long times, times much beyond which the initial transients have died out. What is wrong with this picture? It is a mess. As we shall show here, the attractor is built up by a nice 'stretch & fold' action, but that is totally hidden from the view by the continuous symmetry induced drifts. In the rest of this and next chapter's examples we shall investigate various ways of 'quotienting' this $SO(2)$ symmetry, and reducing the dynamics to a 3-dimensional symmetry-reduced state space. We shall not rest until we attain the simplicity and bliss of a 1-dimensional return map. *(N.B. Budanur and P. Cvitanović)* click to return: p. ??

Example 12.6 ***$SO(2)$ rotations for two-modes system:*** *Substituting the Lie algebra generator*

$$\mathbf{T} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \tag{12.39}$$

acting on a 4–dimensional state space (12.38) into (12.5) yields a finite angle $SO(2)$ rotation:

$$g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos 2\phi & -\sin 2\phi \\ 0 & 0 & \sin 2\phi & \cos 2\phi \end{pmatrix} . \tag{12.40}$$

From (12.32) we see that the action of $SO(2)$ on the complex Lorenz equations state space decomposes into $m = 0$ G-invariant $m = 1$ and $m = 2$ subspaces.

Itinerary	$(x_{p,1}, y_{p,1}, x_{p,2}, y_{p,2})$	Period
1	(0.4525719, 0.0, 0.0509257, 0.0335428)	3.6415120
01	(0.4517771, 0.0, 0.0202026, 0.0405222)	7.3459412
0111	(0.4514665, 0.0, 0.0108291, 0.0424373)	14.6795175
01101	(0.4503967, 0.0, -0.0170958, 0.0476009)	18.3874094

Table 12.1: Several short relative periodic orbits of the two-modes system: itineraries, a periodic point in a Poincaré section for each orbit, the period.

The generator $\mathbf{T}$ is indeed anti-hermitian, $\mathbf{T}^\dagger = -\mathbf{T}$, and the group is compact, its elements parametrized by ϕ mod 2π. Locally, at $x \in \mathcal{M}$, the infinitesimal action of the group is given by the group tangent field $t(x) = \mathbf{T}x = (-y_1, x_1, -y_2, x_2)$. In other words, the flow induced by the group action is normal to the radial direction in the (x_1, y_1) and (x_2, y_2) planes.

click to return: p. ??

Example 12.7 ***Equivariance of the two-modes system:*** *That two-modes (12.38) is equivariant under $SO(2)$ rotations (12.40) can be checked by substituting the Lie algebra generator (12.39) and the stability matrix (4.3) for two-modes (12.38),* $A =$

$$\begin{pmatrix} \mu_1 - 3x_1^2 + c_1x_2 - y_1^2 & c_1y_2 - 2x_1y_1 & c_1x_1 & c_1y_1 \\ c_1y_2 - 2x_1y_1 & \mu_1 - x_1^2 - c_1x_2 - 3y_1^2 & -c_1y_1 & c_1x_1 \\ 2x_1 + 2a_2x_1x_2 & 2a_2x_2y_1 - 2y_1 & 1 + a_2(x_1^2 + y_1^2) & 1 \\ 2y_1 + 2a_2x_1y_2 & 2x_1 + 2a_2y_1y_2 & -1 & 1 + a_2(x_1^2 + y_1^2) \end{pmatrix}. \tag{12.41}$$

into the equivariance condition (12.15). Considering that $t(v)$ depends on the full set of equations (12.38), and $A(x)$ is only its linearization, this is not an entirely trivial statement. (N.B. Budanur)

click to return: p. ??

Example 12.8 ***How contracting is the two-modes flow?*** *For the parameter values (12.37) the flow is strongly volume contracting (4.29),*

$$\partial_i v_i = \sum_{i=1}^{4} \lambda_i(x,t) = \mathrm{tr}\,A(x) = 2\,[1 + \mu_1 - (2 - a_2)\,r^2] = -3.6 - 9.32\,r^2\,. \tag{12.42}$$

Note that this quantity depends on the full state space coordinates only through the $SO(2)$-invariant r^2, so the volume contraction rate is symmetry-invariant characterization of the flow, as is should be. The shortest relative periodic orbit $\overline{1}$ has period $T_1 = 3.64\ldots$ and typical $r^2 \approx 1$, (see table 12.1), so in one period a neighborhood of the relative periodic orbit is contracted by factor $\approx \exp(T_1\,\mathrm{tr}\,A(x)) \approx 3.7 \times 10^{-21}$. This is an insanely contracting flow; if we start with mm^4 cube around a periodic point, this volume (remember, two directions are marginal) shrinks to $\approx mm \times mm \times 10^{-11} mm \times 10^{-11} mm \approx mm \times mm\times$ fermi $\times$ fermi. Diameter of a proton is a couple of fermis. This strange attractor is thin*!.*

click to return: p. ??

12.4.3 Symmetries of iconic fluid flows

Example 12.9 ***Discrete symmetries of the plane Couette flow.*** *The plane Couette flow is a fluid flow bounded by two countermoving planes, in a cell periodic in streamwise and spanwise directions. The Navier-Stokes equations for the plane Couette flow*

have two discrete symmetries: reflection through the (streamwise , wall-normal) plane, and rotation by π in the (streamwise , wall-normal) plane. That is why the system has equilibrium and periodic orbit solutions, (as opposed to relative equilibrium and relative periodic orbit solutions discussed in chapter 12). They belong to discrete symmetry subspaces.

Example 12.10 ***Continuous symmetries of the plane Couette flow.*** *Every solution of Navier-Stokes equations belongs, by the $SO(2) \times O(2)$ symmetry, to a 2-torus T^2 of equivalent solutions. Furthermore these tori are interrelated by a discrete D_2 group of spanwise and streamwise flips of the flow cell. (continued in example 12.11)*

click to return: p. ??

Example 12.11 ***Relative orbits in the plane Couette flow.*** *Translational symmetry allows for relative equilibria (traveling waves), characterized by a fixed profile Eulerian velocity $u_{TW}(x)$ moving with constant velocity c, i.e.*

$$u(x, \tau) = u_{TW}(x - c\tau)\,. \tag{12.43}$$

As the plane Couette flow is bounded by two counter-moving planes, it is easy to see where the relative equilibrium (traveling wave) solutions come from. A relative equilibrium solution hugs close to one of the walls and drifts with it with constant velocity, slower than the wall, while maintaining its shape. A relative periodic solution is a solution that recurs at time T_p with exactly the same disposition of the Eulerian velocity fields over the entire cell, but shifted by a 2-dimensional (streamwise,spanwise) translation g_p. By discrete symmetries these solutions come in counter-traveling pairs $u_q(x - c\tau)$, $-u_q(-x + c\tau)$: for example, for each one drifting along with the upper wall, there is a counter-moving one drifting along with the lower wall. Discrete symmetries also imply existence of strictly stationary solutions, or 'standing waves'. For example, a solution with velocity fields antisymmetric under reflection through the midplane has equal flow velocities in opposite directions, and is thus an equilibrium stationary in time.

click to return: p. ??

Example 12.12 ***Traveling, rotating waves:*** *Names 'traveling waves', and 'rotating waves' are descriptive of solutions of some PDEs with simple continuous symmetries. For example, complex Ginzburg Landau equation is equivariant under the action of the group $g(\theta, y) \in G = S^1 \times \mathbb{R}$ on $u(x) \in \mathbb{R}^2$, given by translation in the domain and the rotation of $u(x)$,*

$$g(\theta, y)\,u(x) \;=\; R(\theta)\,u(x + y)\,, \qquad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{12.44}$$

Hence complex Ginzburg Landau equation allows for rotating wave *solutions of form $u(x,t) = R(-\omega\,t)\,\hat{u}(x - c\,t)$ with fixed profile $\hat{u}(x)$, velocity c and angular velocity ω. Traveling waves are typical of translationally invariant systems such as the plane Couette flow, example 12.11.*

Exercises

12.1. **SO(2) rotations in a plane:** Show by exponentiation (12.5) that the SO(2) Lie algebra element **T** generates rotation g in a plane,

$$\begin{aligned} g(\theta) &= e^{\mathbf{T}\theta} = \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} . \end{aligned} \qquad (12.45)$$

12.2. **Invariance under fractional rotations.** Argue that if the isotropy group of the velocity field $v(x)$ is the discrete subgroup C_m of SO(2) rotations about an axis (let's say the 'z-axis'),

$$C^{1/m} v(x) = v(C^{1/m} x) = v(x) , \qquad (C^{1/m})^m = e ,$$

the only non-zero components of Fourier-transformed equations of motion are a_{jm} for $j = 1, 2, \cdots$. Argue that the Fourier representation is then the quotient map of the dynamics, $\mathcal{M}/C_m$. (Hint: this sounds much fancier than what is - think first of how it applies to the Lorenz system and the 3-disk pinball.)

12.3. **U(1) equivariance of two-modes system for finite angles:** Show that the vector field in two-modes (12.35) is equivariant under (12.5), the unitary group U(1) acting on $\mathbb{R}^4 \cong \mathbb{C}^2$ by

$$g(\theta)(z_1, z_2) = (e^{i\theta} z_1, e^{i2\theta} z_2) , \quad \theta \in [0, 2\pi) . \qquad (12.46)$$

12.4. **SO(2) equivariance of two-modes system for finite angles:** Show that two-modes (12.38) are equivariant under rotation for finite angles.

12.5. **Stability matrix of two-modes system:** Compute the stability matrix (12.41) for two-modes system (12.38).

12.6. **SO(2) equivariance of two-modes system for infinitesimal angles.** Show that two-modes equations are equivariant under infinitesimal SO(2) rotations. Compute the volume contraction rate (4.29), verify (12.42). Period of the shortest relative periodic orbit of this system is $T_1 = 3.6415120$. By how much a small volume centered on the relative periodic orbit contracts in that time?

12.7. **Integrate the two-modes system:** Integrate (12.38) and plot a long trajectory of two-modes in the 4d state space, (x_1, y_1, y_2) projection, as in Figure 12.1.

12.8. **Classify possible symmetries of solutions for your research problem.** Classify types of solutions you expect in your research problem by their symmetries. Literature examples: plane Couette flow [13.42], pipe flow (sect. 2.2 and appendix A of ref. [12.23]), Kuramoto-Sivashinsky (see symmetry discussions of ref. [A1.79, A1.84], and probably many better papers out there that we are less familiar with), Euclidean symmetries of doubly-periodic 2D models of cardiac tissue, 2D Kolmogorov flow [12.25], two-modes flow (Dangelmayr [12.15]; Armbruster, Guckenheimer and Holmes [12.16]; Jones and Proctor [12.17]; Porter and Knobloch [12.18]; Golubitsky *et al.* [12.19], Sect. XX.1), 2D ABC flow [12.26]; perturbed Coulomb systems [12.27]; systems with discrete symmetries [25.10, 25.11]; example 10.5 reflection symmetric 1d map; refexamexmp:HamHenonMap Hamiltonian Hénon map; Hamiltonian Lozi map, etc..

12.9. **Discover the equivariance of a given flow:** Suppose you were given two-modes system, but nobody told you that the equations are SO(2)-equivariant. More generally, you might encounter a flow without realizing that it has a continuous symmetry - how would you discover it?

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