Chapter 10

Relativity for cyclists

Physicists like symmetry more than Nature
— Rich Kerswell

What if the laws of motion retain their form for a family of coordinate frames related by *continuous* symmetries? The finite groups intuition is of little use here.

Instead of writing yet another tome on group theory, in what follows we continue to serve group theoretic nuggets on need-to-know basis, through a series of pedestrian examples (but take a slightly higher, cyclist road in the text proper).

10.1 Continuous symmetries

First of all, why worry about continuous symmetries ? Figure 10.1 illustrates the effect a continuous symmetry has on dynamics of simple, two-modes flow (10.35) (for background, see remark 10.2).

exercise 10.7

example 10.5
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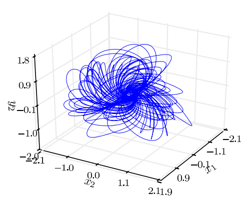

Figure 10.1: A typical $\{x_1, x_2, z\}$ trajectory of the two-modes flow (10.35). Study example 10.5; see also figure 10.7. (N.B. Budanur)

We shall refer to the component of the dynamics along the continuous symmetry directions as a ‘drift’. In a presence of a continuous symmetry an orbit explores the manifold swept by combined action of the dynamics and the symmetry induced drifts. Further problems arise when we try to determine whether an orbit shadows another orbit (see the figure 13.1 for a sketch of a close pass to a periodic orbit), or develop symbolic dynamics (partition the state space, as in chapter 11): here a 1-dimensional trajectory is replaced by a $(N+1)$-dimensional ‘sausage’, a dimension for each continuous symmetry (N being the total number of parameters specifying the continuous transformation, and ‘1’ for the time parameter t). How are we to measure distances between such objects? In this chapter we shall learn here how to develop visualizations of such flows, ‘quotient’ symmetries, and offer computationally straightforward methods of reducing the dynamics to lower-dimensional, reduced state spaces. The methods should also be applicable to high-dimensional flows, such as translationally invariant fluid flows bounded by pipes or planes (study example 10.10).

But first, a lightning review of the theory of Lie groups. The group-theoretical concepts of sect. 9.1 apply to compact continuous groups as well, and will not be repeated here. All the group theory that we shall need is in principle contained in the *Peter-Weyl theorem*, and its corollaries: A compact Lie group G is completely reducible, its representations are fully reducible, every compact Lie group is a closed subgroup of a unitary group $U(n)$ for some n, and every continuous, unitary, irreducible representation of a compact Lie group is finite dimensional.

example 10.1
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example 10.2
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Let G be a group, and $g\mathcal{M} \longrightarrow \mathcal{M}$ a group action on the state space $\mathcal{M}$. The $[d \times d]$ matrices g acting on vectors in the d-dimensional state space $\mathcal{M}$ form a linear representation of the group G. If the action of every element g of a group G commutes with the flow

$$gv(x) = v(gx)\,, \qquad g\,f^{\tau}(x) = f^{\tau}(gx)\,, \tag{10.1}$$

G is a symmetry of the dynamics, and, as in (9.4), the dynamics is said to be invariant under G, or G-*equivariant*.

In order to explore the implications of equivariance for the solutions of dynamical equations, we start by examining the way a compact Lie group acts on state space $\mathcal{M}$.

Definition: Group orbit For any $x \in \mathcal{M}$, the *group orbit* $\mathcal{M}_x$ of x is the set of all points that x is mapped to under the groups actions,

$$\mathcal{M}_x = \mathrm{Orb}(x) = \{g\,x \mid g \in G\}\,. \tag{10.2}$$

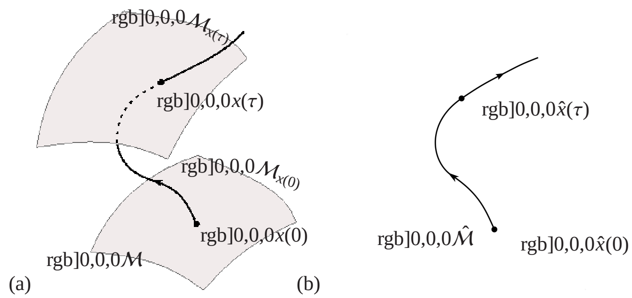

Figure 10.2: (a) The group orbit $\mathcal{M}_{x(0)}$ of state space point $x(0)$, and the group orbit $\mathcal{M}_{x(t)}$ reached by the trajectory $x(t)$ time t later. As any point on the manifold $\mathcal{M}_{x(t)}$ is physically equivalent to any other, the state space is foliated into the union of group orbits. (b) Symmetry reduction $\mathcal{M} \to \hat{\mathcal{M}}$ replaces each full state space group orbit $\mathcal{M}_x$ by a single point $\hat{x} \in \hat{\mathcal{M}}$.

The points in the fixed-point subspace $\mathcal{M}_G$ are those points whose group orbit consists of only the point itself ($\mathcal{M}_x = \{x\}$). See page 167 and figure 10.2 (a).

Definition: Fixed-point subspace $\mathcal{M}_H$ or a 'centralizer' of a subgroup $H \subset G$, G a symmetry of dynamics, is the set of all state space points that are *H-fixed*, *point-wise* invariant under action of the subgroup

$$\mathcal{M}_H = \text{Fix}\,(H) = \{x \in \mathcal{M} \mid h\,x = x \text{ for all } h \in H\}\,. \tag{10.3}$$

Points in the *fixed-point subspace* $\mathcal{M}_G$ are fixed points of the full group action. They are called *invariant points*,

$$\mathcal{M}_G = \text{Fix}\,(G) = \{x \in \mathcal{M} \mid g\,x = x \text{ for all } g \in G\}\,. \tag{10.4}$$

If a point is an invariant point of the symmetry group, by the definition of equivariance (10.10) the velocity at that point is also in $\mathcal{M}_G$, so the trajectory through that point will remain in $\mathcal{M}_G$. $\mathcal{M}_G$ is disjoint from the rest of the state space since no trajectory can ever enter or leave it.

As we saw in example 10.2, the time evolution itself is a noncompact 1-parameter Lie group. Thus the time evolution and the continuous symmetries can be considered on the same Lie group footing. For a given state space point x a symmetry group of N continuous transformations together with the evolution in time sweeps out, in general, a smooth $(N{+}1)$-dimensional manifold of equivalent solutions (if the solution has a nontrivial symmetry, the manifold may have a dimension less than $N+1$). For solutions p for which the group orbit of x_p is periodic in time T_p, the group orbit sweeps out a *compact* invariant manifold $\mathcal{M}_p$. The simplest example is the $N = 0$, no symmetry case, where the invariant manifold $\mathcal{M}_p$ is the 1-torus traced out by a periodic trajectory p. If $\mathcal{M}$ is a smooth C^∞ manifold, and G is compact and acts smoothly on $\mathcal{M}$, the reduced state space can be realized as a 'stratified manifold', meaning that each group orbit (a 'stratum') is represented by a point in the reduced state space, see sect. 10A.7. Generalizing the description of a non–wandering set of sect. 2.1.1, we say that for flows with

continuous symmetries the non–wandering set Ω of dynamics (2.3) is the closure of the set of compact invariant manifolds $\mathcal{M}_p$. Without symmetries, we visualize the non–wandering set as a set of points; in presence of a continuous symmetry, each such 'point' is a group orbit.

10.1.1 Lie groups for pedestrians

> [...] which is an expression of consecration of 'angular momentum'.
>
> — Mason A. Porter's student

Definition: A Lie group is a topological group G such that (i) G has the structure of a smooth differential manifold, and (ii) the composition map $G \times G \to G$: $(g, h) \to gh^{-1}$ is smooth, i.e., $\mathbb{C}^\infty$ differentiable.

Do not be mystified by this definition. Mathematicians also have to make a living. Historically, the theory of compact Lie groups that we will deploy here emerged as a generalization of the theory of SO(2) rotations, i.e., Fourier analysis. By a 'smooth differential manifold' one means objects like the circle of angles that parameterize continuous rotations in a plane, example 10.1, or the manifold swept by the three Euler angles that parameterize SO(3) rotations.

An element of a compact Lie group continuously connected to identity can be written as

$$g(\phi) = e^{\phi\cdot\mathbf{T}}\,, \qquad \phi\cdot\mathbf{T} = \sum \phi_a \mathbf{T}_a,\;\; a = 1, 2, \cdots, N\,, \tag{10.5}$$

where $\phi \cdot \mathbf{T}$ is a *Lie algebra* element, and ϕ_a are the parameters of the transformation. Repeated indices are summed throughout this chapter, and the dot product refers to a sum over Lie algebra generators. The Euclidian product of two vectors x, y will be indicated by x-transpose times y, i.e., $x^\top y = \sum_i^d x_i y_i$. Unitary transformations $\exp(\phi \cdot \mathbf{T})$ are generated by sequences of infinitesimal steps of form

$$g(\delta\phi) \simeq 1 + \delta\phi\cdot\mathbf{T}\,, \quad \delta\phi \in \mathbb{R}^N\,, \quad |\delta\phi| \ll 1\,, \tag{10.6}$$

where $\mathbf{T}_a$, the *generators* of infinitesimal transformations, are a set of linearly independent $[d\times d]$ anti-hermitian matrices, $(\mathbf{T}_a)^\dagger = -\mathbf{T}_a$, acting linearly on the d-dimensional state space $\mathcal{M}$. In order to streamline the exposition, we postpone discussion of combining continuous coordinate transformations with the discrete ones to sect. 10.2.1. . exercise 10.1

For continuous groups the Lie algebra, i.e., the set of N generators $\mathbf{T}_a$ of infinitesimal transformations, takes the role that the $|G|$ group elements play in the theory of discrete groups. The flow field at the state space point x induced by the action of the group is given by the set of N *tangent fields*

$$t_a(x)_i = (\mathbf{T}_a)_{ij} x_j \,, \tag{10.7}$$

which span the *tangent space*. Any representation of a compact Lie group G is fully reducible, and invariant tensors constructed by contractions of $\mathbf{T}_a$ are useful for identifying irreducible representations. The simplest such invariant is

$$\mathbf{T}^\top \cdot \mathbf{T} = \sum_\alpha C_2^{(\alpha)} \, \mathbb{1}^{(\alpha)} \,, \tag{10.8}$$

where $C_2^{(\alpha)}$ is the quadratic Casimir for irreducible representation labeled α, and $\mathbb{1}^{(\alpha)}$ is the identity on the α-irreducible subspace, 0 elsewhere. The dot product of two tangent fields is thus a sum weighted by Casimirs,

$$t(x)^\top \cdot t(x') = \sum_\alpha C_2^{(\alpha)} x_i \, \delta_{ij}^{(\alpha)} x'_j \,. \tag{10.9}$$

example 10.3
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example 10.6
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The really interesting Lie groups are the non-abelian semisimple ones -but- as we will discuss nothing much more complicated than the abelian special orthogonal group SO(2) of rotations in a plane, we shall not discuss the non-abelian case here.

fast track:
sect. 10.2, p. 195

10.1.2 Equivariance under infinitesimal transformations

A flow $\dot{x} = v(x)$ is G-equivariant (10.1), if symmetry transformations commute with time evolution

$$v(x) = g^{-1}\, v(g\, x) \,, \qquad \text{for all } g \in G \,. \tag{10.10}$$

For an infinitesimal transformation (10.6) the G-equivariance condition becomes

$$v(x) = (1 - \phi \cdot \mathbf{T})\, v(x + \phi \cdot \mathbf{T}x) + \cdots = v(x) - \phi \cdot \mathbf{T}v(x) + \frac{dv}{dx}\, \phi \cdot \mathbf{T}x + \cdots \,.$$

The $v(x)$ cancel, and ϕ_a are arbitrary. Denote the *group flow tangent field* at x by $t_a(x)_i = (\mathbf{T}_a)_{ij} x_j$. Thus the infinitesimal, Lie algebra G-equivariance condition is

$$t_a(v) - A(x)\, t_a(x) = 0 \,, \tag{10.11}$$

where $A = \partial v / \partial x$ is the stability matrix (4.3). If case you find such learned remarks helpful: the left-hand side of (10.11) is the *Lie derivative* of the dynamical flow field v along the direction of the infinitesimal group-rotation induced flow $t_a(x) = \mathbf{T}_a x$,

$$\mathcal{L}_{t_a} v = \left(\mathbf{T}_a - \frac{\partial}{\partial y} (\mathbf{T}_a x) \right) v(y) \bigg|_{y=x} \,. \tag{10.12}$$

exercise 10.9

The equivariance condition (10.11) states that the two flows, one induced by the dynamical vector field v, and the other by the group tangent field t, commute if their Lie derivatives (or the 'Lie brackets ' or 'Poisson brackets') vanish.

example 10.7
p. 206

example 10.8
p. 206

Checking equivariance as a Lie algebra condition (10.11) is easier than checking it for global, finite angle rotations (10.10).

10.2 Symmetries of solutions

Let $v(x)$ be the dynamical flow, and f^τ the trajectory or 'time-τ forward map' of an initial point x_0,

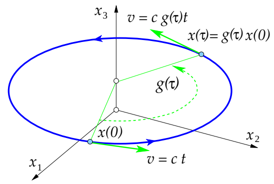
$$\frac{dx}{dt} = v(x) \,, \qquad x(\tau) = f^\tau(x_0) = x_0 + \int_0^\tau d\tau' \, v(x(\tau')) \,. \tag{10.13}$$

As discussed in sect. 9A.5, solutions $x(\tau)$ of an equivariant system can satisfy all of the system's symmetries, a subgroup of them, or have no symmetry at all . For a given solution $x(\tau)$, the subgroup that contains all symmetries that fix x

(that satisfy $gx = x$) is called the isotropy (or stabilizer) subgroup of x. A generic ergodic trajectory $x(\tau)$ has no symmetry beyond the identity, so its isotropy group is $\{e\}$, but recurrent solutions often do. At the other extreme is equilibrium, whose isotropy group is the full symmetry group G.

The simplest solutions are the *equilibria* or *steady* solutions (2.8).

Figure 10.3: (a) A *relative equilibrium orbit* starts out at some point $x(0)$, with the dynamical flow field $v(x) = c \cdot t(x)$ pointing along the group tangent space. For the SO(2) symmetry depicted here, the flow traces out the group orbit of $x(0)$ in time $T = 2\pi/c$. (b) An *equilibrium* lives either in the fixed $\mathrm{Fix}(G)$ subspace (x_3 axis in this sketch), or on a group orbit as the one depicted here, but with zero angular velocity c. In that case the circle (in general, N-torus) depicts a continuous family of fixed equilibria, related only by the group action.



Definition: Equilibrium $x_{EQ} = \mathcal{M}_{EQ}$ is a fixed, time-invariant solution,

$$\begin{aligned} v(x_{EQ}) &= 0\,, \\ x(x_{EQ},\tau) &= x_{EQ} + \int_0^\tau d\tau'\, v(x(\tau')) = x_{EQ}\,. \end{aligned} \tag{10.14}$$

An *equilibrium* with full symmetry,

$$g\, x_{EQ} = x_{EQ} \qquad \text{for all } g \in G\,,$$

lies, by definition, in $\mathrm{Fix}(G)$ subspace, for example the x_3 axis in figure 10.3 (a). The multiplicity of such solution is one. An equilibrium x_{EQ} with symmetry G_{EQ} smaller than the full group G belongs to a group orbit G/G_{EQ}. If G is finite there are $|G|/|G_{EQ}|$ equilibria in the group orbit, and if G is continuous then the group orbit of x is a continuous family of equilibria of dimension $\dim G - \dim G_{EQ}$. For example, if the angular velocity c in figure 10.3 (b) equals zero, the group orbit consists of a circle of (dynamically static) equivalent equilibria.

Definition: Relative equilibrium solution $x_{TW}(\tau) \in \mathcal{M}_{TW}$: the dynamical flow field points along the group tangent field, with constant 'angular' velocity c, and the trajectory stays on the group orbit, see figure 10.3 (a):

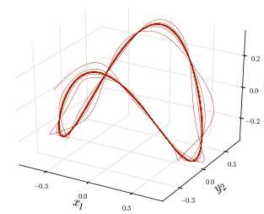
$$\begin{aligned} v(x) &= c \cdot t(x)\,, \qquad x \in \mathcal{M}_{TW} \\ x(\tau) &= g(-\tau\, c)\, x(0) \;=\; e^{-\tau\, c \cdot \mathbf{T}} x(0)\,. \end{aligned} \tag{10.15}$$

A *traveling wave* remark 10A.4

$$x(\tau) = g(-c\tau)\, x_{TW} = x_{TW} - c\,\tau\,, \quad c \in \mathbb{R}^d \tag{10.16}$$

is a special type of a relative equilibrium of equivariant evolution equations, where the action is given by translation (10.26), $g(y)\, x(0) = x(0) + y$. A *rotating wave* is another special case of relative equilibrium, with the action is given by angular

Figure 10.4: $\{x_1, y_1, x_2\}$ plot of the two-modes with initial point close to TW_1. In figure 10.1 this relative equilibrium is superimposed over the strange attractor. (N.B. Budanur)



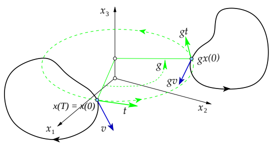


Figure 10.5: A periodic orbit starts out at $x(0)$ with the dynamical v and group tangent t flows pointing in different directions, and returns after time T_p to the initial point $x(0) = x(T_p)$. The group orbit of the temporal orbit of $x(0)$ sweeps out a $(1+N)$-dimensional torus, a continuous family of equivalent periodic orbits, two of which are sketched here. For SO(2) this is topologically a 2-torus.

rotation. By equivariance, all points on the group orbit are equivalent, the magnitude of the velocity c is same everywhere along the orbit, so a 'traveling wave' moves at a constant speed. For an $N > 1$ trajectory traces out a line within the group orbit. As the c_a components are generically not in rational ratios, the trajectory explores the N-dimensional group orbit (10.2) quasi-periodically. In other words, the group orbit $g(\tau)\,x(0)$ coincides with the dynamical orbit $x(\tau) \in \mathcal{M}_{TW}$ and is thus flow invariant.

Definition: Periodic orbit. Let x be a periodic point on the periodic orbit p of period T,

$$f^T(x) = x\,, \qquad x \in \mathcal{M}_p.$$

By equivariance, $g\,x$ is another periodic point, with the orbits of x and gx either identical or disjoint.

If gx lands on the same orbit, g is an element of periodic orbit's symmetry group G_p. If the symmetry group is the full group G, we are back to (10.15), i.e., the periodic orbit is the group orbit traced out by a relative equilibrium. The other option is that the isotropy group is discrete, the orbit segment $\{x, gx\}$ is pre-periodic (or eventually periodic), $x(0) = g_p x(T_p)$, where T_p is a fraction of the full period, $T_p = T/m$, and thus

$$\begin{aligned} x(0) &= g_p x(T_p)\,, \qquad x \in \mathcal{M}_p\,, \qquad g_p \in G_p \\ x(0) &= g_p^m x(m\,T_p) = x(T) = x(0)\,. \end{aligned} \tag{10.17}$$

If the periodic solutions are disjoint, as in figure 10.5, their multiplicity (if G is finite, see sect. 9.1), or the dimension of the manifold swept under the group

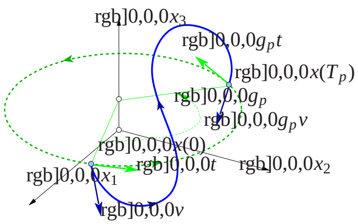


Figure 10.6: A relative periodic orbit starts out at $x(0)$ with the dynamical v and group tangent t flows pointing in different directions, and returns to the group orbit of $x(0)$ after time T_p at $x(T_p) = g_p x(0)$, a rotation of the initial point by g_p. For flows with continuous symmetry a generic relative periodic orbit (not pre-periodic to a periodic orbit) fills out ergodically what is topologically a torus, as in figure 10.5; if you are able to draw such a thing, kindly send us the figure. As illustrated by figure 10.8 (a) this might be a project for Lucas Films.

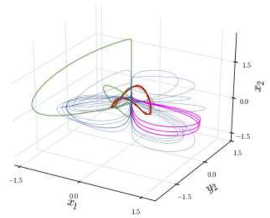


Figure 10.7: (Figure 10.1 continued) A group portrait of the two-modes state space dynamics. Plotted are relative equilibrium TW_1 and its unstable manifold (brown), equilibrium EQ_0 and one trajectory from its unstable manifold (green), three repetitions of relative periodic orbit $\overline{1}$ (magenta) and a generic orbit (blue). Study example 10.5. (N.B. Budanur)

action (if G is continuous) can be determined by applications of $g \in G$. They form a family of conjugate solutions (9A.19),

$$\mathcal{M}_{g\,p} = g\,\mathcal{M}_p\,g^{-1}\,. \qquad (10.18)$$

Definition: Relative periodic orbit p is an orbit $\mathcal{M}_p$ in state space $\mathcal{M}$ which exactly recurs

$$x_p(0) = g_p x_p(T_p)\,, \qquad x_p(\tau) \in \mathcal{M}_p\,, \qquad (10.19)$$

at a fixed *relative period* T_p, but shifted by a fixed group action g_p which brings the endpoint $x_p(T_p)$ back into the initial point $x_p(0)$, see figure 10.6. The group action g_p parameters $\phi = (\phi_1, \phi_2, \cdots \phi_N)$ are referred to as 'phases', or 'shifts'. In contrast to the pre-periodic (10.17), here the phases are irrational, and the trajectory sweeps out ergodically the group orbit without ever closing into a periodic orbit . For dynamical systems with only continuous (no discrete) symmetries, the parameters $\{t, \phi_1, \cdots, \phi_N\}$ are real numbers, ratios π/ϕ_j are almost never rational, likelihood of finding a periodic orbit for such system is zero, and such relative periodic orbits are almost never eventually periodic.

example 10.11
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Relative periodic orbits are to periodic solutions what relative equilibria (traveling waves) are to equilibria (steady solutions). Equilibria satisfy $f^\tau(x) - x = 0$

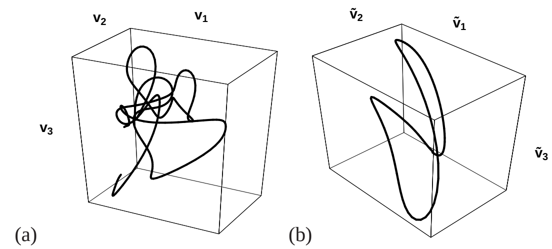

Figure 10.8: A relative periodic orbit of Kuramoto-Sivashinsky flow projected on (a) the stationary state space coordinate frame $\{v_1, v_2, v_3\}$, traced for four periods T_p; (b) the co-moving $\{\tilde v_1, \tilde v_2, \tilde v_3\}$ coordinate frame, moving with the mean angular velocity $c_p = \phi_p/T_p$. (from ref. [10.1])

and relative equilibria satisfy $f^\tau(x) - g(\tau)\,x = 0$ for any τ. In a co-moving frame, i.e., frame moving along the group orbit with velocity $v(x) = c \cdot t(x)$, the relative equilibrium appears as an equilibrium. Similarly, a relative periodic orbit is periodic in its mean velocity $c_p = \phi_p/T_p$ co-moving frame (see figure 10.8), but in the stationary frame its trajectory is quasiperiodic. A co-moving frame is helpful in visualizing a single 'relative' orbit, but useless for viewing collections of orbits, as each one drifts with its own angular velocity. Visualization of all relative periodic orbits as periodic orbits we attain only by global symmetry reductions, to be undertaken in sect. 10A.7.

10.2.1 Discrete and continuous symmetries together

We expect to see relative periodic orbits because a trajectory that starts on and returns to a given torus of a symmetry equivalent solutions is unlikely to intersect it at the initial point, unless forced to do so by a discrete symmetry. This we will make explicit in sect. 10A.7, where relative periodic orbits will be viewed as periodic orbits of the reduced dynamics.

If, in addition to a continuous symmetry, one has a discrete symmetry which is not its subgroup, one does expect equilibria and periodic orbits. However, a relative periodic orbit can be pre-periodic if it is equivariant under a discrete symmetry, as in (10.17): If $g^m = 1$ is of finite order m, then the corresponding orbit is periodic with period mT_p. If g is not of a finite order, a relative periodic orbit is periodic only after a shift by g_p, as in (10.19). Morally, as it will be shown in chapter 21, such orbit is the true 'prime' orbit, i.e., the shortest segment that under action of G tiles the entire invariant submanifold $\mathcal{M}_p$.

Definition: Relative orbit $\mathcal{M}_{Gx}$ in state space $\mathcal{M}$ is the time evolved *group orbit* $\mathcal{M}_x$ of a state space point x, the set of all points that can be reached from x by all symmetry group actions and evolution of each in time.

$$\mathcal{M}_{x(t)} = \{gx(t) \mid t \in \mathbb{R}, g \in G\}\,. \qquad (10.20)$$

In presence of symmetry, an equilibrium is the set of all equilibria related by symmetries, an relative periodic orbit is the hyper-surface traced by a trajectory in time T and all group actions, etc..

chapter 21

10.3 Stability

A spatial derivative of the equivariance condition (10.1) yields the matrix equivariance condition satisfied by the stability matrix (stated here both for the finite group actions, and for the infinitesimal, Lie algebra generators):

$$gA(x)g^{-1} = A(gx)\,, \qquad [\mathbf{T}_a, A] = \frac{\partial A}{\partial x} t_a(x)\,. \tag{10.21}$$

For a flow within the fixed Fix (G) subspace, $t(x)$ vanishes, and the symmetry imposes strong conditions on the perturbations out of the Fix (G) subspace. As in this subspace stability matrix A commutes with the Lie algebra generators $\mathbf{T}$, the spectrum of its eigenvalues and eigenvectors is decomposed into irreducible representations of the symmetry group. This we have already observed for the EQ_0 of the Lorenz flow in example 9A.13.

A infinitesimal symmetry group transformation maps the initial and the end point of a finite trajectory into a nearby, slightly rotated equivalent points, so we expect the perturbations along to group orbit to be marginal, with unit eigenvalues. The argument is akin to (4.9), the proof of marginality of perturbations along a periodic orbit. Consider two nearby initial points separated by an N-dimensional infinitesimal group transformation (10.6): $\delta x_0 = g(\delta\phi)x_0 - x_0 = \delta\phi \cdot \mathbf{T} x_0 = \delta\phi \cdot t(x_0)$. By the commutativity of the group with the flow, $g(\delta\phi) f^\tau(x_0) = f^\tau(g(\delta\phi)x_0)$. Expanding both sides, keeping the leading term in $\delta\phi$, and using the definition of the Jacobian matrix (4.5), we observe that $J^\tau(x_0)$ transports the N-dimensional group tangent space at $x(0)$ to the rotated tangent space at $x(\tau)$ at time τ:

$$t_a(\tau) = J^\tau(x_0)\, t_a(0)\,, \qquad t_a(\tau) = \mathbf{T}_a\, x(\tau)\,. \tag{10.22}$$

For a relative periodic orbit, $g_p x(T_p) = x(0)$, at any point along cycle p the group tangent vector $t_a(\tau)$ is an eigenvector of the Jacobian matrix with an eigenvalue of unit magnitude,

$$J_p\, t_a(x) = t_a(x)\,, \qquad J_p(x) = g_p J^{T_p}(x)\,, \qquad x \in \mathcal{M}_p\,. \tag{10.23}$$

For a relative equilibrium flow and group tangent vectors coincide, $v = c \cdot t(x)$. Dotting by the velocity c (i.e., summing over $c_a t_a$) the equivariance condition (10.11), $t_a(v) - A(x)\, t_a(x) = 0$, we get

$$(c \cdot \mathbf{T} - A)v = 0\,. \tag{10.24}$$

In other words, in the co-rotating frame the eigenvalues corresponding to group tangent are marginal, and the velocity v is the corresponding right eigenvector.

Two successive points along the cycle separated by $\delta x_0 = \delta\phi \cdot t(\tau)$ have the same separation after a completed period $\delta x(T_p) = g_p \delta x_0$, hence eigenvalue of magnitude 1. In presence of an N-dimensional Lie symmetry group, N eigenvalues equal unity.

Résumé

The message: If a dynamical systems has a symmetry, use it!

We conclude with a few general observations: Higher dimensional dynamics requires study of compact invariant sets of higher dimension than 0-dimensional equilibria and 1-dimensional periodic orbits studied so far. In sect. 2.1.1 we made an attempt to classify 'all possible motions:' (1) equilibria, (2) periodic orbits, (3) everything else. Now one can discern in the fog of dynamics an outline of a more serious classification - long time dynamics takes place on the closure of a set of all invariant compact sets preserved by the dynamics, and those are: (1) 0-dimensional equilibria $\mathcal{M}_{EQ}$, (2) 1-dimensional periodic orbits $\mathcal{M}_p$, (3) global symmetry induced N-dimensional relative equilibria $\mathcal{M}_{TW}$, (4) $(N{+}1)$-dimensional relative periodic orbits $\mathcal{M}_p$, (5) terra incognita. We have some inklings of the 'terra incognita:' for example, in symplectic symmetry settings one finds KAM-tori, and in general dynamical settings we encounter *partially hyperbolic invariant M-tori*, isolated tori that are consequences of dynamics, not of a global symmetry. They are harder to compute than anything we have attempted so far, as they cannot be represented by a single relative periodic orbit, but require a numerical computation of full M-dimensional compact invariant sets and their infinite-dimensional linearized Jacobian matrices, marginal in M dimensions, and hyperbolic in the rest. We expect partially hyperbolic invariant tori to play important role in high-dimensional dynamics. In this chapter we have focused on the simplest example of such compact invariant sets, where invariant tori are a robust consequence of a global continuous symmetry of the dynamics. The direct product structure of a global symmetry that commutes with the flow enables us to reduce the dynamics to a desymmetrized $(d{-}1{-}N)$-dimensional reduced state space $\mathcal{M}/G$.

Relative equilibria and relative periodic orbits are the hallmark of systems with continuous symmetry. Amusingly, in this extension of 'periodic orbit' theory from unstable 1-dimensional closed periodic orbits to unstable $(N{+}1)$-dimensional compact manifolds $\mathcal{M}_p$ invariant under continuous symmetries, there are either no or proportionally few periodic orbits. In presence of a continuous symmetry, likelihood of finding a periodic orbit is *zero*. Relative periodic orbits are almost never eventually periodic, i.e., they almost never lie on periodic trajectories in the full state space, so looking for periodic orbits in systems with continuous symmetries is a fool's errand.

However, dynamical systems are often equivariant under a combination of continuous symmetries and discrete coordinate transformations of chapter 9, for example the orthogonal group O(n). In presence of discrete symmetries relative periodic orbits within discrete symmetry-invariant subspaces are eventually periodic. Atypical as they are (no generic chaotic orbit can ever enter these discrete invariant subspaces) they will be important for periodic orbit theory, as there the shortest orbits dominate, and they tend to be the most symmetric solutions. chapter 21

Commentary

Remark 10.1 Ideal is not real. (continued from remark 9.1): The literature on symmetries in dynamical systems is immense, most of it deliriously unintelligible. Would it kill them to say 'symmetry of orbit p' instead of carrying on about 'isotropies, quotients, factors, normalizers, centralizers and stabilizers?' [10.9, 10.10, 10.8, 9A.4] Group action being 'free, faithful, proper, regular?' Symmetry-reduced state space being 'orbitfold?' For the dynamical systems applications at hand we need only the basic Lie group facts, on the level of any standard group theory textbook [10.2]. We found Roger Penrose [10.3] introduction to the subject both enjoyable and understandable. Chapter 2. of ref. [10.4] offers a pedagogical introduction to Lie groups of transformations, and Nakahara [10.5] to Lie derivatives and brackets. The presentation given here is in part based on Siminos thesis [10.6] and ref. [10.7]. The reader is referred to the monographs of Golubitsky and Stewart [10.8], Hoyle [10.9], Olver [10.11], Bredon [10.12], and Krupa [10.13] for more depth and rigor than would be wise to wade into here.

Remark 10.2 Two-modes equations. Dangelmayr [10.14], Armbruster, Guckenheimer and Holmes [10.15], Jones and Proctor [10.16], and Porter and Knobloch [10.17] (see Golubitsky *et al.* [10.18], Sect. XX.1) have investigated bifurcations in 1:2 resonance ODE normal form models to third order in the amplitudes. Our toy model (10.32) is a particular case of Danglmayr [10.14] and Porter and Knobloch [10.17] 2-Fourier mode SO(2)-equivariant ODEs that we here refer to as the 'two-modes' system:

10.4 Examples

10.4.1 Special orthogonal group SO(2)

Example 10.1 ***Special orthogonal group*** $SO(2)$ *(or S^1) is a group of length-preserving rotations in a plane. 'Special' refers to requirement that $\det g = 1$, in contradistinction to the orthogonal group $O(n)$ which allows for length-preserving inversions through the origin, with $\det g = -1$. A group element can be parameterized by angle ϕ, with the group multiplication law $g(\phi')g(\phi) = g(\phi' + \phi)$, and its action on smooth periodic functions $u(\phi + 2\pi) = u(\phi)$ generated by*

$$g(\phi') = e^{\phi' \mathbf{T}}, \quad \mathbf{T} = \frac{d}{d\phi}. \tag{10.25}$$

Expand the exponential, apply it to a differentiable function $u(\phi)$, and you will recognize a Taylor series. So $g(\phi')$ shifts the coordinate by ϕ', $g(\phi')\,u(\phi) = u(\phi' + \phi)$. click to return: p. ??

Example 10.2 ***Translation group:*** *Differential operator $\mathbf{T}$ in (10.25) is reminiscent of the generator of spatial translations. The 'constant velocity field' $v(x) = v = c \cdot \mathbf{T}$' acts on x_j by replacing it by the velocity vector c_j. It is easy to verify by Taylor expanding a function $u(x)$ that the time evolution is nothing but a coordinate translation by (time× velocity):*

$$e^{-\tau c \cdot \mathbf{T}} u(x) = e^{-\tau c \cdot \frac{d}{dx}} u(x) = u(x - \tau\, c). \tag{10.26}$$

As x is a point in the Euclidean $\mathbb{R}^d$ space, the group is not compact. A sequence of time steps in time evolution always forms an abelian Lie group, albeit *never as trivial as this free ballistic motion.*

If the group actions consist of N rotations which commute, for example act on an N-dimensional cell with periodic boundary conditions, the group is an abelian group that acts on a torus T^N. click to return: p. ??

Example 10.3 $SO(2)$ ***irreducible representations:*** *Consider the action (10.25) of the one-parameter rotation group $SO(2)$ on a smooth periodic function $u(\phi + 2\pi) = u(\phi)$ defined on a $1D$-dimensional configuration space domain $x \in [0, 2\pi)$. The state space matrix representation of the $SO(2)$ counter-clockwise (right-handed) rotation $g(\phi')u(\phi) = u(\phi + \phi')$ by angle ϕ' is block-diagonal, acting on the kth Fourier coefficient pair (a_k, b_k) in the Fourier series,*

$$u(\phi) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\phi + b_k \sin k\phi)\,. \tag{10.27}$$

by multiplication by

$$g^{(k)}(\phi') = \begin{pmatrix} \cos k\phi' & -\sin k\phi' \\ \sin k\phi' & \cos k\phi' \end{pmatrix}, \qquad \mathbf{T}^{(k)} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}. \tag{10.28}$$

with the Lie group generator

$$\mathbf{T}^{(k)} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}. \tag{10.29}$$

The $SO(2)$ group tangent (10.7) to state space point $u(\phi)$ on the kth invariant subspace is

$$t^{(k)}(u) = k \begin{pmatrix} -b_k \\ a_k \end{pmatrix} . \tag{10.30}$$

The L^2 norm of $t(u)$ is weighted by the $SO(2)$ quadratic Casimir (10.8), $C_2^{(k)} = k^2$,

$$\langle t(u)^\top | t(u) \rangle = \oint \frac{d\phi}{2\pi} u(\phi)^\top \mathbf{T}^\top \mathbf{T} u(2\pi - \phi) = \sum_{k=1}^{\infty} k^2 \left(a_k^2 + b_k^2 \right) , \tag{10.31}$$

and converges only for sufficiently smooth $u(\phi)$. What does that mean? We saw in (10.26) that $\mathbf{T}$ generates translations, and by (10.29) the velocity of the kth Fourier mode is k times higher than for the $k = 1$ component. If $|u^{(k)}|$ does not fall off faster the $1/k$, the action of $SO(2)$ is overwhelmed by the high Fourier modes. click to return: p. ??

Example 10.4 ***Invariance under fractional rotations:*** *Consider a velocity field $v(x)$ equivariant (10.1) under discrete cyclic subgroup $C_m = \{e, C^{1/m}, C^{2/m}, \cdots, C^{(m-1)/m}\}$ of $SO(2)$ rotations by $2\pi/m$),* exercise 10.2

$$C^{1/m} v(x) = v(C^{1/m} x) , \qquad (C^{1/m})^m = e .$$

The field $v(x)$ on the fundamental domain $2\pi/m$ is now a tile whose m copies tile the entire domain. It is periodic on the fundamental domain, and thus has Fourier expansion with Fourier modes $\cos(2\pi m j x)$, $\sin(2\pi m j x)$. The Fourier expansion on the full interval $(0, 2\pi)$ cannot have any other modes, as they would violate the C_m symmetry. This means that $SO(2)$ always has an infinity of discrete subgroups $C_2, C_3, \cdots, C_m, \cdots$; for each the non-vanishing coefficients are only for Fourier modes whose wave numbers are multiples of m.

10.4.2 Two-modes SO(2)-equivariant flow

Example 10.5 ***Two-modes flow:*** *Consider the pair of $U(1)$-equivariant complex ODEs*

$$\begin{aligned} \dot{z}_1 &= (\mu_1 - \mathrm{i}\, e_1)\, z_1 + a_1\, z_1 |z_1|^2 + b_1\, z_1 |z_2|^2 + c_1\, \bar{z}_1\, z_2 \\ \dot{z}_2 &= (\mu_2 - \mathrm{i}\, e_2)\, z_2 + a_2\, z_2 |z_1|^2 + b_2\, z_2 |z_2|^2 + c_2\, z_1^2 , \end{aligned} \tag{10.32}$$

with z_1, z_2 complex, and all parameters real valued.

We shall refer to this toy model as the two-modes system. It belongs to the family of simplest ODE systems that we know that (a) have a continuous $U(1)$ / $SO(2)$, but no discrete symmetry (if at least one of $e_j \neq$). (b) models 'weather', in the same sense that Lorenz equation models 'weather', (c) exhibits chaotic dynamics, (d) can be easily visualized, in the dimensionally lowest possible setting required for chaotic dynamics, with the full state space of dimension $d = 4$, and the $SO(2)$-reduced dynamics taking place in 3 dimensions, and (e) for which the method of slices reduces the symmetry by a single global slice hyperplane.

For parameters far from the bifurcation values, this is a merely a toy model with no physical interpretation, just like the iconic Lorenz flow (2.18): we use it to illustrate the effects of continuous symmetry on chaotic dynamics. We have not found a second order truncation of such models that exhibits interesting dynamics, hence the third order in the amplitudes, and the unreasonably high number of parameters. After some experimentation we fix or set to zero various parameters, and in the numerical examples that follow, we settle for parameters set to

$$\begin{aligned} \mu_1 &= -2.8 ,\ \mu_2 = 1 ,\ e_1 = 0 ,\ e_2 = 1 , \\ a_1 &= -1 ,\ a_2 = -2.66 ,\ b_1 = 0 ,\ b_2 = 0 ,\ c_1 = -7.75 ,\ c_2 = 1 , \end{aligned} \tag{10.33}$$

unless explicitly stated otherwise. For these parameter values the system exhibits chaotic behavior. Experiment. If you find a more interesting behavior for some other parameter values, please let us know. The simplified system of equations can now be written as a 3-parameter $\{\mu_1, c_1, a_2\}$ two-modes system,

$$\begin{aligned} \dot{z}_1 &= \mu_1\, z_1 - z_1 |z_1|^2 + c_1\, \bar{z}_1\, z_2 \\ \dot{z}_2 &= (1 - \mathrm{i})\, z_2 + a_2\, z_2 |z_1|^2 + z_1^2 . \end{aligned} \tag{10.34}$$

In order to numerically integrate and visualize the flow, we recast the equations in real variables by substitution $z_1 = x_1 + i\, y_1$, $z_2 = x_2 + i\, y_2$. The two-modes system (10.32) is now a set of four coupled ODEs

$$\begin{aligned} \dot{x}_1 &= (\mu_1 - r^2)\, x_1 + c_1\, (x_1 x_2 + y_1 y_2) , \qquad r^2 = x_1^2 + y_1^2 \\ \dot{y}_1 &= (\mu_1 - r^2)\, y_1 + c_1\, (x_1 y_2 - x_2 y_1) \\ \dot{x}_2 &= x_2 + y_2 + x_1^2 - y_1^2 + a_2 x_2 r^2 \\ \dot{y}_2 &= -x_2 + y_2 + 2\, x_1 y_1 + a_2 y_2 r^2 . \end{aligned} \tag{10.35}$$

Try integrating (10.35) with random initial conditions, for long times, times much beyond which the initial transients have died out. What is wrong with this picture? It is a mess. As we shall show here, the attractor is built up by a nice 'stretch & fold' action, but that is totally hidden from the view by the continuous symmetry induced drifts. In the rest of this and next chapter's examples we shall investigate various ways of 'quotienting' this $SO(2)$ symmetry, and reducing the dynamics to a 3-dimensional symmetry-reduced state space. We shall not rest until we attain the simplicity and bliss of a 1-dimensional return map. *(N.B. Budanur and P. Cvitanović)* click to return: p. ??

Example 10.6 ***SO(2) rotations for two-modes equations:*** *Substituting the Lie algebra generator*

$$\mathbf{T} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \tag{10.36}$$

acting on a 4–dimensional state space (10.35) into (10.5) yields a finite angle $SO(2)$ rotation:

$$g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos 2\phi & -\sin 2\phi \\ 0 & 0 & \sin 2\phi & \cos 2\phi \end{pmatrix} . \tag{10.37}$$

Itinerary	$(x_{p,1}, y_{p,1}, x_{p,2}, y_{p,2})$	Period
1	(0.4525719, 0.0, 0.0509257, 0.0335428)	3.6415120
01	(0.4517771, 0.0, 0.0202026, 0.0405222)	7.3459412
0111	(0.4514665, 0.0, 0.0108291, 0.0424373)	14.6795175
01101	(0.4503967, 0.0, -0.0170958, 0.0476009)	18.3874094

Table 10.1: Several short relative periodic orbits of the two-modes system: itineraries, a periodic point in a Poincaré section for each orbit, the period.

From (10.28) we see that the action of $SO(2)$ on the complex Lorenz equations state space decomposes into $m = 0$ G-invariant $m = 1$ and $m = 2$ subspaces.

The generator $\mathbf{T}$ is indeed anti-hermitian, $\mathbf{T}^\dagger = -\mathbf{T}$, and the group is compact, its elements parametrized by ϕ mod 2π. Locally, at $x \in \mathcal{M}$, the infinitesimal action of the group is given by the group tangent field $t(x) = \mathbf{T}x = (-x_2, x_1, -y_2, y_1)$. In other words, the flow induced by the group action is normal to the radial direction in the (x_1, x_2) and (y_1, y_2) planes. click to return: p. ??

Example 10.7 ***Equivariance of the two-modes flow:*** *That two-modes (10.35) is equivariant under $SO(2)$ rotations (10.37) can be checked by substituting the Lie algebra generator (10.36) and the stability matrix (4.3) for two-modes (10.35), $A =$*

$$\begin{pmatrix} \mu_1 - 3x_1^2 + c_1 x_2 - y_1^2 & c_1 y_2 - 2x_1 y_1 & c_1 x_1 & c_1 y_1 \\ c_1 y_2 - 2x_1 y_1 & \mu_1 - x_1^2 - c_1 x_2 - 3y_1^2 & -c_1 y_1 & c_1 x_1 \\ 2x_1 + 2a_2 x_1 x_2 & 2a_2 x_2 y_1 - 2y_1 & 1 + a_2(x_1^2 + y_1^2) & 1 \\ 2y_1 + 2a_2 x_1 y_2 & 2x_1 + 2a_2 y_1 y_2 & -1 & 1 + a_2(x_1^2 + y_1^2) \end{pmatrix} . \tag{10.38}$$

into the equivariance condition (10.11). Considering that $t(v)$ depends on the full set of equations (10.35), and $A(x)$ is only its linearization, this is not an entirely trivial statement.
(N.B. Budanur) click to return: p. ??

Example 10.8 ***How contracting is the two-modes flow?*** *For the parameter values (10.34) the flow is strongly volume contracting (4.29),*

$$\partial_i v_i = \sum_{i=1}^{4} \lambda_i(x,t) = \mathrm{tr}\, A(x) = 2\,[1 + \mu_1 - (2 - a_2)\, r^2] = -3.6 - 9.32\, r^2 . \tag{10.39}$$

Note that this quantity depends on the full state space coordinates only through the $SO(2)$-invariant r^2, so the volume contraction rate is symmetry-invariant characterization of the flow, as is should be. The shortest relative periodic orbit $\overline{1}$ has period $T_1 = 3.64\ldots$ and typical $r^2 \approx 1$, (see table 10.1), so in one period a neighborhood of the relative periodic orbit is contracted by factor $\approx \exp(T_1\, \mathrm{tr}\, A(x)) \approx 3.7 \times 10^{-21}$. This is an insanely contracting flow; if we start with mm^4 cube around a periodic point, this volume (remember, two directions are marginal) shrinks to $\approx mm \times mm \times 10^{-11} mm \times 10^{-11} mm \approx mm \times mm \times$ fermi $\times$ fermi. Diameter of a proton is a couple of fermis. This strange attractor is thin*!.* click to return: p. ??

10.4.3 Symmetries of iconic fluid flows

Example 10.9 ***Discrete symmetries of the plane Couette flow.*** *The plane Couette flow is a fluid flow bounded by two countermoving planes, in a cell periodic in streamwise and spanwise directions. The Navier-Stokes equations for the plane Couette flow have two discrete symmetries: reflection through the (streamwise , wall-normal) plane, and rotation by π in the (streamwise , wall-normal) plane. That is why the system has equilibrium and periodic orbit solutions, (as opposed to relative equilibrium and relative periodic orbit solutions discussed in chapter 10). They belong to discrete symmetry subspaces.*

Example 10.10 ***Continuous symmetries of the plane Couette flow.*** *Every solution of Navier-Stokes equations belongs, by the $SO(2) \times O(2)$ symmetry, to a 2-torus T^2 of equivalent solutions. Furthermore these tori are interrelated by a discrete D_2 group of spanwise and streamwise flips of the flow cell. (continued in example 10.11)* click to return: p. ??

Example 10.11 ***Relative orbits in the plane Couette flow.*** *Translational symmetry allows for relative equilibria (traveling waves), characterized by a fixed profile Eulerian velocity $u_{TW}(x)$ moving with constant velocity c, i.e.*

$$u(x,\tau) = u_{TW}(x - c\tau) . \tag{10.40}$$

As the plane Couette flow is bounded by two counter-moving planes, it is easy to see where the relative equilibrium (traveling wave) solutions come from. A relative equilibrium solution hugs close to one of the walls and drifts with it with constant velocity, slower than the wall, while maintaining its shape. A relative periodic solution is a solution that recurs at time T_p with exactly the same disposition of the Eulerian velocity fields over the entire cell, but shifted by a 2-dimensional (streamwise,spanwise) translation g_p. By discrete symmetries these solutions come in counter-traveling pairs $u_q(x - c\tau)$, $-u_q(-x + c\tau)$: for example, for each one drifting along with the upper wall, there is a counter-moving one drifting along with the lower wall. Discrete symmetries also imply existence of strictly stationary solutions, or 'standing waves'. For example, a solution with velocity fields antisymmetric under reflection through the midplane has equal flow velocities in opposite directions, and is thus an equilibrium stationary in time. click to return: p. ??

Exercises

10.1. **SO(2) rotations in a plane:** Show by exponentiation (10.5) that the SO(2) Lie algebra element **T** generates rotation g in a plane,

$$g(\theta) = e^{\mathbf{T}\theta} = \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \quad (10.41)$$

10.2. **Invariance under fractional rotations.** Argue that if the isotropy group of the velocity field $v(x)$ is the discrete subgroup C_m of SO(2) rotations about an axis (let's say the 'z-axis'),

$$C^{1/m}v(x) = v(C^{1/m}x) = v(x), \qquad (C^{1/m})^m = e,$$

the only non-zero components of Fourier-transformed equations of motion are a_{jm} for $j = 1, 2, \cdots$. Argue that the Fourier representation is then the quotient map of the dynamics, $\mathcal{M}/C_m$. (Hint: this sounds much fancier than what is - think first of how it applies to the Lorenz system and the 3-disk pinball.)

10.3. **U(1) equivariance of two-modes system for finite angles:** Show that the vector field in two-modes (10.32) is equivariant under (10.5), the unitary group U(1) acting on $\mathbb{R}^4 \cong \mathbb{C}^2$ by

$$g(\theta)(z_1, z_2) = (e^{i\theta}z_1, e^{i2\theta}z_2), \quad \theta \in [0, 2\pi). \quad (10.42)$$

10.4. **SO(2) equivariance of two-modes system for finite angles:** Show that two-modes (10.35) are equivariant under rotation for finite angles.

10.5. **Stability matrix of two-modes system:** Compute the stability matrix (10.38) for two-modes system (10.35).

10.6. **SO(2) equivariance of two-modes system for infinitesimal angles.** Show that two-modes equations are equivariant under infinitesimal SO(2) rotations. Compute the volume contraction rate (4.29), verify (10.39). Period of the shortest relative periodic orbit of this system is $T_1 = 3.6415120$. By how much a small volume centered on the relative periodic orbit contracts in that time?

10.7. **Integrate the two-modes system:** Integrate (10.35) and plot a long trajectory of two-modes in the 4d state space, (x_1, y_1, y_2) projection, as in Figure 10.1.

10.8. **Classify possible symmetries of solutions for your research problem.** Classify types of solutions you expect in your research problem by their symmetries. Literature examples: plane Couette flow [10A.45], pipe flow (sect. 2.2 and appendix A of ref. [**?**]), Kuramoto-Sivashinsky (see symmetry discussions of ref. [**?**, **?**], and probably many better papers out there that we are less familiar with), Euclidean symmetries of doubly-periodic 2D models of cardiac tissue, 2D Kolmogorov flow [**?**], two-modes flow (Dangelmayr [10.14]; Armbruster, Guckenheimer and Holmes [10.15]; Jones and Proctor [10.16]; Porter and Knobloch [10.17]; Golubitsky *et al.* [10.18], Sect. XX.1), 2D ABC flow [**?**]; perturbed Coulomb systems [**?**]; systems with discrete symmetries [9A.20, 9A.16]; example 9.5 reflection symmetric 1d map; example 7.5 Hamiltonian Hénon map; Hamiltonian Lozi map, etc..

10.9. **Discover the equivariance of a given flow:**

Suppose you were given two-modes system, but nobody told you that the equations are SO(2)-equivariant. More generally, you might encounter a flow without realizing that it has a continuous symmetry - how would you discover it?

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Chapter 10A

Slice & dice

Physicists like symmetry more than Nature
— Rich Kerswell

If the symmetry is continuous, the notion of fundamental domain is not applicable. Instead, the dynamical system should be reduced to a lower-dimensional, desymmetrized system, with 'ignorable' coordinates separated out (but not forgotten).

We shall describe here two ways of reducing a continuous symmetry. In the 'method of slices' of sect. 10A.7 we slice the state space in such a way that an entire class of symmetry-equivalent points is represented by a single point. In the Hilbert polynomial basis approach of sect. 10A.10 we replace the equivariant dynamics by the dynamics rewritten in terms of invariant coordinates. In either approach we retain the option of computing in the original coordinates, and then, when done, projecting the solution onto the symmetry reduced state space, or 'post-processing'.

In the method of slices symmetry reduction is achieved by cutting the group orbits with a finite set of slice hyperplanes, one for each continuous group parameter, with each group orbit of symmetry-equivalent points represented by a single point, its intersection with the slice. The procedure is akin to (but distinct from) cutting across continuous-time parametrized trajectories by means of Poincaré sections. As is the case for Poincaré sections, choosing a 'good' slice is a dark art. We describe two strategies: (i) Foliation of state space by group orbits is a purely group-theoretic phenomenon that has nothing to do with dynamics, so we construct slices based on a decomposition of state space into irreducible linear representations of the symmetry group G. (ii) Nonlinear dynamics strongly couples such linear symmetry eigenmodes, so locally optimal slices should be constructed from physically important recurrent states, or 'templates'. Our guiding principle is to chose a slice such that the distance between a 'template' state $\hat{x}'$ and nearby group orbits is *minimized*, i.e., we identify the point $\hat{x}$ on the group orbit (10.2) of a nearby state x which is the closest match to the template point $\hat{x}'$.

We start our discussion by explaining the freedom of redefining dynamics in a moving frame.

10A.5 Only dead fish go with the flow: moving frames

The idea : As the symmetries commute with dynamics, we can evolve a solution

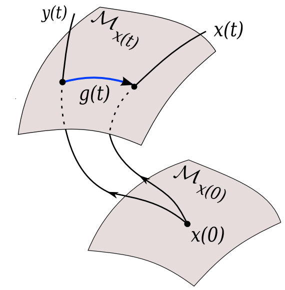

Figure 10A.9: The freedom to pick a moving frame: A point x on the full state space trajectory $x(t)$ is equivalent up to a group rotation $g(t)$ to the point $\hat{x}$ on the curve $\hat{x}(t)$ if the two points belong to the same group orbit $\mathcal{M}_x$, see (10.2).

$x(\tau)$ for as long as we like, and then rotate it to any equivalent point (see figure 10A.9) on its group orbit. We can map each point along any solution $x(\tau)$ to the unique representative $\hat{x}(\tau)$ of the associated group orbit equivalence class, by a coordinate transformation

$$x(\tau) = g(\tau)\,\hat{x}(\tau)\,. \tag{10A.43}$$

Equivariance guarantees that the two states are physically equivalent.

Definition: Moving frame. For a given $x \in \mathcal{M}$ and a given space of 'representative shapes' $\hat{\mathcal{M}}$ there exists a unique group element $g = g(x,t)$ that at instant t rotates x into $gx = \hat{x} \in \hat{\mathcal{M}}$. The map that associates to a state space point x a Lie group action $g(x,t)$ is called a *moving frame*. exercise B.1 exercise 10A.1

Using decomposition (10A.43) one can always write the full state space trajectory as $x(\tau) = g(\tau)\,\hat{x}(\tau)$, where the $(d-N)$-dimensional reduced state space trajectory $\hat{x}(\tau)$ is to be fixed by some condition, and $g(\tau)$ is then the corresponding curve on the N-dimensional group manifold of the group action that rotates $\hat{x}$ into x at time τ. The time derivative is then $\dot{x} = v(g\hat{x}) = \dot{g}\hat{x} + g\hat{v}$, with the reduced state space velocity field given by $\hat{v} = d\hat{x}/dt$. Rewriting this as $\hat{v} = g^{-1}v(g\,\hat{x}) - g^{-1}\dot{g}\,\hat{x}$ and using the equivariance condition (10.10) leads to

$$\hat{v} = v - g^{-1}\dot{g}\,\hat{x}\,.$$

The Lie group element (10.5) and its time derivative describe the group tangent flow

$$g^{-1}\dot{g} = g^{-1}\frac{d}{dt}e^{\phi\cdot\mathbf{T}} = \dot{\phi}\cdot\mathbf{T}\,.$$

This is the group tangent velocity $g^{-1}\dot{g}\,\hat{x} = \dot{\phi}\cdot t(\hat{x})$ evaluated at the point $\hat{x}$, i.e., with $g = 1$. The flow $\hat{v} = d\hat{x}/dt$ in the $(d-N)$ directions transverse to the group flow is now obtained by subtracting the flow along the group tangent direction,

$$\hat{v}(\hat{x}) = v(\hat{x}) - \dot{\phi}(\hat{x})\cdot t(\hat{x})\,. \tag{10A.44}$$

We can pick any coordinate transformation (10A.43) between the ‘lab’ and ‘moving frame’, any time, any way we like; equivariance guarantees that the states and the equations of motion (10A.44) in the two frames are physically equivalent. This is a immense freedom, and with freedom comes responsibility, the responsibility of choosing a good frame.

But the frame that is good for you might not be a frame that is good for your people. We start by describing the frame for space cowboys, the unique frame where you ride into the sunset with no drift in symmetry directions whatsoever. This is no symmetry reduction, just the most natural local ‘proper’ frame for a single trajectory. In sect. 10A.8 we do the right thing, and define a ‘slice’, a fixed frame into which we can reel in all group orbits and compare different physical states or ‘shapes’.

10A.6 Go with the flow: comoving frame

For the time being, this section only exists as a

10A.7 Symmetry reduction

> Maybe when I’m done with grad school I’ll be able to figure it all out . . .
> — Rebecca Wilczak, undergraduate

Given Lie group G acting smoothly on a C^∞ manifold $\mathcal{M}$, we can think of each group orbit as an equivalence class. *Symmetry reduction* is the identification of a unique point on a group orbit as the representative of its equivalence class. We call the set of all such group orbit representatives the *reduced state space* $\mathcal{M}/G$. In the literature this space is often rediscovered, and thus has many names - it is alternatively called ‘desymmetrized state space’, ‘symmetry-reduced space’, ‘orbit space’ (because every group orbit in the original space is mapped to a single point in the orbit space), ‘base manifold’, ‘shape-changing space’ or ‘quotient space’ (because the symmetry has been ‘divided out’), obtained by mapping equivariant dynamics to invariant dynamics (‘image’) by methods such as ‘moving frames’, ‘cross sections’, ‘slices’, ‘freezing’, ‘Hilbert bases’, ‘quotienting’, ‘lowering the degree’, ‘lowering the order’, or ‘desymmetrization’. remark 10A.3

Symmetry reduction replaces a dynamical system $(\mathcal{M}, f)$ with a symmetry by a ‘desymmetrized’ system $(\hat{\mathcal{M}}, \bar{f})$ of figure 10.2 (b), a system where each group orbit is replaced by a point, and the action of the group is trivial, $g\hat{x} = \hat{x}$ for all $\hat{x} \in \hat{\mathcal{M}}, g \in G$. The reduced state space $\hat{\mathcal{M}}$ is sometimes called the ‘quotient space’ $\mathcal{M}/G$ because the symmetry has been ‘divided out’. For a discrete symmetry, the reduced state space $\mathcal{M}/G$ is given by the fundamental domain of sect. 9A.7. In presence of a continuous symmetry, the reduction to $\mathcal{M}/G$ amounts to a change

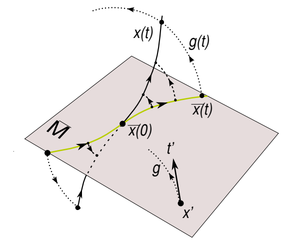

Figure 10A.10: Slice $\hat{\mathcal{M}}$ is a hyperplane (10A.45) passing through the slice-fixing point $\hat{x}'$, and normal to the group tangent t' at $\hat{x}'$. It intersects all group orbits (indicated by dotted lines here) in an open neighborhood of $\hat{x}'$. The full state space trajectory $x(\tau)$ and the reduced state space trajectory $\hat{x}(\tau)$ belong to the same group orbit $\mathcal{M}_{x(\tau)}$ and are equivalent up to a group rotation $g(\tau)$, defined in (10A.43).

of coordinates where the ‘ignorable angles’ $\{\phi_1, \cdots, \phi_N\}$ that parameterize N continuous coordinate transformations are separated out.

10A.8 Bringing it all back home: method of slices

In the ‘method of slices’ the reduced state space representative $\hat{x}$ of a group orbit equivalence class is picked by slicing across the group orbits by a fixed hypersurface (here is a sophorific, look ma, no hands overview). We start by describing how the method works for a finite segment of a full state space trajectory.

Definition: Slice. Let G act regularly on a d-dimensional manifold $\mathcal{M}$, i.e., with all group orbits N-dimensional. A *slice* through point $\hat{x}'$ is a $(d-N)$-dimensional submanifold $\hat{\mathcal{M}}$ such that all group orbits in an open neighborhood of the 'template' point $\hat{x}'$ intersect $\hat{\mathcal{M}}$ transversally once and only once (see figure 10A.10).

The simplest *slice condition* defines a linear slice as a $(d-N)$-dimensional hyperplane $\hat{\mathcal{M}}$ normal to the N group rotation tangents t'_a at point $\hat{x}'$:

$$(\hat{x}-\hat{x}')^\top t'_a = 0\,, \qquad t'_a = t_a(\hat{x}') = \mathbf{T}_a\,\hat{x}'\,. \tag{10A.45}$$

In other words, 'slice' is a Poincaré section (3.14) for group orbits. Each 'big circle' –group orbit tangent to t'_a– intersects the hyperplane exactly twice, with the two solutions separated by π. As for a Poincaré section (3.4), we add an orientation condition, and select the intersection with the clockwise rotation angle into the slice.

As $(\hat{x}')^\top t'_a = 0$ by the antisymmetry of $\mathbf{T}_a$, the slice condition (10A.45) fixes ϕ for a given x by

$$0 = \hat{x}^\top t'_a = x^\top g(\phi)^\top t'_a\,, \tag{10A.46}$$

where $g^\top$ denotes the transpose of g. The method of moving frames can be interpreted as a change of variables

$$\hat{x}(\tau) = g^{-1}(\tau)\,x(\tau)\,, \tag{10A.47}$$

that is passing to a frame of reference in which condition (10A.46) is identically satisfied, see example 10A.12. Therefore the name 'moving frame'. A moving frame should not be confused with the comoving frame, such as the one illustrated in figure 10.8. Each relative equilibrium, relative periodic orbit and general ergodic trajectory has its own comoving frame. In the method of slices one fixes a stationary slice, and rotates all solutions back into the slice.

Comoving frames can be utilized in post-processing methods; trajectories are computed in the full state space, then rotated into the slice whenever desired, with the slice condition easily implemented. The slice group tangent t' is a given vector, and $g(\phi)\,x$ is another vector, linear in x and a function of group parameters ϕ. Rotation parameters ϕ are determined numerically, by a Newton method, through the slice condition (10A.46).

Figure 10A.11 illustrates the method of moving frames for an SO(2) slice normal to the x_2 axis. Looks innocent, except there is nothing to prevent a trajectory from going through $(x_1, x_2) = (0, 0)$, and what ϕ is one to use then? We can always chose a finite time step that hops over this singularity, but in the continuous time formulation we will not be so lucky.

How does one pick a slice point $\hat{x}'$? A generic point $\hat{x}'$ not in an invariant subspace should suffice to fix a slice. The rules of thumb are much like the ones for picking Poincaré sections, sect. 3.1.2. The intuitive idea is perhaps best visualized in the context of fluid flows. Suppose the flow exhibits an unstable coherent structure that –approximately– recurs often at different spatial dispositions. One can fit a 'template' to one recurrence of such structure, and describe other recurrences as its translations. A well chosen slice point belongs to such dynamically important equivalence class (i.e., group orbit). A slice is locally isomorphic to $\mathcal{M}/G$, in an open neighborhood of $\hat{x}'$. As is the case for the dynamical Poincaré sections, in general a single slice does not suffice to reduce $\mathcal{M} \to \mathcal{M}/G$ globally.

The Euclidian product of two vectors x, y is indicated in (10A.45) by x-transpose times y, i.e., $x^\top y = \sum_i^d x_i y_i$. More general bilinear norms $\langle x, y\rangle$ can be used, as long as they are G-invariant, i.e., constant on each irreducible subspace. An example is the quadratic Casimir (10.9).

example 10A.12
p. 224

The slice condition (10A.45) fixes N directions; the remaining vectors $(\hat{x}_{N+1} \ldots \hat{x}_d)$ span the slice hyperplane. They are $d - N$ *fundamental invariants*, in the sense that any other invariant can be expressed in terms of them, and they are functionally independent. Thus they serve to distinguish orbits in the neighborhood of the slice-fixing point $\hat{x}'$, i.e., two points lie on the same group orbit if and only if all the fundamental invariants agree.

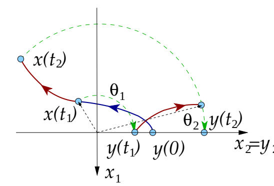

Figure 10A.11: Method of moving frames for a flow SO(2)-equivariant under (10.37) with slice through $\hat{x}' = (0, 1, 0, 0)$, group tangent $t' = (1, 0, 0, 0)$. The clockwise orientation condition restricts the slice to half-hyperplane $\hat{x}_1 = 0$, $\hat{x}_2 > 0$. A trajectory started on the slice at $\hat{x}(0)$, evolves to a state space point with a non-zero $x_1(t_1)$. Compute the polar angle ϕ_1 of $x(t_1)$ in the (x_1, x_2) plane. Rotate $x(t_1)$ clockwise by ϕ_1 to $\hat{x}(t_1) = g(-\phi_1)\,x(t_1)$, so that the equivalent point on the circle lies on the slice, $\hat{x}_1(t_1) = 0$. Thus after every finite time step followed by a rotation the trajectory restarts from the $\hat{x}_1(t_k) = 0$ reduced state space.

10A.9 Dynamics within a slice

I made a wrong mistake.
—Yogi Berra

So far we have taken the post-processing approach: evolve the trajectory in the full state space, than rotate all its points into the slice. You can also split up the time integration into a sequence of finite time steps, each followed by a rotation of the end point into the slice, see figure 10A.11. It is tempting to see what happens if the steps are taken infinitesimal. As we shall see, we do get a flow restricted to the slice, but at a price.

The relation (10A.44) between the 'lab' and 'moving frame' state space velocity holds for any factorization (10A.43) of the flow of form $x(\tau) = g(\tau)\,\hat{x}(\tau)$. To integrate these equations we first have to fix a particular flow factorization by imposing conditions on $\hat{x}(\tau)$, and then integrate phases $\phi(\tau)$ on a given reduced state space trajectory $\hat{x}(\tau)$.

Here we demand that the reduced state space is confined to a slice hyperplane. Substituting (10A.44) into the time derivative of the fixed slice condition (10A.46),

$$\hat{v}(\hat{x})^\top t'_a = v(\hat{x})^\top t'_a - \dot{\phi}_a \cdot t(\hat{x})^\top t'_a = 0\,,$$

yields the equation for the group phases flow $\dot{\phi}$ for the slice fixed by $\hat{x}'$, together with the reduced state space $\hat{\mathcal{M}}$ flow $\hat{v}(\hat{x})$:

$$\hat{v}(\hat{x}) = v(\hat{x}) - \dot{\phi}(\hat{x}) \cdot t(\hat{x})\,, \qquad \hat{x} \in \hat{\mathcal{M}} \tag{10A.48}$$

$$\dot{\phi}_a(\hat{x}) = \frac{v(\hat{x})^\top t'_a}{t(\hat{x})^\top \cdot t'}\,. \tag{10A.49}$$

Each group orbit $\mathcal{M}_x = \{g\,x \,|\, g \in G\}$ is an equivalence class; method of slices represents the class by its single slice intersection point $\hat{x}$. By construction $\hat{v}^\top t' = 0$, and the motion stays in the $(d{-}N)$-dimensional slice. We have thus replaced the original dynamical system $\{\mathcal{M}, f\}$ by a reduced system $\{\hat{\mathcal{M}}, \hat{f}\}$.

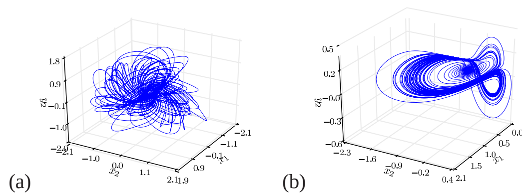

Figure 10A.12: A slice fixed by taking as a template a point on the two-modes relative equilibrium group orbit, $\hat{x}' = x_{TW1}$. The strange attractor of figure 10.1 in the (a) the equivariant state space $\{x_1, x_2, y_2\}$ projection; (b) in the $\{x_1, x_2, y_2\}$ symmetry-reduced state space of (10A.48).

In the pattern recognition and 'template fitting' settings (10A.49) is called the 'reconstruction equation'. Integrated together, the reduced state space trajectory (10A.48) and $g(\tau) = \exp\{\phi(\tau) \cdot \mathbf{T}\}$, the integrated phase (10A.49), reconstruct the full state space trajectory $x(\tau) = g(\tau)\, \hat{x}(\tau)$ from the reduced state space trajectory $\hat{x}(\tau)$, so no information about the flow is lost in the process of symmetry reduction. exercise 10A.2

Slice flow equations (10A.48) and (10A.49) are pretty, but there is a trouble in the paradise. The slice flow encounters singularities in subsets of state space, with phase velocity $\dot{\phi}$ divergent whenever the denominator in (10A.49) changes sign, see $\{x_2, y_2, z\}$ projection of figure 10A.12.

10A.10 Method of images: Hilbert bases

(E. Siminos and P. Cvitanović)

Erudite reader might wonder: why all this slicing and dicing, when the problem of symmetry reduction had been solved by Hilbert and Weyl nearly a century ago? Indeed, the most common approach to symmetry reduction is by means of a Hilbert invariant polynomial bases (9A.22), motivated intuitively by existence of such nonlinear invariants as the rotationally-invariant length $r^2 = x_1^2 + x_2^2 + \cdots + x_d^2$, or, in Hamiltonian dynamics, the energy function. One trades in the equivariant state space coordinates $\{x_1, x_2, \cdots, x_d\}$ for a non-unique set of $m \geq d$ polynomials $\{u_1, u_2, \cdots, u_m\}$ invariant under the action of the symmetry group. These polynomials are linearly independent, but functionally dependent through $m - d + N$ relations called *syzygies*.

The dynamical equations follow from the chain rule

$$\dot{u}_i = \frac{\partial u_i}{\partial x_j}\, \dot{x}_j\,, \qquad (10A.50)$$

upon substitution $\{x_1, x_2, \cdots, x_d\} \to \{u_1, u_2, \cdots, u_m\}$. One can either rewrite the dynamics in this basis or plot the 'image' of solutions computed in the original, equivariant basis in terms of these invariant polynomials.

Nevertheless we can now easily identify a suitable Poincaré section, guided by the Lorenz flow examples of chapter 9.4, as one that contains the z-axis and

the image of the relative equilibrium TW_1, here defined by the condition $u_1 = u_4$. As in example 11.4, we construct the first return map using as coordinate the Euclidean arclength along the intersection of the unstable manifold of TW_1 with the Poincaré surface of section. Thus the goals set into the introduction to this chapter are attained: we have reduced the messy strange attractor of figure 10.1 to a 1-dimensional return map. As will be explained in example 11.4 for the Lorenz attractor, we now have the symbolic dynamics and can compute as many relative periodic orbits of the complex Lorenz flow as we wish, missing none.

Reducing dimensionality of a dynamical system by explicit elimination of variables through inclusion of syzygies introduces singularities. Such elimination of variables, however, is not needed for visualization purposes; syzygies merely guarantee that the dynamics takes place on a $(d - N)$-dimensional submanifold in the projection on the $\{u_1, u_2, \cdots, u_m\}$ coordinates. However, when one *reconstructs* the dynamics in the original space $\mathcal{M}$ from its image $\mathcal{M}/G$, the transformations have singularities at the fixed-point subspaces of the isotropy subgroups in $\mathcal{M}$.

What limits the utility of Hilbert basis methods are not such singularities, but rather the fact that the algebra needed to determine a Hilbert basis becomes computationally prohibitive as the dimension of the system and/or the symmetry group increases. Moreover, even if such basis were available, rewriting the equations in an invariant polynomial basis seems impractical, so in practice Hilbert basis computations appear not feasible beyond state space dimension of order ≈ ten. When the goal is to quotient continuous symmetries of high-dimensional flows, such as the Navier-Stokes flows, one needs a workable framework. The method of slices of sect. 10A.7 is one such minimalist alternative.

Résumé

Here we have described how, and offered two approaches to continuous symmetry reduction. In the *method of slices* one fixes a 'slice' $(\hat{x} - \hat{x}')^\top t' = 0$, a hyperplane normal to the group tangent t' that cuts across group orbits in the neighborhood of the slice-fixing point $\hat{x}'$. Each class of symmetry-equivalent points is represented by a single point, with the symmetry-reduced dynamics in the reduced state space $\mathcal{M}/G$ given by (10A.48):

$$\hat{v} = v - \dot{\phi} \cdot t\,, \qquad \dot{\phi}_a = (v^\top t'_a)/(t \cdot t')\,.$$

In practice one runs the dynamics in the full state space, and post-processes the trajectory by the method of moving frames. In the *Hilbert polynomial basis* approach one transforms the equivariant state space coordinates into invariant ones, by a nonlinear coordinate transformation

$$\{x_1, x_2, \cdots, x_d\} \to \{u_1, u_2, \cdots, u_m\} \quad + \quad \{syzygies\}\,,$$

and studies the invariant 'image' of dynamics (10A.50) rewritten in terms of invariant coordinates.

Continuous symmetry reduction is considerably more involved than the discrete symmetry reduction to a fundamental domain of chapter 9.4. Slices are only local sections of group orbits, and Hilbert polynomials are non-unique and difficult to compute for high-dimensional flows. However, there is no need to actually recast the dynamics in the new coordinates: either approach can be used as a visualization tool, with all computations carried out in the original coordinates, and then, when done, rotating the solutions into the symmetry reduced state space by post-processing the data. The trick is to construct a good set of symmetry invariant Poincaré sections (see sect. 3.1), and that is always a dark art, with or without a symmetry.

Relative equilibria and relative periodic orbits are the hallmark of systems with continuous symmetry. Amusingly, in this extension of 'periodic orbit' theory from unstable 1-dimensional closed periodic orbits to unstable $(N+1)$-dimensional compact manifolds $\mathcal{M}_p$ invariant under continuous symmetries, there are either no or proportionally few periodic orbits. Relative periodic orbits are almost never eventually periodic, i.e., they almost never lie on periodic trajectories in the full state space, so looking for periodic orbits in systems with continuous symmetries is a fool's errand.

However, dynamical systems are often equivariant under a combination of continuous symmetries and discrete coordinate transformations of chapter 9. An example is the orthogonal group $O(n)$. In presence of discrete symmetries relative periodic orbits within discrete symmetry-invariant subspaces are eventually periodic. Atypical as they are (no generic chaotic orbit can ever enter these discrete invariant subspaces) they will be important for periodic orbit theory, as there the shortest orbits dominate, and they tend to be the most symmetric solutions.

The message: If a dynamical systems has a symmetry, use it!

chapter 21

Commentary

Remark 10A.3 A brief history of relativity, or, 'Desymmetrization and its discontents' (after Civilization and its discontents; continued from remark 10.1): The literature on symmetry reduction in dynamical systems is immense, most of it deliriously unintelligible.

Relative equilibria and relative periodic solutions are related by symmetry reduction to equilibria and periodic solutions of the reduced dynamics. They appear in many physical applications, such as celestial mechanics, molecular dynamics, motion of rigid bodies, nonlinear waves, spiralling patterns, and fluid mechanics. A relative equilibrium is a solution which travels along an orbit of the symmetry group at constant speed; an introduction to them is given, for example, in Marsden [10A.1]. According to Cushman, Bates [10A.2] and Yoder [10A.3], C. Huygens [10A.4] understood the relative equilibria of a spherical pendulum many years before publishing them in 1673. A reduction of the translation symmetry was obtained by Jacobi (for a modern, symplectic implementation, see Laskar *et al.* [10A.5]). In 1892 German sociologist Vierkandt [10A.6] showed that on a symmetry-reduced space (the constrained velocity phase space modulo the action of the group of Euclidean motions of the plane) all orbits of the rolling disk system are periodic [10A.7]. According to Chenciner [10A.8], the first attempt to find (relative) periodic solutions of the N-body problem was the 1896 short note by Poincaré [10A.9], in the context of the 3-body problem. Poincaré named such solutions 'relative'. Relative equilibria of the N-body problem (known in this context as the Lagrange points, stationary in the co-rotating frame) are circular motions in the inertial frame, and relative periodic orbits correspond to quasiperiodic motions in the inertial frame. For relative periodic orbits in celestial mechanics see also ref. [10A.10]. A striking application of relative periodic orbits has been the discovery of "choreographies" in the N-body problems [10A.11, 10A.12, 10A.13].

The modern story on equivariance and dynamical systems starts perhaps with S. Smale [10A.14] and M. Field [10A.15], and on bifurcations in presence of symmetries with Ruelle [10A.16]. Ruelle proves that the stability matrix/Jacobian matrix evaluated at an equilibrium/fixed point $x \in \mathcal{M}_G$ decomposes into linear irreducible representations of G, and that stable/unstable manifold continuations of its eigenvectors inherit their symmetry properties, and shows that an equilibrium can bifurcate to a rotationally invariant periodic orbit (i.e., relative equilibrium).

Gilmore and Lettelier monograph [10A.17] offers a very clear, detailed and user friendly discussion of symmetry reduction by means of Hilbert polynomial bases (do not look for 'Hilbert' in the index, though). Vladimirov, Toronov and Derbov [10A.18] use an invariant polynomial basis to study bounding manifolds of the symmetry reduced complex Lorenz flow and its homoclinic bifurcations. There is no general strategy how to construct a Hilbert basis; clever low-dimensional examples have been constructed case-by-case. The determination of a Hilbert basis appears computationally prohibitive for state space dimensions larger than ten [10A.19, 10A.20], and rewriting the equations of motions in invariant polynomial bases appears impractical for high-dimensional flows. Thus, by 1920's the problem of rewriting equivariant flows as invariant ones was solved by Hilbert and Weyl, but at the cost of introducing largely arbitrary extra dimensions, with the reduced flows on manifolds of lowered dimensions, constrained by sets of syzygies. Cartan found this unsatisfactory, and in 1935 he introduced [10A.21] the notion of a *moving frame*, a map from a manifold to a Lie group, which seeks no invariant polynomial basis, but instead rewrites the reduced $\mathcal{M}/G$ flow in terms of $d - N$ *fundamental invariants* defined by a generalization of the Poincaré section, a slice that cuts across all group orbits in some open neighborhood. Fels and Olver view the method as an alternative to the Gröbner bases methods for computing Hilbert polynomials, to compute functionally independent fundamental invariant bases for general group actions (with no explicit connection to dynamics, differential equations or symmetry reduction). 'Fundamental' here means that they can be used to generate all other invariants. Olver's monograph [10.11] is pedagogical, but does not describe the original Cartan's method. Fels and Olver papers [10A.22, 10A.23] are lengthy and technical. They refer to Cartan's method as method of 'moving frames' and view it as a special and less rigorous case of their 'moving coframe' method. The name 'moving coframes' arises through the use of Maurer-Cartan form which is a coframe on the Lie group G, i.e., they form a pointwise basis for the cotangent space. In refs. [10.6, 10.7] the invariant bases generated by the moving frame method are used as a basis to project a full state space trajectory to the slice (i.e., the $\mathcal{M}/G$ reduced state space).

The basic idea of the 'method of slices' is intuitive and frequently reinvented, often under a different name; for example, it is stated without attribution as the problem 1. of Sect. 6.2 of Arnol'd *Ordinary Differential Equations* [10A.24]. The factorization (10A.43) is stated on p. 31 of Anosov and Arnol'd [10A.25], who note, without further elaboration, that in the vicinity of a point which is not fixed by the group one can reduce the order of a system of differential equations by the dimension of the group. Ref. [10A.26] refers to symmetry reduction as 'lowering the order'. For the definition of 'slice' see, for example, Chossat and Lauterbach [10A.20]. Briefly, a submanifold $\mathcal{M}_{\hat{x}'}$ containing $\hat{x}'$ is called a *slice* through $\hat{x}'$ if it is invariant under isotropy $G_{\hat{x}'(\mathcal{M}_{\hat{x}'})} = \mathcal{M}_{\hat{x}'}$. If $\hat{x}'$ is a fixed point of G, than slice is invariant under the whole group. The slice theorem is explained, for example, in Encyclopaedia of Mathematics. Slices tend to be discussed in contexts much more difficult than our application - symplectic groups, sections in absence of global charts, non-compact Lie groups. We follow refs. [10A.27] in referring to a local group-orbit section as a 'slice'. Refs. [10.12, 10A.28] and others refer to global group-orbit sections as 'cross-sections', a term that we would rather avoid, as it already has a different and well established meaning in physics. Duistermaat and Kolk [10A.29] refer to 'slices', but the usage goes back at least to Guillemin and Sternberg [10A.28] in 1984, Palais [10A.30] in 1961 and Mastow [10A.31] in 1957. Bredon [10.12] discusses both cross-sections and slices. Guillemin and Sternberg [10A.28] define the 'cross-section', but emphasize that finding it is very rare: "existence of a global section is a very stringent condition on a group action. The notion of 'slice' is weaker but has a much broader range of existence."

Several important fluid dynamics flows exhibit continuous symmetries which are either SO(2) or products of SO(2) groups, each of which act on different coordinates of the state space. The Kuramoto-Sivashinsky equations [10A.32, 10A.33], plane Couette flow [D.31, 10A.48, 10A.45, **?**], and pipe flow [10A.46, 10A.47] all have continuous symmetries of this form. In the 1982 paper Rand [10A.49] explains how presence of continuous symmetries gives rise to rotating and modulated rotating (quasiperiodic) waves in fluid dynamics. Haller and Mezić [10A.50] reduce symmetries of three-dimensional volume-preserving flows and reinvent method of moving frames, under the name 'orbit projection map'. There is extensive literature on reduction of symplectic manifolds with symmetry; Marsden and Weinstein 1974 article [10A.51] is an important early reference. Then there are studies of the reduced phase spaces for vortices moving on a sphere such as ref. [10A.52], and many, many others.

Reaction-diffusion systems are often equivariant with respect to the action of a finite dimensional (not necessarily compact) Lie group. Spiral wave formation in such nonlinear excitable media was first observed in 1970 by Zaikin and Zhabotinsky [10A.34]. Winfree [10A.35, 10A.36] noted that spiral tips execute meandering motions. Barkley and collaborators [10A.37, 10A.38] showed that the noncompact Euclidean symmetry of this class of systems precludes nonlinear entrainment of translational and rotational drifts and that the interaction of the Hopf and the Euclidean eigenmodes leads to observed quasiperi-

odic and meandering behaviors. Fiedler, in the influential 1995 talk at the Newton Institute, and Fiedler, Sandstede, Wulff, Turaev and Scheel [10A.39, 10A.40, 10A.41, 10A.42] treat Euclidean symmetry bifurcations in the context of spiral wave formation. The central idea is to utilize the semidirect product structure of the Euclidean group $E(2)$ to transform the flow into a 'skew product' form, with a part orthogonal to the group orbit, and the other part within it, as in (10A.48). They refer to a linear slice $\hat{\mathcal{M}}$ near relative equilibrium as a *Palais slice*, with Palais coordinates. As the choice of the slice is arbitrary, these coordinates are not unique. According to these authors, the skew product flow was first written down by Mielke [10A.43], in the context of buckling in the elasticity theory. However, this decomposition is no doubt much older. For example, it was used by Krupa [10.13, 10A.20] in his local slice study of bifurcations of relative equilibria. Biktashev, Holden, and Nikolaev [10A.44] cite Anosov and Arnol'd [10A.25] for the 'well-known' factorization (10A.43) and write down the slice flow equations (10A.48).

Neither Fiedler *et al.* [10A.39] nor Biktashev *et al.* [10A.44] implemented their methods numerically. That was done by Rowley and Marsden for the Kuramoto-Sivashinsky [10A.27] and the Burgers [10A.53] equations, and Beyn and Thümmler [10A.54, 10A.55] for a number of reaction-diffusion systems, described by parabolic partial differential equations on unbounded domains. We recommend the Barkley paper [10A.38] for a clear explanation of how the Euclidean symmetry leads to spirals, and the Beyn and Thümmler paper [10A.54] for inspirational concrete examples of how 'freezing'/'slicing' simplifies the dynamics of rotational, traveling and spiraling relative equilibria. Beyn and Thümmler write the solution as a composition of the action of a time dependent group element $g(t)$ with a 'frozen', in-slice solution $\hat{u}(t)$ (10A.43). In their nomenclature, making a relative equilibrium stationary by going to a comoving frame is 'freezing' the traveling wave, and the imposition of the phase condition (i.e., slice condition (10A.45)) is the 'freezing ansatz'. They find it more convenient to make use of the equivariance by extending the state space rather than reducing it, by adding an additional parameter and a phase condition. The 'freezing ansatz' [10A.54] is identical to the Rowley and Marsden [10A.53] and our slicing, except that 'freezing' is formulated as an additional constraint, just as when we compute periodic orbits of ODEs we add Poincaré section as an additional constraint, i.e., increase the dimensionality of the problem by 1 for every continuous symmetry (see sect. 6A.5). section 6A.5

Several symmetry reduction schemes are reviewed in ref. [10.7]. Here we describe the method of slices [10A.27, 10A.54, 10A.56], the only method that we find practical for a symmetry reduction of chaotic solutions of highly nonlinear and possibly also high-dimensional flows. Derivation of sect. 10A.9 follows most closely Rowley and Marsden [10A.53] who, in the pattern recognition setting refer to the slice point as a 'template', and call (10A.49) the 'reconstruction equation' [10A.1, 10A.57]. They also describe the 'method of connections' (called 'orthogonality of time and group orbit at successive times' in ref. [10A.54]), for which the reconstruction equation (10A.49) denominator is $t(\hat{x})^\top \cdot t(\hat{x})$ and thus non-vanishing as long as the action of the group is regular. This avoids the spurious slice singularities, but it is not clear what the 'method of connections' buys us otherwise. It does not reduce the dimensionality of the state space, and it accrues 'geometric phases' which prevent relative periodic orbits from closing into periodic orbits. Geometric phase in laser equations, including complex Lorenz equations, has been studied in ref. [10A.58, 10A.59, 10A.61, 10A.62, 10A.63]. Another theorist's temptation is to hope that a continuous symmetry would lead us to a conserved quantity. However, Noether theorem requires that equations of motion be cast in Lagrangian form and that the Lagrangian exhibits variational symmetries [10A.64, 10A.65]. Such variational symmetries are hard to find for dissipative systems.

In general relativity 'symmetry reduction' is a method of finding exact solutions by imposing symmetry conditions to obtain a reduced system of equations, i.e., restricting the set of solutions considered to an invariant subspace. This is not what we mean by 'symmetry reduction' in this monograph.

References to 'cyclists' are bit of a joke in more ways than one. First, 'cyclist', 'pedestrian' throughout ChaosBook.org refer jokingly both to the title of Lipkin's *Lie groups for pedestrians* [10A.66] and to our preoccupations with actual cycling. Lipkin's 'pedestrian' is fluent in Quantum Field Theory, but wobbly on Dynkin diagrams. More to the point, it is impossible to dispose of Lie groups in a page of text. As a antidote to the brevity of exposition here, consider reading Gilmore's monograph [10A.67] which offers a quirky, personal and enjoyable distillation of a lifetime of pondering Lie groups. As seems to be the case with any textbook on Lie groups, it will not help you with the problem at hand, but it is the only place you can learn both what Galois actually did when he invented the theory of finite groups in 1830, and what, inspired by Galois, Lie actually did in his 1874 study of symmetries of ODEs. Gilmore also explains many things that we pass over in silence here, such as matrix groups, group manifolds, and compact groups.

One would think that with all this literature the case is shut and closed, but not so. Applied mathematicians are inordinately fond of bifurcations, and almost all of the previous work focuses on equilibria, relative equilibria, and their bifurcations, and for these problems a single slice works well. Only when one tries to describe the totality of chaotic orbits does the non-global nature of slices become a serious nuisance.

(E. Siminos and P. Cvitanović)

Remark 10A.4 Velocity *vs.* speed *Velocity* is a vector, the rate at which the object changes its position. *Speed*, or the magnitude of the velocity, is a scalar quantity which describes how fast an object moves. We denote the rate of change of group phases, or the *phase velocity* by the vector $c = (\dot{\phi}_1, \cdots, \dot{\phi}_N) = (c_1, \cdots, c_N)$, a component for each of the N continuous symmetry parameters. These are converted to state space velocity components along the group tangents by

$$v(x) = c(t) \cdot t(x) \,. \tag{10A.51}$$

For rotational waves these are called 'angular velocities'.

Remark 10A.5 Killing fields. The symmetry tangent vector fields discussed here are a special case of Killing vector fields of Riemannian geometry and special relativity. If this poetry warms the cockles of your heart, hang on. From wikipedia (this wikipedia might also be useful): A Killing vector field is a set of infinitesimal generators of isometries on a Riemannian manifold that preserve the metric. Flows generated by Killing fields are continuous isometries of the manifold. The flow generates a symmetry, in the sense that moving each point on an object the same distance in the direction of the Killing vector field will not distort distances on the object. A vector field X is a Killing field if the Lie derivative with respect to X of the metric g vanishes:

$$\mathcal{L}_X g = 0 \,. \tag{10A.52}$$

Killing vector fields can also be defined on any (possibly nonmetric) manifold $\mathcal{M}$ if we take any Lie group G acting on it instead of the group of isometries. In this broader sense,

a Killing vector field is the pushforward of a left invariant vector field on G by the group action. The space of the Killing vector fields is isomorphic to the Lie algebra $\mathfrak{g}$ of G.

If the equations of motion can be cast in Lagrangian form, with the Lagrangian exhibiting variational symmetries [10A.64, 10A.65], Noether theorem associates a conserved quantity with each Killing vector.

10A.11 Examples

10A.11.1 Special orthogonal group SO(2)

Example 10A.12 ***An SO(2) moving frame:*** *(continued from example 10.1) The SO(2) action*

$$(\hat{x}_1, \hat{x}_2) = (x_1 \cos\theta + x_2 \sin\theta,\ -x_1 \sin\theta + x_2 \cos\theta) \tag{10A.53}$$

is regular on $\mathbb{R}^2\backslash\{0\}$. Thus we can define a slice as a 'hyperplane' (here a mere line), through $\hat{x}' = (0, 1)$, with group tangent $t' = (1, 0)$, and ensure uniqueness by clockwise rotation into positive x_2 axis. Hence the reduced state space is the half-line $x_1 = 0$, $\hat{x}_2 = x_2 > 0$. The slice condition then simplifies to $\hat{x}_1 = 0$ and yields the explicit formula for the moving frame parameter

$$\theta(x_1, x_2) = \tan^{-1}(x_1/x_2)\,, \tag{10A.54}$$

i.e., the angle which rotates the point (x_1, x_2) back to the slice, taking care that $\tan^{-1}$ distinguishes (x_1, x_2) plane quadrants correctly. Substituting (10A.54) back to (10A.53) and using $\cos(\tan^{-1} a) = (1 + a^2)^{-1/2}$, $\sin(\tan^{-1} a) = a(1 + a^2)^{-1/2}$ confirms $\hat{x}_1 = 0$. It also yields an explicit expression for the transformation to variables on the slice,

$$\hat{x}_2 = \sqrt{x_1^2 + x_2^2}\,. \tag{10A.55}$$

This was to be expected as SO(2) preserves lengths, $x_1^2 + x_2^2 = \hat{x}_1^2 + \hat{x}_2^2$. If dynamics is in plane and SO(2) equivariant, the solutions can only be circles of radius $(x_1^2 + x_2^2)^{1/2}$, so this is the "rectification" of the harmonic oscillator by a change to polar coordinates, example B.1. Still, it illustrates the sense in which the method of moving frames yields group invariants. *(E. Siminos)*

click to return: p. ??

10A.11.2 Hilbert polynomial bases

Exercises

10A.1. **SO(2) or harmonic oscillator slice:** Construct a moving frame slice for action of SO(2) on $\mathbb{R}^2$

$$(x, y) \mapsto (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$$

by, for instance, the positive y axis: $x = 0$, $y > 0$. Write out explicitly the group transformation that brings any point back to the slice. What invariant is preserved by this construction? (E. Siminos)

10A.2. **The moving frame flow stays in the reduced state space:** Show that the flow (10A.48) stays in a $(d-1)$-dimensional slice hyperplane.

10A.3. **Stability of a relative equilibrium in the reduced state space:** Find an expression for the stability matrix of the system at a relative equilibrium when a linear slice is used to reduce the symmetry of the flow.

10A.4. **Stability of a relative periodic orbit in the reduced state space:** Find an expression for the Jacobian matrix (monodromy matrix) of a relative periodic orbit when a linear slice is used to reduce the dynamics of the flow.

10A.5. **Determination of invariants by the method of slices:** Show that the $d - N$ reduced state space coordinates determined by the method of slices are independent and invariant under group actions, and that the method of slices allows the determination of (in general non-polynomial) symmetry invariants by a simple algorithm that works well in high-dimensional state spaces.

10A.6. **Invariant subspace of the two-modes system:** Show that $(0, 0, x_2, y_2)$ is a flow invariant subspace of the two-modes system (10.35), i.e., show that a trajectory with the initial point within this subspace remains within it forever.
(N.B. Budanur)

10A.7. **Slicing the two-modes system:** Choose the simplest slice template point that fixes the 1. Fourier mode,

$$\hat{x}' = (1, 0, 0, 0)\,. \tag{10A.56}$$

(a) Show for the two-modes system (10.35), that the velocity within the slice (10A.48), and the phase velocity (10A.49) along the group orbit are

$$\begin{aligned} \hat{v}(\hat{x}) &= v(\hat{x}) - \dot{\phi}(\hat{x})t(\hat{x}) && (10A.57)\\ \dot{\phi}(\hat{x}) &= -v_2(\hat{x})/\hat{x}_1 && (10A.58) \end{aligned}$$

(b) Determine the chart border (the locus of point where the group tangent is either not transverse to the slice or vanishes).

(c) What is its dimension?

(d) What is its relation to the invariant subspace of exercise 10A.7?

(e) Can a symmetry-reduced trajectory cross the chart border?

(N.B. Budanur and P. Cvitanović)

10A.8. **The symmetry reduced two-modes flow:** Pick an initial point $\hat{x}(0)$ that satisfies the slice condition (10A.45) for the template choice (10A.56) and integrate (10A.57) & (10A.58). Plot the three dimensional slice hyperplane spanned by (x_1, x_2, y_2) to visualize the symmetry reduced dynamics. Does it look like figure 10A.12 (b)?
(N.B. Budanur)

10A.9. **Visualize the relative equilibrium of the two-modes system:** Starting the initial condition

$$x_0 = (0.439966, 0, -0.386267, 0.070204) \tag{10A.59}$$

integrate the full state space SO(2)-equivariant (10.35) and the symmetry reduced (10A.57) two-modes system for $t = 250$ time units. Plot the (x_1, x_2, y_1) projection of both trajectories. Explain your results.
(N.B. Budanur)

10A.10. **Relative equilibria of the two-modes system:** Write down an expression for the reduced velocity (10A.57) of the two-modes system explicitly by substituting (10A.58) and solve $\hat{v} = 0$ to find the relative equilibria. Part of this might be doable analytically (you have an invariant subspace). If that does not work out for you, solve the system numerically, for the parameter values (10.34). Check that x_0 of exercise 10A.59 is among your solutions. Mark the relative equilibria that you have found on the strange attractor plot of exercise 10A.8, interpret the role they play in the dynamics, if any. (N.B. Budanur)

10A.11. **Stability of the two-modes relative equilibrium:**

(a) Write down the stability matrix of the two-modes system in the reduced state space by computing derivatives of (10A.57).

(b) Compute eigenvalues and eigenvectors of this stability matrix at the relative equilibrium (10A.59)

(c) Indicate the direction along which the nearby trajectories expand.

(d) Compute the stability eigenvalues and eigenvectors of all relative equilibria of exercise 10A.10

(N.B. Budanur)

10A.12. **Relative periodic orbits of the two-modes system:** Initial conditions and periods for 4 relative periodic orbitof the two-modes system are listed in the table 10.1. Integrate (10.35) and (10A.57) with these initial conditions for 3-4 periods, and plot the four trajectories. Explain what you see.
(N.B. Budanur)

10A.13. **Poincaré section in the slice** Construct a Poincaré section for the two-modes system in the slice hyperplane, such that the relative equilibrium (10A.59) and its expanding direction that you found in (10A.11) is in this Poincaré section. Interpolate this Poincaré section with a smooth curve, and compute the arclengths positions of each crossing of the symmetry-reduced flow with the Poincaré section. (N.B. Budanur)

10A.14. **Finding relative equilibria from a Poincaré return map.** Produce a return map of the arclengths that you found in exercise 10A.13. Plot this return map. Note that its derivative is discontinuous at its critical point - why? Interpolate to this return map in two pieces and find its fixed point. Take the fixed point as the initial point to integrate the reduced two-modes system (10A.57) for $t = 3.7$. What do you see? (N.B. Budanur)

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