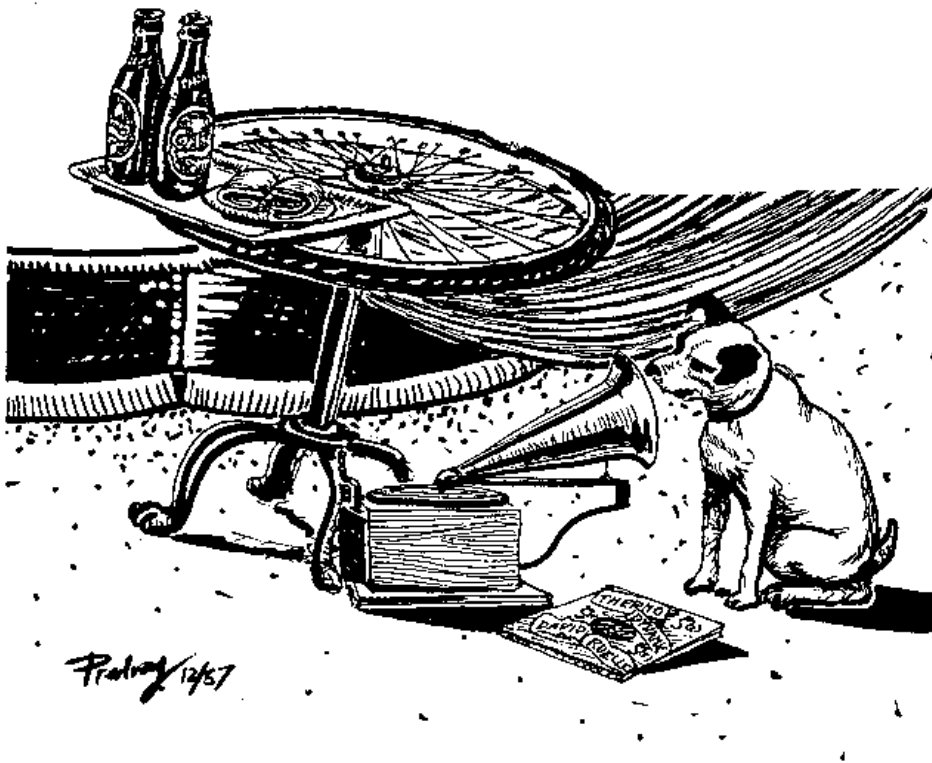


Chaos: Classical and Quantum

I: Deterministic Chaos



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No man but a blockhead ever wrote except for money
—Samuel Johnson

This book is a result of collaborative labors of many people over a span of several decades. Coauthors of a chapter or a section are indicated in the byline to the chapter/section title. If you are referring to a specific coauthored section rather than the entire book, cite it as (for example):

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Daniel Borrero Oct 23 2008, `soluCycles.tex`

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