

# Appendix G

## Counting itineraries

### G.1 Counting curvatures

ONE CONSEQUENCE of the finiteness of topological polynomials is that the contributions to curvatures at every order are even in number, half with positive and half with negative sign. For instance, for complete binary labeling (20.7),



$$c_4 = -t_{0001} - t_{0011} - t_{0111} - t_0 t_{01} t_1 + t_0 t_{001} + t_0 t_{011} + t_{001} t_1 + t_{011} t_1. \quad (\text{G.1})$$

We see that  $2^3$  terms contribute to  $c_4$ , and exactly half of them appear with a negative sign - hence if all binary strings are admissible, this term vanishes in the counting expression.

exercise G.2

Such counting rules arise from the identity

$$\prod_p (1 + t_p) = \prod_p \frac{1 - t_p^2}{1 - t_p}. \quad (\text{G.2})$$

Substituting  $t_p = z^{np}$  and using (15.18) we obtain for unrestricted symbol dynamics with  $N$  letters

$$\prod_p (1 + z^{np}) = \frac{1 - Nz^2}{1 - Nz} = 1 + Nz + \sum_{k=2}^{\infty} z^k (N^k - N^{k-1})$$

The  $z^n$  coefficient in the above expansion is the number of terms contributing to  $c_n$  curvature, so we find that for a complete symbolic dynamics of  $N$  symbols and

$n > 1$ , the number of terms contributing to  $c_n$  is  $(N - 1)N^{k-1}$  (of which half carry a minus sign).

exercise G.4

We find that for complete symbolic dynamics of  $N$  symbols and  $n > 1$ , the number of terms contributing to  $c_n$  is  $(N - 1)N^{n-1}$ . So, superficially, not much is gained by going from periodic orbits trace sums which get  $N^n$  contributions of  $n$  to the curvature expansions with  $N^n(1 - 1/N)$ . However, the point is not the number of the terms, but the cancelations between them.



### Exercises

**G.1. Lefschetz zeta function.** Elucidate the relation between the topological zeta function and the Lefschetz zeta function.

Substituting into the identity

$$\prod_p (1 + t_p) = \prod_p \frac{1 - t_p^2}{1 - t_p}$$

**G.2. Counting the 3-disk pinball counterterms.** Verify that the number of terms in the 3-disk pinball curvature expansion (21.29) is given by

we obtain

$$\begin{aligned} \prod_p (1 + t_p) &= \frac{1 - 3z^4 - 2z^6}{1 - 3z^2 - 2z^3} = 1 + 3z^2 + 2z^3 + \frac{z^4(6 + 12z + 2z^2)}{1 - 3z^2 - 2z^3} \\ &= 1 + 3z^2 + 2z^3 + 6z^4 + 12z^5 + 20z^6 + 48z^7 + 84z^8 + 184z^9 + \dots \end{aligned}$$

This means that, for example,  $c_6$  has a total of 20 terms, in agreement with the explicit 3-disk cycle expansion (21.30).

$$= 1 + t_0 + t_1 + \frac{2t_0t_1}{1 - t_0 - t_1} = 1 + t_0 + t_1 + \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} 2 \binom{n-2}{k-1} t_0^k t_1^{n-k}$$

**G.3. Cycle expansion denominators\*\*.** Prove that the denominator of  $c_k$  is indeed  $D_k$ , as asserted (F.14).

Hence for  $n \geq 2$  the number of terms in the expansion ?! with  $k$  0's and  $n - k$  1's in their symbol sequences is  $2 \binom{n-2}{k-1}$ . This is the degeneracy of distinct cycle eigenvalues in fig.?!; for systems with non-uniform hyperbolicity this degeneracy is lifted (see fig. ?!).

**G.4. Counting subsets of cycles.** The techniques developed above can be generalized to counting subsets of cycles. Consider the simplest example of a dynamical system with a complete binary tree, a repeller map (11.4) with two straight branches, which we label 0 and 1. Every cycle weight for such map factorizes, with a factor  $t_0$  for each 0, and factor  $t_1$  for each 1 in its symbol string. The transition matrix traces (15.7) collapse to  $tr(T^k) = (t_0 + t_1)^k$ , and  $1/\zeta$  is simply

In order to count the number of prime cycles in each such subset we denote with  $M_{n,k}$  ( $n = 1, 2, \dots$ ;  $k = \{0, 1\}$  for  $n = 1$ ;  $k = 1, \dots, n - 1$  for  $n \geq 2$ ) the number of prime  $n$ -cycles whose labels contain  $k$  zeros, use binomial string counting and Möbius inversion and obtain

$$\prod_p (1 - t_p) = 1 - t_0 - t_1 \tag{G.4}$$

$$\begin{aligned} M_{1,0} &= M_{1,1} = 1 \\ nM_{n,k} &= \sum_{m|\frac{n}{k}} \mu(m) \binom{n/m}{k/m}, \quad n \geq 2, k = 1, \dots, n-1 \end{aligned}$$

where the sum is over all  $m$  which divide both  $n$  and  $k$ .