

Chapter 17

Walkabout: Transition graphs

I think I'll go on a walkabout
find out what it's all about [...] take a ride to the other side
—Red Hot Chili Peppers, 'Walkabout'

IN CHAPTERS 14 AND 15 we learned that invariant manifolds partition the state space in invariant way, and how to name distinct orbits. We have established and related the *temporally* and *spatially* ordered topological dynamics for a class of 'stretch & fold' dynamical systems, and discussed pruning of inadmissible trajectories.

Here we shall use these results to generate the totality of admissible itineraries. This task will be particularly easy for repellers with complete Smale horseshoes and for subshifts of finite type, for which the admissible itineraries are generated by finite transition matrices, and the topological dynamics can be visualized by means of finite transition graphs. We shall then turn topological dynamics into a linear multiplicative operation on the state space partitions by means of transition matrices, the simplest examples of 'evolution operators.' They will enable us – in chapter 18 – to *count* the distinct orbits.



17.1 Matrix representations of topological dynamics



The allowed transitions between the regions of a partition $\{\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_m\}$ are encoded in the $[m \times m]$ -dimensional transition matrix whose elements take values

$$T_{ij} = \begin{cases} 1 & \text{if the transition } \mathcal{M}_j \rightarrow \mathcal{M}_i \text{ is possible} \\ 0 & \text{otherwise.} \end{cases} \quad (17.1)$$

The transition matrix is an explicit linear representation of topological dynamics. If the partition is a dynamically invariant partition constructed from sta-



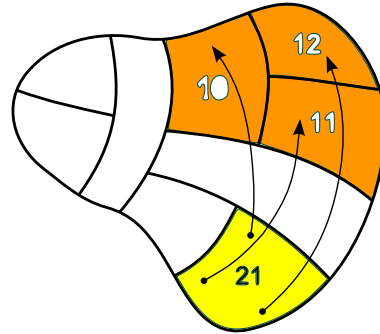


Figure 17.1: Points from the region \mathcal{M}_{21} reach regions $\{\mathcal{M}_{10}, \mathcal{M}_{11}, \mathcal{M}_{12}\}$, and no other regions, in one time step. Labeling exemplifies the ‘shift map’ of example 14.6 and (14.13).

ble/unstable manifolds, it encodes the topological dynamics as an invariant law of motion, with the allowed transitions at any instant independent of the trajectory history, requiring no memory.

Several related matrices as well will be needed in what follows. Often it is convenient to distinguish between two or more paths connecting the same two regions; that is encoded by the *adjacency* matrix with non-negative integer entries,

$$A_{ij} = \begin{cases} k & \text{if a transition } \mathcal{M}_j \rightarrow \mathcal{M}_i \text{ is possible in } k \text{ ways} \\ 0 & \text{otherwise.} \end{cases} \quad (17.2)$$

More generally, we shall encounter $[m \times m]$ matrices which assign different real or complex weights to different transitions, ▶

$$L_{ij} = \begin{cases} L_{ij} \in \mathbb{R} \text{ or } \mathbb{C} & \text{if } \mathcal{M}_j \rightarrow \mathcal{M}_i \text{ is allowed} \\ 0 & \text{otherwise.} \end{cases} \quad (17.3)$$

As in statistical physics, we shall refer to these as *transfer* matrices.

\mathcal{M}_i is *accessible* from \mathcal{M}_j in k steps if $(L^k)_{ij} \neq 0$. A matrix L is called *reducible* if there exists one or more index pairs $\{i, j\}$ such that $(L^k)_{ij} = 0$ for all k , otherwise the matrix is *irreducible*. This means that a trajectory starting in any partition region eventually reaches all of the partition regions, i.e., the partition is dynamically transitive or indecomposable, as assumed in (2.3). The notion of topological *transitivity* is crucial in ergodic theory: a mapping is transitive if it has a dense orbit. If that is not the case, state space decomposes into disconnected pieces, each of which can be analyzed separately by a separate irreducible matrix. Region \mathcal{M}_i is said to be *transient* if no trajectory returns to it. Region \mathcal{M}_j is said to be *absorbing* if no trajectory leaves it, $L_{jj} \neq 0, L_{ij} = 0$ for all $i \neq j$. Hence it suffices to restrict our considerations to irreducible matrices.

If L has strictly positive entries, $L_{ij} > 0$, the matrix is called *positive*; if $L_{ij} \geq 0$, the matrix is called *non-negative*. Matrix L is said to be *eventually positive* or *Perron-Frobenius* if L^k is positive for some power k (as a consequence, the matrix is transitive as well). A non-negative matrix whose columns conserve probability, $\sum_i L_{ij} = 1$, is called *Markov, probability* or *stochastic* matrix.

graphs, i.e., graphs for which there is a path from any node to any other node. (In a *connected* graph one may reach node j from node k , but not node k from node j .)

A transition graph compactly describes the ways in which the state space regions map into each other, accounts for finite memory effects in dynamics, and generates the totality of admissible trajectories as the set of all possible walks along its links. Construction of a good transition graph is, like combinatorics, unexplainable (check page 245). The only way to learn is by some diagrammatic gymnastics, so we recommend that you work your way through the examples, exercises in lieu of plethora of baffling definitions.



example 17.1
p. 315



example 17.2
p. 316



example 17.3
p. 316

The complete unrestricted symbolic dynamics is too simple to be illuminating, so we turn next to the simplest example of pruned symbolic dynamics, the finite subshift obtained by prohibition of repeats of one of the symbols, let us say $_11_$. This situation arises, for example, in a billiard, and in studies of the circle maps, where this kind of symbolic dynamics describes “golden mean” rotations.

exercise 18.6
exercise 18.8



example 17.4
p. 316



example 17.5
p. 317

In the complete N -ary symbolic dynamics case (see example 17.2) the choice of the next symbol requires no memory of the previous ones. However, any further refinement of the state space partition requires finite memory.



example 17.6
p. 317

For M -step memory the only nonvanishing matrix elements are of the form $T_{s_1 s_2 \dots s_{M+1}, s_0 s_1 \dots s_M}$, $s_{M+1} \in \{0, 1\}$. This is a sparse matrix, as the only non vanishing entries in the $a = s_0 s_1 \dots s_M$ column of T_{ba} are in the rows $b = s_1 \dots s_M 0$ and $b = s_1 \dots s_M 1$. If we increase the number of remembered steps, the transition matrix grows large quickly, as the N -ary dynamics with M -step memory requires an $[N^{M+1} \times N^{M+1}]$ matrix. Since the matrix is very sparse, it pays to find a compact representation for T . Such a representation is afforded by transition graphs, which are not only compact, but also give us an intuitive picture of the topological dynamics.

exercise 18.1


17.3 Transition graphs: stroll from link to link

(P. Cvitanović and Matjaž Gomilšek)


What do finite graphs have to do with infinitely long trajectories? To understand the main idea, let us construct an infinite rooted tree graph that explicitly enumerates all possible itineraries. In this construction the nodes are unlabeled, and the links labeled (or colored, or dotted in different ways), signifying different kinds of transitions.

A *tree graph* is an undirected graph (its links have no sense of direction, $j \leftrightarrow i$) in which there exists exactly one path between any two of its nodes. A tree graph is thus connected (irreducible) and contains no loops, i.e., it is not possible to return to any of its nodes by a walk along a sequence of distinct links. A *rooted tree graph* is a directed graph (its links are directed, $j \rightarrow i$), obtained from an undirected tree graph by picking a distinguished node, called the *root*, and orienting all links in the tree so that they point away from the root.

Each node in a directed graph has an *in-degree* (number of links pointing towards it, or the number of ‘parents’), and an *out-degree* (number of links pointing away from it, or the number of ‘children’). An *internal node* has both in- and out-degree ≥ 1 . In a rooted tree graph, all nodes have exactly one parent (in-degree = 1), except for the root, which is the single “parentless” node (in-degree = 0), with all links pointing away from it. An *external node (leaf)* is a “childless” node, with in-degree ≥ 1 , out-degree = 0. We shall refer to a node with known ancestors, but as yet unspecified descendants, as a *free node*.

 example 17.7
p. 317

We illustrate how trees are related to transition graphs by first working out the simplest example of pruned symbolic dynamics, the finite subshift obtained by prohibition of repeats of one of the symbols, let us say $_00_$. As we shall see, for finite grammars a rooted tree (and, by extension, but less obviously, the associated transition graph) is the precise statement of what is meant topologically by a “self-similar” fractal; supplemented by scaling information, such a rooted tree generates a self-similar fractal. Any slightly more complicated grammar merits a full section of its own, here sect. 17.3.1.

 example 17.8
p. 318

17.3.1 Converting pruning blocks into transition graphs

Suppose now that, by hook or crook, you have been so lucky fishing for pruning rules that you now know the grammar (14.16) in terms of a finite set of pruning blocks $\mathcal{G} = \{b_1, b_2, \dots, b_k\}$, of lengths $\leq m$. Our task is to generate all admissible itineraries. What to do?

We have already seen the main ingredients of a general algorithm: (1) the transition graph encodes the self-similarities of the tree of all itineraries, and (2) if



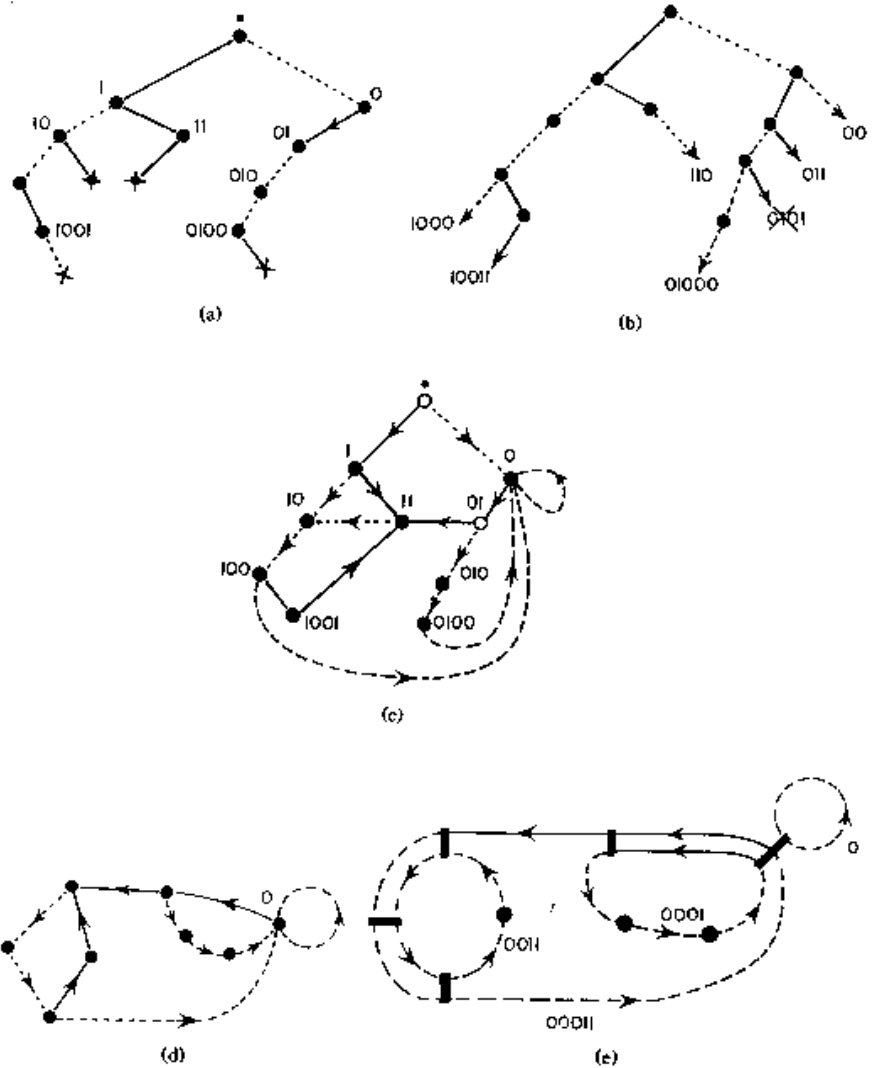


Figure 17.3: Conversion of the pruning front of figure 15.11 (b) into a finite transition graph. (a) Starting with the root node “.”, delineate all pruning blocks on the binary tree. A solid line stands for “1” and a dashed line for “0”. The ends of forbidden strings (i.e., the external nodes) are marked with \times . Label all internal nodes by reading the bits connecting “.”, the root of the tree, to the node. (b) Indicate all admissible starting blocks by arrows. (c) Recursively drop the leading bits in the admissible blocks; if the truncated string corresponds to an internal node in (a), identify them. (d) Delete the transient, non-circulating nodes; all admissible sequences are generated as walks on this finite transition graph. (e) Identify all distinct non-intersecting (products of) loops and construct the determinant (18.32).

we have a pruning block of length m , we need to descend m levels before we can start identifying the self-similar sub-trees.

Finite grammar transition graph algorithm.

1. Starting with the root of the tree, delineate all branches that correspond to all pruning blocks; implement the pruning by removing the last node in each pruning block (marked ‘ \times ’ in figure 17.3 (a)).
2. Label all nodes internal to pruning blocks by the itinerary connecting the root node to the internal node, figure 17.3 (a). Why? So far we have pruned forbidden branches by looking m_b steps into future for a given pruning block, let’s say $b = 10110$. However, the blocks with the right combination of past and future $[1.0110]$, $[10.110]$, $[101.10]$ and $[1011.0]$ are also pruned. In other words, any node whose near past coincides with the beginning of a pruning block is potentially dangerous - a branch further down the tree might get pruned.

3. Add to each remaining node, including the root, all remaining branches allowed by the alphabet, and label their top (free) nodes, figure 17.3 (b). Why? Each one of the free nodes is the beginning point of an infinite tree, a tree that should be similar to another one originating closer to the root of the whole tree.
4. Check that the labels of the newly added free nodes do not themselves contain any pruning blocks. If they do, remove them (marked 'x' in figure 17.3 (b)).
5. Pick one of the remaining free nodes (e.g. closest to the root of the entire tree), forget the most distant symbol in its past. Does the truncated itinerary correspond to an internal node? If yes, identify the two nodes. If not, forget the next symbol in its past, repeat. If no such truncated past corresponds to any internal node, identify with the root of the tree.

This is a little bit abstract, so let's say the free node in question is [1010.]. Three time steps back the past is [010.]. That is not dangerous, as no pruning block in this example starts with 0. Now forget the third step in the past: [10.] is dangerous, as that is the start of the pruning block [10.110]. Hence the free node [1010.] should be identified with the internal node [10.].

6. Repeat until all free nodes have been tied back into internal nodes or the root.
7. Clean up: check whether every node can be reached from every other node. Remove the transient nodes, i.e., the nodes to which dynamics never returns.
8. The result is a transition graph. There is no guarantee that this is the smartest, most compact transition graph possible for a given pruning (if you have a better algorithm, teach us), but walks around it do generate all admissible itineraries, and nothing else.



example 17.9
p. 318

Résumé

The set of all admissible itineraries is generated multiplicatively by transition matrices, diagrammatically by transition graphs. Pruning rules for inadmissible sequences are implemented by constructing corresponding transition matrices and/or transition graphs. These matrices are the simplest examples of evolution operators, prerequisite to developing a theory of averaging over chaotic flows. From our initial chapters 2 to 4 fixation on things local: a representative point, a short-time trajectory, a neighborhood, in this chapter and the next we make a courageous leap, and go global.

Commentary

Remark 17.1 Transition graphs. We enjoyed studying Lind and Marcus [?] introduction to symbolic dynamics and transition graphs. Alligood, Sauer and Yorke [1.10] discussion of baker’s maps, Smale horseshoes and their symbolic dynamics is simple and clear. Finite transition graphs or finite automata are discussed in refs. [18.3, 18.4, 18.5]. They belong to the category of regular languages. Transition graphs for unimodal maps are discussed in refs. [17.8, 27.3, 17.10]. (see also remark 14.1)

Remark 17.2 Inflating transition graphs. In the above examples the symbolic dynamics has been encoded by labeling links in the transition graph. Alternatively one can encode the dynamics by labeling the nodes, as in example 17.6, where the 4 nodes refer to 4 Markov partition regions $\{M_{00}, M_{01}, M_{10}, M_{11}\}$, and the 8 links to the 8 non-zero entries in the 2-step memory transition matrix (17.11).

Remark 17.3 The unbearable growth of transition graphs. A construction of finite Markov partitions is described in refs. [17.11, 17.12, 17.13], as well as in the innumerable many other references.

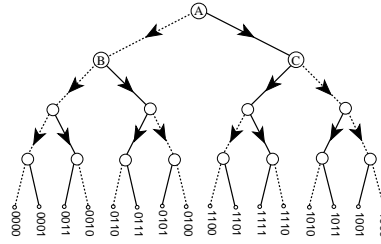
If two regions in a Markov partition are not disjoint but share a boundary, the boundary trajectories require special treatment in order to avoid overcounting, see sect. 25.4.3. If the image of a trial partition region cuts across only a part of another trial region and thus violates the Markov partition condition (14.2), a further refinement of the partition is needed to distinguish distinct trajectories.

The finite transition graph construction sketched above is not necessarily the minimal one; for example, the transition graph of figure 17.3 does not generate only the “fundamental” cycles (see chapter 23), but shadowed cycles as well, such as t_{00011} in (18.32). For methods of reduction to a minimal graph, consult refs. [17.8, 15.48, 27.3]. Furthermore, when one implements the time reversed dynamics by the same algorithm, one usually gets a graph of a very different topology even though both graphs generate the same admissible sequences, and have the same determinant. The algorithm described here makes some sense for 1-dimensional dynamics, but is unnatural for 2-dimensional maps whose dynamics it treats as 1-dimensional. In practice, generic pruning grows longer and longer, and more plentiful pruning rules. For generic flows the refinements might never stop, and almost always we might have to deal with infinite Markov partitions, such as those that will be discussed in sect. 18.5. Not only do the transition graphs get more and more unwieldy, they have the unpleasant property that every time we add a new rule, the graph has to be constructed from scratch, and it might look very different from the previous one, even though it leads to a minute modification of the topological entropy. The most determined effort to construct such graphs may be the one of ref. [15.14]. Still, this is the best technology available, until the day when a reader alerts us to something superior.

17.4 Examples

Example 17.1 Full binary shift. Consider a full shift on two-state partition $\mathcal{A} = \{0, 1\}$, with no pruning restrictions. The transition matrix and the corresponding transi-

Figure 17.5: The self-similarity of the complete binary symbolic dynamics represented by a rooted binary tree: trees originating in nodes B, C, \dots (actually - any node) are the same as the tree originating in the root node A . Level $m = 4$ partition is labeled by 16 binary strings, coded by dotted (0) and full (1) links read down the tree, starting from A . See also figure 14.12.



Example 17.5 ‘Golden mean’ pruning. Consider a subshift on two-state partition $\mathcal{A} = \{0, 1\}$, with the simplest grammar \mathcal{G} possible, a single pruned block $b = _11_$ (consecutive repeat of symbol 1 is inadmissible): the state M_0 maps both onto M_0 and M_1 , but the state M_1 maps only onto M_0 . The transition matrix and the corresponding transition graph are

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{array}{c} \text{---} \textcircled{0} \text{---} \textcircled{1} \text{---} \\ \text{---} \textcircled{1} \text{---} \textcircled{0} \text{---} \end{array} \quad (17.10)$$

Admissible itineraries correspond to walks on this finite transition graph. (continued in example 17.8) [click to return: p. ??](#)

Example 17.6 Finite memory transition graphs. For the binary labeled repeller with complete binary symbolic dynamics, we might chose to partition the state space into four regions $\{M_{00}, M_{01}, M_{10}, M_{11}\}$, a 1-step refinement of the initial partition $\{M_0, M_1\}$. Such partitions are drawn in figure 15.3, as well as figure 1.9. Topologically f acts as a left shift (15.7), and its action on the rectangle $[.01]$ is to move the decimal point to the right, to $[0.1]$, forget the past, $[.1]$, and land in either of the two rectangles $\{[.10], [.11]\}$. Filling in the matrix elements for the other three initial states we obtain the 1-step memory transition matrix/graph acting on the 4-regions partition exercise 14.7

$$T = \begin{bmatrix} T_{00,00} & 0 & T_{00,10} & 0 \\ T_{01,00} & 0 & T_{01,10} & 0 \\ 0 & T_{10,01} & 0 & T_{10,11} \\ 0 & T_{11,01} & 0 & T_{11,11} \end{bmatrix} = \begin{array}{c} \text{---} \textcircled{00} \text{---} \textcircled{10} \text{---} \textcircled{11} \text{---} \\ \text{---} \textcircled{01} \text{---} \textcircled{10} \text{---} \textcircled{11} \text{---} \\ \text{---} \textcircled{00} \text{---} \textcircled{01} \text{---} \end{array} \quad (17.11)$$

(continued in example 18.7) [click to return: p. ??](#)

Example 17.7 Complete binary topological dynamics. Mark a dot ‘.’ on a piece of paper. That will be the root of our tree. Draw two short directed lines out of the dot, end each with a dot. The full line will signify that the first symbol in an itinerary is ‘1,’ and the dotted line will signifying ‘0.’ Repeat the procedure for each of the two new dots, and then for the four dots, and so on. The result is the binary tree of figure 17.5. Starting at the top node, the tree enumerates exhaustively all distinct finite itineraries of lengths $n = 1, 2, 3, \dots$

$$\{0, 1\} \quad \{00, 01, 10, 11\} \\ \{000, 001, 010, 011, 100, 101, 111, 110\} \dots$$

The $n = 4$ nodes in figure 17.5 correspond to the 16 distinct binary strings of length 4, and so on. By habit we have drawn the tree as the alternating binary tree of figure 14.12, but that has no significance as far as enumeration of itineraries is concerned

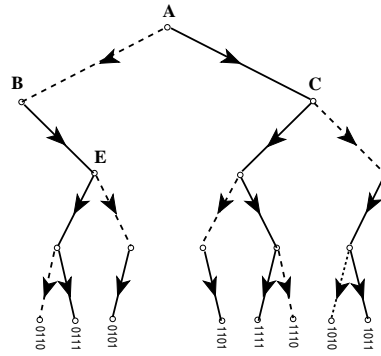


Figure 17.6: The self-similarity of the $_00_$ pruned binary tree: trees originating from nodes C and E are the same as the entire tree.

- a binary tree with labels in the natural order, as increasing binary 'decimals' would serve just as well.

The trouble with an infinite tree is that it does not fit on a piece of paper. On the other hand, we are not doing much - at each node we are turning either left or right. Hence all nodes are equivalent. In other words, the tree is self-similar; the trees originating in nodes B and C are themselves copies of the entire tree. The result of identifying $B = A$, $C = A$ is a single node, 2-link transition graph with adjacency matrix (17.2)

$$A = \begin{bmatrix} 2 \end{bmatrix} = \text{graph with node } A=B=C \text{ and two self-loops.} \quad (17.12)$$

An itinerary generated by the binary tree figure 17.5, no matter how long, corresponds to a walk on this graph. This is the most compact encoding of the complete binary symbolic dynamics. Any number of more complicated transition graphs such as the 2-node (17.6) and the 4-node (17.11) graphs generate all itineraries as well, and might sometimes be preferable.

exercise 18.6
exercise 18.5
click to return: p. ??

Example 17.8 'Golden mean' pruning. (a link-to-link version of example 17.5) Now the admissible itineraries are enumerated by the pruned binary tree of figure 17.6. Identification of nodes $A = C = E$ leads to the finite 2-node, 3-links transition graph

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \text{graph with nodes } B \text{ and } A=C=E \text{ and three links.} \quad (17.13)$$

As 0 is always followed by 1, the walks on this graph generate only the admissible itineraries. This is the same graph as the 2-node graph (17.10), with full and dotted lines interchanged. (continued in example 18.4)

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Example 17.9 Heavy pruning.

We complete this training by examples by implementing the pruning of figure 15.11 (b). The pruning blocks are

$$[100.10], [10.1], [010.01], [011.01], [11.1], [101.10]. \quad (17.14)$$


Blocks 01101, 10110 contain the forbidden block 101, so they are redundant as pruning rules. Draw the pruning tree as a section of a binary tree with 0 and 1 branches and

label each internal node by the sequence of 0's and 1's connecting it to the root of the tree, figure 17.3 (a). These nodes are the potentially dangerous nodes - beginnings of blocks that might end up pruned. Add the side branches to those nodes, figure 17.3 (b). As we continue down such branches we have to check whether the pruning imposes constraints on the sequences so generated: we do this by knocking off the leading bits and checking whether the shortened strings coincide with any of the internal pruning tree nodes: $00 \rightarrow 0$; $110 \rightarrow 10$; $011 \rightarrow 11$; $0101 \rightarrow 101$ (pruned); $1000 \rightarrow 00 \rightarrow 00 \rightarrow 0$; $10011 \rightarrow 0011 \rightarrow 011 \rightarrow 11$; $01000 \rightarrow 0$.

The trees originating in identified nodes are identical, so the tree is "self-similar." Now connect the side branches to the corresponding nodes, figure 17.3 (d). Nodes "." and 1 are transient nodes; no sequence returns to them, and as you are interested here only in infinitely recurrent sequences, delete them. The result is the finite transition graph of figure 17.3 (d); the admissible bi-infinite symbol sequences are generated as all possible walks on this graph.

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Exercises

17.1. **Time reversibility.**  Hamiltonian flows are time reversible. Does that mean that their transition graphs are symmetric in all node \rightarrow node links, their transition matrices are adjacency matrices, symmetric and diagonalizable, and that they have only real eigenvalues?

17.2. **Alphabet $\{0,1\}$, prune $_1000_, _00100_, _01100_.$** This example is motivated by the pruning front description of the symbolic dynamics for the Hénon-type map - remark 15.3.

step 1. $_1000_$ prunes all cycles with a $_000_$ subsequence with the exception of the fixed point $\bar{0}$; hence we factor out $(1 - t_0)$ explicitly, and prune $_000_$ from the rest. This means that x_0 is an isolated fixed point - no cycle stays in its vicinity for more than 2 iterations. In

the notation of sect. 17.3.1, the alphabet is $\{1, 2, 3; \bar{0}\}$, and the remaining pruning rules have to be rewritten in terms of symbols $2=10, 3=100$:

step 2. alphabet $\{1, 2, 3; \bar{0}\}$, prune $_33_, _213_, _313_.$ This means that the 3-cycle $\bar{3} = \bar{100}$ is pruned and no long cycles stay close enough to it for a single $_100_$ repeat. Prohibition of $_33_$ is implemented by dropping the symbol "3" and extending the alphabet by the allowed blocks 13, 23:

step 3. alphabet $\{1, 2, \underline{13}, \underline{23}; \bar{0}\}$, prune $_213_, _23 \underline{13}_, _13 \underline{13}_.$ where $\underline{13} = 13, \underline{23} = 23$ are now used as single letters. Pruning of the repetitions $_13 \underline{13}_$ (the 4-cycle $\bar{13} = \bar{1100}$ is pruned) yields the

result: alphabet $\{1, 2, \underline{23}, \underline{113}; \bar{0}\}$, unrestricted 4-ary dynamics. The other remaining possible blocks $_213_, _2313_$ are forbidden by the rules of step 3.

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